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Uniquely pairable graphs

Zhongyuan Che

Department of Mathematics, Penn State University, Beaver Campus, Monaca, PA 15061, USA

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Abstract

The concept of a k-pairable graph was introduced by Z. Chen [On k-pairable graphs, Discrete Mathematics 287 (2004), 11–15] as an extension of hypercubes and graphs with an antipodal isomorphism. In the present paper we generalize further this concept of a k-pairable graph to the concept of a semi-pairable graph. We prove that a graph is semi-pairable if and only if its prime factor decomposition contains a semi-pairable prime factor or some repeated prime factors. We also introduce a special class of k-pairable graphs which are called uniquely k-pairable graphs. We show that a graph is uniquely pairable if and only if its prime factor decomposition has at least one pairable prime factor, each prime factor is either uniquely pairable or not semi-pairable, and all prime factors which are not semi-pairable are pairwise non-isomorphic. As a corollary we give a characterization of uniquely pairable Cartesian product graphs.
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1. Introduction

All graphs considered in this paper are finite, connected and simple. The set of vertices of a graph G is denoted as $V(G)$; the distance between two vertices x and y in graph G is denoted as $d_G(x, y)$ or simply as $d(x, y)$ if it causes no confusion; x adj y is used to mean that x is adjacent to y.

An *involution* of a set X is a bijection $f : X \to X$ such that $f(f(x)) = x$ for all $x \in X$. The *support* of an involution f is a subset S of X such that an element x of X is contained in S if and only if x is not fixed by f. We say that an involution is *nontrivial* if its support is not empty. An *isomorphism* between two graphs G and H is a bijection f from $V(G)$ to $V(H)$ such that u adj v in G if and only if $f(u)$ adj $f(v)$ in H for any vertices u and v of G. If there is an isomorphism between graphs G and H, then we say that G and H are *isomorphic*, and write $G \cong H$. Otherwise, we say that G and H are *non-isomorphic*, and write $G \ncong H$. An *automorphism* of a graph G is an isomorphism from G onto itself. The set of all automorphisms of a graph G forms a group, which is called the *automorphism group* of G.

In [2], Chen introduced the concept of a k-pairable graph as an extension of hypercubes and graphs with an antipodal isomorphism. Here we give an equivalent definition of a k-pairable graph.

E-mail address: zxc10@psu.edu.

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Definition 1.1 (*[2]*). Let k be a positive integer. A graph G is said to be *k-pairable* if its automorphism group contains an involution ϕ with support $V(G)$ such that $d(x, \phi(x)) \geq k$ for any $x \in V(G)$.

It was proved [2] that if G is a k-pairable graph ($k > 1$), then for every spanning tree T of G, there exists an edge e of G outside T whose addition to T forms a cycle of length at least $2k$.

Definition 1.2 (*[2]*). The *pair length* of a graph G, denoted as $p(G)$, is the maximum k such that G is k-pairable; $p(G) = 0$ if G is not k-pairable for any positive integer k.

It was shown [1,2] that any tree has pair length either 0 or 1. A characterization of 1-pairable trees was given in [1]: a tree T has $p(T) = 1$ if and only if there is an edge $e = xy$ of T such that there exists an isomorphism f between the two connected components of $T - e$ satisfying $f(x) = y$.

The *Cartesian product* of two graphs G and H, denoted as $G \Box H$, has the vertex set $V(G) \times V(H)$ and (g_1, h_1) adj (g_2, h_2) if either $g_1 = g_2$ in G and h_1 adj h_2 in H or g_1 adj g_2 in G and $h_1 = h_2$ in H. It is well known [5] that the Cartesian product is associative and commutative, and that $G \Box H$ is connected if and only if both G and H are connected.

Theorem 1.3 (*[3]*). *For any graphs G and H, $p(G \Box H) = p(G) + p(H)$.*

In this paper, we first generalize the concept of a k-pairable graph to the concept of a semi-pairable graph.

Definition 1.4. A graph G is said to be *semi-pairable* if its automorphism group contains a nontrivial involution.

When there is no confusion, we may simply call a k-pairable graph a *pairable graph*. By definition, any pairable graph is semi-pairable, but not vice versa. For example, a path with $2n + 1$ vertices is semi-pairable but not pairable. If a graph G is not semi-pairable, then it is not pairable, and so $p(G) = 0$.

A graph is called *prime* if it cannot be written as a Cartesian product of nontrivial graphs. It is well known [5] that for each connected graph G, there is a decomposition of G into prime factors with respect to the Cartesian product, and the decomposition is unique up to the order of the prime factors, that is, $G = G_1 \Box G_2 \Box \cdots \Box G_n$, where G_i's are prime factors of G and they are unique up to the order of G_i's. In this paper, we prove that a graph is semi-pairable if and only if its prime factor decomposition contains a semi-pairable prime factor or some repeated prime factors.

Secondly, we introduce a special class of k-pairable graphs, which are called uniquely k-pairable graphs.

Definition 1.5. A graph G is *uniquely k-pairable* if $p(G) = k > 0$ and its automorphism group contains exactly one involution ϕ with support $V(G)$ such that $d(x, \phi(x)) \geq k$ for any $x \in V(G)$.

For example, any path P_{2n} with $2n$ vertices is uniquely 1-pairable, and any hypercube Q_n is uniquely n-pairable.

Note that:

(1) When we say that G and H are pairable graphs, it means that both $p(G)$ and $p(H)$ are positive, but it is not necessary for them to be the same.

(2) When we say that a graph G is uniquely pairable, it means that G is uniquely k-pairable where $k = p(G) > 0$.

It is easy to see that the Cartesian product graph $P_{2m} \Box P_{2n}$ of two uniquely pairable graphs P_{2m} and P_{2n} is still uniquely pairable. It is natural to ask the following questions: Is it true that the set of uniquely pairable graphs is closed under Cartesian product operation? On the other hand, if $G \Box H$ is uniquely pairable, is it necessary that both G and H are uniquely pairable?

In this paper, we show that a graph is uniquely pairable if and only if its prime factor decomposition has at least one pairable prime factor, each prime factor is either uniquely pairable or not semi-pairable, and all prime factors which are not semi-pairable are pairwise non-isomorphic. Then the answers to the above questions follow immediately.

2. Preliminaries

In this section, we give some basic results to be used in the proof of our main results. The key tool we use here is the following theorem given by Sabidussi [6].

Theorem 2.1 ([6]). *Let ϕ be an automorphism of a connected graph G with the prime factor decomposition $G = G_1 \Box G_2 \Box \cdots \Box G_n$. Then there exists a permutation π of $\{1, 2, \ldots, n\}$ together with isomorphisms $\psi_i : G_i \to G_{\pi i}$ such that*

$$\phi(v_1, v_2, \ldots, v_n) = (\psi_{\pi^{-1}1} v_{\pi^{-1}1}, \psi_{\pi^{-1}2} v_{\pi^{-1}2}, \ldots, \psi_{\pi^{-1}n} v_{\pi^{-1}n}),$$

where $v_i \in V(G_i)$, $1 \le i \le n$.

Therefore, the automorphism group of a connected graph G with a prime factor decomposition is generated by automorphisms and transpositions of the prime factors.

Given a connected Cartesian product graph $G \Box H$, for a vertex h of H, we use $G \Box \{h\}$ to denote the induced subgraph of $G \Box H$ generated by the set $\{(g, h) : g \in V(G)\}$ and call it the *G-layer* at position h. Similarly, for a vertex g of G, we use $\{g\} \Box H$ to denote the induced subgraph generated by the set $\{(g, h) : h \in V(H)\}$ and call it the *H-layer* at position g.

Assume that $G \Box H = (S_1 \Box S_2 \Box \cdots \Box S_m) \Box (S_{m+1} \Box S_{m+2} \Box \cdots \Box S_n)$ is the prime factor decomposition of $G \Box H$, where $G = S_1 \Box S_2 \Box \cdots \Box S_m$ is the prime factor decomposition of G and $H = S_{m+1} \Box S_{m+2} \Box \cdots \Box S_n$ is the prime factor decomposition of H for some integers $m \ge 1$ and $n > m$. Let f be an automorphism of $G \Box H$. By Theorem 2.1, there is a permutation π of $\{1, 2, \ldots, n\}$ together with isomorphisms $\psi_i : S_i \to S_{\pi i}$ such that $f(v_1, v_2, \ldots, v_n) = (\psi_{\pi^{-1}1} v_{\pi^{-1}1}, \psi_{\pi^{-1}2} v_{\pi^{-1}2}, \ldots, \psi_{\pi^{-1}n} v_{\pi^{-1}n})$.

If a G-layer is mapped onto a G-layer by f, then the permutation $\pi = \pi_G \circ \pi_H$ where π_G is a permutation of $\{1, 2, \ldots, m\}$ and π_H is a permutation of $\{m + 1, m + 2, \ldots, n\}$. This implies that each G-layer is mapped onto a G-layer, and each H-layer is mapped onto an H-layer by f. If a G-layer is mapped into an H-layer by f, then the permutation π sends $\{1, 2, \ldots, m\}$ into $\{m + 1, m + 2, \ldots, n\}$. This implies that each G-layer is mapped into an H-layer by f. If a G-layer is neither mapped onto any G-layer nor mapped into any H-layer by f, then it is trivial that G is not prime.

Therefore, if f is an automorphism of $G \Box H$, then one of the following three cases must occur.

- Case 1. Each G-layer is mapped onto a G-layer, and each H-layer is mapped onto an H-layer by f.
- Case 2. Each G-layer is mapped onto an H-layer by f.
- Case 3. A G-layer is neither mapped onto any G-layer nor mapped onto any H-layer by f, and G is not prime.

Now we consider pairable graphs and semi-pairable graphs. By definition, a graph G is k-pairable if $V(G)$ admits a partition $\mathcal{P}$ into parts each with size 2 such that (i) the involution ϕ which interchanges two vertices in each part of $\mathcal{P}$ is an automorphism of G, and (ii) $d(x, \phi(x)) \ge k$ for each part of $\mathcal{P}$. The above partition $\mathcal{P}$ of $V(G)$ is called a *k-pair partition* of G, and $x' (= \phi(x))$ is called the *mate* of x for each $x \in V(G)$.

A graph G is semi-pairable if $V(G)$ admits a partition $\mathcal{P}$ into parts each with size 1 or 2 such that (i) at least one part of $\mathcal{P}$ has size 2, and (ii) the involution ϕ which interchanges the two vertices in each part of $\mathcal{P}$ of size 2 and fixes the vertex in each part of size 1 is an automorphism of G. The above partition $\mathcal{P}$ of $V(G)$ is called a *semi-pair partition* of G, and $x' (=\phi(x))$ is called the *semi-mate* of x. If $x = \phi(x)$, then x is said to be the semi-mate of itself.

Let $\mathcal{P}$ be an arbitrary semi-pair partition of a graph G. Then there is an induced automorphism f of G that maps each vertex x of G to its semi-mate x', i.e., $f(x) = x'$ and $f(x') = x$. If $\mathcal{P}$ has a part with just one vertex x, then x is fixed by f. It is clear that f cannot be the identity map of G because there are at least two vertices of G not fixed by f.

Assume that f is an automorphism of $G \Box H$ induced by a semi-pair partition of $G \Box H$. We will show that if there is a G-layer mapped onto a different G-layer by f, then H is semi-pairable. In particular, if each G-layer is mapped onto a different G-layer by f, then H is pairable. Moreover, we show that if there is a G-layer mapped into an H-layer by f and f is induced by a k-pair partition of $G \Box H$, then H is k-pairable.

Lemma 2.2. *Let f be an automorphism of $G \Box H$ such that a G-layer is mapped onto a G-layer by f. Then*

(i) *the map $f_H : V(H) \to V(H)$ defined by $f_H(h) = \overline{h}$ if $f(G \Box \{h\}) = G \Box \{\overline{h}\}$ is an automorphism of H;*

(ii) *if f is induced by a semi-pair partition of $G \Box H$ and there is a G-layer that is mapped onto a different G-layer by f, then H is semi-pairable. In particular, if each G-layer is mapped onto a different G-layer by f, then the map f_H corresponds to a k_H-pair partition of H where $k_H = \min_{h \in V(H)} d(h, \overline{h}) > 0$ and H is k_H-pairable.*

Proof. (i) It is trivial by Theorem 2.1. We only need to prove (ii). Assume that f is induced by a semi-pair partition of $G\Box H$. Then $f_H(h)=\overline{h}$ if and only if $f_H(\overline{h})=h$ since $f(G\Box\{h\})=G\Box\{\overline{h}\}$ if and only if $f(G\Box\{\overline{h}\})=G\Box\{h\}$. If there is a G-layer that is mapped onto a different G-layer, then f_H does not fix all vertices of H. Hence f_H corresponds to a semi-pair partition of H and so H is semi-pairable. If each G-layer is mapped onto a different G-layer, then $k_H=\min_{h\in V(H)} d(h,\overline{h})>0$. This implies that f_H corresponds to a k_H-pair partition of H and so H is k_H-pairable. $\Box$

Remark. By the commutativity of the Cartesian product, Lemma 2.2 also holds when the roles of G and H are interchanged.

A *homomorphism* from a graph G to a graph H is a mapping from $V(G)$ to $V(H)$ that preserves the adjacency between vertices. In the proof of the following lemma, we will use the well known fact [4] that a homomorphism from a graph to itself is an automorphism if and only if it is one-to-one.

Lemma 2.3. *Let f be an automorphism of $G\Box H$ induced by a k-pair partition of $G\Box H$. If a G-layer is mapped into an H-layer by f, then $p(H)\geq k$.*

Proof. If there is a G-layer that is mapped into an H-layer by f, then by Theorem 2.1, each G-layer is mapped into an H-layer by f. Let $G\Box\{h\}$ be an arbitrary G-layer and assume that $f(G\Box\{h\})\subseteq\{g_h\}\Box H$ where $g_h\in V(G)$ is uniquely determined by h. Let $\overline{h}$ be the vertex of H such that $(g_h,\overline{h})=f(g_h,h)$. Then $\overline{h}$ is uniquely determined by h. So, we can define a mapping $f_H: V(H)\to V(H)$ by letting $f_H(h)=\overline{h}$ if $f(G\Box\{h\})\subseteq\{g_h\}\Box H$ and $f(g_h,h)=(g_h,\overline{h})$. It is clear that f_H is well defined. We will prove $p(H)\geq k$ by showing that f_H is an automorphism of H induced by a k-pair partition of H.

In order to prove that f_H is an automorphism of H, we only need to show that $f_H(h_1)\neq f_H(h_2)$ for any $h_1\neq h_2$ and $f_H(h_1)$ adj $f_H(h_2)$ if h_1 adj h_2.

For arbitrarily given $h_1\neq h_2$ in H, we may distinguish two cases.

Case 1. The two G-layers $G\Box\{h_1\}$ and $G\Box\{h_2\}$ are mapped into the same H-layer by f, that is, $f(G\Box\{h_1\})\subseteq\{g_h\}\Box H$ and $f(G\Box\{h_2\})\subseteq\{g_h\}\Box H$ for some vertex g_h in G. Assume that $f(g_h,h_1)=(g_h,\overline{h_1})$ and $f(g_h,h_2)=(g_h,\overline{h_2})$ where $\overline{h_1}$ and $\overline{h_2}$ are some vertices in H. Since f is one-to-one, we have $\overline{h_1}\neq\overline{h_2}$, that is, $f_H(h_1)\neq f_H(h_2)$.

If h_1 adj h_2 in H, then (g_h,h_1)adj(g_h,h_2) in $\{g_h\}\Box H$. Since f is an automorphism, then $f(g_h,h_1)$ adj $f(g_h,h_2)$ in $\{g_h\}\Box H$. That is, $(g_h,\overline{h_1})$ adj$(g_h,\overline{h_2})$ in $\{g_h\}\Box H$. So, we have $\overline{h_1}$ adj $\overline{h_2}$ in H. Thus, we have shown that $f_H(h_1)$ adj $f_H(h_2)$ if h_1 adj h_2.

Case 2. The two G-layers $G\Box\{h_1\}$ and $G\Box\{h_2\}$ are mapped into distinct H-layers by f, that is, $f(G\Box\{h_1\})\subseteq\{g_{h_1}\}\Box H$ and $f(G\Box\{h_2\})\subseteq\{g_{h_2}\}\Box H$ where $g_{h_1}\neq g_{h_2}$ in G. We can show that $f_H(h_1)\neq f_H(h_2)$ by contradiction. Suppose otherwise, say, $f_H(h_1)=f_H(h_2)=\overline{h}$ for some $\overline{h}$ in H. Then $f(g_{h_1},h_1)=(g_{h_1},\overline{h})$ and $f(g_{h_2},h_2)=(g_{h_2},\overline{h})$. Since $f=f^{-1}$, we have $f(g_{h_1},\overline{h})=(g_{h_1},h_1)$ and $f(g_{h_2},\overline{h})=(g_{h_2},h_2)$. Recall that each G-layer is mapped into an H-layer by f. Then (g_{h_1},h_1) and (g_{h_2},h_2) must belong to the same H-layer. This contradicts the assumption that $g_{h_1}\neq g_{h_2}$. Thus, we have shown that $f_H(h_1)\neq f_H(h_2)$.

If h_1 adj h_2 in H, then (g_{h_1},h_1) adj(g_{h_1},h_2) in $G\Box H$ so that $f(g_{h_1},h_1)$ adj $f(g_{h_1},h_2)$ in $G\Box H$. Since f maps $G\Box\{h_1\}$ and $G\Box\{h_2\}$ into distinct H-layers $\{g_{h_1}\}\Box H$ and $\{g_{h_2}\}\Box H$, $f(g_{h_1},h_1)$ and $f(g_{h_1},h_2)$ must be in the same G-layer. Then $f(g_{h_1},h_2)=(g_{h_2},\overline{h_1})$ since $f(g_{h_1},h_1)=(g_{h_1},\overline{h_1})$. Thus, we have $(g_{h_1},\overline{h_1})$ adj $(g_{h_2},\overline{h_1})$ in $G\Box H$, which implies g_{h_1} adj g_{h_2} in G. Then we have

$$\begin{aligned}
h_1 \text{ adj } h_2 \text{ in } H &\Longrightarrow g_{h_1} \text{ adj } g_{h_2} \text{ in } G\\
&\Longrightarrow (g_{h_1},h_2)\text{ adj}(g_{h_2},h_2) \text{ in } G\Box\{h_2\}\\
&\Longrightarrow f(g_{h_1},h_2) \text{ adj } f(g_{h_2},h_2) \text{ in } G\Box H\\
&\Longrightarrow (g_{h_2},\overline{h_1})\text{ adj}(g_{h_2},\overline{h_2}) \text{ in } \{g_{h_2}\}\Box H\\
&\Longrightarrow \overline{h_1} \text{ adj } \overline{h_2} \text{ in } H\\
&\Longrightarrow f_H(h_1) \text{ adj } f_H(h_2) \text{ in } H.
\end{aligned}$$

In the above we have proved that f_H is an automorphism of H. Note that f does not fix any vertex of $G\Box H$. This implies that f_H does not fix any vertex of H, that is, $f_H(h)=\overline{h}\neq h$. And it is easy to see that $f_H(\overline{h})=h$ if and only if $f_H(h) = \overline{h}$, since $f(g_h, h) = (g_h, \overline{h})$ if and only if $f(g_h, \overline{h}) = (g_h, h)$. Therefore, $f_H = f_H^{-1}$ and f_H is an automorphism of H that does not fix any vertex.

Now we can show that f_H is induced by a k-pair partition of H as follows. Since f is an automorphism of $G\Box H$ induced by a k-pair partition of $G\Box H$, we have $d_H(h, \overline{h}) = d_{G\Box H}((g_h, h), (g_h, \overline{h})) = d_{G\Box H}((g_h, h), f(g_h, h)) \geq k$. It follows that f_H is induced by a k-pair partition of H and $p(H) \geq k$. $\Box$

Given any nontrivial connected graph G, it is easy to see that $G\Box G$ always has a semi-pair partition $V(G\Box G) = \bigcup_{i\neq j}\{(u_i, u_j), (u_j, u_i)\} \cup \bigcup_i\{(u_i, u_i)\}$, and so $G\Box G$ is semi-pairable. Hence if a graph $G\Box H$ is semi-pairable, then it is not necessary that at least one factor is semi-pairable.

Let $V(G) = \bigcup_{i=1}^m\{u_i, u_i'\}$ and $V(H) = \bigcup_{i=1}^n\{v_j, v_j'\}$ be semi-pair partitions of G and H respectively. Then $G\Box H$ has at least three different semi-pair partitions:

(1) $V(G\Box H) = \bigcup_{i,j}\{(u_i, v_j), (u_i', v_j)\}$,
(2) $V(G\Box H) = \bigcup_{i,j}\{(u_i, v_j), (u_i, v_j')\}$, and
(3) $V(G\Box H) = \bigcup_{i,j}(\{(u_i, v_j), (u_i', v_j')\} \cup \{(u_i', v_j), (u_i, v_j')\})$.

In particular, if G is uniquely k-pairable and H is semi-pairable, then $G\Box H$ has at least two different k-pair partitions (1) and (3).

Consider a uniquely pairable graph $G\Box H$. Then $p(G\Box H) = p(G)+p(H) > 0$. If both $p(G) > 0$ and $p(H) > 0$, then both G and H are uniquely pairable; if $p(G) > 0$ and $p(H) = 0$, then G is uniquely pairable and H cannot be semi-pairable.

We will end this section with a special class of uniquely pairable graphs, namely the graphs with an antipodal isomorphism. The *eccentricity* of a vertex u in a graph G is $e(u) = \max_{v\in V(G)} d(u, v)$ and the *diameter* of G is $d(G) = \max_{u\in V(G)} e(u)$. A graph G has an *antipodal isomorphism* if (i) $e(v) = d(G)$ for each vertex $v \in V(G)$, (ii) for each $v \in V(G)$, there exists a unique $\overline{v} \in V(G)$ such that $d(v, \overline{v}) = d(G)$, and (iii) the map $\phi : V(G) \to V(G)$ defined by $\phi(v) = \overline{v}$ is an isomorphism of G. Hence a graph with an antipodal isomorphism is a uniquely k-pairable graph where $k = d(G)$.

It is well known that $d_{G\Box H}((g, h), (x, y)) = d_G(g, x) + d_H(h, y)$ for any nontrivial graphs G and H. Then $d(G\Box H) = d(G) + d(H)$ and $d_{G\Box H}((g, h), (x, y)) = d(G\Box H)$ if and only if $d_G(g, x) = d(G)$ and $d_H(h, y) = d(H)$. Therefore, $G\Box H$ has an antipodal isomorphism if and only if both G and H have an antipodal isomorphism. Hence the set of graphs with an antipodal isomorphism is closed under the operation of Cartesian product, and so is the uniquely pairable property of this set of graphs.

3. Main results

Theorem 3.1. *A graph is semi-pairable if and only if its prime factor decomposition contains a semi-pairable prime factor or some repeated prime factors.*

Proof. Let $G = G_1\Box G_2\Box\cdots\Box G_n$ be the prime factor decomposition of a graph G. Sufficiency is trivial by the definition of semi-pairable graphs. We only need to show necessity. It is trivial when $n = 1$. Assume that it is true for any graph with less than n prime factors. Let f be an automorphism of G induced by a semi-pair partition of G. We may write $G = G_1\Box H$ where $H = G_2\Box\cdots\Box G_n$. Assume that G has no repeated prime factors. Then by Theorem 2.1, each G_1-layer is mapped onto a G_1-layer by f, and each H-layer is mapped onto an H-layer by f since G_1 is prime and different from G_i for all $2 \leq i \leq n$.

Case 1. If there is a G_1-layer mapped onto itself by f, then either there is a G_1-layer mapped onto a different G_1-layer, or there is an H-layer mapped onto a different H-layer since f is not the identity map of $G = G_1\Box H$. If the former is true, then H is semi-pairable by Lemma 2.2, and so at least one G_i $(2 \leq i \leq n)$ is semi-pairable by induction hypothesis; if the latter is true, then G_1 is semi-pairable by the remark of Lemma 2.2.

Case 2. If each G_1-layer is mapped onto a different G_1-layer by f, then H is pairable by Lemma 2.2, and so semi-pairable. Hence, at least one G_i $(2 \leq i \leq n)$ is semi-pairable by induction hypothesis. $\Box$

Remark. Theorem 3.1 is equivalent to say that the automorphism group of a graph G contains a nontrivial involution if and only if either the prime factor decomposition of G contains repeated prime factors, or there exists a prime factor of G whose automorphism group contains a nontrivial involution.

Theorem 3.2. *A graph is uniquely pairable if and only if the following hold:*

(i) *its prime factor decomposition has at least one pairable prime factor;*
(ii) *each prime factor is either uniquely pairable or not semi-pairable; and*
(iii) *all prime factors which are not semi-pairable are pairwise non-isomorphic.*

To prove the above theorem, we need the following two lemmas.

Lemma 3.3. *Let $G = G_1 \square G_2 \square \cdots \square G_n$ be the prime factor decomposition of a graph G. If G is uniquely pairable, then the following hold:* (i) *at least one G_i has $p(G_i) > 0$,* (ii) *for $1 \le i \le n$, G_i is uniquely pairable when $p(G_i) > 0$ and G_i is not semi-pairable when $p(G_i) = 0$, and* (iii) *$p(G_i) = p(G_j) = 0$ for some $i \ne j$ implies that $G_i \ncong G_j$.*

Proof. We prove by mathematical induction on n. It is trivial when $n = 1$. Assume that it is true for any uniquely pairable graph with less than n prime factors. If G is uniquely pairable, then $p(G) = p(G_1) + p(G_2) + \cdots + p(G_n) = k > 0$, and so at least one of the prime factors has $p(G_i) > 0$. Without loss of generality, we can assume that $p(G_1) > 0$. Note that $G = G' \square G_n$ where $G' = G_1 \square \cdots \square G_{n-1}$. Then G' must be uniquely pairable since $p(G') \ge p(G_1) > 0$ and G is uniquely pairable. By the induction hypothesis, for $1 \le i \le n-1$, G_i is uniquely pairable when $p(G_i) > 0$ and G_i is not semi-pairable when $p(G_i) = 0$, and $p(G_i) = p(G_j) = 0$ for some $1 \le i \ne j \le n-1$ implies that $G_i \ncong G_j$. Now if $p(G_n) > 0$, then G_n must be uniquely pairable. If $p(G_n) = 0$, then G_n cannot be semi-pairable. Suppose that $p(G_n) = p(G_j) = 0$ for some $1 \le j \le n-1$ and $G_n \cong G_j$. Without loss of generality, we can assume that $j = n-1$ and $G_n \cong G_{n-1}$. Then $G_{n-1} \square G_n$ is semi-pairable. This implies that $G = (G_1 \square G_2 \square \cdots \square G_{n-2}) \square (G_{n-1} \square G_n)$ cannot be uniquely pairable, which contradicts the hypothesis. Therefore, $p(G_n) = p(G_j) = 0$ for some $1 \le j \le n-1$ implies that $G_n \ncong G_j$. $\square$

Lemma 3.4. *Let $G = G_1 \square G_2 \square \cdots \square G_n$ be the prime factor decomposition of a graph G. If* (i) *at least one G_i has $p(G_i) > 0$,* (ii) *for $1 \le i \le n$, G_i is uniquely pairable when $p(G_i) > 0$ and G_i is not semi-pairable when $p(G_i) = 0$, and* (iii) *$p(G_i) = p(G_j) = 0$ for some $i \ne j$ implies that $G_i \ncong G_j$, then G is uniquely pairable.*

Proof. We prove by induction on n. It is trivial when $n = 1$. Assume that it is true for a graph with less than n prime factors. Consider $G = G_1 \square G_2 \square \cdots \square G_n$ with n prime factors. By hypothesis (i), we can assume that $p(G_1) > 0$ without loss of generality. Then G_1 is uniquely pairable by hypothesis (ii). Let $V(G_1) = \bigcup_{i=1}^{m} \{u_i, u_i'\}$ be the unique $p(G_1)$-pair partition $\mathcal{P}_1$ of G_1. We may write G as $G = G_1 \square H$ where $H = G_2 \square G_3 \square \cdots \square G_n$.

Case 1. If $p(H) > 0$, then H is uniquely pairable by the induction hypothesis. Let $V(H) = \bigcup_{v_j \in V(H)} \{v_j, v_j'\}$ be the unique $p(H)$-pair partition $\mathcal{P}_2$ of H. Assume that $p(G) = k$. Then $k = p(G_1) + p(H)$ and $V(G_1 \square H) = \bigcup_{i,j} (\{(u_i, v_j), (u_i', v_j')\} \cup \{(u_i', v_j), (u_i, v_j')\})$ is a k-pair partition $\mathcal{P}_{G_1 \square H}$ of $G_1 \square H$. Let $\mathcal{P}$ be an arbitrary k-pair partition of $G_1 \square H$. We will show that $\mathcal{P}$ is just $\mathcal{P}_{G_1 \square H}$, and so $G = G_1 \square H$ is uniquely pairable. Let f be the automorphism of $G_1 \square H$ induced by $\mathcal{P}$. By Theorem 2.1, a G_1-layer is either mapped onto a G_1-layer by f, or mapped into an H-layer by f since G_1 is prime. By Lemma 2.3, a G_1-layer must be mapped onto a G_1-layer by f since $p(H) < k$. Hence, f maps each G_1-layer (resp., each H-layer) onto a G_1-layer (resp., an H-layer). We claim that each G_1-layer must be mapped onto a different G_1-layer by f. Otherwise, if there is a G_1-layer, say $G_1 \square \{v\}$, that is mapped onto itself, then $f(G_1 \square \{v\}) = G_1 \square \{v\}$ induces a k'-pair partition of G_1 where $p(G_1) \ge k' \ge p(G_1 \square H) = p(G_1) + p(H) > p(G_1)$. This is a contradiction. Now by Lemma 2.2, the map $f_H : V(H) \to V(H)$ defined as $f_H(v_j) = \overline{v_j}$ if $f(G_1 \square \{v_j\}) = G_1 \square \{\overline{v_j}\}$ corresponds to a k_2-pair partition of H, where $k_2 = \min_{v_j \in V(H)} d(v_j, \overline{v}_j) > 0$. Similarly, we can show that each H-layer must be mapped onto a different H-layer by f. The map $f_{G_1} : V(G_1) \to V(G_1)$ defined as $f_{G_1}(u_i) = \overline{u_i}$ if $f(\{u_i\} \square H) = \{\overline{u_i}\} \square H$ corresponds to a k_1-pair partition of G_1, where $k_1 = \min_{u_i \in V(G_1)} d(u_i, \overline{u}_i) > 0$. Recall that $d_{G_1 \square H}((u, v), (\overline{u}, \overline{v})) = d_{G_1}(u, \overline{u}) + d_H(v, \overline{v})$. Then $p(G_1 \square H) \le k_1 + k_2 \le p(G_1) + p(H) = p(G_1 \square H)$. Hence $k_1 = p(G_1)$ and $k_2 = p(H)$. Both G_1 and H are uniquely pairable, the induced k_1-pair partition of G_1 from f_{G_1} must be $\mathcal{P}_1$, and the induced k_2-pair partition of H from f_H must be $\mathcal{P}_2$. Hence, $\mathcal{P}$ is just $\mathcal{P}_{G_1 \square H}$.

Case 2. If $p(H) = 0$, then $p(G_i) = 0$ for $2 \le i \le n$. It follows that each G_i is not semi-pairable for $2 \le i \le n$ by hypothesis (ii) and all of them are pairwise non-isomorphic by the hypothesis (iii). Hence H is not semi-pairable by Theorem 3.1. It is easy to see that $V(G_1 \square H) = \bigcup_{i,j} \{(u_i, v_j), (u_i', v_j)\}$ is a k-pair partition $\mathcal{P}_{G_1 \square H}$ of $G_1 \square H$. Let $\mathcal{P}$ be an arbitrary k-pair partition of $G_1 \square H$. We will show that $\mathcal{P}$ is just $\mathcal{P}_{G_1 \square H}$, and so $G_1 \square H$ is uniquely pairable. Let f be the automorphism of $G_1 \square H$ induced by $\mathcal{P}$. By Theorem 2.1, a G_1-layer is either mapped onto a G_1-layer by f, or mapped into an H-layer by f since G_1 is prime. By Lemma 2.3, a G_1-layer must be mapped onto a G_1-layer by f since $p(H) = 0$. Hence, f maps each G_1-layer (resp., each H-layer) onto a G_1-layer (resp., an H-layer). We claim that each G_1-layer must be mapped onto itself by f. Otherwise, if there is a G_1-layer that is mapped onto a different G_1-layer, then H is semi-pairable by Lemma 2.2. This is a contradiction. Now $f(G_1\Box\{v\}) = G_1\Box\{v\}$ induces a k'-pair partition of G where $p(G_1) \geq k' \geq k = p(G_1\Box H) = p(G_1)$. This implies that $k' = p(G_1)$. Hence, the induced k'-pair partition is just $\mathcal{P}_1$ since G_1 is uniquely pairable, and so $\mathcal{P}$ is just $\mathcal{P}_{G_1\Box H}$. $\Box$

Now Theorem 3.2 follows from Lemmas 3.3 and 3.4 immediately.

We will give a characterization of the uniquely pairable Cartesian product graphs and apply the result to Cartesian product graphs with some special factors.

Theorem 3.5. *A Cartesian product graph $G\Box H$ is uniquely pairable if and only if exactly one of the following holds:*

(i) *both G and H are uniquely pairable and any prime factor of G with pair length zero is different from any prime factor of H with pair length zero, or*
(ii) *G is uniquely pairable and H is not semi-pairable, and any prime factor of G with pair length zero is different from any prime factor of H, or*
(iii) *H is uniquely pairable and G is not semi-pairable, and any prime factor of H with pair length zero is different from any prime factor of G.*

Proof. By Theorem 3.2. $\Box$

Our first application of Theorem 3.5 is to the Cartesian product of two relatively prime graphs. Two graphs are called *relatively prime* if they do not have any prime factors in common.

Corollary 3.6. *Let G and H be relatively prime graphs. Then $G\Box H$ is uniquely pairable if and only if either* (i) *both G and H are uniquely pairable, or* (ii) *G (resp. H) is uniquely pairable and H (resp. G) is not semi-pairable.*

We introduce a new concept called *properly pairable graphs* for the second application of Theorem 3.5.

Definition 3.7. Let $G = G_1\Box G_2\Box\cdots\Box G_n$ be the prime factor decomposition of a graph G. Then G is called *properly pairable* if each prime factor of G is pairable.

By the definition of properly pairable graphs, it is easy to see that the set of properly pairable graphs is closed under the Cartesian product operation. Hence $G\Box H$ is properly pairable if and only if both G and H are properly pairable.

For example, all prime graphs with positive pair length are properly pairable; hypercubes Q_n, lattices $P_{2n}\Box P_{2m}$ are properly pairable; graphs with an antipodal isomorphism are also properly pairable.

Corollary 3.8. *Let G and H be graphs such that at least one of them is properly pairable. Then $G\Box H$ is uniquely pairable if and only if either* (i) *both G and H are uniquely pairable, or* (ii) *G (resp. H) is uniquely pairable, and H (resp. G) is not semi-pairable.*

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