On the positive almost periodic type solutions for some nonlinear delay integral equations

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Abstract

In this paper, we discuss the existence of positive almost periodic type solutions for some nonlinear delay integral equations, by constructing a new fixed point theorem in the cone. Some known results are extended.

Keywords: Almost periodicity; Delay integral equation; Mixed monotone operators

1. Introduction

In epidemic problems, a mathematical model is described by the following nonlinear integral equation:

\[ x(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds, \quad (1) \]
in which \(x(t)\) is the proportion of infectious individuals present in the population at time \(t\), \(\tau\) is the length of time an individual remains infective, and \(f(t, x(t))\) is the proportion of new infective individuals per unit of time. Equation (1) has been studied by various authors (see [3,4,6,7,9,11,12,14] and references therein). D. Guo and V. Lakshmikantham [7], L.R. Williams and R.W. Leggett [9,14] studied the conditions of the existence of positive periodic solution to Eq. (1) by means of the theorems of the fixed point index and the extension and compression of the cone, respectively. Nussbaum [11] and Smith [12] in particular demonstrated the bifurcation of positive periodic solutions from the zero solution for sufficiently large \(\tau\), provided \(f\) is periodic in \(t\). A.M. Fink and J.A. Gatica [6] were first to consider the positive almost periodic solution of Eq. (1) in the case of delay \(\tau\) being a constant. R. Torrejón [13] studied the existence of the positive almost periodic solutions of Eq. (1) in the case when the delay is state-dependent, namely the following equation:

\[ x(t) = \int_{t-\tau(t)}^{t} f(s, x(s)) \, ds. \quad (1') \]

In 1996, K. Ezzinbi and M.A. Hachimi [4] studied the existence of positive almost periodic solutions for Eqs. (1) and (1') via Hilbert’s Projective metric. In 2000, E. Ait Dads and K. Ezzinbi [2] also used the Hilbert’s projective metric to present the conditions of existence of positive pseudo almost periodic solutions for the following infinite delay integral equation:

\[ x(t) = \int_{-\infty}^{t} a(t-s) \left[ f(s, x(s)) + g(s, x(s)) \right] \, ds. \quad (1'') \]

The conditions in [2,4] contain the following two items:

\( (H'_1) \) for every \( t \in \mathbb{R} \), \( f(t, \cdot) \) is nondecreasing in \( \mathbb{R}^+ \);

\( (H'_2) \) there exists a positive continuous function \( \varphi \) defined on \((0,1)\) such that

\[ f(t, \lambda x) \geq \varphi(\lambda) f(t, x) \quad \text{and} \quad \varphi(\lambda) > \lambda \quad \text{for all} \quad t \in \mathbb{R}, \; x > 0, \; \lambda \in (0, 1). \]

In this paper, we consider the following generalized equations:

\[ x(t) = \int_{t-\tau(t)}^{t} \left[ f(s, x(s)) + g(s, x(s)) \right] \, ds, \quad (2) \]

\[ x(t) = \int_{-\infty}^{t} a(t-s) \left[ f(s, x(s)) + g(s, x(s)) \right] \, ds. \quad (3) \]

We would like to construct a new fixed point theorem in the cone, through which we relax the above \((H'_1)\)–\((H'_2)\) to the following assumptions \((H_1)\)–\((H_2)\) to present the conditions of existence of positive almost periodic type solutions to Eqs. (2) and (3):

\( (H_1) \) for every \( t \in \mathbb{R} \), \( f(t, \cdot) \) is nondecreasing and \( g(t, \cdot) \) is nonincreasing in \( \mathbb{R}^+ \);
(H2) there exists a positive function \( \varphi \) defined on \( (0, 1) \to \mathbb{R}^+ \) such that for every \( \lambda \in (0, 1) \), one has \( \varphi(\lambda) > \lambda \), and
\[
f(t, \lambda x) \geq \varphi(\lambda)f(t, x), \quad g(t, \lambda^{-1}y) \geq \varphi(\lambda)g(t, y), \quad \forall t \in \mathbb{R}, \ x, y > 0.
\]

The paper is organized as follows. In Section 2, we recall some definitions, lemmas and establish a new fixed point theorem in the cone. In the following three sections, we give our main results and corresponding proofs, respectively. In the final section, we provide two examples for presenting the applications of our theorems obtained in this paper.

2. Some definitions and lemmas

In this paper, we denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{R}^+ \) the set of nonnegative real numbers, and by \( \Omega \) an open subset of \( \mathbb{R}^q \). \( BC(\mathbb{R}, \mathbb{R}^q) \) (respectively \( BC(\mathbb{R} \times \Omega, \mathbb{R}^q) \)) stands for the Banach space of bounded continuous functions \( \phi(t) \) (respectively \( \phi(t, x) \)) from \( \mathbb{R} \) (respectively \( \mathbb{R} \times \Omega \)) to \( \mathbb{R}^q \) with norm \( \| \phi \| = \sup_{t \in \mathbb{R}} \| \phi(t) \| \) (respectively \( \| \phi \| = \sup_{t \in \mathbb{R}, x \in \Omega} \| \phi(t, x) \| \)), where \( \| . \| \) is the Euclidean norm.

Definition 1 [5]. A function \( f \in BC(\mathbb{R}, \mathbb{R}^q)(BC(\mathbb{R} \times \Omega, \mathbb{R}^q)) \) is called an almost periodic function (an almost periodic function in \( t \in \mathbb{R} \) uniformly on \( x \in \Omega \)) (denote by \( f \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)(\mathcal{AP}(\mathbb{R} \times \Omega, \mathbb{R}^q)) \)), if the \( \varepsilon \)-translation set of \( f \)
\[
T(\varepsilon) = \{ \tau \in \mathbb{R} : \| f(t + \tau) - f(t) \| < \varepsilon, \ \forall t \in \mathbb{R} \},
\]
\[
(\varepsilon) = \{ \tau \in \mathbb{R} : \| f(t + \tau, x) - f(t, x) \| < \varepsilon, \ \forall (t, x) \in \mathbb{R} \times \Omega, \ \forall \text{ compact set } W \subset \Omega \}
\]
is a relatively dense set in \( \mathbb{R} \) for all \( \varepsilon > 0 \) (for any \( \varepsilon > 0 \) and for any compact subsets \( W \) of \( \Omega \)). Each \( \tau \in T(\varepsilon) \) is called an \( \varepsilon \)-period for \( f \).

Suppose that \( f \) belongs to \( \mathcal{AP}(\mathbb{R}, \mathbb{R}) \). Let \( \{ \lambda_j \} \) denote the set of all real numbers such that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(t) \exp(-i\lambda t) dt \neq 0.
\]
It is well known that the set of numbers \( \{ \lambda_j \} \) in the above formula is countable. The set \( \{ \sum_{1}^{N} n_j \lambda_j \} \) for all integers \( N \) and integers \( n_j \) is called the module of \( f(t) \), denoted by \( \text{mod}(f) \). For the module containment, we have the following lemma.

Lemma 1 [5]. Suppose that \( f \) and \( g \) are almost periodic. Then the following statements are equivalent:

(1) \( \text{mod}(f) \supset \text{mod}(g) \);
(2) For any sequence \( \{ t_n^* \} \), if \( \lim_{n \to \infty} f(t + t_n^*) = f(t) \) for each \( t \in \mathbb{R} \), then there exists a subsequence \( \{ t_n \} \subseteq \{ t_n^* \} \) such that \( \lim_{n \to \infty} g(t + t_n) = g(t) \) for each \( t \in \mathbb{R} \).
If $\alpha = \{\alpha_n\}$ is a sequence in $\mathbb{R}$, we shall write $\alpha^* \subset \alpha$ to indicate that $\alpha^* = \{\alpha^*_n\}$ is a subsequence of $\alpha$. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $T_\alpha f = g$ shall indicate that $\lim_{n \rightarrow \infty} f(t + \alpha_n)$ exists and $g(t) = \lim_{n \rightarrow \infty} f(t + \alpha_n)$. It can be found from [13, Lemma 2.2] that the following lemma holds.

**Lemma 2.** Let $x(t) \in BC(\mathbb{R}, \mathbb{R})$, $f(t, x)$ be almost periodic in $t$ uniformly with respect to $x$ in compact subsets of $\mathbb{R}^+$, and $\tau(t)$ be almost periodic on $\mathbb{R}$. If $\alpha$ is a sequence for which $T_\alpha f(t, x)$ exists uniformly on $\mathbb{R}$ and $T_\alpha \tau(t), T_\alpha x(t)$ exist uniformly on compact subsets of $\mathbb{R}$, then

$$T_\alpha \int_{t - \tau(t)}^t f(s, x(s)) \, ds = \int_{t - T_\alpha \tau(t)}^t T_\alpha f(s, T_\alpha x(s)) \, ds.$$ 

Set

$$C_0(\mathbb{R}^+, \mathbb{R}^q) = \left\{ w \in BC(\mathbb{R}^+, \mathbb{R}^q) : \lim_{t \rightarrow +\infty} w(t) = 0 \right\},$$

$$C_0(\mathbb{R}^+ \times \Omega, \mathbb{R}^q) = \left\{ w \in BC(\mathbb{R}^+ \times \Omega, \mathbb{R}^q) : \lim_{t \rightarrow +\infty} w(t, x) = 0, \right.\left. \text{uniformly for } x \in W, W \text{ is any compact subset of } \Omega \right\}.$$ 

**Definition 2** [5]. A function $f \in BC(\mathbb{R}^+, \mathbb{R}^q)(BC(\mathbb{R}^+ \times \Omega, \mathbb{R}^q))$ is called an asymptotically almost periodic function (an asymptotically almost periodic function in $t \in \mathbb{R}^+$ uniformly for $x$ in compact subsets of $\Omega$) (denote by $f \in \mathcal{AAP}(\mathbb{R}^+, \mathbb{R}^q)(\mathcal{AAP}(\mathbb{R}^+ \times \Omega, \mathbb{R}^q))$, if it can be written as a sum

$$f = f^{ap} + f^{c_0},$$

where

$$f^{ap} \in \mathcal{AAP}(\mathbb{R}^+ \times \Omega, \mathbb{R}^q) \quad \text{and} \quad f^{c_0} \in C_0(\mathbb{R}^+, \mathbb{R}^q)(C_0(\mathbb{R}^+ \times \Omega, \mathbb{R}^q)).$$

Set

$$\mathcal{AAP}_0(\mathbb{R}, \mathbb{R}^q) = \left\{ w \in BC(\mathbb{R}, \mathbb{R}^q) : \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |w(s)| \, ds = 0 \right\},$$

$$\mathcal{AAP}_0(\mathbb{R} \times \Omega, \mathbb{R}^q) = \left\{ w \in BC(\mathbb{R} \times \Omega, \mathbb{R}^q) : \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |w(s, x)| \, ds = 0, \right.\left. \text{uniformly for } x \in W, W \text{ is any compact subset of } \Omega \right\}.$$ 

**Definition 3** [15]. A function $f \in BC(\mathbb{R}, \mathbb{R}^q)(BC(\mathbb{R} \times \Omega, \mathbb{R}^q))$ is called a pseudo almost periodic function (a pseudo almost periodic function in $t \in \mathbb{R}$ uniformly for $x$ in compact
subsets of $\mathbb{R}$) (denote by $f \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)(\mathcal{AP}(\mathbb{R} \times \Omega, \mathbb{R}^q))$, if $f$ can be written as a sum
$$f = f^{ap} + f^e,$$
where
$$f^{ap} \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)(\mathcal{AP}(\mathbb{R} \times \Omega, \mathbb{R}^q))$$
and
$$f^e \in \mathcal{AP}_0(\mathbb{R}, \mathbb{R}^q)(\mathcal{AP}_0(\mathbb{R} \times \Omega, \mathbb{R}^q)).$$

**Lemma 3.** Let $x, \tau \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$. Then $y(t) = \int_{t-\tau(t)}^{t} x(s) \, ds$ is an almost periodic function.

The proof is obvious, so we omit it. It has been pointed out in [2, Theorem 4] that the following lemma holds.

**Lemma 4.** Suppose $f(t, x) \in \mathcal{AP}(\mathbb{R} \times \Omega, \mathbb{R}^q)$ is continuous in $x$ $\in W$ uniformly in $t \in \mathbb{R}$ for every compact subset $W \subset \Omega$. If $H \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$ and $H(t) \in \Omega$ for all $t \in \mathbb{R}$, then $t \rightarrow f(t, H(t))$ is pseudo almost periodic.

**Definition 4** [10]. Let $E$ be a real Banach space. A closed convex set $K$ in $E$ is called a convex cone if the following conditions are satisfied:

1. if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$;
2. if $x \in K$ and $-x \in K$, then $x = 0$.

A cone $K$ induces a partial ordering $\leq$ in $E$ by $x \leq y$ if and only if $y - x \in K$. A cone $K$ is called normal if there exists a constant $k > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k \|y\|$, where $\|\|$ is the norm on $E$. We denote by $K^0$ the interior set of $K$. A cone $K$ is called a solid cone, if $K^0 \neq \emptyset$. The following definition will be used in this paper for providing our main results.

**Definition 5** [10]. Suppose that $E$ is a real Banach space and $K$ is a cone in $E$. An operator $A : K \times K \rightarrow E$ is called mixed monotone, if $A(x, y)$ is nondecreasing in $x$ and nonincreasing in $y$.

Choosing $w \in K \setminus \{\theta\}$, let
$$K^0_w = \{x \in E : \text{there exist } \mu > \nu > 0 \text{ such that } \nu w \leq x \leq \mu w\},$$
where $\theta$ denotes the zero element, $K, E$ are the same as in Definition 4 in this paper. Observe that $\theta \notin K^0_w$ and that $K^0_w \subseteq K$. We formulate a theorem as follows.

**Theorem 1.** Let $K$ be a normal cone in a real Banach space $E$, $w \in K \setminus \{\theta\}$, and $A : K^0 \times K^0_w \rightarrow E$ be a mixed monotone operator. Suppose that $A(w, w) \in K^0_w$, and there exists a function $\phi : (0, 1) \rightarrow (0, 1)$ such that $\phi(t) > t$ for each $t \in (0, 1)$ and
Remark 1. If the function \( \phi \) in Theorem 1 is left-hand lower semi-continuous in \( t \), then Theorem 1 yields [10, Theorem 1] to some extent. Hence Theorem 1 generalizes [10, Theorem 1] to some extent.

Proof of Theorem 1. First we show that

\[
A(tx, t^{-1}y) \geq \phi(t)A(x, y), \quad \forall t \in (0, 1) \text{ and } \forall x, y \in K_w^0. \tag{4}
\]

Then \( A \) has a unique fixed point \( u^* \) in \( K_w^0 \). Moreover, if constructing the iterative sequences

\[
x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots)
\]

for any initial \( (x_0, y_0) \in K_w^0 \times K_w^0 \), we have

\[
\|x_n - u^*\|, \|y_n - u^*\| \to 0 \quad (n \to \infty). \tag{6}
\]

Remark 1. If the function \( \phi \) in Theorem 1 is left-hand lower semi-continuous in \( t \), then Theorem 1 yields [10, Theorem 1] to some extent. Hence Theorem 1 generalizes [10, Theorem 1] to some extent.

Proof of Theorem 1. First we show that \( A : K_w^0 \times K_w^0 \to K_w^0 \). In fact, letting \( x, y \in K_w^0 \) be arbitrarily given, we can choose \( t \in (0, 1) \) sufficiently small such that \( tw \leq x \leq t^{-1}w \) and \( tw \leq y \leq t^{-1}w \). Since \( A(w, w) \in K_w^0 \), there exist \( A > \lambda > 0 \) such that \( \lambda w \leq A(w, w) \leq \lambda w \), by (4) and its equivalent formulation that

\[
A(t^{-1}w, tw) \leq \left[ \phi(t) \right]^{-1} A(w, w), \tag{7}
\]

we obtain

\[
\lambda \phi(t)w \leq A(tw, t^{-1}w) \leq A(x, y) \leq A(t^{-1}w, tw) \leq A \left[ \phi(t) \right]^{-1} w,
\]

namely

\[
A(x, y) \in K_w^0.
\]

Now choose \( t_0 \) in \( (0, 1) \) such that

\[
0 < t_0 \leq m_0 = \min[\lambda, 1/A]; \tag{8}
\]

then \( 0 < t_0 \leq m_0 < 1 \) and \( tw \leq A(w, w) \leq t_0^{-1}w \). Since \( [\phi(t_0)/m]^{k_0} \to +\infty(k \to \infty) \), there exists positive integer \( k_0 \) such that

\[
\left[ \phi(t_0) \right]^{k_0} t_0 \geq \frac{1}{t_0}. \tag{9}
\]

Taking \( u_0 = t_0^{k_0} w, v_0 = t_0^{-k_0} w \) and constructing the iterative sequences

\[
u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}) \quad (n = 1, 2, \ldots),
\]

it is clear that \( u_0, v_0 \in K_w^0 \), \( u_0 = t_0^{2k_0} v_0 \leq v_0 \), and \( u_1 = A(u_0, v_0) \leq A(v_0, u_0) = v_1 \). Repeatedly using (4), and taking notice of (9), we have

\[
u_1 = A(t_0^{k_0} w, t_0^{-k_0} w) \geq \phi(t_0) A((t_0^{k_0-1} w, t_0^{-(k_0-1)} w) \geq \cdots \geq t_0^{k_0} w = u_0.
\]

Likewise, repeatedly using (7), we have \( v_1 = A(t_0^{-k_0} w, t_0^{k_0} w) \leq v_0 \), and so \( u_0 \leq u_1 \leq v_1 \leq v_0 \). It is easy to show by induction that

\[
u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{10}
\]
Set
\[ t_n = \sup\{ t > 0 : u_n \geq t v_n \}, \quad n = 1, 2, \ldots \]
We can infer that
\[ u_n \geq t_n v_n, \quad n = 1, 2, \ldots \]
(11)
It follows that
\[ u_{n+1} \geq u_n \geq t_n v_n \geq t_n v_{n+1}, \quad n = 1, 2, \ldots \]
Thus there exists \( \varepsilon_0 > 0 \) such that
\[ 0 < \varepsilon_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \cdots \leq 1. \]
Set \( t^* = \lim_{n \to +\infty} t_n \). We claim that \( t^* = 1 \).
Otherwise, if \( t^* \in [\varepsilon_0, 1) \), we should have \( \phi(t^*)/t^* - 1 > 0 \). We consider two cases:

(a) there exists \( N \) such that \( t_N = t^* \);
(b) \( t_n < t^*, \quad \forall n = 1, 2, \ldots \)

Under case (a), we can obtain that \( t_n = t^*, \ u_n \geq t^* v_n, \forall n > N \), and
\[ u_{n+1} = A(u_n, v_n) \geq A(t^* v_n, (t^*)^{-1} u_n) \geq \phi(t^*) A(v_n, u_n) = \phi(t^*) v_{n+1}, \quad \forall n > N. \]
Thus
\[ t_{n+1} \geq \phi(t^*) > t^*, \quad \forall n > N. \]
Letting \( n \to \infty \), it can follow that
\[ t^* \geq \phi(t^*) > t^*, \]
which is a contradiction.

Under the case (b), setting \( \eta(t) = \phi(t) / t - 1, \forall t \) in \( (0, 1) \), we have that
\[ u_{n+1} = A(u_n, v_n) \geq A(t_n v_n, (t_n)^{-1} u_n) \geq A \left( \frac{t_n}{t^*} t^* v_n, \frac{t^* - 1}{t^*} u_n \right) \]
\[ \geq \frac{t_n}{t^*} \left[ 1 + \eta \frac{t_n}{t^*} \right] A(t^* v_n, (t^*)^{-1} u_n) \]
\[ \geq \frac{t_n}{t^*} \times t^* [1 + \eta(t^*)] A(v_n, u_n) = t_n [1 + \eta(t^*)] v_{n+1}. \]
Thus
\[ t_{n+1} \geq t_n [1 + \eta(t^*)] \geq t_n. \]
Letting \( n \to \infty \), we have
\[ t^* \geq t^* [1 + \eta(t^*)] \geq t^*, \]
which is a contradiction. Hence \( t^* = 1 \).
From the formulas (10) and (11), it follows that for any natural number \( k \),
\[ 0 \leq u_{n+k} - u_n, \quad v_n - v_{n+k} \leq v_n - u_n \leq (1 - t_n) v_1, \quad n = 1, 2, \ldots \]
Since \( K \) is a normal cone, there exist \( u^*, v^* \in [u_0, v_0] \) such that
\[
\|u_n - u^*\| \to 0, \quad \|v_n - v^*\| \to 0 \quad (n \to \infty),
\]
\[
0 \leq v_n^* - u_n^* \leq v_n - u_n \leq (1 - t_n) v_1, \quad n = 1, 2, \ldots. \tag{12}
\]
Thus \( u^* = v^* \in [u_0, v_0] \). If the operator \( A \) has other fixed point \( u_n \in K^0_w \), let \( t_n = \sup \{0 < t < 1: t u_n \leq u^* \leq t^{-1} u_n \} \). Clearly, \( 0 < t_n \leq 1 \) and \( t_n u_n \leq u^* \leq t_n^{-1} u_n \). If \( t_n < 1 \), then
\[
u_n^* = A(u^*, u^*) \geq A(t_n u_n, t_n^{-1} u_n) \geq \phi(t_n) A(u_n, u_n) = \phi(t_n) u_n,
\]
\[
u_n^* = A(u^*, u^*) \leq A(t_n^{-1} u_n, t_n u_n) \leq [\phi(t_n)]^{-1} A(u_n, u_n) = [\phi(t_n)]^{-1} u_n,
\]
i.e., \( \phi(t_n) u_n \leq u^* \leq [\phi(t_n)]^{-1} u_n \). These contradict the definition of \( t_n \), since \( 1 > \phi(t_n) > t_n \).

Hence \( t_n = 1 \) and \( u^* = u_n \). This, at the same time, proves the uniqueness of fixed point of \( A \) in \( K^0_w \). We proceed to show that (6) holds. Let \( (x_0, y_0) \in K^0_w \times K^0_w \) be given. There exist \( \mu_1 > \gamma_1 > 0 \) and \( \mu_2 > \gamma_2 > 0 \) such that \( \gamma_1 w \leq x_0 \leq \mu_1 w \) and \( \gamma_2 w \leq y_0 \leq \mu_2 w \).

Taking \( t_0 = \min \{\gamma_1, 1/\mu_1, \gamma_2, 1/\mu_2, m_0\} \) in (8), then \( u_0 \leq x_0 \leq v_0 \) and \( u_0 \leq y_0 \leq v_0 \), and the iterative sequences constructed according to (5) satisfy \( u_n \leq x_n \leq v_n \) and \( u_n \leq y_n \leq v_n \) (\( n = 1, 2, \ldots \)). Thus by (12) we get
\[
0 \leq x_n - u_n, \quad u_n^* - u_n \leq v_n - u_n \leq (1 - t_n) v_0. \tag{13}
\]

By virtue of the normality of \( K \) and (13), we have \( \|x_n - u^*\| \to 0 \) (\( n \to \infty \)). Similarly we can prove that \( \|y_n - u^*\| \to 0 \) (\( n \to \infty \)), and hence (6) holds. The proof is finished. \( \square \)

Note that if the cone \( K \) is solid, denote its interior by \( K^0 \) and choose \( w \in K^0 \), then \( K^0_w = K^0 \).

**Corollary 1.** Let \( K \) be a normal and solid cone in a real Banach space \( E \), \( A : K^0 \times K^0 \to K^0 \) be a mixed monotone operator. Suppose there exists \( \phi(t) > 1 \) for each \( t \in (0, 1) \) such that
\[
A(t x, t^{-1} y) \geq \phi(t) A(x, y), \quad \forall x, y \in K^0,
\]
where \( \phi(t) : (0, 1) \to (0, 1) \). Then \( A \) has a unique fixed point \( u^* \) in \( K^0 \). Moreover, if constructing the iterative sequences
\[
x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots) \tag{14}
\]
for any initial \( (x_0, y_0) \in K^0 \times K^0 \), we have
\[
\|x_n - u^*\|, \quad \|y_n - u^*\| \to 0 \quad (n \to +\infty). \tag{15}
\]

3. Almost periodic case

**Theorem 2.** For Eq. (2), suppose (\( H_1 \))–(\( H_2 \)) hold, in addition, we assume that \( f, g, \tau \) satisfy the following conditions:
\((H_3)\) \(f, g : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+\) are almost periodic in \(t\) uniformly for \(x\) in any compact subset of \(\mathbb{R}^+\);

\((H_4)\) \(\tau(t)\) is a positive almost periodic function on \(\mathbb{R}\);

\((H_5)\) there exist \(\rho > 0, \delta > 0\) such that

\[
\inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} f(s, \rho) \, ds \geq \delta.
\]

Then Eq. (2) has exactly one almost periodic solution \(x^*\) with a positive infimum on \(\mathbb{R}\) satisfying \(\text{mod}(x^*) \subset \text{mod}(\tau, f, g)\). Furthermore, for each \(x_0, y_0 \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^+)\) such that \(\inf_{t \in \mathbb{R}} x_0(t) > 0\) and \(\inf_{t \in \mathbb{R}} y_0(t) > 0\), if we construct the iterative sequences

\[
x_n(t) = \int_{t-\tau(t)}^{t} \left[ f(s, x_{n-1}(s)) + g(s, y_{n-1}(s)) \right] ds, \quad (n = 1, 2, \ldots),
\]

\[
y_n(t) = \int_{t-\tau(t)}^{t} \left[ f(s, y_{n-1}(s)) + g(s, x_{n-1}(s)) \right] ds
\]

we have

\[
\|x_n - x^*\|, \|y_n - x^*\| \to 0 \quad (n \to \infty).
\]

**Proof.** Let \(X\) be the Banach space consisting of almost periodic functions with the norm defined by \(\|x\| = \sup_{t \in \mathbb{R}} |x(t)|\). Let \(W = \{x \in \mathcal{AP}(\mathbb{R}, \mathbb{R}): x(t) \geq 0, \forall t \in \mathbb{R}\}\). We can easily prove that \(W\) is a normal cone (in fact, if \(f_1 \geq f_2 \geq 0\) for \(f_1, f_2 \in W\), then \(\|f_1\| \geq \|f_2\|\), and hence, by Definition 4, \(W\) is normal). Furthermore, \(W^0 = \{x \in W: \text{there exists} \lambda > 0 \text{ such that} x(t) \geq \lambda, \forall t \in \mathbb{R}\}\). For every \(x \in W^0\), we consider the operator

\[
A_f[x](t) = \int_{t-\tau(t)}^{t} f(s, x(s)) \, ds,
\]

and it is well known that \(t \to f(t, x(t))\) is almost periodic. It follows from Lemma 3 that \(A_f[x](t)\) is almost periodic, namely \(A_f[W^0] \subset \mathcal{AP}(\mathbb{R}, \mathbb{R})\). Moreover, we can prove that \(A_f[W^0] \subset W^0\). In fact, for every \(x \in W^0\), there exists \(a > 0\) such that \(x(t) > a\) for all \(t \in \mathbb{R}\). If \(a \geq \rho\), the conclusion follows from \((H_1)\) and \((H_5)\). If \(a < \rho\), by \((H_1)\), \((H_2)\), and \((H_5)\), we have

\[
A_f[x](t) = \int_{t-\tau(t)}^{t} f(s, x(s)) \, ds \geq \int_{t-\tau(t)}^{t} f(s, a) \, ds
\]

\[
= \int_{t-\tau(t)}^{t} f(s, \rho^{-1}a) \, ds \geq \varphi(\rho^{-1}a) \int_{t-\tau(t)}^{t} f(s, \rho) \, ds \geq \rho^{-1}a\delta.
\]
Hence $A_f[W^0] \subset W^0$. Setting

$$\phi(\lambda) = \frac{\varphi(\lambda) + \lambda}{2}, \quad \forall \lambda \in (0, 1),$$

$$A_{(f,g)}[x,y](t) = \int_{t-T(t)}^{t} \left[ f(s,x(s)) + g(s,y(s)) \right] ds, \quad \forall t \in \mathbb{R}, \forall x, y \in W^0,$$

we can show by using $(H_1)$ that the operator $A_{(f,g)}$ is mixed monotone. Taking into account the fact that $A_f[W^0] \subseteq W^0$, the nonnegativity of the function $g$, and $0 < \phi(\lambda) \leq 1$, we have

$$A_{(f,g)} : W^0 \times W^0 \to W^0,$$

where $A_{(f,g)}[\lambda x, \lambda^{-1} y] \geq \phi(\lambda) A_{(f,g)}[x, y], \quad \forall x, y \in W^0$ and $\lambda \in (0, 1)$, and

$$0 < \lambda < \frac{\phi(\lambda) + \lambda}{2} = \phi(\lambda) \leq \frac{1 + \lambda}{2} < 1, \quad \forall \lambda \in (0, 1).$$

From Corollary 1, we obtain that the operator $A_{(f,g)}$ has exactly one fixed point (denote by $x^*$) in $W^0$, namely $A_{(f,g)}[x^*, x^*] = x^*$. For any real sequence $\alpha = (\alpha_n)$ such that $T_{\alpha}f = f$, $T_{\alpha}\tau = \tau$, $T_{\alpha}g = g$, it follows from Lemma 2 that

$$T_{\alpha}x^*(t) = T_{\alpha} \int_{t-T(t)}^{t} \left[ f(s,x^*(s)) + g(s,x^*(s)) \right] ds$$

$$= \int_{t-T_\tau(t)}^{t} T_{\alpha} \left[ f(s,T_{\alpha}x^*(s)) + g(s,T_{\alpha}x^*(s)) \right] ds$$

$$= \int_{t-T_\tau(t)}^{t} \left[ f(s,T_{\alpha}x^*(s)) + g(s,T_{\alpha}x^*(s)) \right] ds.$$

From the uniqueness of the fixed point, we have $T_{\alpha}x^* = x^*$. By Lemma 1, we can get $\text{mod}(x^*) \subseteq \text{mod}(f, g, \tau)$. The proof of the formula (18) can be obtained by Corollary 1 easily. □

**Corollary 2.** Suppose that all conditions in Theorem 2 hold. Furthermore, we assume that $f$, $g$, and $\tau$ are $T$-periodic in $t$ ($T > 0$). Then Eq. (2) has exactly one $T$-periodic solution $x^*$ with a positive infimum on $\mathbb{R}$.

**Proof.** It suffices to take $X$ as the Banach space consisting of $T$-periodic continuous functions with norm defined by $\|x\| = \sup_{t \in \mathbb{R}} |x(t)|$. The rest is the same, so we omit it. □

**Remark 2.** If the functions $\varphi$, $f$, and $g$ in Theorem 2 are endowed with the following assumptions: $\varphi$ is continuous, $g \equiv 0$, and $f(t, 0) \equiv 0$, $\forall t \in \mathbb{R}$, then Theorem 2 will coincide with [4, Theorem 2.3].
From the proofs of Theorems 1 and 2, we should indicate that the following Theorem 3 holds.

**Theorem 3.** For Eq. (2), suppose (H₁), (H₂), and (H₄) hold, in addition, we assume that the following conditions are satisfied:

\( \forall \xi > 0, \) there exists a positive function \( \varphi_\xi \) defined on \((0, 1)\) satisfying 
\[
\varphi_\xi (\lambda) > \lambda \quad \text{and} 
\]
\[
f(t, \lambda x) \geq \varphi_\xi (\lambda)f(t, x), \quad g(t, \lambda^{-1}y) \geq \varphi_\xi (\lambda)g(t, y),
\]
for any \( \lambda \in (0, 1), \) \( t \in \mathbb{R}, \) \( x, y \geq \xi; \)

(\( \bar{H}_5 \)) there exist \( w_1, w_2 \in W^0 \) such that
\[
\int_{t-\tau(t)}^{t} \left[ f(s, w_1(s)) + g(s, w_2(s)) \right] ds \geq w_1,
\]
\[
\int_{t-\tau(t)}^{t} \left[ f(s, w_2(s)) + g(s, w_1(s)) \right] ds \leq w_2,
\]
where \( W^0 \) is the same as the proof of Theorem 2.

Then Eq. (2) has exactly one almost periodic solution \( u^* \) with a positive infimum on \( \mathbb{R}. \)

**Proof.** Let \( X, W, \) and \( W^0 \) be the same as in the proof of Theorem 2. We define the operator
\[
A(f, g)[x, y](t) = \int_{t-\tau(t)}^{t} \left[ f(s, x(s)) + g(s, y(s)) \right] ds, \quad \forall t \in \mathbb{R}, \forall x, y \in W^0.
\]

It follows from Lemma 3 that \( A(f, g)[W^0 \times W^0] \subset \mathcal{AP}(\mathbb{R}, \mathbb{R}). \) Moreover, we can prove that \( A(f, g)[W^0 \times W^0] \subset W^0. \) In fact, for every \( z_1, z_2 \in W^0, \) setting
\[
\inf_{t \in \mathbb{R}} w_i(t) = \underline{w}_i, \quad \sup_{t \in \mathbb{R}} w_i(t) = \overline{w}_i, \quad \inf_{t \in \mathbb{R}} z_i(t) = \underline{z}_i, \quad \sup_{t \in \mathbb{R}} z_i(t) = \overline{z}_i, \quad i = 1, 2,
\]
it follows from (H₁), (H₃), and (H₄) that
\[
A(f, g)[z_1, z_2](t) \geq A(f, g)[\underline{z}_1, \overline{z}_1](t).
\]
If \( \underline{z}_1 \geq \overline{w}_1 \) and \( \overline{z}_2 \leq \underline{w}_2, \) using (H₅), we have
\[
A(f, g)[\underline{z}_1, \overline{z}_2](t) \geq A(f, g)[\underline{w}_1, \overline{w}_2](t) \geq A(f, g)[w_1, w_2](t) \geq w_1 \geq \underline{w}_1.
\]
If \( \underline{z}_1 \geq \overline{w}_1 \) and \( \overline{z}_2 \geq \overline{w}_2, \) we will know from (H₂), and (H₅) that
Analogous to the proof of Theorem 1 (from (10) to (12), replacing the function \( \phi(t) \) by \( (\phi(t) + t) / 2 \), \( \forall t \in (0, 1) \)), we could obtain the existence of the fixed point \( u^* \) of the operator \( A \) in \( W^0 \), so we omit it.

If the operator \( A \) has other fixed point \( u_* \in W^0 \), we let \( t_* = \sup \{0 < t < 1: tu_* \leq u_* \leq t^{-1}u_* \} \). Clearly, \( 0 < t_* \leq 1 \) and \( t_*u_* \leq u_* \leq t_*^{-1}u_* \). If \( t_* < 1 \), setting \( \rho = t_* \min \{\inf_{t \in R} u^*(t), \inf_{t \in R} u_*(t)\} \), then

\[
A(u_*, u_*) = A(t_*^{-1}u_*, t_*^{-1}u_*) \geq \left[ \frac{\varphi(t_*) + t_*}{2} \right] A(t_*^{-1}u_*, t_*^{-1}u_*) .
\]

\[
u_* = A(u^*, u^*) \geq A(t_*u_*, t_*^{-1}u_*) \geq \left[ \frac{\varphi(t_*) + t_*}{2} \right] A(t_*u_*, t_*^{-1}u_*) .
\]

\[
u_* = A(u^*, u^*) \leq A(t_*^{-1}u_*, t_*u_*) \leq \left[ \frac{\varphi(t_*) + t_*}{2} \right]^{-1} A(t_*u_*, t_*^{-1}u_*) .
\]

i.e.,

\[
\frac{\varphi(t_*) + t_*}{2} u_* \leq u_* \leq \left[ \frac{\varphi(t_*) + t_*}{2} \right]^{-1} u_*. 
\]

These contradict the definition of \( t_* \), since

\[
1 > \frac{\varphi(t_*) + t_*}{2} > t_* .
\]
Hence $t_0 = 1$ and $u^* = u_0$. This, at the same time, proves the uniqueness of fixed point of $A$ in $W^0$. □

4. Asymptotically almost periodic case

We begin with two propositions.

Proposition 1 [8, p. 36]. Definition 2 in this paper is equivalent with the following statement:

Suppose $f(t) \in BC(\mathbb{R}^+, \mathbb{R}^q)(BC(\mathbb{R}^+ \times \Omega, \mathbb{R}^q))$. For all $\varepsilon > 0$ (and for any compact subset $K$ of $\Omega$), there exist $l(\varepsilon) > 0$ ($l(\varepsilon, K) > 0$) and $T(\varepsilon) > 0$ ($T(\varepsilon, K) > 0$) such that any interval with length $l(\varepsilon)$ ($l(\varepsilon, K)$) contains $\sigma$ satisfying

$$
|f(t + \sigma) - f(t)| < \varepsilon, \quad \text{for all } t \geq T(\varepsilon),
$$

$$
|f(t + \sigma, x) - f(t, x)| < \varepsilon, \quad \text{for all } t \geq T(\varepsilon, K), \ x \in K.
$$

Proposition 2. If $f(t, x) \in \mathcal{AAP}(\mathbb{R}^+ \times \Omega, \mathbb{R}^q)$, then $f(t, x) : \mathbb{R}^+ \times K \to \mathbb{R}^q$ is bounded and uniformly continuous, where $K$ is any compact subset of $\Omega$.

Proof. Denote $f(t, x) = f_1(t, x) + f_2(t, x)$, where $f_1(t, x) \in \mathcal{AAP}(\mathbb{R}^+ \times \Omega, \mathbb{R}^q)$, $f_2(t, x) \in C_0(\mathbb{R}^+ \times \Omega, \mathbb{R}^q)$. Since $f_1(t, x)$ is bounded and uniformly continuous on $\mathbb{R}^+ \times K$, it suffices to prove that $f_2(t, x) : \mathbb{R}^+ \times K \to \mathbb{R}$ is bounded and uniformly continuous. The proof is easy, so we omit it. □

Theorem 4. Suppose (H1)–(H5) hold, in addition, $f$, $g$, and $\tau$ satisfy the following conditions:

(H6) $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are asymptotically almost periodic in $t$ uniformly for $x$ in any compact subset of $\mathbb{R}^+$;

(H7) $\tau(t)$ is a positive asymptotically almost periodic function on $\mathbb{R}^+$.

Then Eq. (2) has exactly one asymptotically almost periodic solution $x^*$ with a positive infimum on $\mathbb{R}^+$. Furthermore, for any initial $x_0, y_0 \in \mathcal{AAP}(\mathbb{R}^+, \mathbb{R})$ such that $\inf_{t \in \mathbb{R}^+} x_0(t) > 0$, and $\inf_{t \in \mathbb{R}^+} y_0(t) > 0$, the iterative sequences (16) and (17) satisfy (18).

Proof. Let $Y$ be the Banach space consisting of asymptotically almost periodic functions on $\mathbb{R}^+$ with the norm

$$
\|f\| = \sup_{t \in \mathbb{R}^+} |p(t)| + \sup_{t \in \mathbb{R}^+} |q(t)|,
$$

where $p \in \mathcal{AAP}(\mathbb{R}^+, \mathbb{R})$, $q \in C_0(\mathbb{R}^+, \mathbb{R})$, and $f = p + q$. Let $W$ be a cone of nonnegative functions in $\mathcal{AAP}(\mathbb{R}^+, \mathbb{R})$. Then $W$ is a normal cone. Furthermore, we have

$$
W^0 = \{x \in W: \text{there exists } \lambda > 0, \text{ with } x(t) \geq \lambda, \text{ for all } t \in \mathbb{R}^+\}.
$$
For every $x \in W^0$, set
\[ A_f[x](t) = \int_{t-\tau(t)}^t f(s, x(s)) \, ds, \tag{21} \]
Clearly, for any $x \in W^0$, $t \in \mathbb{R}^+$, and $\sigma \in \mathbb{R}^+$,
\[ |A_f[x](t + \sigma) - A_f[x](t)| \]
\[ = \left| \int_{t + \sigma - \tau(t+\sigma)}^{t + \sigma} f(s, x(s)) \, ds - \int_{t - \tau(t)}^t f(s, x(s)) \, ds \right| \]
\[ \leq \int_{t - \tau(t)}^{t + \sigma} f(s + \sigma, x(s + \sigma)) \, ds - \int_{t - \tau(t)}^t f(s, x(s)) \, ds \]
\[ + \int_{t - \tau(t)}^{t + \tau(t)} f(s, x(s + \sigma)) \, ds - \int_{t - \tau(t)}^t f(s, x(s)) \, ds \]
\[ \leq |f(t + \sigma, y) - f(t, y)| + |x(t + \sigma) - x(t)| + |\tau(t + \sigma) - \tau(t)| \leq 2M + S \varepsilon, \]
\[ \forall t \geq T(f, x, \tau, \delta, [0, N]), \text{ and } y \in [0, N]. \]
Hence,
\[ A_f[W^0] \subset A_d \mathcal{D}(\mathbb{R}^+, \mathbb{R}). \]
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the rest of the proof is analogous to the proof of Theorem 2, so we omit it. □

5. Pseudo almost periodic case

We begin with a proposition.

**Proposition 3.** If \( f : \mathbb{R} \to \mathbb{R} \) is a pseudo almost periodic function and \( \tau(t) \) is a positive almost periodic function in \( \mathbb{R} \), then the function

\[
    h(t) = \int_{t-\tau(t)}^{t} f(s) \, ds
\]

is also a pseudo almost periodic function.

**Proof.** Since \( f(t) = g(t) + w(t) \), where \( g \in AP(\mathbb{R}, \mathbb{R}) \) and \( w(t) \in \mathcal{AP}_0(\mathbb{R}, \mathbb{R}) \), we have

\[
    h(t) = \int_{t-\tau(t)}^{t} g(s) \, ds + \int_{t-\tau(t)}^{t} w(s) \, ds = I_1(t) + I_2(t). \tag{24}
\]

By Lemma 3, we know that \( I_1(t) \) is almost periodic. It suffices to consider \( I_2(t) \). Let \( q = \|\tau\| \). Clearly,

\[
    \frac{1}{2T} \int_{-T}^{T} |I_2(t)| \, dt = \frac{1}{2T} \int_{-T}^{T} \int_{t-\tau(t)}^{t} |w(s)| \, ds \, dt \leq \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} |w(s)| \, ds \, dt
\]

\[
    = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{0} |w(s + t)| \, ds \, dt.
\]

By Fubini’s theorem, we have

\[
    \frac{1}{2T} \int_{-T}^{0} \int_{-T}^{0} |w(s + t)| \, ds \, dt = \int_{-q}^{0} \frac{1}{2T} \int_{-T}^{T} |w(s + t)| \, ds \, dt \, ds = \int_{-q}^{0} F_T(s) \, ds,
\]

where

\[
    F_T(s) = \frac{1}{2T} \int_{-T}^{T} |w(s + t)| \, dt.
\]
∀s ∈ [−q, 0] and ∀T > 0. Since w ∈ \(\mathcal{PA}\mathcal{P}_0(\mathbb{R}, \mathbb{R}^+))\), we obtain lim\(_{T\to\infty}\) \(F_T(s) = 0\), ∀s ∈ [−q, 0]. Clearly, \(F_T\) is bounded. From the Lebesgue dominated convergent theorem, we have

\[
\lim_{T\to\infty} \int_{-q}^0 \frac{1}{2T} \int_{-T}^T |w(s + t)| \, ds \, dt = 0,
\]

which implies

\[
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T |I_2(t)| \, dt = 0.
\]

By Definition 3, \(h\) is pseudo almost periodic. □

**Theorem 5.** Suppose (H\(_1\))–(H\(_2\)), (H\(_4\))–(H\(_5\)) in Theorem 2 hold, in addition, we assume that \(f, g\) satisfy the following conditions:

- (H\(_3\)) \(f, g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+\) are pseudo almost periodic uniformly in \(t\) for \(x\) in any compact subset of \(\mathbb{R}^+\);
- (H\(_6\)) \(f, g\) are continuous in \(x \in G\) uniformly in \(t \in \mathbb{R}\) for every compact subset \(G \subset \mathbb{R}^+\).

Then Eq. (2) has a unique pseudo almost periodic solution \(x^*\) with a positive infimum on \(\mathbb{R}\). Furthermore, for each \(x_0, y_0 \in \mathcal{PA}\mathcal{P}(\mathbb{R}, \mathbb{R}^+)\) such that inf\(_{t \in \mathbb{R}}\) \(x_0(t) > 0\), and inf\(_{t \in \mathbb{R}}\) \(y_0(t) > 0\), the iterative sequences (16) and (17) satisfy (18).

**Proof.** Let \(Y\) be the Banach space consisting of pseudo almost periodic functions endowed with the norm defined by

\[
\|x(t)\| = \sup_{t \in \mathbb{R}} |p(t)| + \sup_{t \in \mathbb{R}} |q(t)|, \quad x(t) = p(t) + q(t),
\]

where \(p \in \mathcal{AP}(\mathbb{R}, \mathbb{R})\) and \(q \in \mathcal{PA}\mathcal{P}_0(\mathbb{R}, \mathbb{R})\). Let \(W\) be the cone consisting of nonnegative functions in \(\mathcal{PA}\mathcal{P}(\mathbb{R}, \mathbb{R})\). It is easy to know that \(W\) is a normal and solid cone. The interior of \(W\) is given by \(W^0 = \{x \in W : \text{there exists } \lambda > 0, \text{ with } x(t) \geq \lambda, \text{ for all } t \in \mathbb{R}\}\). For every \(x \in W^0\), we consider the operator

\[
A_f[x](t) = \int_{t - \tau(t)}^t f(s, x(s)) \, ds.
\]

From (H\(_3\)), Lemma 4, and Proposition 3, we obtain that

\[
A_f[x] \in \mathcal{AP}(\mathbb{R}, \mathbb{R}),
\]

thus,

\[
A_f[W^0] \subset \mathcal{AP}(\mathbb{R}, \mathbb{R}).
\]
Setting

\[ A(f,g)[x,y](t) = \int_{t-\tau(t)}^{t} \left[ f(s,x(s)) + g(s,y(s)) \right] ds, \quad \forall x, y \in W^{0}, \]

the rest of the proof is analogous to the proof of Theorem 2, so we omit it. \(\square\)

6. Infinite delay case

We begin with two propositions.

**Proposition 4** [2, Lemma 11]. If \( f \in \mathcal{AP}(\mathbb{R}, \mathbb{R}) \) and \( a \in L^{1}(\mathbb{R}^{+}) \), then the function

\[ h(t) = \int_{-\infty}^{t} a(t-s)f(s)ds \]

is almost periodic.

**Proposition 5** [2, Lemma 12]. If \( f \in \mathcal{APP}_{0}(\mathbb{R}, \mathbb{R}) \) and \( a \in L^{1}(\mathbb{R}^{+}) \), then the function

\[ h(t) = \int_{-\infty}^{t} a(t-s)f(s)ds \]

is also an element of \( \mathcal{APP}_{0}(\mathbb{R}, \mathbb{R}) \).

**Theorem 6.** Suppose the assumptions \((H_{1})-(H_{2}), (H_{8})-(H_{9})\) hold and the following condition is satisfied:

\[(H_{10}) a(s) \text{ is a positive integrable function on } \mathbb{R}^{+} \text{ and there exists } \rho > 0 \text{ such that } \]

\[ \delta = \inf_{t \in \mathbb{R}} \int_{0}^{+\infty} a(s)f(t-s, \rho)ds > 0. \]

Then Eq. (3) has a unique pseudo almost periodic solution \( x^{\ast} \) with a positive infimum. Furthermore, for each \( x_{0} \) and \( y_{0} \in \mathcal{AP}(\mathbb{R}, \mathbb{R}) \) such that \( \inf_{t \in \mathbb{R}} x_{0}(t) > 0 \) and \( \inf_{t \in \mathbb{R}} y_{0}(t) > 0 \), we construct the iterative sequences

\[ x_{n}(t) = \int_{-\infty}^{t} a(t-s) \left[ f\left(s, x_{n-1}(s)\right) + g\left(s, y_{n-1}(s)\right) \right] ds, \quad (n = 1, 2, \ldots), \]

\[ y_{n}(t) = \int_{-\infty}^{t} a(t-s) \left[ f\left(s, y_{n-1}(s)\right) + g\left(s, x_{n-1}(s)\right) \right] ds, \quad (n = 1, 2, \ldots), \]

then we have
\[ \|x_n - x^*\|, \|y_n - x^*\| \to 0 \quad (n \to \infty). \] (27)

**Proof.** Let \( Y, W, \) and \( W^0 \) be the same as in the proof of Theorem 4. For every \( u \in W^0, \)
denote
\[ A_f[u](t) = \int_{-\infty}^{t} a(t-s)f(s, u(s)) \, ds, \]
from \((H_9)\) and Lemma 4, Propositions 4, 5, we have that \( A_f[W^0] \subset Y. \) Choosing \( u \in W^0, \)
then there exists \( 0 < \lambda < \rho, \) such that \( u(t) \geq \lambda, \) for all \( t \in \mathbb{R}. \) By the monotonicity of \( f, \)
\((H_2),\) and \((H_{10}),\) one has
\[ A_f[u](t) \geq \int_{-\infty}^{t} a(t-s)f(s, \lambda) \, ds \geq \int_{-\infty}^{t} a(t-s)f\left(s, \frac{\lambda}{\rho} \right) \, ds \]
\[ \geq \varphi\left(\frac{\lambda}{\rho}\right) \int_{-\infty}^{t} a(t-s)f(s, \rho) \, ds \geq \delta \varphi\left(\frac{\lambda}{\rho}\right). \]

Thus
\[ A_f[W^0] \subset W^0. \]
Setting
\[ A_{(f,g)}[x, y](t) = \int_{t-\tau(t)}^{t} a(t-s)\left[f(s, x(s)) + g(s, y(s))\right] \, ds, \quad \forall t \in \mathbb{R}, \ \forall x, y \in W^0, \]
the rest of the proof is analogous to the proof of Theorem 5, so we omit it. \( \square \)

**Remark 3.** If the assumptions \((H_1)\)–\((H_2)\) are strengthened as \((H'_1)\)–\((H'_2)\), then the conclusion of Theorem 6 will coincide with \([2, \text{Theorem 9}].\)

7. Applications

**Example 1.** Consider the functions
\[ f_1(t, x) = \left[1 + \sin^2(\pi t) + \sin^2(t)\right] \log(1 + x), \] (28)
\[ f(t, x) = f_1(t, x) + 1, \] (29)
\[ g(t, x) = c[3 + \sin t + \sin \sqrt{3}t]x^{-1/2}, \quad c \geq 0, \] (30)
\[ \tau(t) = 1 + |\sin t|. \] (31)
In [1, p. 1488], E. Ait Dads and K. Ezzinbi presented the following conclusion: $f_1(t, x)$ in the formula (28) satisfies $(\overline{H}_2)$ of Theorem 3, i.e., for every $\xi > 0$, there exists positive functions $\phi_\xi$ defined on $(0, 1)$, such that $\forall \lambda \in (0, 1), t \in \mathbb{R}, x \geq \xi$, one has

$$f_1(t, \lambda x) \geq \phi_\xi(\lambda) f_1(t, x) \quad \text{and} \quad \phi_\xi(\lambda) > \lambda.$$  

Taking into account the fact $0 < \phi_\xi(\lambda) \leq 1$, it is easy to see that $\forall \lambda \in (0, 1), \forall t \in \mathbb{R}, x, y \geq \xi$,

$$f(t, \lambda x) \geq \phi_\xi(\lambda) f(t, x), \quad g(t, \lambda^{-1} y) \geq \phi_\xi(\lambda) g(t, y), \quad \text{and} \quad \phi_\xi(\lambda) > \lambda,$$

where $\phi_\xi(\lambda) = \min \{\phi_\xi(\lambda), \lambda^{1/2}\}$, for every $\lambda \in (0, 1)$. Thus Eq. (2) in which $f$ and $g$ are as in the formulas (29)–(30) satisfies $(\overline{H}_2)$. It is easy to see that

$$\lim_{x \to +\infty} \left( f(t, x)/x \right) = 0, \quad \text{uniformly in } t.$$  

We can take $\alpha > 1$ large enough such that

$$f(t, \alpha) \leq \frac{1}{4^\alpha} < \frac{1}{2^\alpha} + 10c \leq \alpha, \quad \text{uniformly in } t,$$

which implies

$$\int_{t-\tau(t)}^{t} \left[ f(s, 1) + g(s, \alpha) \right] ds \geq 1, \quad \int_{t-\tau(t)}^{t} \left[ f(s, \alpha) + g(s, 1) \right] ds \leq \alpha.$$  

Other assumptions in Theorem 3 are shown easily. From Theorem 3, we know that Eq. (2) in which $f, g, \tau$ are as in the formulas (29)–(31) has a unique positive almost periodic solution with a positive infimum.

**Example 2.** Consider the functions $f(t, x) = x^{1/2}$, $g \equiv 0$, $\tau(t) = 1$. Then Eq. (2) will yield

$$x(t) = \int_{t-1}^{t} \left( x(s) \right)^{1/2} ds. \quad (32)$$

The function $\psi$ of $(H_2)$ in Theorem 2 can be chosen as $\lambda^{1/2}$, where $\lambda \in (0, 1)$. We easily notice that $\int_{t-1}^{t} 1 ds = 1$. Next, we test the utility of formula (16). Choose $x_0 = 2$, we can obtain $x_1 = 2^{1/2}$, $x_2 = 2^{1/4}$, ..., $x_n = 2^{1/n}$, ... Hence, $\|x_n - 1\| \to 0$ ($n \to \infty$).

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References