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Original Article

Fixed point theorems in complex valued metric spaces

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Abstract The aim of this paper is to establish and prove several results on common fixed point for a pair of mappings satisfying more general contraction conditions portrayed by rational expressions having point-dependent control functions as coefficients in complex valued metric spaces. Our results generalize and extend the results of Azam et al. (2011) [1], Sintunavarat and Kumam (2012) [2], Rouzkard and Imdad (2012) [3], Sitthikul and Saejung (2012) [4] and Dass and Gupta (1975) [5]. To substantiate the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also furnished.

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1. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. This idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis cannot be generalized to cone metric spaces but to complex valued metric spaces.

Complex valued metric space is useful in many branches of Mathematics, including algebraic geometry, number theory,

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applied Mathematics as well as in physics including hydrodynamics, mechanical engineering, thermodynamics and electrical engineering. After the establishment of complex valued metric spaces, Rouzkard et al. [3] established some common fixed point theorems satisfying certain rational expressions in these spaces to generalize the result of [1]. Subsequently Sintunavarat et al. [2,6] obtained common fixed point results by replacing the constant of contractive condition to control functions. Recently, Sitthikul et al. [4] established some fixed point results by generalizing the contractive conditions in the context of complex valued metric spaces. Many researchers have contributed with different concepts in this space. One can see in [7–13].

In what follows, we recall some notations and definitions due to Azam et al. [1], that will be used in our subsequent discussion.

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the followings conditions is satisfied:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 1.1 ([1]). Let X be a non-empty set. A mapping d: $X \times X \to C$ is called a complex valued metric on X if the following conditions are satisfied:

(CM1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, we say that (X, d) is a complex valued metric space.

Example 1.1. Let $X = C$ be a set of complex number. Define d: $C \times C \to C$. By

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C, d) is a complex valued metric space.

Example 1.2 (inspired by [2]). Let $X = C$. Define a mapping d: $X \times X \to C$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in [0, \frac{\pi}{2}]$. Then (X, d) is a complex valued metric space.

Definition 1.2. [1] Suppose that (X, d) is a complex valued metric space.

1. We say that a sequence $\{x_n\}$ is a *Cauchy sequence* if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x_m) \prec c$ for all $n, m \geq N$.
2. We say that $\{x_n\}$ converges to an element $x \in X$ if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x) \prec c$ for all $n \geq N$. In this case, we write $x_n \xrightarrow{d} x$.
3. We say that (X, d) is complete if every Cauchy sequence in X converge to a point in X.

Lemma 1.1. *[1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2. *[1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

2. Main result

We start to this section with the following observation.

Proposition 2.1. *Let (X, d) be a complex valued metric space and S, T: $X \to X$. Let $x_0 \in X$ and defined the sequence $\{x_n\}$ b;y*

$$x_{2n+1} = Sx_{2n},$$
$$x_{2n+2} = Tx_{2n+1}, \ \forall\, n = 0, 1, 2, \ldots \tag{2.1}$$

Assume that there exists a mapping λ: $X \times X \times X \to [0, 1)$ such that $\lambda(TSx, y, a) \leq \lambda(x, y, a)$ and $\lambda(x, STy, a) \leq \lambda(x, y, a), \forall x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2, \ldots$. Then

$$\lambda(x_{2n}, y, a) \leq \lambda(x_0, y, a) \quad \text{and} \quad \lambda(x, x_{2n+1}, a) \leq \lambda(x, x_1, a).$$

Proof. Let $x, y \in X$ and $n = 0, 1, 2, \ldots$. Then we have

$$\lambda(x_{2n}, y, a) = \lambda(TSx_{2n-2}, y, a) \leq \lambda(x_{2n-2}, y, a)$$
$$= \lambda(TSx_{2n-4}, y, a) \leq \cdots \leq \lambda(x_0, y, a).$$

Similarly, we have

$$\lambda(x, x_{2n+1}, a) = \lambda(x, STx_{2n-1}, a) \leq \lambda(x, x_{2n-1}, a)$$
$$= \lambda(x, STx_{2n-3}, a) \leq \cdots \leq \lambda(x, x_1, a). \quad \square$$

The subsequent example illustrates the preceding proposition.

Example 2.1. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. Define d: $X \times X \to C$ as $d(x, y) = i|x - y|$ then clearly (X, d) is a complex valued metric space also define self-mappings S and T by

$$S\Big(\frac{1}{n+1}\Big) = \frac{1}{n+2} = T\Big(\frac{1}{n+1}\Big), \ n = 0, 1, 2, 3, \ldots.$$

Choosing sequence $\{x_n\}$ as $x_n = \frac{1}{n+1}, n = 0, 1, 2, 3, \ldots$ Then $x_0 = 1 \in X$.

Clearly $Sx_{2n} = x_{2n+1}$ and $Tx_{2n+1} = x_{2n+2}$.

Consider a mapping λ: $X \times X \times X \to [0, 1)$ by $\lambda(x, y, a) = \frac{x}{6} + \frac{y}{8} + a$, for all $x, y \in X$ and for fixed $a = \frac{1}{2} \in X$, then $\lambda(x, y, a) = \frac{x}{6} + \frac{y}{8} + \frac{1}{2}$.

Undoubtedly

$$\lambda(TSx, y, a) \leq \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \leq \lambda(x, y, a),$$

for all $x, y \in X$ and for fixed $a \in X$.

Consider

$$\lambda(x_{2n}, y, a) = \frac{1}{6(2n+1)} + \frac{y}{8} + \frac{1}{2} \ \leq \ \frac{1}{6} + \frac{y}{8} + \frac{1}{2}$$
$$= \lambda(x_0, y, a),$$

that is $\lambda(x_{2n}, y, a) \leq \lambda(x_0, y, a), n = 0, 1, 2 \ldots, \forall y \in X$ and for $a = \frac{1}{2} \in X$. Also consider

$$\lambda(x, x_{2n+1}, a) = \frac{x}{6} + \frac{1}{8(2n+2)} + \frac{1}{2} \le \frac{x}{6} + \frac{\frac{1}{2}}{8} + \frac{1}{2} = \lambda(x, x_1, a),$$

that is $\lambda(x, x_{2n+1}, a) \le \lambda(x, x_1, a), n = 0, 1, 2 \ldots, \forall x \in X$ and for fixed $a = \frac{1}{2} \in X$.

Thus Proposition 2.1 is verified.

Lemma 2.1. *[4] Let $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = \mid d(x_n, x_{n+1}) \mid$ satisfies*

$$a_n \le h a_{n-1}, \forall\, n \in N,$$

then $\{x_n\}$ is a Cauchy sequence.

Our main theorem runs as follows.

Theorem 2.1. *Let (X, d) be a complete complex valued metric space and $S, T\colon X \to X$. If $\exists$ mappings $\lambda, \mu, \gamma, \delta\colon X \times X \times X \to [0, 1)$ such that $\forall x, y \in X$,*

(a) $\lambda(TSx, y, a) \le \lambda(x, y, a)$ *and* $\lambda(x, STy, a) \le \lambda(x, y, a)$,
$\mu(TSx, y, a) \le \mu(x, y, a)$ *and* $\mu(x, STy, a) \le \mu(x, y, a)$,
$\gamma(TSx, y, a) \le \gamma(x, y, a)$ *and* $\gamma(x, STy, a) \le \gamma(x, y, a)$,
$\delta(TSx, y, a) \le \delta(x, y, a)$ *and* $\delta(x, STy, a) \le \delta(x, y, a)$;

(b)

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a) \times \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y, a)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} + \delta(x, y, a)\left\{\frac{d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)}{1 + d(x, Ty) + d(y, Sx)}\right\}; \tag{2.2}$$

(c)

$$\lambda(x, y, a) + \mu(x, y, a) + \lambda(x, y, a) + \delta(x, y, a) < 1, \tag{2.3}$$

then S and T have a unique common fixed point.

Proof. Let $x, y \in X$, from (2.2), we have

$$\begin{aligned} d(Sx, TSx) &\precsim \lambda(x, Sx, a)d(x, Sx) + \mu(x, Sx, a)\frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)} \\ &\quad + \gamma(x, Sx, a)\frac{d(Sx, Sx)d(x, TSx)}{1 + d(x, Sx)} + \delta(x, Sx, a) \\ &\quad \times\left\{\frac{d(x, Sx)d(x, TSx) + d(Sx, TSx)d(Sx, Sx)}{1 + d(x, TSx) + d(Sx, Sx)}\right\} \\ &= \lambda(x, Sx, a)d(x, Sx) + \mu(x, Sx, a)\frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)} \\ &\quad + \delta(x, Sx, a)\frac{d(x, Sx)d(x, TSx)}{1 + d(x, TSx)}. \end{aligned}$$

So that

$$\begin{aligned} |d(Sx, TSx)| &\le \lambda(x, Sx, a)|d(x, Sx)| + \mu(x, Sx, a)\left|\frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)}\right| \\ &\quad + \delta(x, Sx, a)\left|\frac{d(x, Sx)d(x, TSx)}{1 + d(x, TSx)}\right| \\ &= \lambda(x, Sx, a)|d(x, Sx)| + \mu(x, Sx, a)\left|\frac{d(x, Sx)}{1 + d(x, Sx)}\right| |d(Sx, TSx)| + \delta(x, Sx, a)\left|\frac{d(x, TSx)}{1 + d(x, TSx)}\right| |d(x, Sx)|. \end{aligned}$$

$$\begin{aligned} \Rightarrow \mid d(Sx, TSx) \mid &\le \lambda(x, Sx, a) \mid d(x, Sx) \mid \\ &\quad + \mu(x, Sx, a) \mid d(Sx, TSx) \mid \\ &\quad + \delta(x, Sx, a) \mid d(x, Sx) \mid . \end{aligned} \tag{2.4}$$

Similarly, from (2.2) we have

$$\begin{aligned} d(STy, Ty) &\precsim \lambda(Ty, y, a)d(Ty, y) + \mu(Ty, y, a) \\ &\quad \times \frac{d(Ty, STy)d(y, Ty)}{1 + d(y, Ty)} + \gamma(Ty, y, a)\frac{d(y, STy)d(Ty, Ty)}{1 + d(Ty, y)} \\ &\quad + \delta(Ty, y, a)\left\{\frac{d(Ty, STy)d(Ty, Ty) + d(y, Ty)d(y, STy)}{1 + d(Ty, Ty) + d(y, STy)}\right\}. \end{aligned}$$

Applying the same treatment as above, we get,

$$\begin{aligned} \mid d(STy, Ty) \mid &\le \lambda(Ty, y, a)\, |d(Ty, y)| \\ &\quad + \mu(Ty, y, a) \mid d(Ty, STy) \mid \\ &\quad + \delta(Ty, y, a) \mid d(y, Ty) \mid . \end{aligned} \tag{2.5}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (2.1). We show that $\{x_n\}$ is a Cauchy sequence. From Proposition 2.1 and inequalities (2.4), (2.5) and for all $k = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned} &\mid d(x_{2k+1}, x_{2k}) \mid \\ &= \mid d(STx_{2k-1}, Tx_{2k-1}) \mid \\ &\le \lambda(Tx_{2k-1}, x_{2k-1}, a) \mid d(Tx_{2k-1}, x_{2k-1}) \mid \\ &\quad + \mu(Tx_{2k-1}, x_{2k-1}, a) \mid d(Tx_{2k-1}, STx_{2k-1}) \mid \\ &\quad + \delta(Tx_{2k-1}, x_{2k-1}, a) \mid d(Tx_{2k-1}, x_{2k-1}) \mid \\ &= \lambda(x_{2k}, x_{2k-1}, a)|d(x_{2k-1}, x_{2k})| \\ &\quad + \mu(x_{2k}, x_{2k-1}, a) \mid d(x_{2k}, x_{2k+1}) \mid \\ &\quad + \delta(x_{2k}, x_{2k-1}, a) \mid d(x_{2k-1}, x_{2k}) \mid \\ &\le \lambda(x_0, x_{2k-1}, a) \mid d(x_{2k-1}, x_{2k}) \mid \\ &\quad + \mu(x_0, x_{2k-1}, a) \mid d(x_{2k+1}, x_{2k}) \mid \\ &\quad + \delta(x_0, x_{2k-1}, a) \mid d(x_{2k-1}, x_{2k}) \mid \\ &\le \lambda(x_0, x_1, a) \mid d(x_{2k-1}, x_{2k}) + \mu(x_0, x_1, a) \mid d(x_{2k+1}, x_{2k}) \mid \\ &\quad + \delta(x_0, x_1, a) \mid d(x_{2k-1}, x_{2k}) \mid, \end{aligned}$$

which yeilds that

$$\mid d(x_{2k+1}, x_{2k}) \mid \le \frac{\{\lambda(x_0, x_1, a) + \delta(x_0, x_1, a)\}}{1 - \mu(x_0, x_1, a)}|d(x_{2k-1}, x_{2k})|.$$

Similarly, one can obtain

$$\mid d(x_{2k+2}, x_{2k+1}) \mid \le \frac{\{\lambda(x_0, x_1, a) + \delta(x_0, x_1, a)\}}{1 - \mu(x_0, x_1, a)}|d(x_{2k}, x_{2k+1})|.$$

Let $P = \frac{\lambda(x_0, x_1, a) + \delta(x_0, x_1, a)}{1 - \mu(x_0, x_1, a)} < 1$.

Since $\lambda(x_0, x_1, a) + \mu(x_0, x_1, a) + \delta(x_0, x_1, a) + \gamma(x_0, x_1, a) < 1$,
thus we have, $\mid d(x_{2k+2}, x_{2k+1}) \mid \le P \mid d(x_{2k}, x_{2k+1}) \mid$,
or in fact $\mid d(x_{n+1}, x_n) \mid \le P \mid d(x_{n-1}, x_n) \mid, \forall\, n \in N$.

Now from Lemma 2.1, we have $\{x_n\}$ is a Cauchy sequence in (X, d).

By the completeness of X there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next we show that z is a fixed point of S.

Now by (2.2) and Proposition 2.1, we have

$$\begin{aligned}
&d(z, Sz)\\
&\precsim d(z, Tx_{2n+1}) + d(Tx_{2n+1}, Sz)\\
&= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1})\\
&\precsim d(z, x_{2n+2}) + \lambda(z, x_{2n+1}, a)\, d(z, x_{2n+1}) + \mu(z, x_{2n+1}, a)\\
&\quad \times \frac{d(z, Sz)d(x_{2n+1}, Tx_{2n+1})}{1 + d(z, x_{2n+1})} + \gamma(z, x_{2n+1}, a)\\
&\quad \times \frac{d(x_{2n+1}, Sz)d(z, Tx_{2n+1})}{1 + d(z, x_{2n+1})} + \delta(z, x_{2n+1}, a)\\
&\quad \times \frac{\{d(z, Sz)d(z, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d(x_{2n+1}, Sz)\}}{1 + d(z, Tx_{2n+1}) + d(x_{2n+1}, Sz)}\\
&\precsim d(z, x_{2n+2}) + \lambda(z, x_1, a)d(z, x_{2n+1}) + \mu(z, x_1, a)\\
&\quad \times \frac{d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})}\\
&\quad + \gamma(z, x_1, a)\frac{d(x_{2n+1}, Sz)d(z, x_{2n+2})}{1 + d(z, x_{2n+2})} + \delta(z, x_1, a)\\
&\quad \times \frac{\{d(z, Sz)d(z, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, Sz)\}}{1 + d(z, x_{2n+2}) + d(x_{2n+1}, Sz)},
\end{aligned}$$

which on letting $n \to \infty$, give rise $d(z, Sz) = 0 \;\Rightarrow Sz = z$.

Now we shall show that z is fixed point of T. Utilizing inequality (2.2), we have

$$\begin{aligned}
&d(z, Tz)\\
&\precsim d(z, Sx_{2n}) + d(Sx_{2n}, Tz)\\
&\precsim (z, x_{2n+1}) + \lambda(x_{2n}, z, a)d(x_{2n}, z) + \mu(x_{2n}, z, a)\\
&\quad \times \frac{d(x_{2n}, Sx_{2n})d(z, Tz)}{1 + d(x_{2n}, z)} + \gamma(x_{2n}, z, a)\frac{d(z, Sx_{2n})d(x_{2n}, Tz)}{1 + d(x_{2n}, z)}\\
&\quad + \delta(x_{2n}, z, a)\left\{\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tz)+d(z, Tz)d(z, Sx_{2n})}{1 + d(x_{2n}, Tz) + d(z, Sx_{2n})}\right\}\\
&\precsim d(z, x_{2n+1}) + \lambda(x_0, z, a)d(x_{2n}, z) + \mu(x_0, z, a)\\
&\quad \times \frac{d(x_{2n}, x_{2n+1})d(z, Tz)}{1 + d(x_{2n}, z)} + \gamma(x_0, z, a)\frac{d(z, x_{2n+1})d(x_{2n}, Tz)}{1 + d(x_{2n}, z)}\\
&\quad + \delta(x_0, z, a)\left\{\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, Tz)+d(z, Tz)d(z, x_{2n+1})}{1 + d(x_{2n}, Tz) + d(z, x_{2n+1})}\right\},
\end{aligned}$$

which on making $n \to \infty$, we get $d(z, Tz) = 0$ and hence $Tz = z$.

This implies that z is a common fixed point of S and T.

Uniqueness of common fixed point is an easy consequence of the inequality (2.2) in view of condition (2.3).

This concludes the theorem. □

Following example demonstrates the validity of genuineness and degree of generality of our main theorem over comparable ones from the existing literature.

Example 2.2. Let $X = [0, 1]$ and $d\colon X \times X \to C$ be defined by $d(x, y) = |x - y|e^{i\frac{\pi}{6}}$.

Then (X, d) is a complex valued metric space. Now we define self-mappings $S, T\colon X \to X$ by $S(x) = \frac{x}{4}$ and $T(y) = \frac{y}{4}$. Further, for all $x, y \in X$ and for fixed $a = \frac{1}{3} \in X$, we define the functions $\lambda, \mu, \gamma, \delta\colon X \times X \times X \to [0, 1)$ by

$$\lambda(x, y, a) = \left(\frac{x}{4} + \frac{y}{5} + a\right), \quad \mu(x, y, a) = \frac{xya}{10},$$

$$\gamma(x, y, a) = \frac{x^2y^2a^2}{10}, \quad \delta(x, y, a) = \frac{x^3y^3a^3}{10}$$

Clearly $\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1$ for all $x, y \in X$ and for a fixed $a = \frac{1}{3} \in X$.

Now consider

$$\begin{aligned}
\lambda(TSx, y, a) &= \lambda\left(T\left(\frac{x}{4}\right), y, a\right) = \lambda\left(\frac{x}{16}, y, a\right)\\
&= \frac{x}{64} + \frac{y}{5} + a \le \frac{x}{4} + \frac{y}{5} + a = \lambda(x, y, a).
\end{aligned}$$

That is $\lambda(TSx, y, a) \le \lambda(x, y, a)$, for all $x, y \in X$ and for a fixed $a = \frac{1}{3} \in X$.

And

$$\begin{aligned}
\lambda(x, STy, a) &= \lambda\left(x, S\left(\frac{y}{4}\right), a\right) = \lambda\left(x, \frac{y}{16}, a\right)\\
&= \frac{x}{4} + \frac{y}{80} + a \le \frac{x}{4} + \frac{y}{5} + a = \lambda(x, y, a).
\end{aligned}$$

That is $\lambda(x, STy, a) \le \lambda(x, y, a)$, for all $x, y \in X$ and for a fixed $a = \frac{1}{3} \in X$.

Similarly we can show that

$$\mu(TSx, y, a) \le \mu(x, y, a) \text{ and } \mu(x, STy, a) \le \mu(x, y, a);$$

$$\gamma(TSx, y, a) \le \gamma(x, y, a) \text{ and } \gamma(x, STy, a) \le \gamma(x, y, a);$$

$$\delta(TSx, y, a) \le \delta(x, y, a) \text{ and } \delta(x, STy, a) \le \delta(x, y, a).$$

Finally we assert that inequality (2.2) is also satisfied.

Before discussing different cases one needs to notice that for all $x, y \in X$,

$$\begin{aligned}
0 \precsim\; & d(x, y), d(Sx, Ty), \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}, \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)},\\
& \frac{d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)}{1 + d(x, Ty) + d(y, Sx)}.
\end{aligned}$$

It is sufficient to show that

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y).$$

Consider

$$\begin{aligned}
&d(Sx, Ty)\\
&= d\left(\frac{x}{4}, \frac{y}{4}\right) = \left|\frac{x}{4} - \frac{y}{4}\right|e^{i\frac{\pi}{6}} = \frac{1}{4}\left|x - y\right|e^{i\frac{\pi}{6}}\\
&\precsim \frac{1}{3}\left|x - y\right|e^{i\frac{\pi}{6}} \precsim \left(\frac{x}{4} + \frac{y}{5} + \frac{1}{3}\right)\left|x - y\right|e^{i\frac{\pi}{6}} \quad \text{for all } x, y \in X.\\
&= \lambda(x, y, a)d(x, y), \quad \text{for all } x, y \in X \text{ and for } a = \frac{1}{3} \in X.
\end{aligned}$$

That is $d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y)$, for all $x, y \in X$ and for $a = \frac{1}{3} \in X$.

Therefore all the conditions of Theorem 2.1 are satisfied, also $x = 0$ remains fixed under S and T and is indeed unique.

By choosing point dependent control function $\lambda, \mu, \gamma, \delta$ and mappings S and T suitably, one can deduce subsequent corollaries.

Choosing $\mu = 0,\ \gamma = 0,\ \delta = 0$ in Theorem 2.1 results in following corollary.

Corollary 2.1. *Let* (X, d) *be a complete complex valued metric space and* $S, T: X \to X$. *If* $\exists$ *mapping* $\lambda: X \times X \times X \to [0, 1)$ *such that*

$$\lambda(TSx, y, a) \leq \lambda(x, y, a) \ \text{ and } \ \lambda(x, STy, a) \leq \lambda(x, y, a),$$

satisfying

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y),$$
$$\forall x, y \in X \text{ and for a fixed } a \in X,$$

then S and T have a unique common fixed point.

Opting $\mu = \gamma = 0$ in Theorem 2.1, we get the following observation.

Corollary 2.2. *Let* (X, d) *be a complete complex valued metric space and* $S, T: X \to X$. *If* $\exists$ *mapping* $\lambda, \delta: X \times X \times X \to [0, 1)$ *such that for all* $x, y \in X$ *and for a fixed* $a \in X$,

$$\lambda(TSx, y, a) \leq \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \leq \lambda(x, y, a),$$

$$\delta(TSx, y, a) \leq \delta(x, y, a) \text{ and } \delta(x, STy, a) \leq \delta(x, y, a)$$

and

$$\lambda(x, y, a) + \delta(x, y, a) < 1,$$

also satisfying

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \delta(x, y, a) \times \frac{d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)}{1 + d(x, Ty) + d(y, Sx)},$$

then S and T have a unique common fixed point.

Setting $\mu = \delta = 0$ in Theorem 2.1, we get another corollary.

Corollary 2.3. *Let* (X, d) *be a complete complex valued metric space and* $S, T: X \to X$. *If* $\exists$ *mappings* $\lambda, \gamma: X \times X \times X \to [0, 1)$ *such that for all* $x, y \in X$ *and for a fixed* $a \in X$,

$$\lambda(x, y, a) + \gamma(x, y, a) < 1$$

and

$$\lambda(TSx, y, a) \leq \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \leq \lambda(x, y, a),$$

$$\gamma(TSx, y, a) \leq \gamma(x, y, a) \text{ and } \gamma(x, STy, a) \leq \gamma(x, y, a);$$

also satisfying

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \gamma(x, y, a)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)},$$

then S and T have a unique common fixed point.

In Theorem 2.1, if we choose $\gamma = \delta = 0$, then we deduce the following corollary.

Corollary 2.4. *Let* (X, d) *be a complete complex valued metric space and* $S, T: X \to X$. *If* $\exists$ *mappings* $\lambda, \mu: X \times X \times X \to [0, 1)$ *such that for all* $x, y \in X$ *and for a fixed* $a \in X$,

$$\lambda(x, y, a) + \mu(x, y, a) < 1$$

and

$$\lambda(TSx, y, a) \leq \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \leq \lambda(x, y, a),$$

$$\mu(TSx, y, a) \leq \mu(x, y, a) \text{ and } \mu(x, STy, a) \leq \mu(x, y, a);$$

also satisfying

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)},$$

then S and T have a unique common fixed point.

Remark 2.1. In Corollary 2.4, if we replace $\lambda, \mu: X \times X \times X \to [0, 1)$ by $\Lambda, \Xi: X \to [0, 1)$ with $\lambda(x, y, a) = \Lambda(x)$ and $\mu(x, y, a) = \Xi(x)$, $\forall\, x, y \in X$ and so $\Lambda(x) + \Xi(x) < 1$ and

$$\Lambda(Sx) \leq \Lambda(x) \text{ and } \Xi(Sx) \leq \Xi(x),$$

$$\Lambda(Tx) \leq \Lambda(x) \text{ and } \Xi(Tx) \leq \Xi(x),$$

now condition (2.2) becomes

$$d(Sx, Ty) \precsim \Lambda(x)d(x, y) + \Xi(x)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}. \tag{2.6}$$

Then S and T have a unique common fixed point.

Thus we obtain Theorem 3.1 of Sintunavarat et al. [2].

Following example demonstrates the superiority of Theorem 2.1 over Theorem 3.1 of [2] as slight changes in the setting of λ and μ in Example 2.2 give rise to the verification of Theorem 3.1 of Sintunavarat et al. [2].

Example 2.3. In the setting of Example 2.2, replace the mappings $\lambda, \mu, \delta, \gamma: X \times X \times X \to [0, 1)$ by the following besides retaining the rest:

$$\lambda(x, y, a) = \Lambda(x) = \frac{x+1}{3}, \quad \mu(x, y, a) = \Xi(x) = \frac{x}{10} \quad \text{and}$$
$$\gamma(x, y, a) = \delta(x, y, a) = 0.$$

Clearly $\Lambda(x) + \Xi(x) < 1$ and

$$\Lambda(Sx) \leq \Lambda(x) \quad \text{and} \quad \Xi(Sx) \leq \Xi(x)$$

$$\lambda(Tx) \leq \Lambda(x) \ \text{ and } \ \Xi(Tx) \leq \Xi(x).$$

By routine calculation, one can easily verify inequality (2.6). Thus all the conditions of Theorem 3.1 of [2] are satisfied and $x = 0$ is a unique common fixed point of (S, T).

Remark 2.2. In Corollary 2.4 if we set mappings $\lambda, \mu: X \times X \times X \to [0, 1)$ as

$$\lambda(x, y, a) = \lambda_1 \text{ and } \mu(x, y, a) = \mu_1, \quad \text{where} \quad \lambda_1, \mu_1 \in [0, 1)$$

such that $\lambda_1 + \mu_1 < 1$ and for all $x, y \in X$,

$$d(Sx, Ty) \precsim \lambda_1 d(x, y) + \mu_1\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)},$$

then S and T have a unique common fixed point.

Thus we get Theorem 4 of Azam et al. [1].

Restricting δ to zero in Theorem 2.1, one gets the following corollary.

Corollary 2.5. *Let (X, d) be a complete complex valued metric space and $S, T: X \to X$. If $\exists$ mappings $\lambda, \mu, \gamma: X \times X \times X \to [0, 1)$ such that for all $x, y \in X$ and for a fixed $a \in X$,*

(a) $\lambda(TSx, y, a) \le \lambda(x, y, a)$ *and* $\lambda(x, STy, a) \le \lambda(x, y, a)$,
$\mu(TSx, y, a) \le \mu(x, y, a)$ *and* $\mu(x, STy, a) \le \mu(x, y, a)$,
$\gamma(TSx, y, a) \le \gamma(x, y, a)$ *and* $\gamma(x, STy, a) \le \gamma(x, y, a)$;
(b) $\gamma(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) < 1$;
(c) $d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x, y, a)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$,

then S and T have a unique common fixed point.

Remark 2.3. In Corollary 2.5, we replace $\lambda, \mu, \gamma: X \times X \times X \to [0, 1)$ by $\lambda, \mu, \gamma: X \times X \to [0, 1)$ with

$$\lambda(x, y, a) = \lambda(x, y);\ \ \mu(x, y, a) = \mu(x, y);\ \ \gamma(x, y, a) = \gamma(x, y);$$

(a) $\lambda(TSx, y) \le \lambda(x, y)$ and $\lambda(x, STy) \le \lambda(x, y)$,
$\mu(TSx, y) \le \mu(x, y)$ and $\mu(x, STy) \le \mu(x, y)$,
$\gamma(TSx, y) \le \gamma(x, y)$ and $\gamma(x, STy) \le \gamma(x, y)$;
(b) $\lambda(x, y) + \mu(x, y) + \gamma(x, y) < 1$;
(c) $d(Sx, Ty) \precsim \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x, y)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$,

for all $x, y \in X$ and for fixed $a \in X$.

Then S and T have a unique common fixed point.

This coincides with Theorem 2.4 of Sitthikul et al. [4]. Thus our Corollary 2.5 extends the result of Sitthikul et al. [4].

Following example shows that Theorem 2.4 of Sitthikul et al. [4] is a consequence of Theorem 2.1.

Example 2.4. In Example 2.1 if we set the mapping $\lambda, \mu, \delta, \gamma: X \times X \times X \to [0, 1)$ by the subsequent functions besides preserving the rest: $\lambda(x, y, a) = \lambda(x, y) = \frac{x+1}{3} + \frac{y}{5}$, $\mu(x, y, a) = \mu(x, y) = \frac{xy}{20}$, $\gamma(x, y, a) = \gamma(x, y) = \frac{x^2y^2}{20}$ and $\delta(x, y, a) = \delta(x, y) = 0$,then as in Example 2.2 all the conditions of Theorem 2.4 of [4] are satisfied immediately and $x = 0$ is the unique common fixed point of the mappings S and T.

Remark 2.4. In Corollary 2.5, if we define $\lambda, \mu, \gamma: X \times X \times X \to [0, 1)$ by

$$\lambda(x, y, a) = \lambda;\ \ \mu(x, y, a) = \mu;\ \ \gamma(x, y, a) = \gamma$$

where $\lambda, \mu, \gamma \in [0, 1)$ such that $\lambda + \mu + \gamma < 1$ and for all $x, y \in X$,

$$d(Sx, Ty) \precsim \lambda d(x, y) + \mu\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)},$$

then S and T have a unique common fixed point. Thus Theorem2.1 of Rouzkard et al. [3] is obtained.

Now setting $\gamma = 0$ in Theorem 2.1, we get another Corollary.

Corollary 2.6. *Let (X, d) be a complete complex valued metric space and $S, T: X \to X$. If $\exists$ mappings $\lambda, \mu, \gamma: X \times X \times X \to [0, 1)$ such that*

(a) $\lambda(TSx, y, a) \le \lambda(x, y, a)$ *and* $\lambda(x, STy, a) \le \lambda(x, y, a)$,
$\mu(TSx, y, a) \le \mu(x, y, a)$ *and* $\mu(x, STy, a) \le \mu(x, y, a)$,
$\delta(TSx, y, a) \le \delta(x, y, a)$ *and* $\delta(x, STy, a) \le \delta(x, y, a)$;
(b) $\gamma(x, y, a) + \mu(x, y, a) + \delta(x, y, a) < 1$;
(c) $d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \delta(x, y, a)\{\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{1+d(x,Ty)+d(y,Sx)}\}$,

for all $x, y \in X$ and for a fixed $a \in X$. Then S and T have a unique common fixed point.

Now setting $S = T$ in Theorem 2.1, we get the following corollary.

Corollary 2.7. *Let (X, d) be a complete complex valued metric space and $S: X \to X$. If $\exists$ mappings $\lambda, \mu, \gamma, \delta: X \times X \times X \to [0, 1)$ such that for all $x, y \in X$ and for fixed $a \in X$,*

(a) $\lambda(S^2x, y, a) \le \lambda(x, y, a)$ *and* $\lambda(x, S^2y, a) \le \lambda(x, y, a)$,
$\mu(S^2x, y, a) \le \mu(x, y, a)$ *and* $\mu(x, S^2y, a) \le \mu(x, y, a)$,
$\gamma(S^2x, y, a) \le \gamma(x, y, a)$ *and* $\gamma(x, S^2y, a) \le \gamma(x, y, a)$,
$\delta(S^2x, y, a) \le \delta(x, y, a)$ *and* $\delta(x, S^2y, a) \le \delta(x, y, a)$;
(b) $\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1$;
(c) $d(Sx, Sy) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\frac{d(x,Sx)d(y,Sy)}{1+d(x,y)} + \gamma(x, y, a)\{\frac{d(y,Sx)d(x,Sy)}{1+d(x,y)}\} + \delta(x, y, a)\{\frac{d(x,Sx)d(x,Sy)+d(y,Sy)d(y,Sx)}{1+d(x,Sy)+d(y,Sx)}\}$,

then S has a unique fixed point.

In Theorem 2.1, if we replace mappings $\lambda, \mu, \gamma, \delta: X \times X \times X \to [0, 1)$ by mappings $\lambda, \mu, \gamma, \delta: X \times X \to [0, 1)$ using relations

$$\lambda(x, y, a) = \lambda(x, y);\ \ \mu(x, y, a) = \mu(x, y);$$
$$\gamma(x, y, a) = \gamma(x, y);\ \delta(x, y, a) = \delta(x, y),$$

we get following corollary.

Corollary 2.8. *Let (X, d) be a complete complex valued metric space and $S, T: X \to X$. If there exists mappings $\lambda, \mu, \gamma, \delta: X \times X \to X$ such that for all $x, y \in X$*

(a) $\lambda(TSx, y) \le \lambda(x, y)$ *and* $\lambda(x, STy) \le \lambda(x, y)$,
$\mu(TSx, y) \le \mu(x, y)$ *and* $\mu(x, STy) \le \mu(x, y)$,
$\gamma(TSx, y) \le \gamma(x, y)$ *and* $\gamma(x, STy) \le \gamma(x, y)$,
$\delta(TSx, y) \le \delta(x, y)$ *and* $\delta(x, STy) \le \delta(x, y)$;
(b) $d(Sx, Ty) \precsim \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x, y)\{\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}\} + \delta(x, y)\{\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{1+d(x,Ty)+d(y,Sx)}\}$;
(c) $\lambda(x, y) + \mu(x, y) + \gamma(x, y) + \delta(x, y) < 1$,

then S and T have a unique common fixed point .

Remark 2.5. In Corollary 2.8 if we set $\delta(x, y) = \gamma(x, y) = 0$ then, we get Corollary 2.6 of Sitthikul et al. [4] and if we set $\mu(x, y) = \delta(x, y) = 0$ then we get the Corollary 2.7 of Sitthikul et al. [4].

Next theorem is presented for single mapping satisfying slightly different conditions.

Theorem 2.2. *Let (X, d) be a complete complex valued metric space and $T: X \to X$. If there exists mappings $\lambda, \mu: X \times X \times X \to [0, 1)$ such that for all $x, y \in X$ and for fixed $a \in X$*

(a) $\lambda(Tx, y, a) \le \lambda(x, y, a)$ *and* $\lambda(x, Ty, a) \le \lambda(x, y, a)$,
$\mu(Tx, y, a) \le \mu(x, y, a)$ *and* $\mu(x, Ty, a) \le \mu(x, y, a)$;

(b) $d(Tx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\dfrac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)};$ (2.7)

(c) $\lambda(x, y, a) + \mu(x, y, a) < 1$,

then T has a unique fixed point.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by $x_{n+1} = Tx_n$, where $n = 0, 1, 2 \ldots$ Now we show that $\{x_n\}$ is a Cauchy sequence. From condition (2.7), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\precsim \lambda(x_n, x_{n+1}, a)d(x_n, x_{n+1}) + \mu(x_n, x_{n+1}, a) \\ &\quad\times \frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})} \\ &= \lambda(x_n, x_{n+1}, a)d(x_n, x_{n+1}) + \mu(x_n, x_{n+1}, a) \\ &\quad\times \frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})} \end{aligned} \tag{2.8}$$

i.e.

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim \lambda(x_n, x_{n+1}, a)d(x_n, x_{n+1}) \\ &\quad + \mu(x_n, x_{n+1}, a)d(x_{n+1}, x_{n+2}). \end{aligned} \tag{2.9}$$

Now

$$\begin{aligned} \lambda(x_n, x_{n+1}, a) &= \lambda(Tx_{n-1}, x_{n+1}, a) \\ &\le \lambda(x_{n-1}, x_{n+1}, a) = \lambda(Tx_{n-2}, x_{n+1}, a) \\ &\le \lambda(x_{n-2}, x_{n+1}, a) = \lambda(Tx_{n-3}, x_{n+1}, a) \\ &\ldots\ldots \\ &\le \lambda(x_0, x_{n+1}, a), \end{aligned}$$

and similarly

$$\mu(x_n, x_{n+1}, a) \le \mu(x_0, x_{n+1}, a).$$

Then from (2.9), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim \lambda(x_0, x_{n+1}, a)d(x_n, x_{n+1}) \\ &\quad + \mu(x_0, x_{n+1}, a)d(x_{n+1}, x_{n+2}). \end{aligned}$$

Arguing the same as above, we obtain

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim \lambda(x_0, x_0, a)d(x_n, x_{n+1}) \\ &\quad + \mu(x_0, x_0, a)d(x_{n+1}, x_{n+2}). \end{aligned}$$

Therefore

$$\begin{aligned} &| d(x_{n+1}, x_{n+2}) | \le \lambda(x_0, x_0, a) | d(x_n, x_{n+1}) | \\ &\quad + \mu(x_0, x_0, a) | d(x_{n+1}, x_{n+2}) | \\ &\Rightarrow | d(x_{n+1}, x_{n+2}) | \le \frac{\lambda(x_0, x_0, a)}{1 - \mu(x_0, x_0, a)} | d(x_n, x_{n+1}) |, \end{aligned}$$

for all n = 0,1,2,....

Let $k = \frac{\lambda(x_0, x_0, a)}{1-\mu(x_0, x_0, a)} < 1$, then

$$| d(x_{n+1}, x_{n+2}) | \le k | d(x_n, x_{n+1}) |, \ \forall n = 0, 1, 2, \ldots.$$

Then utilizing Lemma 2.1, we have $\{x_n\}$ is a Cauchy sequence in (X, d).

Since X is complete, so $\exists\, z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next to show that z is a fixed point of T.

From (2.7), we have

$$\begin{aligned} d(z, Tz) &\precsim d(z, Tx_n) + d(Tx_n, Tz) \\ &\precsim d(z, Tx_n) + \lambda(x_n, z, a)d(x_n, z) \\ &\quad + \mu(x_n, z, a)\frac{d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(z, x_n)} \\ &\precsim d(z, x_{n+1}) + \lambda(x_0, z, a)d(x_n, z) \\ &\quad + \mu(x_0, z, a)\frac{d(z, Tz)[1 + d(x_n, x_{n+1})]}{1 + d(z, x_n)}, \end{aligned}$$

which on making $n \to \infty$ reduces to

$$d(z.Tz) \precsim \mu(x_0, z, a)d(z, Tz),$$

so that

$$| d(z.Tz) | \le \mu(x_0, z, a) | d(z, Tz) |,$$

which is a contradiction since $\mu(x_0, z, a) < 1$.

Therefore $| d(z, Tz) | = 0 \Rightarrow z = Tz$.

This implies that z is a fixed point of T.

Uniqueness of fixed point is an easy consequence of condition (2.9). This completes the proof. □

Following example substantiates the validity of theorem hypothesis of Theorem 2.2.

Example 2.5. Let $X = [0, 1]$ and d: $X \times X \to C$ be defined by $d(x, y) = |x - y|e^{i\frac{\pi}{6}}$.

Then (X, d) is a complex valued metric space. Let T: $X \to X$ be defined by $T(x) = \frac{x}{6}$.

Functions λ, μ: $X \times X \times X \to [0, 1]$ are defined as $\lambda(x, y, a) = (\frac{x}{3} + \frac{y}{4} + a)$, $\mu(x, y, a) = \frac{x^2y^2a^2}{50}$, for all $x, y \in X$ and for fixed $a = \frac{2}{5} \in X$.

Clearly $\lambda(x, y, a) + \mu(x, y, a) < 1$.

Consider

$$\begin{aligned} \lambda(Tx, y, a) &= \lambda\Big(\frac{x}{6}, y, a\Big) = \frac{x}{18} + \frac{y}{4} + \frac{2}{5} \le \frac{x}{3} + \frac{y}{4} + \frac{2}{5} \\ &= \lambda(x, y, a) \end{aligned}$$

also

$$\begin{aligned} \lambda(x, Ty, a) &= \lambda\Big(x, \frac{y}{6}, \frac{2}{5}\Big) \\ &= x + \frac{y}{24} + \frac{2}{5} \le x + \frac{y}{4} + \frac{2}{5} \le \lambda(x, y, a). \end{aligned}$$

and similarly we can show that

$$\mu(Tx, y, a) \le \mu(x, y, a) \quad \text{and} \quad \mu(x, Ty, a) \le \mu(x, y, a).$$

Now for the verification of inequality (2.7), one needs to note that

$$0 \precsim d(x, y), d(Tx, Ty), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \quad \forall x, y \in X.$$

Now it is sufficient to show that $d(Tx, Ty) \precsim \lambda(x, y, a)d(x, y)$.

Consider

$$d(Tx, Ty) = d\Big(\frac{x}{6}, \frac{y}{6}\Big) = \frac{1}{6}\Big|x - y\Big|e^{i\frac{\pi}{6}} \precsim \frac{2}{5}\Big|x - y\Big|e^{i\frac{\pi}{6}}$$
$$\precsim \Big(\frac{x}{3} + \frac{y}{4} + \frac{2}{5}\Big)\Big|x - y\Big|e^{i\frac{\pi}{6}} = \lambda(x, y, a)d(x, y),$$

for all $x, y \in X$ and for $a = \frac{2}{5} \in X$.

Thus all the conditions of Theorem 2.2 are satisfied. And T has a fixed point $x = 0 \in X$, which is unique.

Corollary 2.9. *In Theorem 2.2, if we define mappings* λ, μ: $X \times X \times X \to [0, 1)$ *such that*

$$\lambda(x, y, a) = \lambda(x, y) \text{ and } \mu(x, y, a) = \mu(x, y),$$

then for all $x, y \in X$,

(a) $\lambda(Tx, y) \le \lambda(x, y)$ *and* $\lambda(x, Ty) \le \lambda(x, y)$,
$\mu(Tx, y) \le \mu(x, y)$ *and* $\mu(x, Ty) \le \mu(x, y)$;
(b) $d(Tx, Ty) \precsim \lambda(x, y)d(x, y) + \mu(x, y)\frac{(d(y,Ty)[1+d(x,Tx)])}{[1+d(x,y)]}$;
(c) $\lambda(x, y) + \mu(x, y) < 1$.

Then T has a unique fixed point.

Above corollary is exactly Theorem 2.8 of Sitthikul et al. [4]. From Theorem 2.2, we can deduce the result of Dass and Gupta [5] in the context of real valued metric spaces.

For this we set the mappings λ, μ: $X \times X \times X \to [0, 1)$ as

$$\lambda(x, y, a) = \lambda \text{ and } \mu(x, y, a) = \mu, \ \forall\, x, y \in X \text{ and for fixed } a \in X,$$

then all the conditions of Theorem 2.2 are satisfied. This follows $\{x_n\}$ is a Cauchy sequence. By (ii) of Dass and Gupta [5], sequence $\{x_n\} \to z$ as $n \to \infty$.

It follows from the proof of Theorem 2.2 that z is a unique fixed point of T. Thus we obtain result of Dass and Gupta [5], which is stated as follows.

Theorem 2.3. *[5] Let* (X, d) *be a real valued metric space. Let* T: $X \to X$ *be such that*

(i) $d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$ *for all* $x, y \in X, \lambda > 0, \lambda + \mu < 1$ *and*
(ii) *for some* $x_0 \in X$, *the sequence treats* $\{T^n(x_0)\}$ *has a subsequence* $\{T^{n_k}(x_0)\}$
with $z = \lim_{k\to\infty} T^{n_k}(x_0)$,

then z is unique fixed point of T.

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