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The h-vector of coned graphs

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The coned graph $\hat{G}$ on a finite graph G is obtained by joining each vertex of G to a new vertex p with a simple edge. In this work we show a combinatorial interpretation of each term in the h-vector of $\hat{G}$ in terms of partially edge-rooted forests in the base graph G. In particular, our interpretation does not require edge ordering. For an application, we will derive an exponential generating function for the sequence of h-polynomials for the complete graphs. We will also give a new proof for the number of spanning trees of the wheels.

1. Introduction

The f-polynomial of a finite graph G with $n+1$ vertices is defined by

$$f_G(\lambda) = f_0\lambda^n + f_1\lambda^{n-1} + \cdots + f_n,$$

where each f_i is the number of spanning forests in G with i edges. When G is connected, f_n is positive and equals the number of spanning trees in G.

An interesting variation of $f_G(\lambda)$ is the h-polynomial of G, denoted as $h_G(x)$:

$$h_G(x) = f_G(x-1) = h_0x^n + h_1x^{n-1} + \cdots + h_n.$$

Define the *h-vector* of G to be the sequence $(h_0, h_1, \ldots, h_n)$. In matroid theoretic terms, it is the h-vector of the independent set complex of the cycle matroid of G. It has the obvious property $\sum_{i\geq 0} h_i = f_n$. However, the meaning of h_i's is not as plain as that of f_i's. The main goal of this work is to give a simple combinatorial interpretation of the terms in the h-vector for *coned graphs* (see Section 2 for the definition.) For example, the complete graphs K_{n+1} and the wheels W_{n+1} are coned graphs on K_n and C_n circuits of length n, respectively.

The following well-known formula for $h_G(x)$ provides a classical interpretation of the h-vector of a graph G which is important for our purpose:

$$h_G(x) = \sum_T x^{i(T)},$$

where the sum is over all spanning trees T in G and $i(T)$ is the number of the internally active edges in T with respect to a given ordering of the edges of G (see Section 2). This is a consequence of the definition of the Tutte polynomial $t_G(x, y)$ via basis activities and the identity $t_G(x, 1) = f_G(x-1)$ [1,2]. Now we have the following interpretation of the h-vectors for graphs.

Theorem 1. *Let G be a graph with $n+1$ vertices and a linear ordering ω of the edges. Let $(h_0, h_1, \ldots, h_n)$ denote its h-vector. Then each h_i is the number of spanning trees in G with $n-i$ internally active edges with respect to ω.*

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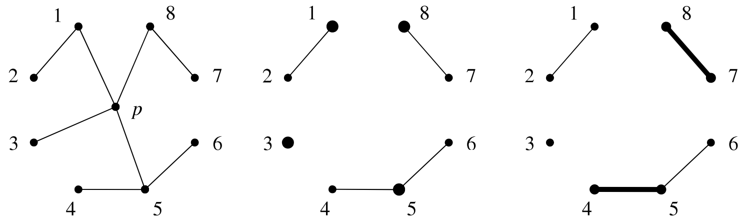

(a) Spanning tree. (b) Rooted forest. (c) Partially edge-rooted forest.

Fig. 1. The spanning tree (a) in $W_9 = \hat{C}_8$ corresponds to a rooted forest (b) in C_8, which in turn corresponds to a partially edge-rooted forest (c) in C_8.

Our goal is to find the meaning of each h_i without edge ordering for coned graphs. For example, the identities $h_G(0) = f_G(-1) = \sum_{i=0}^{n}(-1)^{n-i}f_i$ imply that h_n is the signless reduced Euler characteristic of the independent set complex of the cycle matroid of G [3]. Hence, it was shown that for the complete graph K_{n+1} with $n+1$ vertices, h_n is the number of *edge-rooted forests* in K_n [4], where an edge-rooted forest is a spanning forest with exactly one edge in each component marked as an edge-root. More generally, if $\hat{G}$ is the coned graph on G, then $h_{\hat{G}}(0)$ equals the number of edge-rooted forests in G [4,5]. Note that the notion of edge-rooted forests does not require edge ordering.

The main result (Theorem 4) of this work will extend these results to give a new interpretation of the terms in the h-vector of a coned graph $\hat{G}$ via *partially edge-rooted forests* in the base G (see Section 2). For application, we will derive an exponential generating function for the sequence of h-polynomials of the complete graphs $K_{n+1} (n \geq 0)$. We will also give a new proof for the number of spanning trees in W_{n+1} being $L_{2n} - 2$, where L_{2n} is the 2nth Lucas number.

2. Partially edge-rooted forests and main results

In the work we assume that a finite graph G is loopless and undirected with vertex set $V(G) = [n] = \{1, \ldots, n\}$. We allow multiple edges in G, and $E(G)$ denotes the multiset of its edges. A coned graph $\hat{G}$ on the base G is obtained by adding a new vertex p, called the cone point, and a simple edge pv for each vertex v in G. Hence $V(\hat{G}) = V(G) \cup \{p\}$ and $E(\hat{G}) = E(G) \cup \{pv | v \in V(G)\}$.

Given a spanning tree T in a coned graph $\hat{G}$, define the *support* of T to be $T \cap G$. Note that $T \cap G$ is a spanning forest in G, and there is exactly one vertex in each component, called a *connecting vertex*, that is adjacent to p in T. Regarding the connecting vertices as roots, we observe that the spanning trees in $\hat{G}$ correspond bijectively to the rooted spanning forests in G. (Refer to (a) and (b) in Fig. 1.)

A tree is called *simple* if it is not rooted. A forest is called simple if every component is. A tree with at least one edge is called *edge-rooted* if exactly one edge is marked as an edge-root. For $s, r \geq 0$, a *partially edge-rooted forest of type* (s, r), or simply a *forest of type* (s, r), in a graph G is a spanning forest in G with s simple components and r edge-rooted components. Hence, a forest of type (s, r) has a total of $s + r$ components. If $s = 0$ and $r \geq 1$, it is called an edge-rooted forest.

Theorem 2. *The set of all spanning trees in a coned graph $\hat{G}$ corresponds bijectively to the set of all partially edge-rooted forests in the base G.*

Proof. It suffices to show that a rooted forest F in G corresponds uniquely to a partially edge-rooted forest in G, and vice versa. Assume that the vertices of G are ordered. Now, if a rooted vertex v in F is the smallest vertex in its component C_v, then we replace it by a simple vertex, which makes C_v a simple component. If a rooted vertex v is not the smallest vertex in C_v, then we turn C_v into an edge-rooted component with the edge-root being the first edge in the unique path from v to the smallest vertex in C_v. The result is a partially edge-rooted forest in G that corresponds to F. (Refer to (b) and (c) in Fig. 1.) Similarly, this correspondence can be easily reversed, mapping a partially edge-rooted forest in G to a unique rooted forest in G. □

Next, we will review the notion of internal activity for a spanning tree in a graph G [2]. Suppose that a linear ordering ω of the edges in G is given. Deleting an edge e from a spanning tree T in G creates a forest with two components $T \setminus e = T_1 \cup T_2$. The *basic bond* of e with respect to T is the set $E_G(T_1, T_2)$ of all edges in G with one vertex in T_1 and the other in T_2. In particular, e is always in its own basic bond. The edge $e \in T$ is *internally active* if e is ω-smallest in its basic bond. The internal activity of T is the number of internally active edges in T.

For a coned graph $\hat{G}$, let its vertices be ordered, $p < 1 < 2 < \cdots < n$. Note that each edge in the star $\{pv | v \in [n]\}$ of p is simple. If a pair $\{i, j\} \subset [n]$ induces parallel edges, fix an ordering $ij_1 < ij_2 < \cdots$ of those edges. Now let ω be the resulting "lexicographic" ordering of the edges in $\hat{G}$:

$$p1 <_\omega p2 <_\omega \cdots <_\omega pn <_\omega \cdots <_\omega ij_1 <_\omega ij_2 <_\omega \cdots.$$

Hence every edge in G is ω-larger then any edge in the star of p.

Lemma 3. *An edge e in a spanning tree T in $\hat{G}$ is internally active with respect to ω iff e is in the star of p and its connecting vertex $v \in T \cap G$ is the smallest vertex in its component C_v in $T \cap G$.*

Proof. For any spanning tree T in $\hat{G}$, the basic bond of any edge $e \in T$ contains some edge that is incident to p. Hence, no edge in $T \cap G$ is internally active. Now let $C_1, \ldots, C_d$ be the components in $T \cap G$, where d is the degree of p in T. Let v_i be the connecting root in C_i so that pv_i is an edge in T for each i. Now an edge pv is in the basic bond of pv_i iff v is a vertex in $V(C_i)$. Hence, pv_i is ω-smallest in its basic bond iff v_i is the smallest vertex in $V(C_i)$. □

The following is the main result of the work.

Theorem 4. *Suppose G is a graph with n vertices. Let $(h_0, h_1, \ldots, h_n)$ denote the h-vector of its coned graph $\hat{G}$. For each $0 \le s \le n$, the term h_{n-s} equals the number of partially edge-rooted forests in G with exactly s simple components.*

Proof. It is clear from Theorem 1 and Lemma 3 that h_{n-s} counts the rooted spanning forests in G such that exactly s of the roots are the smallest vertices in their respective components. From the proof of Theorem 2, it is also clear that these rooted forests correspond to partially edge-rooted forests in G with exactly s simple components. Hence the result. □

3. Examples and applications

3.1. Complete graphs

As an application of Theorem 4 to complete graphs, we will derive an exponential generating function for the sequence $(h_{K_{n+1}}(x))$ of h-polynomials of $K_{n+1} (n \ge 0)$ as a coned graph on K_n. We refer the reader to [6] for exponential generating functions and necessary operations. Let

$$T(y) = \sum_{m \ge 1} m^{m-2} \frac{y^m}{m!} \quad \text{and} \quad R(y) = \sum_{m \ge 1} (m-1) m^{m-2} \frac{y^m}{m!}.$$

Note that m^{m-2} is the number of spanning trees and $(m-1)m^{m-2}$ the number of edge-rooted trees both on m vertices.

Theorem 5. $\sum_{n \ge 0} h_{K_{n+1}}(x) \frac{y^n}{n!} = \exp(xT(y) + R(y))$.

Proof. For $n, s \ge 0$, let $h_{n,s}$ denote the $(n-s)$th term in the h-vector of K_{n+1} where we define $h_{n,s} = 0$ for $n < s$. By fixing $s \ge 0$ and letting n vary, we get a sequence $(h_{n,s})$ for $n \ge 0$. By Theorem 4, an exponential generating function for the sequence $(h_{n,s})$ is given by

$$\sum_{n \ge 0} h_{n,s} \frac{y^n}{n!} = (T(y)^s / s!) \exp(R(y)).$$

Therefore we have

$$\begin{aligned} \sum_{n \ge 0} h_{K_{n+1}}(x) \frac{y^n}{n!} &= \sum_{n \ge 0} \sum_{s \ge 0} h_{n,s} x^s \frac{y^n}{n!} \\ &= \sum_{s \ge 0} x^s \sum_{n \ge 0} h_{n,s} \frac{y^n}{n!} \\ &= \sum_{s \ge 0} x^s (T(y)^s / s!) \exp(R(y)) \\ &= \exp(xT(y)) \exp(R(y)). \quad \square \end{aligned}$$

3.2. Wheels

A wheel W_{n+1} of order $n+1$ is the coned graph on the circuit C_n of order n. We will derive a formula for each term in its h-vector via *partial matchings* in C_n. Recall that a partial matching in a graph is a collection of disjoint non-loop edges in the graph including the empty collection. Let $g(m, r)$ denote the number of partial matchings of cardinality r in C_m for $m \ge 1$. One can show that $g(m, r) = \frac{m}{m-r} \binom{m-r}{r}$. This is also the number of ways to pick r non-consecutive objects from m objects that are arranged in a circle. Define $g(0, 0) = 0$.

Theorem 6. *Let $(h_0, h_1, \ldots, h_n)$ be the h-vector of W_{n+1}. For each $0 \le s \le n$,*

$$h_{n-s} = \sum_{r \ge 0} \binom{n}{s+2r} g(s+2r, r).$$

Proof. We claim that the number of the forests of type (s, r) in C_n is

$$\binom{n}{s+2r} g(s+2r, r).$$

Then the theorem follows by Theorem 4. Suppose that the edges in C_n are ordered counterclockwise (or clockwise). A partially edge-rooted forest in C_n is determined by a pair of disjoint subsets D and R of $E(C_n)$, where D consists of the edges deleted from C_n, creating a forest with $|D|$ components, and R the edges that are marked as edge-roots. Furthermore, the union $M = D \cup R$ must satisfy the condition that there is at least one element in D between any two elements in R. Otherwise there will be a component with two edge-roots, which is impossible.

Since a forest of type (s, r) in C_n has $s + r$ components and r edge-roots, it corresponds to a disjoint pair (D, R) with $|D| = s + r$ and $|R| = r$ satisfying the above condition for $M = D \cup R$. Equivalently, it corresponds to a pair of subsets $R \subset M \subset E(C_n)$ with $|M| = s + 2r$ and $|R| = r$ such that no two consecutive elements from M are in R. Since there are $\binom{n}{s+2r}$ ways to choose M from $E(C_n)$ and $g(s + 2r, r)$ ways to choose R from M, the claim follows. □

Example. We have $h_n = \sum_{r\geq 0} \binom{n}{2r} g(2r, r) = \sum_{r\geq 1} 2\binom{n}{2r} = 2(2^{n-1} - 1)$.

We will apply this theorem to give a new proof for a formula of the number of spanning trees in W_{n+1}, which we denote by $\tau(W_{n+1})$. To do this, we will need the following facts concerning the Lucas numbers. They are defined by the recursions $L_m = L_{m-1} + L_{m-2}$ for $m \geq 2$ with $L_0 = 2$ and $L_1 = 1$. Also, L_m for $m \geq 1$ is the total number of partial matchings in C_m. Since $g(m, r)$ is the number of partial matchings of cardinality r in C_m, we have $L_m = \sum_{r\geq 0} g(m, r)$ for $m \geq 1$.

Corollary 7. *$\tau(W_{n+1}) = L_{2n} - 2$ for $n \geq 1$.*

Proof. Since $\sum_{s=0}^{n} h_{n-s}$ equals the number of spanning trees, we have

$$\begin{aligned}
\tau(W_{n+1}) = \sum_{s=0}^{n} h_{n-s} &= \sum_{s=0}^{n} \sum_{r\geq 0} \binom{n}{s+2r} g(s+2r, r) \\
&= \sum_{m\geq 1} \sum_{r\geq 0} \binom{n}{m} g(m, r) \\
&= \sum_{m\geq 1} \binom{n}{m} L_m = L_{2n} - L_0,
\end{aligned}$$

where the second equality is by Theorem 6, and the third uses the change of variable $m = s + 2r$ and the fact $g(0, 0) = 0$ by definition. The last equality is a simple consequence of repeated applications of the recursions $L_m = L_{m-1} + L_{m-2}$ for $m \geq 2$. Since $L_0 = 2$, the result follows. □

For a bijective proof of this corollary, refer to [7]. We also wish to remark that this corollary can be seen as a direct consequence of Theorem 2 by an argument similar to that in the proof of Theorem 6. We omit the details.

3.3. Fans

The *fan* of order $n + 1$, denoted as Fan_{n+1}, is the coned graph on the path P_n with n vertices (and hence $n - 1$ edges). Most of the discussion concerning Fan_{n+1} is as regards a "linearization" of W_{n+1}. For example, instead of the Lucas numbers, we need the Fibonacci numbers defined by the recursions $F_m = F_{m-1} + F_{m-2}$ for $m \geq 2$ with $F_0 = F_1 = 1$. Also, F_m is the total number of partial matchings in P_m for $m \geq 1$. If we let $f(m, r)$ denote the number of partial matchings of cardinality r in P_{m+1}, then we have $F_{m+1} = \sum_{r\geq 0} f(m, r)$ for $m \geq 0$. Note that $f(0, 0) = 1$, and we define $f(m, r) = 0$ for $m < 0$.

Using these facts, one can show the following results concerning Fan_{n+1}.

Theorem 8. *Let $(h_0, h_1, \ldots, h_n)$ be the h-vector of Fan_{n+1}. For each $0 \leq s \leq n$,*

$$h_{n-s} = \sum_{r\geq 0} \binom{n-1}{s+2r-1} f(s+2r-1, r).$$

Corollary 9. *$\tau(\mathrm{Fan}_{n+1}) = F_{2n-1}$ for $n \geq 1$.*

The proofs of these are similar to those of Theorem 6 and Corollary 7 except for the use of F_m and $f(m, r)$ in place of L_m and $g(m, r)$, respectively. The details are omitted.

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