JOURNAL OF ALGEBRA 62, 430–449 (1980)

A Generalized Frobenius Structure for Coalgebras with Applications to Character Theory

H. P. ALLEN*

Ohio State University, Columbus, Ohio 43210

AND

D. S. TRUSHIN†

Northern Illinois University, De Kalb, Illinois 60115

Received May 31, 1977

TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

In this paper we introduce and study Frobenius coalgebras. These are coalgebras which are not only comodule isomorphic to their rational duals, but coproper as well, i.e., coalgebras whose rational duals are dense (cf. [2]).

The structure of Frobenius coalgebras parallels that of Frobenius algebras in many respects. In particular such a coalgebra C possesses a nondegenerate C^*-associative form; all projective comodules are injective (and finite-dimensional injectives are projective). Moreover, from a representational point of view, C^* possess a computational structure which appears to behave like an integral for the case of a Hopf algebra. Using these results we obtain an elementary extention of Sullivan's results on the dimensions of the spaces of integrals for a Hopf algebra to Frobenius bialgebras.

We also introduce a general character theory from a coalgebraic point of view and give applications to cosemisimple involutorial Hopf algebras to obtain results of Larson on character relations, character formulas, and the dimension of absolutely irreducible comodules. Finally we present several versions of the first orthogonality relations for rational characters on Frobenius coalgebras.

1. Let k be a field and let C be a coalgebra over k. We will freely use the notation of [11]. In particular $C^{\square}$ (resp. ${}^{\square}C$) will denote the unique maximal rational left (resp. right) ideal of C^* and $\iota: V \to V^{**}$ the canonical inclusion of a vector space in its second dual.

* This research was supported by NSF Grant 4471-A1.

† Present address: Bell Telephone Laboratories, Inc., Naperville, Ill. 60540.

0021-8693/80/020430–20$02.00/0

1.1. PROPOSITION. *Let C be a coalgebra over the field k, and $\eta\colon C \to C^*$ a linear map.*

(a) *η is a left (right) C^*-morphism $\Leftrightarrow \eta^* \circ i$ is a right (left) C^*-morphism;*

(b) *η has dense image $\Leftrightarrow \eta^* \circ i$ is 1–1;*

(c) η is 1–1 $\Leftrightarrow \eta^* \circ i$ has dense image.

The proof of (a) and (b) is straightforward. Parts (b) and (c) are equivalent since $\eta = (\eta^* \circ i)^* \circ i$.

1.2. COROLLARY. *Let C be a coalgebra over the field k. Then $\exists$ a left C^*-monomorphism $\eta\colon C \to C^*$ with dense image $\Leftrightarrow \exists$ a right C^*-monomorphism $\zeta\colon C \to C^*$ with dense image.*

1.3. DEFINITION. A coalgebra C over the field k is called Frobenius if either condition of Corollary 1.2 holds.

We observe that a Frobenius coalgebra is automatically coproper. Indeed, if $\eta\colon C \to C^*$ is a left C^*-monomorphism with dense image then $\eta(C) \subseteq C^\square$ and is a dense left ideal. Then $C^\square$ is dense [2, Proposition 1.8]. Similarly, ${}^\square C$ is dense. In particular, $C^\square = C^* \cdot C^\square = {}^\square C C^\square \subseteq {}^\square C$ [2, Proposition 1.1], and by symmetry, $C^\square = {}^\square C$.

We will find it convenient to refer to a pair (C, η) (resp. (C, ζ)) as a Frobenius coalgebra if C is Frobenius and η (resp. ζ) is a left (resp. right) C^*-morphism with dense image.

1.4. PROPOSITION. *Let C be a coalgebra over the field k. Then the following are equivalent.*

(a) *$\exists$ a C^*-associative bilinear form $B^*\colon C^* \times C^\square \to k$ which is left nondegenerate and whose restriction to $C^\square \times C^\square$ is right nondegenerate.*

(b) *$\exists\ \lambda \in C^{\square *}$ whose kernel contains no (right or left) ideals of the form $c^* \cdot C^\square$, $c^* \neq 0$ or $C^\square \cdot c^\square$, $c^\square \neq 0$.*

(c) *C is Frobenius.*

(d) *$\exists$ a nondegenerate C^*-associative bilinear form on C.*

Proof. (a) $\Rightarrow$ (b). Let λ_{B^*} be the functional defined on $C^\square$ by setting $\lambda_{B^*}(x) = B^*(\epsilon, x)\ \forall x \in C^\square$. Now $c^* \cdot C^\square \subseteq \ker \lambda_{B^*} \Leftrightarrow B^*(c^*, C^\square) = 0 \Leftrightarrow c^* = 0$. Similarly, $C^\square \cdot c^\square \subseteq \ker \lambda_{B^*} \Leftrightarrow B^*(C^\square, c^\square) = 0 \Leftrightarrow c^\square = 0$. Thus the nondegeneracy conditions yield the desired information about $\ker \lambda_{B^*}$.

(b) $\Rightarrow$ (c). Let $\psi\colon C^\square \to C^\square \otimes_k C$ be the right C-comodule structure on $C^\square$. Define $\varphi_\lambda\colon C^\square \to C$ by $\varphi_\lambda = \lambda \otimes I \circ \psi$. It is clear that φ_λ is a left C^*-morphism. We note that

$$\langle y, \varphi_\lambda(x)\rangle = \sum_{(x)} \lambda(x_{(0)})\langle y, x_{(1)}\rangle = \lambda(y \cdot x)$$

$\forall x \in C^\square, y \in C^*$.

Now $y \in \varphi_\lambda(C^\square)^\perp \Leftrightarrow (0) = \langle y, \varphi_\lambda(C^\square)\rangle = \lambda(y \cdot C^\square)$. Thus $y = 0$ and φ_λ is an epimorphism. Finally if either $\varphi_\lambda(x) = 0$ or $\varphi_\lambda(x) \in (C^\square)^\perp$ we obtain $(0) = \langle C^\square, \varphi_\lambda(x)\rangle = \lambda(C^\square \cdot x)$, so $x = 0$. Thus φ_λ is an isomorphism and $C^\square$ is dense.

(c) $\Rightarrow$ (d). Let (C, η) be a Frobenius coalgebra and define $B_\eta: C \times C \to k$ by $B_\eta(c, d) = \langle \eta(d), c\rangle$. A rapid computation shows that B_η is bilinear, nondegenerate, and C^*-associative.

(d) $\Rightarrow$ (c). Given a C^*-associative nondegenerate bilinear form B on C we obtain a linear mapping $\eta_B: C \to C^*$ which satisfies $\langle \eta_B(d), c\rangle = B(c, d)$. η_B is easily seen to be a left C^*-monomorphism with dense image.

(c) $\Rightarrow$ (a). Let (C, η) be a Frobenius coalgebra and define $B_\eta^*: C^* \times C^\square \to C$ by $B_\eta^*(c^*, c^\square) = \langle c^*, \eta^{-1}(c^\square)\rangle$. B_η is clearly bilinear and C^*-associative. Left nondegeneracy is equivalent to $\eta^{-1}(C^\square) = C$ and right nondegeneracy on $C^\square \times C^\square$ is equivalent to the denseness of $C^\square$, and the injectivity of η^{-1}. Q.E.D.

EXAMPLES. (1) A finite-dimensional associative algebra is Frobenius [3] $\Leftrightarrow$ the dual coalgebra A^* is Frobenius. In particular every simple coalgebra is Frobenius.

(2) Let $C = \bigoplus C_\alpha$ be a direct sum of coalgebras. We know that C is coproper $\Leftrightarrow$ each C_α is coproper and that $C^\square = \bigoplus C_\alpha{}^\square$. If $\eta_\alpha: C_\alpha \to C_\alpha^*$ is a left C_α^*-morphism then $\eta = \bigoplus \eta_\alpha: C \to \bigoplus C_\alpha^* \to C^*$ is a left C^*-morphism. In particular C is Frobenius if each C_α is Frobenius. On the other hand suppose that C is a Frobenius coalgebra and that $C = D \oplus E$, D, E subcoalgebras. Choose $\lambda \in C^{\square *}$ as in Proposition 1.4(b). Now $d^* \cdot D^\square = d^* \cdot C^\square\ \forall d^* \in D^* \subseteq C^*$ (C^* is the ideal direct sum $D^* \oplus E^*$) so $(0) = \lambda\,|_{D^\square}\,(d^* \cdot D^\square) = \lambda(d^* \cdot C^\square) \Rightarrow d^* = 0$. Similarly $\lambda\,|_{D^\square}\,(D^\square \cdot d^\square) = 0 \Rightarrow d^\square = 0$. Thus D is Frobenius. Thus $C = \bigoplus C_\alpha$ is Frobenius $\Leftrightarrow$ each C_α is Frobenius. In particular any cosemisimple coalgebra is Frobenius.

(3) Let H be a Hopf algebra with nonzero left integral $\lambda_l \in H^*$. The map $h \to (\lambda_l \leftharpoonup h)$ defines a left H^*-monomorphism $\eta: H \to H^\square$ [11, 5.13]. By 1.1′, $(\eta^* \circ \iota)(H) \subseteq {}^\square H$ is dense in H^*. Thus ${}^\square H \neq (0)$ and since ${}^\square H \cong H \otimes \int_r$, H has a nonzero right integral also. Repeating the above argument on the right, we see that $H^\square$ is dense so H is coproper. Since $\int_l$ is one dimensional, η is surjective so H is Frobenius.

1.5. PROPOSITION. *Let C be a Frobenius coalgebra, e an idempotent in $C^\square$. Then*

(a) *$C^\square \cdot e$ and $C \cdot e$ are finite-dimensional projective and injective C^*-modules. Moreover, if $\{e_\alpha\}$ is a maximal set of orthogonal primitive idempotents in $C^\square$, then $C = \bigoplus C \cdot e_\alpha$.*

(b) *Let P be a finite-dimensional left $C^\square$-summand of $C^{(n)}$. Then P is projective and injective and there are orthogonal idempotents $\{e_1,..., e_s\} \subseteq C^\square$, $\{f_1,...,f_t\} \subseteq C^\square$ with $\oplus C \cdot e_i \cong P \cong \oplus C^\square \cdot f_j$.*

Proof. (a) From [2, 2.3] we know that any $C^\square$-summand of $C^\square$ is C^*-projective. Hence $C^\square \cdot e$ is finite dimensional and projective. Now $(C^\square \cdot e)^*$ is a finite-dimensional injective right C^*-module, hence isomorphic to a right C^*-summand of $C^{(n)}$ for some n. If ζ: $C \to {}^\square C = C^\square$ is a right C^*-isomorphism then by applying $\zeta^{(n)}$ we see that $(C^\square \cdot e)^*$ is isomorphic to a right ${}^\square C$-summand of $({}^\square C)^{(n)}$. Again by [2, 2.3] we have that $(C^\square \cdot e)^*$ is C^*-projective, which establishes the injectivity of $C^\square \cdot e$. $C \cdot e$ is injective [2, 2.17], hence also projective.

For the second assertion [2] shows that $C^\square = \oplus C^\square \cdot e_\alpha$, where $\{e_\alpha\}$ is any maximal set of orthogonal primitive idempotents. If we apply ζ^{-1} (ζ: $C^\square \to C$ a right C^*-isomorphism) we obtain $C = \oplus C \cdot e_\alpha$ as desired.

(b) Let P be a finite-dimensional left $C^\square$-summand of $C^{(n)}$. Since C is a direct sum of finite-dimensional injectives it is (left) C^*-injective [7], hence P is injective. By applying η^n we see that P is isomorphic to a finite-dimensional left summand of $C^{\square(n)}$; hence by [2] P is projective. By [2] we know that $P \cong C^\square \cdot e_1 \oplus \cdots \oplus C^\square \cdot e_s$, where $\{e_1,..., e_s\} \subseteq C^\square$ consists of orthogonal primitive idempotents. Similarly since P^* is projective, $P^* \cong f_1 \cdot C^\square \oplus \cdots \oplus f_t \cdot C^\square$, where $\{f_1,..., f_t\} \subseteq C^\square$ consists of orthogonal primitive idempotents. Since $f_j \cdot C^\square$ is finite dimensional it is closed in C^*; therefore $f_j \cdot C^\square = I_j^\perp$, where I_j is a right coideal in C. Let $\{f_\beta\}$ be a maximal set of orthogonal primitive idempotents in $C^\square$ containing $\{f_1,..., f_t\}$. Then it is clear that $I_j = (f_j \cdot C^\square)^\perp \supseteq \oplus_{\beta \neq j} C \cdot f_\beta$. However, $(f_j \cdot C^\square)^\perp \cap C \cdot f_j = (0)$ so $I_j = \oplus_{\beta \neq j} C \cdot f_\beta$. This implies that $(f_j \cdot C^\square)^* \cong C/I_j \cong C \cdot f_j$. Now $P \cong P^{**} \cong (f_1 \cdot C^\square)^* \oplus \cdots \oplus (f_t \cdot C^\square)^* \cong C \cdot f_1 \oplus \cdots \oplus C \cdot f_t$. Q.E.D.

We turn next to the internal structure of $C^\square$.

1.6. Definition. A subset $A \subseteq C^*$ is a right TA-set if for every finite-dimensional right C^*-module N and $n \in N$, $n \cdot a = 0$ for all but finitely many $a \in A$. If $A \subseteq C^\square$ we will call A a right rational TA-set.

Clearly a subset $A \subseteq C^*$ is a right TA-set if $N \cdot a = (0)$ for all but finitely many $a \in A$, whenever N is a finite-dimensional right C^*-module. We note that nontrivial right TA-sets always exist. Indeed any set of orthogonal idempotents is clearly a right (and left) TA-set.

Suppose that C is a coproper coalgebra, $\{e_\alpha\}$ a maximal set of orthogonal primitive idempotents in $C^\square = {}^\square C$. Then $C^\square = \oplus e_\alpha C^\square$ and we let $A = \cup A_\alpha$, where A_α is a basis for the finite-dimensional right ideal $e_\alpha C^\square$. It is clear that A is a k-basis for $C^\square$, which is a TA-set. For the moment we will refer to A as a special TA-basis for $C^\square$.

1.7. LEMMA. *Let* $\{\pi_\alpha\}$ *be the projections corresponding to the decomposition* $C^\square = \bigoplus e_\alpha C^\square$. *For* $B \subseteq C^\square$ *set* $B_\alpha = \{b \in B \mid \pi_\alpha b \neq 0\}$. *Then* B *is a right* TA*-set if and only if* $|B_\alpha| < \infty$ *for all* α.

Proof. $B_\alpha = \{b \in B \mid e_\alpha \cdot b \neq 0\}$ and since $e_\alpha C^\square$ is a finite-dimensional right C^*-module we see that B_α is finite if B is a TA-set. Conversely let N be a finite-dimensional right C^*-module and suppose that $N \cdot e_\alpha = (0)$ if $\alpha \notin \{\alpha_1, ..., \alpha_t\}$. If $b \notin \bigcup B_{\alpha_i}$, then $b = \sum e_\alpha b_\alpha$, where $\alpha \notin \{\alpha_1, ..., \alpha_t\}$. Thus $N \cdot b = (0)$. Q.E.D.

Note that the previous proof shows that B is a right TA-set if and only if $e \cdot b = 0$ for almost all b whenever e is an idempotent in $C^\square$.

1.8. LEMMA. *Let* $T: C^\square \to C^\square$ *be a* k*-morphism. Then the following are equivalent.*

(a) *T carries a special TA-basis to an TA-set.*

(b) *The matrix of T with respect to a special TA-basis is row finite.*

(c) *T carries TA-sets to TA-sets.*

Proof. (a) $\Rightarrow$ (b). Let $A = \{a_i \mid i \in I\}$ be a special TA-basis and write $Ta_j = \sum \lambda_{ij} a_i$. Fixing i we want $\lambda_{ij} = 0$ for all but finitely many j. Suppose that $a_i \in e_\alpha C^\square$ with $\{a_1, ..., a_e\} \subseteq A$ a basis for $e_\alpha C^\square$. Then $Ta_j \in (TA)_\alpha \Leftrightarrow \lambda_{sj} \neq 0$ for some $1 \leqslant s \leqslant e$. Thus if $(TA)_\alpha = \{Ta_{j_1}, ..., Ta_{j_n}\}$ then for $j \neq j_1, ..., j_n$ $Ta_j \notin (TA)_\alpha$ and $\lambda_{ij} = 0$.

(b) $\Rightarrow$ (c). Let B be a TA-set, A a special right TA-basis. For $b \in B$ write $b = \sum \gamma_j(b)\, a_j$. Let $\{e_\alpha\}$ be the maximal set of orthogonal idempotents involved with A, $e_\alpha C^\square \cap A = \{a_1, ..., a_e\}$, and $X = \{a_j \mid \lambda_{ij} \neq 0$ for some $1 \leqslant i \leqslant e\}$. By assumption, X is finite.

Now

$$\begin{aligned} Tb &= \sum\sum \gamma_j(b)\, \lambda_{kj} a_k \in (TB)_\alpha \\ &\Rightarrow \exists k \in \{1, ..., e\} \qquad \text{with } \sum \gamma_j(b)\, \lambda_{kj} \neq 0 \\ &\Rightarrow \exists j \text{ s.t. } \gamma_j(b) \neq 0 \qquad \text{and} \qquad \lambda_{kj} \neq 0 \\ &\Rightarrow a_j \in X \qquad \text{for some } j \text{ involved in } b. \end{aligned}$$

Consequently if $a_j \in e_\beta C^\square$ then $b \in B_\beta$. Hence $(TB)_\alpha = \cup \{B_\beta \mid a_j \in X \cap e_\beta C^\square\}$, which is finite.

(c) $\Rightarrow$ (a) is trivial. Q.E.D.

The set of k-endomorphisms of $C^\square$ just described is a subring of $\mathrm{End}_k\, C^\square$, which contains $\mathrm{End}_{C^*}(C^\square_{C^*})$ and which is stable under the left action of C^*.

Let (C, η) be a Frobenius coalgebra, $A = \{a_i \mid i \in I\}$ a k-basis for $C^\square$, and $B \subseteq C^*$ the dual functionals to the basis $\eta^{-1}A$. Letting B_η^* be the form described in the proof of (d) $\Rightarrow$ (a), Proposition 1.4, we see that $B_\eta^*(b_i, a_j) = \delta_{ij}$ so A and B are dual with respect to B_η^*. We call B the n-dual (or B_η^*-dual) set to A.

1.9. PROPOSITION. *Let (C, η) be a Frobenius coalgebra, A a k-basis for $C^\square$, and B the corresponding B_η^*-dual set. Then B is a right (resp. left) TA-set with respect to rational right (resp. left) C^*-modules.*

Proof. Let (M, ρ) be a finite-dimensional rational right C^*-module, $D = (\ker \rho)^\perp$. D is a finite-dimensional subcoalgebra and $\psi M \subseteq D \otimes M$, ψ the C-comodule structure. Choose $a_1, \ldots, a_t$ s.t. $D \subseteq \sum k\eta^{-1}(a_i)$. Then it is clear that $M \cdot b_j = (0)$ if $j \neq 1, \ldots, t$. A similar proof holds for rational left C^*-modules Q.E.D.

1.10. THEOREM. *Let (C, η) be a Frobenius coalgebra, $A = \{a_i \mid i \in I\}$ a TA-basis for $C^\square$, and $B = \{b_j \mid j \in I\}$ the η-dual set. Then kB is a right ideal.*

Proof. For $x \in C^*$, $c^\square \to xc^\square$ is a right C^* morphism, and hence by the remark following Lemma 1.8 is row finite with respect to A. For $x \in C^*$ write $x \cdot a_i = \sum a_j \lambda_{ji}(x)$ and compute

$$B_\eta^*(b_j x - \sum \lambda_{jk}(x)\, b_k, a_i) = B_\eta^*(b_j x, a_i) - \sum \lambda_{jk}(x)\, B_\eta^*(b_k, a_i)$$
$$= B_\eta^*(b_j, \sum a_s \lambda_{si}(x)) - \lambda_{ji}(x) = 0.$$

By nondegeneracy $b_j \cdot x = \sum \lambda_{jk}(x)\, b_k$ as desired. Q.E.D.

Note that since B is an independent set we see that $b_j \cdot x = \sum \lambda_{jk}(x)\, b_k \Leftrightarrow x \cdot a_i = \sum a_k \lambda_{ki}(x)$.

Before leaving this material we note that the bilinear form B^* arising from a Frobenius coalgebra (C, η) induces a k-monomorphism from C^* to $C^{\square *}$ which is easily seen to be an isomorphism. We also note that

$$B_\eta^*(xy, c^\square) = \sum B_\eta^*(x, c^\square_{(1)}) \langle y, c^\square_{(0)} \rangle$$
$$\forall x, y \in c^*, \qquad c^\square \in C^\square.$$

1.11. THEOREM. *Let (C, η) be a Frobenius coalgebra, $\{a_i \mid i \in I\}$ a TA-basis for $C^\square$, and $\{b_j \mid j \in I\}$ the η-dual set. Let (M, ρ) be a finite-dimensional rational left C^*-module. Then the following are equivalent.*

(1) *M is projective.*

(2) *M is injective.*

(3) $\exists X \in \operatorname{End}_k M$ s.t. $\sum \rho(a_i)\, X \rho(b_i) = I_M$.

Proof. Note that the sum in (3) is well defined since $\rho(b_j)m = 0$ a.e.

(1) $\Leftrightarrow$ (2) is standard.

(3) $\Rightarrow$ (2). Let (N, σ) be a left C^*-module, $\tau: M \to N$ a C^*-monomorphism, and let $\pi: N \to M$ be any k-splitting. Set $P = \sum \rho(a_i) X\pi\sigma(b_i) \in \text{Hom}_k(N, M)$. We compute

$$\begin{aligned} x \cdot P(n) &= \sum x a_i \cdot (X b_i \cdot n) \\ &= \sum a_\gamma \lambda_{\gamma i}(x)(X\pi(b_i n)) \\ &= \sum a_\gamma \cdot (X\pi(\lambda_{\gamma i}(x)\, b_i \cdot n)) \\ &= \sum a_\gamma \cdot (X\pi(b_\gamma \cdot x \cdot n)) \\ &= P(x \cdot n) \qquad \forall x \in C^*, \quad n \in N, \end{aligned}$$

and

$$\begin{aligned} P\tau(m) &= \sum a_i \cdot (X\pi(b_i \cdot \tau m)) \\ &= \sum a_i \cdot (X(b_i \cdot m)) \\ &= m. \end{aligned}$$

(1) $\Rightarrow$ (3). Let $\omega: C^* \otimes M \to M$ be the left C^*-module structure. It is clear that $\omega \mid C^\square \otimes M$ is surjective. Thus $\exists$ a C^*-morphism $\mu: M \to C^\square \otimes M$ such that $\omega\mu = I_M$. We define $\gamma: C^\square \otimes M \to C^* \otimes M$ by setting $\gamma(c^\square \otimes m) = B_\eta^*(\epsilon, c^\square)\epsilon \otimes m$ and compute

$$\begin{aligned} \sum_i a_i \cdot (\gamma(b_i \cdot (a_j \otimes m))) &= \sum_i a_i \cdot B_\eta^*(\epsilon, b_i a_j)\epsilon \otimes m) \\ &= \sum_i a_i \cdot B_\eta^*(b_i, a_j)\epsilon \otimes m \\ &= a_j \otimes m. \end{aligned}$$

Thus $\sum_i a_i \cdot \gamma(b_i \cdot z) = z \; \forall z \in C^\square \otimes M$. Set $X = \omega\gamma\mu: M \to C^\square \otimes M \to C^* \otimes M \to M$ and compute

$$\begin{aligned} \sum a_i \cdot X(b_i \cdot m) &= \zeta \left(\sum a_i \cdot \gamma(b_i \cdot \mu(m))\right)' \\ &= \zeta\mu m \\ &= m \qquad \forall m \in M \end{aligned}$$

as desired. Q.E.D.

Note that the proof of (3) $\Rightarrow$ (2) shows that $\sum a_i \cdot (\text{Hom}(M, N)\, b_i \subseteq \text{Hom}_{C^*}(M, N)$ and that this action is reminiscent of the action of an integral in the case of a Hopf algebra (cf. [1]). Also the implications $1 \Rightarrow 2, 3$, $3 \Rightarrow 2, 1$ are valid even if M is infinite dimensional, provided one restricts attention to the category of rational C^*-modules.

1.12. PROPOSITION. *Let* (C, η) *be a Frobenius coalgebra*, $\{a_i \mid i \in I\}$ *a TA-basis for* $C^\square$, $\{b_j \mid j \in I\}$ *the corresponding* η*-dual set, and* $\rho\colon C^* \to \operatorname{End} M$ *a rational representation. Then*

$$\sum B_\eta^*(\epsilon, a_i)\, \rho(b_i) = I_M\,.$$

Proof. Let $\varphi = \eta^{-1}(C) = \bigoplus k\varphi(a_i)$ and if we identify $k\varphi(a_i)^*$ with kb_i we see that $c^* \to \Pi\langle c^*, \varphi a_i\rangle\, b_i$ is a k-isomorphism between C^* and Πkb_i. If (M, ρ) is a rational C^* module then $\rho b_i = 0$ a.e. so the indicated sum makes sense.

Now let $m \in M$ and writing $\psi_M m = \sum m_j \otimes \varphi a_j$ (ψ_M, the comodule structure map) we compute,

$$\begin{aligned} \sum B_\eta^*(\epsilon, a_i) b_i \cdot m &= \sum \langle \epsilon, \varphi(a_i)\rangle\langle b_i\,, \varphi a_j\rangle\, m_j \\ &= \sum \langle \epsilon, \varphi(a_j)\rangle\, m_j \\ &= m. \end{aligned}$$

Q.E.D.

1.13. PROPOSITION.[1] *Let* (A, η) *be a Frobenius bialgebra. Then the space of left (resp. right) integrals is* 1*-dimensional.*

Proof.

$$\int_l = \{x \in A^\square \mid yx = \langle y, 1_A\rangle x\ \forall y \in A^*\},$$

$$\begin{aligned} \eta^{-1} \int_l &= \{a \in A \mid y \cdot a = \langle y, 1_A\rangle a\ \forall y \in A^*\} \\ &= \{a \in A \mid \Delta a = a \otimes 1_A\} \\ &= k1_A\,. \end{aligned}$$

Q.E.D.

Note that the existence of an integral in the infinite-dimensional case does not guarantee that A is a Hopf algebra as is the case for certain finite-dimensional cases [5]. Indeed if S is an infinite semigroup then kS is a Frobenius bialgebra and hence has a nonzero integral. But kS is a Hopf algebra $\Leftrightarrow$ S is a group.

2. In this section we construct one automorphism of $C^\square$ and derive some of its properties.

[1] As pointed out by the referee, Proposition 1.13 is far more general. If $g \in GC$ and $L_g = \{x \in C^* \mid yx = \langle y, g\rangle x\ \forall y \in C^*\}$, then L_g is a two-sided ideal. If (C, η) is Frobenius then $\eta^{-1}L_g = \{c \in C \mid y \cdot c = \langle y, g\rangle c\ \forall y \in C^*\} = \{c \in C \mid sc = c \otimes g\} = kg$.

2.1. LEMMA. *Let (C, η) be a Frobenius coalgebra over k and $\{e_\alpha \mid \alpha \in \Lambda\}$ a maximal set of primitive orthogonal idempotents in $C^\square$. Then*

(1) *$C \cdot x$ is finite dimensional $\forall x \in C^\square$.*

(2) *The map $\tau: C^\square \to C^\square$ defined by $\tau(x) = \sum_\alpha \eta(\varphi(e_\alpha) \cdot x)$, where $\varphi = \eta^{-1}$, is an algebra morphism.*

Proof. (1) Since $\eta \circ \iota$ is a right C^* isomorphism between C and $C^\square$, $\eta^* \circ \iota(C \cdot x) = \eta^* \circ \iota(C) \cdot x = C^\square \cdot x$ is f.d.

(2) It is clear that $\{\varphi(e_\alpha) \cdot x \mid \varphi(e_\alpha) \cdot x \neq 0\}$ is linearly independent. By (1) this set is finite and so τ is well defined and linear.

Now $\varphi(x \cdot \tau(y)) = \sum_\alpha \varphi(x \cdot \eta(\varphi(e_\alpha) \cdot y)) = \sum_\alpha \varphi(xe_\alpha) \cdot y = \varphi(x) \cdot y \ \forall x \in C^\square$ since $\sum_\alpha x \cdot e_\alpha = x[2]$. Therefore $\tau(x)\,\tau(y) = \eta(\varphi(\tau(x) \cdot \tau(y)) = \eta(\varphi(\tau(x)) \cdot y) = \eta(\varphi(\tau(x)) \cdot y) = \eta(\sum_\alpha \varphi(e_\alpha) \cdot xy) = \tau(xy)$. Q.E.D.

2.2. LEMMA. *Let (C, η) be a Frobenius coalgebra, and τ the algebra endomorphism of 2.1. Then $\varphi \cdot \tau$ is a right C^* isomorphism between $C^\square$ and C.*

Proof. $(\varphi \circ \tau)(xy) = \sum_\alpha \varphi(e_\alpha) \cdot xy = (\varphi \circ \tau(x)) \cdot y$, so $\varphi \circ \tau$ is a right C^*-morphism. Q.E.D.

We have already seen (Proposition 1.1 that $\eta^* \circ \iota$ is a right C^*-morphism between C and $C^\square$ and we now claim that $(\eta^* \circ \iota)^{-1} = \varphi \circ \tau$. It suffices to show that $\varphi \circ \tau$ is a right inverse to $\eta^* \circ \iota$.

For $c, d \in C$, it is evident that $\langle \eta^* \circ \iota(c), d\rangle = \langle \eta(d), c\rangle$. Thus

$$\begin{aligned}\langle(\eta^* \circ \iota)(\varphi \circ \tau)(x), d\rangle &= \langle \eta(d), \varphi \circ \tau(x)\rangle = \left\langle \eta(d), \sum_\alpha \varphi(e_\alpha) \cdot x\right\rangle \\ &= \left\langle 1_{C^*}, \sum_\alpha \varphi(\eta(d) \cdot e_\alpha) \cdot x\right\rangle = \langle 1_{C^*}, \varphi(\eta(d)) \cdot x\rangle \\ &= \langle 1_{C^*}, d \cdot x\rangle = \langle x, d\rangle.\end{aligned}$$

Q.E.D.

These two lemmas yield the following:

2.3. THEOREM. *Let (C, η) be a Frobenius coalgebra, τ as before. Then τ is an algebra automorphism.*

2.4. COROLLARY. *τ is independent of the choice of idempotents $\{e_\alpha \mid \alpha \in \Lambda\}$.*

Proof. $\tau = \eta \circ (\eta^* \circ \iota)^{-1}$. Q.E.D.

Suppose that H is a Hopf algebra and $\lambda \in H^*$ is anonzero left integral. Let $\eta: H \to H^\square$ be defined by $\eta(h) = \lambda \leftharpoonup h$, (cf. Example 3). For $h, k \in H$, $\langle \eta^* \circ \iota(h), k\rangle = \langle \eta(k), h\rangle = \langle \lambda \leftharpoonup k, h\rangle = \langle \lambda, hS(k)\rangle = \langle \lambda \leftharpoonup h, S(k)\rangle = \langle S^*(\lambda \leftharpoonup h), k\rangle$ and so $\eta^* \circ \iota(h) = S^*(\lambda \leftharpoonup h)$. Now if $x \in H^\square$ and $\varphi = \eta^{-1}$ as before, then $\tau^{-1}(x) = (\eta^* \circ \iota)\, \eta^{-1}(x) = (\eta^* \circ \iota(\varphi(x))) = S^*(\lambda \leftharpoonup \varphi(x))$. Without loss of generality we may take $\varphi(\lambda) = 1_H$ and so $\tau^{-1}(\lambda) = S^*(\lambda \leftharpoonup 1_H) = S^*(\lambda)$. S^* is surjective (S is injective) so $S^*(\lambda)$ is a right integral for H, and nonzero since $S^*(\lambda) = \tau^{-1}(\lambda)$.

We conclude this section with the following useful fact.

2.5. LEMMA. *Let (C, η) be a Frobenius coalgebra. Let $z_\tau(C^\square) = \{x \in C^\square \mid yx = x\tau(y)\ \forall y \in C^\square\}$, and let $\chi(C) = \{c \in C \mid \Delta c = T\Delta c\}$; then $\eta(\chi(C)) = z_\tau(C^\square)$.*

Proof. $c \in \chi(C) \Rightarrow c^* \cdot c = c \cdot c^*\ \forall c^* \in C^*$. Then $\eta(c)\,\tau(y) = \eta(c \cdot y) = \eta(y \cdot c) = y\eta(c)$. Thus $\eta(c) \in Z_\tau(C^\square)$. Conversely, if $x \in Z_\tau(C^\square)$ and $y \in C^\square$, then $y\eta^{-1}(x) = \eta^{-1}(yx) = \eta^{-1}(x\tau(y)) = \eta^{-1}(x(\eta(\sum \eta^{-1}(e_x) \cdot y))) = \eta^{-1}(x)y$. Now $C^\square \otimes C^\square$ is dense in $C^* \otimes C^*$ and $(\Delta - T\Delta)\,\eta^{-1}(x) \in (C^\square \otimes C^\square)^\perp$. Thus $(\Delta - T\Delta)\,\eta^{-1}(x) \in (C^* \otimes C^*)^\perp$ so $(\Delta - T\Delta)\,\eta^{-1}(x) = 0$. Q.E.D.

3. This section serves as an introduction to character theory for coalgebras. We begin with a basic view of characters for associative algebras [2].

3.1. DEFINITION. Let C be a coalgebra over k. A character χ of C is simply a cocommutative element. If A is an associative algebra over k then a character on A is simply a character on A^0. The space of characters on A, char A, is thus $[A, A]^\perp \cap A^0$.

A character on C^* is called rational if it lies in $\iota(C) \subseteq C^{**}$. We will always identify a rational character on C^* as a character of C.

3.2. EXAMPLE. Let A be an associative algebra and (M, ρ) be a finite-dimensional A-module. Then $\chi \equiv \chi_M = \mathrm{tr} \circ \rho$ is a character on A. We will also say that χ is afforded by M in this case; similarly for characters afforded by comodules.

Let (M, ρ) be a finite-dimensional rational C^*-module, $\{m_i\}$ a basis for M. Set $\psi m_i = \sum m_j \otimes c_{ji}$. Then $\chi_M = \iota(\sum c_{ii})$. Thus the character associated to a rational module is rational.

Conversely let (M, ρ) be an absolutely irreducible finite dimensional C^*–module whose character is rational. Then M is rational. Indeed let $\{m_i\}$ be a basis for M and choose $c^*_{kj} \in C^*$ with $\rho(c^*_{ij}) = E_{ij}$. Set $c^* \cdot m_i = \sum m_j f_{ji}(c^*)$, where $f_{ji} \in C^{**}$, and let $\chi_M = \iota(c)$. Then $f_{ji} = \sum \langle c^*_{ij}, c_{(1)} \rangle\, \iota c_{(2)} \in \iota(C)$ and M is rational.

3.3. PROPOSITION. *Let χ be a character of the coalgebra C, M a simple comodule. Then χ is afforded by $M \Leftrightarrow \chi$ generates* $\mathrm{Supp}_C(M)$.

The character χ_M is called irreducible, absolutely irreducible, etc., if M is irreducible, absolutely irreducible, etc., and characters in general are class functions since $[xg^{-1}, g] = x - gxg^{-1}$ for any unit in $A(C^*)$. Also characters afforded by representations are clearly additive "in M." The following results also hold for coalgebra characters and are obtained by considering rational $A = C^*$-modules.

3.4. PROPOSITION. *Let* $(M_i, \rho_i) i = 1, 2$ *be absolutely irreducible finite-dimensional A-modules,* χ_i *the corresponding characters. Then*

$$M_1 \simeq M_2 \Leftrightarrow \chi_1 = \chi_2 .$$

Proof. $\chi_1 = \chi_2 \Rightarrow \ker \rho_1 = \ker \rho_2$, for ρ_i is surjective and $\ker \rho_i = \{x \mid \chi_i(xa) = 0, \forall a \in A\}$. Thus $\exists$ an isomorphism θ: End $M_1 \to$ End M_2 s.t. $\rho_1\theta = \rho_2$. If θ is realized as conjugation by the isomorphism T: $M_1 \to M_2$ then the last equation states that T is a module morphism. Q.E.D.

3.5. PROPOSITION. *Let* (M_i, ρ_i) $i = 1,..., t$ *be inequivalent absolutely irreducible finite-dimensional A-modules,* χ_i *the corresponding characters. Then* $\chi_1 ,..., \chi_t$ *are linearly independent.*

Proof. Let $\chi_1 = \sum_{i \geqslant 2} z_i \chi_i$ be a dependence relation. Then $\bigcap_{i \geqslant 2} \ker \rho_i \subseteq \ker \rho_1$. Thus $\exists$ a morphism $\bigoplus_{\geqslant 2}$ End $M_i \to$ End M_1 s.t.

$$\begin{array}{ccc} & A & \\ {\scriptstyle \rho_1} \swarrow & & \searrow {\scriptstyle \oplus \rho_i} \\ \text{End } M_1 & \longleftarrow & \bigoplus_{\geqslant 2} \text{End } M_i . \end{array}$$

commutes. This $\Rightarrow M_1 \simeq M_j$ some $j \geqslant 1$.

3.6. PROPOSITION. *Let A be a* s.s. algebra *with* min-*r* *over an* a.c. *field k. Then* char(A) *is spanned by irreducible characters.*

Proof. Let $f \in$ char(A) and let I be a cofinite ideal in ker f. $\bar{A} = A/I$ is finite dimensional, s.s. over an a.c. field. Thus $\bar{A} =$ End $M_1 \oplus \cdots \oplus$ End M_r, where $(M_i, \bar{\rho}_i)$ are irreducible representations of $\bar{A}$. Then ρ_i: $A \to \bar{A} \to$ End M_i are irreducible representations of A. f induces a functional $\bar{f}$ on $\bar{A}$ and $\bar{f} = \sum \bar{f}_i \bar{\rho}_i$, $\bar{f}_i = \bar{f} \mid$ End M_i. Now ker $\bar{f}_i \supseteq$ [End M_i, End M_i] so $\bar{f}_i = z_i \operatorname{tr}_{\mathrm{End} M_i}$, $z_i \in k$.
Now:

$$\begin{aligned} f(a) &= \bar{f}(a + I) \\ &= \bar{f}\left(\sum \bar{\rho}_i(a + I)\right) \\ &= \sum \bar{f}_i \bar{\rho}_i(a + I) \\ &= \sum z_i \operatorname{tr}_{\mathrm{End} M_i} \bar{\rho}_i(a + I) \\ &= \sum z_i \operatorname{tr}_{\mathrm{End} M_i} \rho_i(a) \\ &= \sum z_i \chi_{M_i}(a). \end{aligned}$$

Q.E.D.

We turn next to some character aspects of coalgebra theory.

Let C be an arbitrary coalgebra. We set $\text{coco}(C) = \ker(\Delta - T\Delta) = \{$cocommutative elts$\}$ and note that $\text{coco}(\oplus C_W) = \oplus \text{ coco } C_W$.

3.7. PROPOSITION. $\text{coco}(C) = [C^*, C^*]^\perp$.

Proof. $c \in \text{coco}(C)$, a^*, $b^* \in C^* \Rightarrow \langle a^*b^*, c\rangle = \langle a^* \otimes b^*, \Delta c\rangle = \langle a^* \otimes b^*, T\Delta c\rangle = \langle b^* \otimes a^*, \Delta c\rangle = \langle b^*a^*, c\rangle$. $\therefore c \in [C^*, C^*]^\perp$. The reverse inclusion is similar. Q.E.D.

3.8. PROPOSITION. *Let C be a coalgebra. Then $\exists$ minimal coideal J with $I_m(\Delta - T\Delta) \subseteq J \otimes J$. Furthermore, $J^\perp = \text{center}(C^*)$. Call J the cocenter of $C(\text{cocen}(C))$.*

Proof. Let $\{c_i\}$ be a basis for C and write $(\Delta - T\Delta)c = \sum m_i(c) \otimes c_i$. Let $J = \text{span}(m_i(c) \mid i, c)$ and choose $c_i^* \in C^*$ with $\langle c_i^*, c_j\rangle = \delta_{ij}$.

For $c^* \in \text{center } C^*$,

$$\begin{aligned} 0 &= ([c^*, c_j^*], c) \\ &= \langle c^* \otimes c_j^*, \Delta c\rangle - \langle c_j^* \otimes c^* \Delta c\rangle \\ &= \langle c^* \otimes c_j^*, (\Delta - T\Delta)c\rangle \\ &= \sum \langle c^* \otimes c_j^*, m_i(c) \otimes c_i\rangle \\ &= \langle c^*, m_j(c)\rangle. \end{aligned}$$

Thus center $(C^*) \subseteq J^\perp$.

Conversely, for $c^* \in J^\perp$, $x^* \in C^*$, $c \in C$, $\langle [c^*, x^*], c\rangle = \langle c^* \otimes x^*, (\Delta - T\Delta)c\rangle = 0$, so $\text{center}(C^*) = J^\perp$.

Now repeat the same argument on the right to obtain a subspace J' with $\text{center}(C^*) = J'^\perp$, and $(\Delta - T\Delta)(C) \subseteq C \otimes J'$. Thus $J = J'$ is a coideal and $(\Delta - T\Delta)(C) \subseteq J \otimes C \cap C \otimes J = J \otimes J$. Q.E.D.

3.9. *Remark.* $C = \oplus C_w \Rightarrow \text{cocen}(C) = \oplus \text{ cocen}(C_w)$.

3.10. THEOREM. *Let C be a* c.s.s. *coalgebra over an* a.c. *field k of characteristic p. Assume that* $(\dim C_w, p) = 1$, $\forall$ *simple subcoalgebra C_w. Then*

$$C = \text{cocen}(C) \oplus \text{coco}(C).$$

Proof. $C = \oplus C_w$, therefore it suffices to establish the result for C simple. In this case $C^* \simeq k_n$ for some n and our hypotheses imply that $(n, p) = 1$. It is trivial to see that $k_n = kI \oplus [k_n, k_n]$, hence $C \simeq (\text{center } k_n)^\perp \oplus [k_n, k_n]^\perp \simeq \text{cocen}(C) \oplus \text{coco}(C)$. Q.E.D.

3.11. Application. Let G be a finite group with $(|G|, p) = 1$. Then $A = kG = \text{center}(A) \oplus [A, A]$ and # inequivalent absolutely irreducible representations $\leqslant \dim \text{center}(A) = $ # conjugacy classes in G. For arbitrary finite G we always have # inequivalent absolutely irreducible representations $\leqslant \dim \text{center}(kG/\text{rad}(kG))$.

3.12. *Remark.* We shall see that the above result holds for c.s.s. Hopf algebras.

Let (M, ρ) be a rational left C^*-module. $D = D_M = (\ker \rho)^{\perp}$ is the unique minimal subcoalgebra of C with $\psi(M) \subseteq D \otimes M$, D is called the cosupport of M, and we observe that M is a faithful rational D^*-module. Fix $m^* \in M^*$. Then $M \to M \otimes C \to^{m^* \otimes 1} k \otimes C \to C$ is a comodule morphism, and hence M is isomorphic to a subcomodule of C^n. In particular every simple C-comodule is isomorphic to a minimal right coideal.

Let λ be a minimal right coideal in C, and set $H(\lambda) = \sum\{\lambda' \subseteq C \mid \lambda' \cong \lambda\}$. Let $\{\lambda_b \mid b \in B\}$ be a representative collection of minimal right coideals. Then $\text{corad}(C) = \oplus H(\lambda_b)$, and $H(\lambda_b)$ is a simple subcoalgebra. If λ is an absolutely simple right coideal of dimension d, then $H(\lambda) \cong \oplus^d \lambda$.

3.13. Example. Let L be a split s.s. Lie algebra over a field of characteristic 0, $U = $ u.e.a. (L), $\{V_t \mid t \in T\}$ a representative collection of irreducible finite-dimensional L-modules. Then each V_t is absolutely irreducible and $\{V_t \mid t \in T\}$ is a representative collection of simple U^0-comodules. This implies that U^0 is the direct sum of simple subcoalgebras $H_t = \oplus^{d_t} V_t$ (as U^0-comodule).

3.14. Proposition (Larson's coordinatization [5]). *Let M be an absolutely simple right C-comodule, D the cosupport. For each choice of basis $\{m_i\}$ for M, there is a basis $\{d_{ij}\}$ for D* s.t.,

(a) $\psi m_k = \sum m_s \otimes d_{sk}$,

(b) $\Delta d_{ij} = \sum d_{ik} \otimes d_{kj}$, $\epsilon(d_{ij}) = \delta_{ij}$.

If M is a rational left H^*-module, H a Hopf algebra, then End M inherits an H^*-module structure which is characterized by $(h^* \rightharpoonup T)(m) = \sum (h^* \leftharpoonup m_{(1)}) \cdot T(m_{(0)})$ [1].

3.15. Theorem. *Let H be a* c.s.s. *Hopf algebra with $S^2 = I$, (M, ρ) an absolutely irreducible rational H^*-module, and λ a nonzero idempotent in $\int_H$. Then* $(\dim M, \text{char } k) = 1$ *and*

$$\lambda \rightharpoonup \rho(h^*) = \frac{1}{\dim M} \chi_M(h^*) I.$$

Proof. We know that $\lambda \rightharpoonup \text{End } M \subseteq \text{End}_H M = kI$, so $\lambda \rightharpoonup T = y(T)I$. Computing $\lambda \rightharpoonup E_{ij}$, we see that

$$\begin{aligned}(\lambda \rightharpoonup E_{ij})m_k &= \sum \lambda \rightharpoonup d_{sk} \cdot E_{ij} m_s \\ &= \lambda \rightharpoonup d_{jk} \cdot m_i \\ &= \sum m_r \langle \lambda \rightharpoonup d_{jk}, d_{ri} \rangle \\ &= \sum m_r \langle \lambda, d_{ri} S d_{jk} \rangle.\end{aligned}$$

Since $\lambda \rightharpoonup E_{ij}$ is diagonal we see that

$$\begin{aligned}\langle \lambda, d_{ri} S d_{jk} \rangle &= 0 \qquad \text{if} \quad r \neq k \\ &= y(E_{ij}) \qquad \text{if} \quad r = k.\end{aligned}$$

Thus $\langle \lambda, d_{ki} S d_{jk} \rangle = y(E_{ij})$. Summing over k yields

$$\left\langle \lambda, \sum d_{ki} S d_{jk} \right\rangle = \dim M y(E_{ij})$$

or

$$\text{tr}(E_{ij}) = \delta_{ij} = \epsilon(d_{ji}) = \dim M y(E_{ij}).$$

Thus $\text{tr}(T) = \dim M y(T)$ for all $T \in \text{End } M$. In particular, $1 = \text{tr}(E_{ii}) = \dim M y(E_{ii})$ so $(\dim M, \text{char } k) = 1$. We thus see that $\lambda \rightharpoonup T = (1/\dim M) \cdot \text{tr}(T)$ for all $T \in \text{End } M$. Q.E.D.

Let H be a c.s.s. Hopf algebra with $S^2 = I$.

3.16. COROLLARIES. (i) $H = \text{cocen}(H) \oplus \text{coco}(H)$ *and* $H^* =$ *center* $H^* \oplus [H^*, H^*]$.

(ii) H^* *has at most* dim(*center* H^*) *inequivalent irreducible representations.*

(iii) H *is* f.d. $\Leftrightarrow$ coco(H) *is* f.d. $\Leftrightarrow$ *center* H^* *is* f.d.

As a final observation we see that if χ is a character of a coalgebra C, then $B(c^*, d^*) = \langle c^* d^*, \chi \rangle$ defines a symmetric bilinear form on C^* (which is nondegenerate precisely when χ generates C).

4. Let (C, η) be a Frobenius coalgebra and (M, ρ) an irreducible left C^*-module. Recall that $\text{socle}_M(C^*) = \sum\{\text{left ideals } I \subseteq C^* \mid I \simeq M\}$ is a two-sided ideal of C^*. If M is a rational module then $\text{socle}_M(C^*) \subseteq C^\square$, $\text{socle}_M(C^*) = \text{socle}_M(C^\square)$, and is a finite-dimensional simple ideal. Furthermore

$$\eta^{-1}(\text{socle}_M(C^\square)) = \text{supp}_C(M) = (\ker \rho)^\perp.$$

4.1. Proposition. *Let* (C, η) *be a Frobenius coalgebra,* (M, ρ) *a rational irreducible left* C^**-module. If* $c \in \operatorname{supp}(M)$ *then* $(\ker \rho) \cdot \eta(c) = (0)$ *and for any left* C^**-module* N, $C^*\eta(c) \cdot N \subseteq \operatorname{socle}_M(N)$.

Proof. The first part is clear since $\ker \rho \cdot c = 0$. For the second part $\ker \rho \cdot C^*\eta(c) \cdot N = (0)$ so $C^*\eta(c) \cdot N$ is a module for the finite-dimensional simple algebra $C^*/\ker \rho$. As such it is a direct sum of irreducibles isomorphic to M.

4.2. Proposition. *Let* (C, η) *be a Frobenius coalgebra and let* (M, ρ) *be an irreducible rational left* C^**-module. Then* $\operatorname{socle}_M(C^{\square})$ *is a nonzero simple two-sided ideal. Moreover the following are equivalent.*

(a) $\operatorname{socle}_M(C^{\square})$ *is a block of* $C^{\square}$.

(b) $(\operatorname{socle}_M(C^{\square}))^2 \neq (0)$.

(c) M *is projective.*

Proof. Since M is simple, $\operatorname{supp}_C(M)$ is a finite-dimensional simple subcoalgebra. By Proposition 4.1 if we take $0 \neq c \in \operatorname{supp}_C(M)$ then $0 \neq \eta(c) \in C^{\square} \cdot \eta(c) \cdot C^{\square} \subseteq \operatorname{socle}_M(C^{\square})$. We claim that $C^{\square}\eta(c) \cdot C^{\square} = \operatorname{socle}_M(C^{\square})$. Indeed let $I \subseteq C^{\square}\eta(c)C$ be a minimal left ideal (hence $I \cong M$) and let $f\colon I \to C^{\square}$ be a left C^*-morphism. Since $C^{\square}$ is injective (Proposition 1.5) $\exists$ a $C^{\square}$-morphism $g\colon C^* \to C^{\square}$ s.t. $g|_I = f$. Then $f(I) = g(I) = I \cdot g(\epsilon) \subseteq C^{\square}\eta(c)\, C^{\square}C^{\square} \subseteq C^{\square}\eta(c)\, C^{\square}$. Moreover this argument shows that C^* (hence $C^{\square}$) acts transitively (from the right) on {minimal left ideals}, and thus $\operatorname{socle}_M C^{\square}$ is simple.

(a) $\Rightarrow$ (b) follows from the fact that $C^{\square}$ has no nil (module) summands [2] while (a) $\Rightarrow$ (c) follows from the fact that M is isomorphic to a $C^{\square}$-summand of $C^{\square}$ [2]. (b) $\Rightarrow$ (c). If M is not projective, then $M \cong I$, I a bottom composition factor of $C^{\square}e_\alpha$ for some α ($\{e_\alpha\}$ a maximal set of orthogonal primitive idempotents) [2]. $_M$ut then $I \subseteq \operatorname{rad}(C^{\square})$, so $\operatorname{socle}_M(C^{\square}) \subseteq \operatorname{rad} C^{\square} \subseteq \ker \rho$.

(c) $\Rightarrow$ (a). Since M is projective, $M \cong C^{\square}e_\alpha$ for some α ($\{e_\alpha\}$ a maximal set of orthogonal primitive idempotents). Now e_β is linked to $e_\alpha \Leftrightarrow e_\beta C^{\square} e_\alpha \neq (0)$ so e_β linked to e_α implies that the top composition factor of $C^{\square}e_\beta$ is isomorphic to M, and hence $C^{\square}e_\beta \cong C^{\square}e_\alpha$ [3]. Q.E.D.

4.3. Proposition. *Let* (C, η) *be a Frobenius coalgebra and* M, N *simple right-comodules. Then* $M \cong N$ *if either* $\eta(\operatorname{supp}_C M) \cdot N \neq (0)$ *or* $\langle \eta(\operatorname{supp}_C M), (\operatorname{supp}_C N)\rangle \neq (0)$.

Proof. The second condition is stronger than the first for $0 \neq \langle \eta(c), d\rangle$, $c \in \operatorname{supp}_C M$, $d \in \operatorname{supp}_C N$ implies $\eta(c) \notin \ker \rho_N$, hence $\eta(c) \cdot N \neq (0)$. Now $\eta(\operatorname{supp}_C M) \cdot N \neq (0)$ together with proposition 4.1 immediately establishes $M \cong N$. Q.E.D.

We note that the above proof shows that $\eta(c)\,\eta(d) \neq (0)$ (c, d, M, N as above) implies $M \cong N$.

We conclude this section with a result on characters.

4.4. THEOREM. *Let M be a simple right comodule for the Frobenius coalgebra (C, η), and let $\chi_M \in C$ be the corresponding rational character (cf. (3.3)). Then $C^{\square}\eta(\chi_M) = \text{socle}_M(C^{\square})$. In particular M is projective $\Leftrightarrow \eta(\chi_M)^2 \neq 0$.*

Proof. By the proof of the first part of Proposition 4.2 $\text{socle}_M(C^{\square}) = C^{\square}\eta(\chi_M)\,C^{\square}$. However, Lemma 2.5 shows that $C^{\square}\eta(\chi_M)\,C^{\square} = C^{\square}\eta(\chi_M)\,\tau(C^{\square}) = C^{\square 2}\eta(\chi_M) = C^{\square}\eta(\chi_M)$ since $C^{\square}$ is dense.

For the second part, if M is projective then $(0) \neq (\text{socle}_M C^{\square})^2 = C^{\square}\eta(\chi_M) \cdot C^{\square}\eta(\chi_M) \subsetneq C^{\square}\eta(\chi_M)^2$ so $\eta(\chi_M)^2 \neq 0$. Conversely $\eta(\chi_M)^2 \neq 0$ implies $(\text{socle}_C M)^2 \neq (0)$ so M is projective. Q.E.D.

Theorem 4.4 shows that a simple comodule M is projective if and only if $\eta(\chi_M)$ is a unit in $\text{socle}_M C^{\square}$. This together with Proposition 4.3 shows that $\eta(\chi_M)$ is a unit $\Leftrightarrow \langle \eta(\chi_M), \chi_M \rangle \neq 0$.

5. This last section develops orthogonality relations for characters for Frobenius coalgebras and parallels the procedure in [4].

5.1. THEOREM. *Let (C, η) be a Frobenius coalgebra, M, N simple C-comodules with characters χ_M and χ_N, respectively. If $\langle \eta(\chi_M), \chi_N \rangle \neq 0$ then $M \cong N$ and $\chi_M = \chi_N$. If $\{a_i\}$ is a TA-basis for $C^{\square}$ and $\{b_i\}$ is the B^*-dual set then $\sum_i \langle a_i, \chi_M \rangle \cdot \langle b_i, \chi_N \rangle \neq 0 \Rightarrow M \cong N$.*

The proof of Theorem 5.1 follows from Proposition 4.3, Theorem 4.4, and Proposition 3.4 and the fact that $\eta(c) = \sum \langle b_i, c \rangle\, a_i$.

5.2. COROLLARY. *If $M \ncong N$ then $\sum_i \langle a_i x, \chi_N \rangle \langle b_i, \chi_M \rangle = 0\ \forall x \in C^*$.*

Proof. $\eta(\chi_M) = \sum_\alpha \langle b_i, \chi_M \rangle\, a_i$ so $\sum_i \langle a_i x, \chi_N \rangle \langle b_i \chi_M \rangle \neq 0$ implies $0 \neq \langle \eta(\chi_M) x, \chi_N \rangle = \langle \eta(\chi_M), x \cdot \chi_N \rangle$ so $N \cong M$ (by Proposition 4.3).

If H is a Hopf algebra with nonzero left integral λ, and we choose the usual morphism η given by $\eta(h) = \lambda \leftharpoonup h$, then $\langle \eta(\chi_M), \chi_M \rangle = \langle \lambda \leftharpoonup \chi_M, \chi_N \rangle = \langle \lambda, \chi_N S \chi_M \rangle$. We thus obtain the following restatement of Larson's character formula [5].

5.3. THEOREM. *Let H be a Hopf algebra with nonzero left integral $\lambda \neq 0$, M, N simple H-comodules with $\langle \lambda, \chi_N S \chi_M \rangle \neq 0$. Then $M \cong N$.*

We list several alternative versions of Theorem 5.3 as

5.4. Corollary. *Let H be a Hopf algebra with nonzero left integral λ, η as above $\{a_i\}$ a TA-basis for $C^\square$, $\{b_i\}$ the η-dual set, and set $c_i = \eta^{-1}a_i$. Let M, N be simple H-comodules. Then $M \cong N$ if either*

(a) $\sum_i \langle a_i, \chi_N\rangle\langle b_i, \chi_M\rangle \neq 0$ *or*

(b) $\sum_i \langle \lambda, \chi_N Sc_i\rangle\langle b_i, \chi_M\rangle \neq 0$ *holds.*

5.5. Definition. Let C be a coalgebra over a field k, M a simple right C-comodule, N an arbitrary right C-comodule. M is said to have finite multiplicity in N if there is a bound to the number of occurrences of M as a composition factor for finite-dimensional subcomodules of N. In the contrary case M is said to have infinite multiplicity in N.

Clearly a coalgebra C is finite dimensional if and only if corad(C) is finite dimensional and every simple C-comodule has finite multiplicity in C.

5.6. Proposition. *Let C be a coalgebra and M a simple right C-comodule, I a right coideal having M as composition factor m times. Then $C^*/I^\perp$ has M^* as composition factor m times.*

Proof. $\perp$ a composition series for I.

5.7. Proposition. *Let C be a left coproper coalgebra, J a closed cofinite right ideal in C^*. Then C^*/J and $C^\square/J \cap C^\square$ are isomorphic C^*-modules.*

Proof. C^*/J is a cyclic rational C^*-module and the canonical map θ: $C^* \to C^*/J(\simeq J^{\perp *})$ is continuous. Then $\theta|_{C^\square}$ is surjective so $C^*/J = \theta(C^\square) \cong (C^\square + J)/J \cong C^\square/J \cap C^\square$. Q.E.D.

Recall that if C is coproper and $C^\square = \bigoplus e_\alpha C^\square$ is an idempotent decomposition for $C^\square$, and if N is an irreducible rational C^*-module, then $\exists\alpha$ s.t. $Ne_\alpha \neq (0)$ for some primitive idempotent e_α, so $N = Ne_\alpha C^\square$ for some α, and hence $N \cong e_\alpha C^\square/e_\alpha \operatorname{rad} C^\square$ [2, 3].

One easily obtains from [3] that the number of composition factors of $e_\gamma C^\square$ isomorphic to N is $\dim e_\alpha C^\square e_\gamma = n_\gamma$. Note that $n_\gamma = 0$ a.e.

5.8. Theorem. *Let C be a coproper coalgebra, M a simple right C-comodule. Then M has finite multiplicity in C.*

Proof. It suffices to produce an integer $m \geqslant 0$ such that M^* appears fewer than m times as a composition factor in $C^\square/J$ for all cofinite right ideals J of $C^\square$.

Let $J \subseteq C^\square$ be a cofinite right ideal. Then $e_\alpha \in J$ a.e. Let $J' = \sum \{e_\alpha C^\square \mid e_\alpha \in J\} \subseteq J$. J' is a cofinite right ideal and the number of occurrences of M^* as a composition factor in $C^\square/J' \cong \bigoplus \{e_\alpha C^\square \mid e_\alpha \notin J\}$ is at most $\sum n_\beta = m$. Q.E.D.

Let (C, η) be Frobenius and $\{M_\mu \mid \mu \in U\}$ a representative collection of simple right C-comodules; $\{\chi_\mu \mid \mu \in U\}$ the corresponding characters. Let N be a finite-dimensional comodule having M_μ as constituent $h_\mu =$ times. Then $\chi_N = \sum_\mu h_\mu \chi_\mu$, and by Corollary 5.2 for fixed μ_0, $x \in C^*$,

$$\begin{aligned} \langle \eta(\chi_N)x, \chi_{\mu_0} \rangle &= \sum h_v \langle \eta(\chi_\mu)\, x, \chi_{\mu_0} \rangle \\ &= h_{\mu_0} \langle (\chi_{\mu_0})\, x, \chi_{\mu_0} \rangle. \end{aligned}$$

5.9. PROPOSITION. *Let (C, η) be a Frobenius coalgebra, $\{a_i\}$ a TA-basis for $C^\square$, $\{b_i\}$ the dual B_η^*-set, M a simple right C-comodule, and N a finite-dimensional C-comodule. Then*

$$\begin{aligned} \langle \eta(\chi_N) \cdot x, \chi_M \rangle &= h_M \langle \eta(\chi_M) \cdot x, \chi_M \rangle \\ &= h_M \sum_i \langle b_i, \chi_M \rangle \langle a_i x, \chi_M \rangle \end{aligned}$$

$\forall x \in C^*$ (h_M, the multiplicity of M in N).

Proof. $\eta(\chi_M) = \sum_i \langle b_i, \chi_M \rangle a_i$, so $\langle \eta(\chi_M) \cdot x, \chi_M \rangle = \sum_i \langle b_i, \chi_M \rangle \langle a_i x, \chi_M \rangle$. The result follows easily from this. Q.E.D.

Let M be a f.d. simple projective C-comodule, (C, η) Frobenius. We know that $\mathrm{socle}_M(C^\square) = A$ is a block so $A \simeq M_h(D)$, where D is a division algebra and h is the multiplicity of M in A. Hence $h^2 \dim D = h \dim M$. In particular, if $h \neq 0$ (in k) we obtain $h \dim D = \dim M$.

5.10. THEOREM. (a) *Let (C, η) be a Frobenius coalgebra, M a simple projective C-comodule, N a finite-dimensional C-comodule in which the multiplicity of M is the same as the multiplicity of M in $C^\square$. Assume that the multiplicity h_M of M in N is not 0 in k. Then $\exists x \in C^\square$ s.t. $\langle \eta(\chi_N) \cdot x, \chi_M \rangle \neq 0$. Furthermore,*

(b) $\dim_k D = \langle \eta(\chi_N) \cdot x, \chi_M \rangle^{-1} \epsilon(\chi_M) \sum_i \langle b_i, \chi_M \rangle \langle a_i \cdot x, \chi_M \rangle$, where $\{a_i\}$ is a TA-basis for $C^\square$, $\{b_i\}$ the corresponding dual set.

Proof. Since $\mathrm{socle}_M(C^\square)$ is a block, $C^\square = \mathrm{socle}_M(C^\square) \oplus (C^\square \cap \ker \rho_M)$. Now $\langle \mathrm{socle}_M(C^\square), \chi_M \rangle \neq (0)$ together with $\mathrm{socle}_M(C^\square) = \eta(\chi_M)\, C^\square$ (Theorem 4.4 and Lemma 2.5) implies that $\langle \eta(\chi_M)\, C^\square, \chi_M \rangle \neq (0)$ so $\exists x \in C^\square$ s.t. $\langle \eta(\chi_M) \cdot x, \chi_M \rangle \neq 0$. $_M$ut now $\langle \eta(\chi_N) \cdot x, \chi_M \rangle = h_M \langle \eta(\chi_M) \cdot x, \chi_M \rangle \neq 0$.

Since $\dim M = \epsilon(\chi_M)$ we need only combine Proposition 5.9 with the observation preceding Theorem 5.10 to obtain the result. Q.E.D.

If M is not projective, then Theorem 4.4 together with Proposition 4.2 shows that $(0) = \eta(\chi_M)\, C^\square \eta(\chi_M) = \eta(\chi_M)(C^* \eta(\chi_M))$. Thus $\forall x \in C^*$, $\eta(\chi_M)\, x \eta(\chi_M) = 0$ so $\langle \eta(\chi_M) \cdot x, \chi_M \rangle = 0$. Thus a necessary and sufficient condition for a simple C-comodule M to be projective is that $\exists$ an $x \in C^*$ with $\langle \eta(\chi_M) \cdot x, \chi_M \rangle \neq 0$.

At this point we translate Proposition 5.9 and Theorem 5.10 in terms of a C^*-associate nondegenerate bilinear form B on C.

COROLLARY. *Let (C, B) be a Frobenius coalgebra, M a simple C-comodule, and N a C-comodule. Then* $B(x \cdot \chi_M, \chi_N) = h_M B(x \cdot \chi_M, \chi_M)$ $\forall x \in C^*$. *In particular* $B(\chi_M, \chi_N) = h_M B(\chi_M, \chi_M)$.

5.12. COROLLARY. *Let (C, B) be a Frobenius coalgebra, M a simple projective C-comodule, N a C-comodule in which the multiplicity of M is the same as the multiplicity of M in C. Let D be the division algebra part of* $\operatorname{socle}_M C^{\square}$. *Then*

$$\dim D = \frac{\epsilon(\chi_M)}{B(x\chi_M, \chi_N)} B(x\chi_M, \chi_M)$$

for all $x \in C^*$ with $B(x\chi_M, \chi_M) \neq 0$.

We conclude with

5.13. THEOREM (First orthogonality relation). *Let (C, η) be a Frobenius coalgebra, $B = B_\eta$, and M be an absolutely simple C-comodule with* $(\dim M, \operatorname{char} k) = 1$. *Then M is projective $\Leftrightarrow \exists x \in C^*$ such that $\langle \eta(\chi_M)x, \chi_M \rangle = B(x \cdot \chi_M, \chi_M) \neq 0$. For each $x \in C^*$ with this property there is a $\lambda = \lambda(x) \in k$, $\lambda \neq 0$ such that for every simple C-comodule N,*

$$\delta_{N,M} = \frac{\dim_k M}{\lambda} \sum_i \langle b_i, \chi_M \rangle \langle a_i x, \chi_N \rangle$$

$$= \frac{\epsilon(\chi_M)}{\lambda} B(x\chi_M, \chi_N),$$

where $\{a_i\}$ is a TA-basis for $C^{\square}$ with $\{b_i\}$ the dual set.

Proof. Clear.

REFERENCES

1. H. P. ALLEN, Invariant radical splittings: A Hopf approach, *J. Pure Appl. Algebra* **3** (1973), 1–19.
2. H. P. ALLEN AND D. S. TRUSHIN, Coproper coalgebras, *J. Algebra* **54** (1978), 203–215.
3. C. CURTIS AND I. REINER, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1966.
4. T. V. FOSSUM, Characters and orthogonality in frobenius algebras, *Pacific J. Math.* **36** (1) (1971).
5. R. G. LARSON, Characters of Hopf algebras, *J. Algebra* **17** (1971), 352–368.
6. R. G. LARSON AND M. E. SWEEDLER, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75–94.

7. B. I-P. Lin, "On Homological Properties of Coalgebras," Ph.D. dissertation, University of Illinois at Chicago Circle, 1973.
8. D. E. Radford, Finiteness conditions for a Hopf algebra with non-zero integral, *J. Algebra*, in press.
9. D. E. Radford and R. G. Heyneman, Reflexivity and coalgebras of finite type, *J. Algebra* **28** (1974), 215–246.
10. J. B. Sullivan, The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras, *J. Algebra* **19** (1971), 426–440.
11. M. E. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.
12. H. Zassenhaus, "TRACE Bilinear Forms."