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A bound on the k-gonality of facets of the hypermetric cone and related complexity problems*

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Abstract

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We give a bound on $g_h(n)$, the largest integer such that there is a $g_h(n)$-gonal facet of the hypermetric cone Hyp_n, $g_h(n) \leqslant 2^{n-2}(n-1)!$ This proves simultaneously the polyhedrality of the hypermetric cone. We give complete description of Delaunay polytopes related to facets of Hyp_n. We prove that the problem determining hypermetricity lies in co-NP and give some related NP-hard problem.

1. Introduction

The hypermetric cone Hyp_n of all hypermetrics on n-point set X is described by *hypermetric* inequalities

$$\sum_{1\leqslant i<j\leqslant n} b_i b_j d_{ij} \leqslant 0 \quad \text{with } \sum_{1}^{n} b_i = 1, \text{ and } b_i \in \mathbb{Z}. \tag{1}$$

The inequality (1) is called *hypermetric* inequality. If the condition $\sum_1^n b_i = 1$ is changed into the condition $\sum_1^n b_i = 0$, then the equality (1) is called inequality of

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negative type. The inequality (1) is called *k-gonal* inequality if $\sum_1^n |b_i| = k$. The hypermetric inequality is called *pure* if $b_i \in \{\pm 1, 0\}$.

It is proved in [4] that Hyp_n is polyhedral. Moreover, each hypermetric $d \in \mathrm{Hyp}_n$ determines uniquely (up to orthogonal transformations of space, that is up to translations, rotations and reflections) an L-polytope (or Delaunay polytope) P_d such that d is represented as

$$d_{ij} = (v_i - v_j)^2, \; i, j \in X \tag{2}$$

where v_i, $1 \leqslant i \leqslant n$, are vertices of the L-polytope P_d. Let $V(P_d)$ be the set of vertices of P_d. We suppose that the origin of the space spanned by P_d is the center of the circumscribed sphere S of P_d with radius r. Since all vertices of P_d lie on S,

$$v^2 = r^2$$

for all $v \in V(P_d)$. We have $X \subseteq V(P_d)$, and X affinely generates a lattice $L(X)$. P_d is an L-polytope of the lattice $L(X)$. Every point $v \in L(X)$ is an affine integral combination of points v_i, $i \in X$, i.e.

$$v = \sum_{i \in X} b_i v_i, \qquad \sum_{i \in X} b_i = 1. \tag{3}$$

Substituting (2) in (1) we obtain that inequality (1) is equivalent to the following inequality

$$v^2 = \left(\sum b_i v_i\right)^2 \geqslant r^2.$$

Hence the inequality (1) is equivalent to the condition that the lattice point $v = \sum b_i v_i$ does not lie inside the sphere S. Moreover, if the inequality (1) is satisfied by d as equality, then the point v lies on the sphere S, i.e. v is a vertex of P_d. Conversely, every vertex $v \in L(X)$ provides an equality of type (1) (may be trivial) since v as a lattice point of the lattice $L(X)$ has the form $v = \sum_{i \in X} b_i v_i$.

Let P be an L-polytope of a lattice L. Let $V(P)$ be the set of vertices of P. A subset $X \subseteq V(P)$ is called *generating* if every vertex $v \in V(P)$ is an affine integral combination (3) of v_i, $i \in X$. An L-polytope of dimension $\dim P = k$ is called *basic* if there is a generating $X \subseteq V(P)$ of cardinality $k + 1$. It is not known whether every L-polytope is basic or not.

Every subset $X \subseteq V(P)$ provides a point d_X (given by (2)) of the cone $\mathrm{Hyp}(X)$ of all hypermetrics on X. For $d \in \mathrm{Hyp}(X)$ we denote by $\mathrm{rank}\, d$ the dimension of the minimal, by inclusion, face of $\mathrm{Hyp}(X)$ containing d. $\mathrm{rank}\, d_{V(P)}$ is called the rank of an L-polytope P and is denoted by $\mathrm{rank}\, P$. The following Proposition is proved in [5].

Proposition 1. *Let P be an L-polytope, and let $X \subseteq V(P)$ be a generating subset of its vertices. Then* $\mathrm{rank}\, d_X$ *does not depend on X, i.e.,* $\mathrm{rank}\, d_X = \mathrm{rank}\, P$.

This Proposition shows that every L-polytope P corresponds to a face of dimension $\operatorname{rank} P$ of the cone Hyp_n for any $n \geq \dim P+1$. More exactly, if $|X| \geq \dim P+1$, then any map of X onto a generating subset of $V(P)$ yields a point $d \in \mathrm{Hyp}(X)$ lying on a face of dimension rank P.

An L-polytope P of $\operatorname{rank} P=1$ corresponds to an extreme ray of Hyp_n. The polytope is called *extreme*. Extreme L-polytopes are studied in [5]. Below we give complete description of L-polytopes related to facets of Hyp_n.

The contents of the paper are as follows. In Section 2 we give a complete description of L-polytopes related to facets. The description allows to prove in Section 3 upper bounds on the k-gonality of facet defining inequalities and on the number of facets of Hyp_n. Section 4 is devoted to complexity questions. It is proved there that the problem of determining hypermetricity is in co-NP, and we give a related co-NP-complete problem and NP-hard problem.

2. L-polytopes related to facets

Let P be a polytope. The convex hull of P and of a point v which does not lie in the affine space spanned by P is called *a pyramid with base P and apex v* and is denoted by $\mathrm{Pyr}_v(P)$.

It is easy to see that

$$\mathrm{Pyr}_u(\mathrm{Pyr}_v(P))=\mathrm{Pyr}_v(\mathrm{Pyr}_u(P)) \equiv \mathrm{Pyr}_u \mathrm{Pyr}_v(P).$$

Hence the polytope $\prod_{i \in Y} \mathrm{Pyr}_{v_i}(P)$ is well defined.

The convex hull of 2 simplices Sim_i, $i=1,2$, such that the affine spaces spanned by Sim_1 and Sim_2 intersect in a point is called *the repartitioning* polytope. The polytope was studied by Voronoi, and the name is given by him (see [2]).

We note that there is only one affine dependency between vertices of the repartitioning polytope:

$$\sum_{v \in V_1} b_v v=\sum_{v \in V_2} b_v v.$$

where $V_i=V(\mathrm{Sim}_i)$ and $\sum_{v \in V_1} b_v=\sum_{v \in V_2} b_v$. Each repartitioning polytope P has the form $\prod_{v \in V_0} \mathrm{Pyr}_v(P_1)$ where $V_0=\{v \in V_1 \cup V_2\colon b_v=0\}$ and P_1 is a repartitioning polytope on vertices of $V_1 \cup V_2-V_0$. Hence $b_v \neq 0$ for all $v \in V(P_1)$. Using the notation of [2] we denote the polytope by $P_{p,q}^m$ where $m=|V_0|$, $p+1=|V_1-V_0|$, $q+1=|V_2-V_0|$. Note that $\dim P_{p,q}^m=m+p+q$, $|V(P_{p,q}^m)|=m+p+q+2$, and $\dim \mathrm{Sim}_1=p$, $\dim \mathrm{Sim}_2=q$.

We underline that the denotation $P_{p,q}^m$ does not describe a concrete polytope but corresponds to a class of affinely equivalent repartitioning polytopes of the same combinatorial type (and the same L-type, too) with only one affine dependency between vertices.

A facet of the hypermetric cone Hyp_n is determined by the equality

$$\sum_{1\leq i<j\leq n} b_i b_j d_{ij} = 0 \quad \text{with} \sum_{i=1}^{n} b_i = 1. \tag{4}$$

Proposition 2. *Let P be an L-polytope corresponding to the facet* (4) *of the hypermetric cone* Hyp_n. *Then P is basic and*

$$P = P^m_{p,q}$$

where

$$m + p + q + 1 = n, \qquad m = |\{i: b_i = 0\}|,$$

$$p = |\{i: b_i > 0\}| - 1, \qquad q = |\{i: b_i < 0\}|.$$

Hence $\dim P = n - 2$, *and P has $n+1$ vertices $v_0, v_1, \ldots, v_n$ where the indices agree with* (4).

Proof. Let d be a hypermetric such that a minimal by inclusion face containing d is a facet. This means that d satisfies only one hypermetric inequality as equality. Let the equality be the equality (4).

Note, first, that $d_{ij} > 0$ for all i, j. In fact, suppose that there is a pair (ij) such that $d_{ij} = 0$. Then the triangle inequalities

$$d_{ik} - d_{ij} - d_{jk} \leq 0,$$

$$d_{jk} - d_{ij} - d_{ik} \leq 0$$

imply that $d_{ik} = d_{jk}$ for all $k \neq i, j$. Hence the above inequalities are, in fact, equalities, and d satisfies at least $2(n-2)$ triangle equalities.

Let $X = \{1, 2, \ldots, n\}$. Recall that the equality (4) is equivalent to the point

$$v_0 = \sum_{i\in X} b_i v_i \tag{5}$$

being a vertex of P_d. We have two cases: (a) $v_0 = v_i$ for some $i \in X$, and (b) $v_0 \neq v_i$ for all $i \in X$. We show that the case (a) is impossible. Suppose $v_0 = v_k$, $k \in X$. Then the equality (5) provides the affine dependency between v_i, $i \in X$:

$$\sum_{i\neq k} b_i v_i + (b_k - 1)v_k = 0.$$

For each $j \in X$ the equality

$$v_j = (b_j + 1)v_j + (b_k - 1)v_k + \sum_{i\neq j,k} b_i v_i$$

provides a hypermetric equality. Hence in the case (a) the hypermetric d satisfies n hypermetric equalities. This is a contradiction, since d satisfies only the equality (4). Therefore we have case (b).

Since d satisfies only one hypermetric equality, and each vertex of P_d provides an equality which is satisfied by d, there is only one vertex $v_0 \in V(P_d)$ different from v_i, $i \in X$.

Besides, the set of vertices v_i, $i \in X$, is affinely independent. In fact, suppose that there is a affine dependency

$$\sum_{i \in X} a_i v_i = 0, \; \sum_{i \in X} a_i = 0$$

between vertices of X (not all a_i are equal to 0). It is not difficult to see that a_i can be taken integral. Hence the point v_0 has another representation

$$v_0 = \sum (b_i + a_i) v_i, \; \sum (b_i + a_i) = 1.$$

The representation provides one more hypermetric equality

$$\sum (b_i + a_i)(b_j + a_j) d_{ij} = 0$$

which is satisfied by d. A contradiction, since d lies on only one facet.

So P_d has $n+1$ vertices, v_0 and v_i for $i \in X$. There is only one affine dependency between vertices of P_d. The set v_i, $i \in X$, is a basis of the lattice $l(X)$, and P_d is a basic L-polytope. We set $V = V(P_d)$ and rewrite the dependency as follows

$$\sum_{v \in V} b_v v = 0, \; \sum_{v \in V} b_v = 0$$

where $b_v = b_i$ if $v = v_i$, $i \in X$, and $b_v = b_0 = -1$ if $v = v_0$.

Let

$$V_0 = \{v \in V : b_v = 0\},$$

$$V_+ = \{v \in V : b_v > 0\},$$

$$V_- = \{v \in V : b_v < 0\},$$

and $m = |V_0|$, $p + 1 = |V_+|$, $q + 1 = |V_-|$. Obviously, any strict subset of V is affinely independent. Hence V_+ and V_- span simplices Sim_+ and Sim_- of dimension p and q, respectively.

Let P_1 be the L-polytope spanned by vertices of Sim_+ and Sim_-. Since there is only one affine dependency between vertices of P_1, $\dim P_1 = p + q = \dim \mathrm{Sim}_+ + \dim \mathrm{Sim}_-$. Hence dimension of intersection of spaces spanned by the simplices equal to 0, i.e. the intersection is a point.

So, P_1 is a convex hull of vertices of 2 simplices such that the spaces spanned by the simplexes intersect in only one point, and $P = P^m_{p,q}$. □

Remark. In fact, we proved that the L-polytope of the type $P^m_{p,q}$ yields a facet of $\mathrm{Hyp}(X)$ if and only if $|X| = m + p + q + 1$ and X is bijectively mapped onto an affine basis of $P^m_{p,q}$.

Recall that if $\sum_{1\leq i\leq n}|b_i| = k$, then the inequality (1) is called k-gonal inequality. As an example, consider 3-gonal (triangle) and 5-gonal (pentagonal) equalities. A triangle inequality is related to the equality $v_0 = v_1 + v_2 - v_3$. Hence $V_+ = \{v_1, v_2\}$ and $V_- = \{v_0, v_3\}$, i.e. simplexes are the segments $[v_1v_2]$ and $[v_0v_3]$. The corresponding L-polytope is a rectangle whose diagonals are these segments.

Similarly, a pentagonal equality provides a 4-dimensional polytope which spans 2 triangles $\{v_1v_2v_3\}$ and $\{v_0v_4v_5\}$ where $v_0 = v_1 + v_2 + v_3 - v_4 - v_5$.

Consider the $(2m+1)$-gonal pure equality with $b_i = 1$ for $i \in x_+$ and $b_i = -1$ for $i \in x_-$ where $|X_+| = m+1$, $|X_-| = m$, and $X_+ \cap X_- = \emptyset$. The corresponding affine dependency is

$$v_0 = \sum_{i\in X_+} v_i - \sum_{i\in X_-} v_i.$$

The equality is satisfied by the following hypermetric d^t with $t = 1/m$ where

$$d^t_{ij} = \begin{cases} 1 & \text{if } |\{ij\} \cap X_+| = 1, \\ 1+t & \text{otherwise.} \end{cases} \tag{6}$$

It is easy to verify that if we add the zero index to the set X_-, then the enlarged distance function d^t satisfies the pure $(2m+1)$-gonal equality and so the pure $(2m+2)$-gonal equality, too. Hence, for $d = d_m \equiv d^{1/m}$, $P_{d_m} = P^0_{m,m}$ is the convex hull of two m-dimensional simplices both with $m+1$ vertices such that squared distances between vertices of the same simplex is $1 + 1/m$, and squared distance between vertices of different simplices is equal to 1. Multiplying by m we obtain that the norm (squared length) of edges of P_{md_m} connecting vertices of the same simplex is equal to $m+1$, and norm of edges connecting vertices of different simplices is equal to m. The dimension of P_{d_m} is equal to $2m$.

In the above examples, the simplices are regular and intersect in the center of the circumscribed sphere of P. Note that the squared radius of the circumscribed sphere of the regular simplex with norm of edges equal to $m+1$ is equal to $r^2_m = m/2$. Recall that the squared Euclidean distance between vertices of P_{md_m} belonging to different simplices is equal to $m = 2r^2_m$. Hence the spaces spanned by the simplices are orthogonal.

It is easy to see that P_{d^t} is an L-polytope if and only if $0 \leq t \leq 1/m$, since otherwise d^t does not satisfy the pure $(2m+1)$-gonal hypermetric inequality. Moreover, if $0 \leq t < 1/m$, then P_{d^t} is a simplex (in fact, a convex hull of $2m+1$ vertices of $P^0_{m,m}$ such that the norm of edges between vertices of the same simplex is equal to $1+t$), and $P_{d^t} = P^0_{m,m}$ if $t - 1/m$.

3. A bound on k-gonality of facets

Let $g_h(n)$ be the largest integer such that there is a $g_h(n)$-gonal facet of the cone Hyp_n.

Theorem 3. *The following bound is valid*

$$g_h(n) < 2^{n-2}(n-1)!.$$

Proof. Let $b_{\max} = \max_i |b_i|$ be maximal coefficient of the Equation (4) determining a facet of Hyp_n. Obviously

$$g_h(n) \leq nb_{\max}.$$

Hence the assertion of theorem follows from the following lemma. □

Lemma 4.

$$b_{\max} < g'(n) \equiv 2^{n-2}(n+1)!/(n+1). \tag{7}$$

Proof. We represent the space of dimension $n-1$ spanned by the L-polytope P_d as a hyperplane $p_0 = 1$ of a space of dimension n. In other words, we represent each vertex $v \in V(P_d)$ by the vector $p(v) = (p_0, v)$ where $p_0 = 1$.

Let B be an $n \times n$ matrix whose rows are the vectors $p(v_i)$ for $i \in X$. Since X is an affine basis, the matrix B is nonsingular. Each point of the lattice $L(X)$ can be written as

$$p(v) = \sum_{i \in X} b_i p(v_i)$$

i.e. as the inner product $p = bB$. Let M_d be the matrix whose rows correspond to vertices of P_d. The matrix M_d has the form

$$M_d = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} B.$$

Some of b_i can be equal to zero. Since P_d spans the hyperplane $p_0 = 1$, $\det B = \det L(X)$.

Take a simplex Sim_i spanned by a set of n vertices of P_d containing the vertex v_0 and not containing a vertex v_i. The vertices of the simplex form a matrix M_d^i which is obtained from the matrix M_d by deleting the row corresponding to the vertex v_i. We have

$$\det M_d^i = b_i \det B = b_i \det L(X).$$

Recall that $\det M_d^i$ is equal to volume of the parallelepiped spanned by vectors $p(v)$. It is well known (and can be easily verified) that volume of any k-dimensional simplex spanned by $k+1$ vertices of a k-dimensional parallelepiped P is equal to $\mathrm{vol}\, P/k!$ where $\mathrm{vol}\, P$ is the volume of P. Hence

$$\mathrm{vol}\,\mathrm{Sim}_i = \det M_d^i/(n-1)! = b_i \det L(X)/(n-1)!.$$

Obviously, the volume of the maximal simplex contained in P_d is less than volume of P_d. Hence

$$b_{\max} \det L(X)/(n-1)! < \operatorname{vol} P_d. \tag{8}$$

The L-polytope P_d belongs to a set $\mathrm{St}(v)$ of L-polytopes having a fixed point v of the lattice $L(X)$ as a vertex. For $v \in V(P_d)$ the star $\mathrm{St}(v)$ contains $2\,|V(P_d)|$ polytopes which are translates of P_d and $-P_d$ by vectors $v - v'$, where v' is a vertex of P_d. (Note that if an L-polytope is symmetric, i.e. $P_d = -P_d$, then the multiple 2 must be omitted.) Since $|V(P_d)| = n+1$, we have

$$2(n+1)\operatorname{vol} P_d \leqslant \operatorname{vol} \mathrm{St}(v). \tag{9}$$

It can be proved (see [4]) that the homothetic contraction of $\mathrm{St}(v)$ with the coefficient $\frac{1}{2}$ is contained in Voronoi polytope P_{Vor} centered at v. Hence

$$\operatorname{vol} \mathrm{St}(v) \leqslant 2^{n-1} \operatorname{vol} P_{\mathrm{Vor}}.$$

(Recall that dimensions of all considered polytopes is equal to $n-1$.) Since

$$\operatorname{vol} P_{\mathrm{Vor}} = \det L(X),$$

we obtain

$$\operatorname{vol} \mathrm{St}(v) \leqslant 2^{n-1} \det L(X).$$

Comparing (9) and (8) with the last inequality, we obtain the bound asserted by Lemma. □

Remark. Note that the proof is an application to a special L-polytope of a refinement of the proof of polyhedrality of Hyp_n given in [4].

Using Stirling's formula we have

$$g_h(n) \leqslant 2^{n(1-\log_2 e)-2} n^{n+1/2}.$$

Recall the bound $g_c(n)$ on the k-gonality of hypermetric facets of the cut cone Cut_n obtained in [1],

$$g_c(n) \leqslant 2^{n-2} n(n-1)^{(n-1)/2} < 2^{n-2} n^{(n+1)/2}.$$

Since Hyp_n coincides with Cut_n for $n \leqslant 6$, and $g_c(n)$ is known for $n \leqslant 7$, we have

$$g_h(3) = g_h(4) = 3, \qquad g_h(5) = 5, \qquad g_h(6) = 7.$$

Since $\mathrm{Cut}_n \subseteq \mathrm{Hyp}_n$, every hypermetric facet of Cut_n is a facet of Hyp_n. Therefore $g_h(n) \geqslant g_c(n)$ (in particular, $g_h(7) \geqslant 9$, $g_h(8) \geqslant 13$, see lower bounds on g_c given in [1]), and the lower bound on $g_c(n)$ obtained in [1] is valid for $g_h(n)$, too, i.e.,

$$g_h(n) \geqslant n^2/4 - 4.$$

Lemma 4 yields an upper bound on the number of facets of Hyp_n. This is the number of integral points in the intersection of the n-cube $-g'(n) \leqslant b_i \leqslant g'(n)$, $1 \leqslant i \leqslant n$, with the hyperplane $\sum_n^1 b_i = 1$, where $g'(n)$ is given in (7).

Theorem 5. *The hypermetric cone* Hyp_n *has at most*

$$f(n) = 2^{(n-1)^2}((n-1)!/(n+1))^{n-1}$$

facets.

4. Applications to computational complexity

In this section we use a RAM model of computational (See [8]). Let $m(s, t)$ be the time required to multiply an s bit integer by a t bit integer. Using basic multiplication, $m(s, t) = \mathrm{O}(st)$, but faster methods are available. Also let $\|d\| = \max_{1 \leq i < j \leq n} \|d_{ij}\|$.

Consider the following computational problem.

P1. Hypermetricity
Instance: An integral distance d.
Question: Is d hypermetric?

As reported in [1], the computational complexity of P1 was previously unknown. Theorems 3 and 5 have corollaries that give complexity bounds for P1.

By Lemma 4, $|b_i| \leq g'(n)$ for any facet defining vector $b = (b_1, \ldots, b_n)$. Therefore b can be represented by $n\lceil \log_2 g'(n) \rceil = \mathrm{O}(n \log n)$ bits. If d is a nonhypermetric integral metric on n points, then it must violate some facet of Hyp_n. The time to check inequality (1) is dominated by the multiplications and is

$$\mathrm{O}(n^2[m(n \log n, n \log n) + m(2n \log n, \log \|d\|)])$$
$$= \mathrm{O}(n^4 \log^2 n + 2n^3 \log n \log_2 \|d\|)$$

which is polynomial in the size of the input. So we have the following.

Corollary 6. *Testing hypermetricity of d is in co-*NP.

Additionally, Theorem 5 immediately gives the following.

Corollary 7. *There is a* $\mathrm{O}(f(n)[n^4 \log^2 n + 2n^3 \log n \log_2 \|d\|])$ *algorithm for* P1.

Finally we remark that Theorem 5 gives (probably very weak) bounds on the number of j-faces of Hyp_n, $1 \leq j \leq \binom{n}{2} - 1$. The Upper Bound Theorem of McMullen (See, e.g. [6, Theorem 18.1]) states that the number of j-faces of a d-dimensional polytope with p facets is at most

$$\Phi_j(d, p) = \sum_{i=1}^{n} \binom{i}{j} \binom{p - d + i - 1}{i} + \sum_{i=0}^{n} \binom{d - i}{j} \binom{p - d + i - 1}{i},$$

$j = 0, \ldots, d-2$.

Since Hyp_n is a pointed convex cone in $\binom{n}{2}$ dimensions, the number of j-faces of Hyp_n is at most

$$\Phi_{j-1}\left(\binom{n}{2} - 1, f(n)\right),$$

$j = 1, \ldots, n-2$.

Any nonnegative matrix $(d_{ij})_1^n$ with zero diagonal is called a *distance* matrix. The element d_{ij} is called the *distance* between the points i and j. A distance matrix d is called *of negative type* if d satisfies all inequalities (1) with $\sum_{i=1}^n b_i = 0$. We note that, in general a distance matrix of negative type does not satisfy triangle inequalities. The cone Neg_n of all distance matrices of negative type contains Hyp_n as a subcone. Besides, Neg_n is a linear transform of the cone of symmetric positive semi-definite $n \times n$ matrices (see, for example, [4]). Hence the following computational problem is in P.

P1N. Negative type testing
Instance: An integral distance d.
Question: Is d of negative type?
Complexity: P.

Note a similarity of P1 to the following problem. (Recall that the norm of a lattice vector a is squared length of a.)

P1M. Testing minimality of a lattice vector
Instance: A lattice vector a_0.
Question: Is a_0 a lattice vector with minimum norm?
Complexity: In co-NP.

Proof. Let μ be the norm of a_0. To prove that a_0 is a minimal lattice vector, it is sufficient to prove that norm a^2 of any lattice vector a is not less than μ. If $\{a_i: 1 \leq i \leq n\}$ is a basis of the lattice, then $a = \sum_i^n z_i a_i$, $z_i \in \mathbb{Z}$, and $a^2 = \sum_{i,j} z_i z_j a_{ij}$, where $a_{ij} = a_i a_j$ is a symmetric positive semi-definite matrix. Hence a_0 is minimal if and only if

$$\sum_{i,j} z_i z_j a_{ij} \geq \mu \quad \text{for all } z_i \in \mathbb{Z}.$$

An upper bound on z_i for a minimal lattice vector can be found in [3, Ch. 5, Proposition 5.3]. Using Cramer's rule, the Hadamard inequality and the Minkowski inequality, one can obtain that

$$|z_i| \leq \frac{2^n}{v_n} \approx \left(\frac{2n}{\pi e}\right)^{n/2}$$

where v_n denotes the volume of the n-dimensional unit ball. This bound is similar to the bound (7). Hence, as above, we see that P1M is in co-NP. □

We are not able to prove that P1 is NP-hard. However we can show this for some closely related problems.

Consider the following computational questions.

P2. (2m + 1)-gonality testing

Instance: An integral distance d on n points and an integer m.
Question: Does d satisfy all $(2m+1)$-gonal hypermetric inequalities?
Complexity: co-NP complete.
Comments: remain co-NP complete for testing pure $(2m+1)$-gonality.

P3. Strong hypermetricity

Instance: An integral distance d on n points.
Question: Is d hypermetric? If not, give smallest k such that d violates a $(2k+1)$-gonal inequality.
Complexity: NP-hard.

We prove the complexity of P2 and P3 by using the known complexity of the following problems.

P4. Complete Bipartite Subgraph

Instance: Graph G on n vertices and an integer m.
Problem: Does G contains an induced complete bipartite subgraph $K_{m+1,m}$.
Complexity: NP-complete. [8]

P4′. Largest Complete Bipartite Subgraph

Instance: Graph G on n vertices.
Problem: Find largest m such that G contains an induced $K_{m,m+1}$ subgraph.
Complexity: NP-hard. [8]

We reduce P4 to P2 and P4′ to at most $k = \lceil (n-1)/2 \rceil$ questions of type P3. Suppose we are given a graph G with the set of edges $E(G)$. Construct the distance

$$d^t_{ij}(G) = \begin{cases} 1 & \text{if } (ij) \in E(G), \\ 1+t & \text{if } (ij) \notin E(G). \end{cases}$$

Note that $d^t(K_{m,m+1})$ coincides with d^t of (6) and $d^t(K_{m,m+1})$ for $t = 1/m$ lies on the facet corresponding to the L-polytope $P^0_{m,m}$. Let $h_G(t, b)$ be the left hand side of the inequality (1) with $d = d^t(G)$. We calculate $h_G(t, b)$.

Let the set of vertices of G be $V(G) = \{1, 2, \ldots, n\}$ We set

$$V_+(b) = \{i : b_i > 0\}, \qquad V_-(b) = \{i : b_i < 0\}, \qquad V(b) = V_+(b) \cup V_-(b)$$
$$n_+ = |V_+(b)|, \qquad n_- = |V_-(b)|, \qquad n_b = n_+ + n_-.$$

Let b determine a $(2k+1)$-gonal inequality. We have

$$\sum_{i=1}^{n} b_i = 1, \qquad \sum_{i=1}^{n} |b_i| = 2k+1, \qquad \sum_{i\in V_+(b)} b_i = k+1, \qquad \sum_{i\in V_-(b)} b_i = -k. \tag{10}$$

We denote by $G(b)$ the subgraph of G induced on the set $V(b)$. Let $K(b)$ be the complete bipartite graph K_{n_+,n_-} on the set $V(b)$ with the partition $(V_+(b), V_-(b))$. Let

$$E_b(G) = E(G(b)) \triangle E(K(b)),$$

where $\triangle$ denoted the symmetric difference of 2 sets.

Lemma 8. *Let b determine a $(2k+1)$-gonal inequality. Then*

$$h_G(t, b) = k^2 t - k - (1+t)\sum_{i=1}^{n} |b_i|\,(|b_i|-1)/2 - t \sum_{(ij)\in E_b(G)} |b_i|\,|b_j|. \tag{11}$$

Proof. Suppose at first that $E_b(G) = \emptyset$, i.e. the set $V(b)$ induces a complete bipartite graph $K(b)$. Then we have

$$h_G(t, b) = h_{K(b)}(t, b) = \sum_{i\in V_+, j\in V_-} b_i b_j + (1+t)\Big(\sum_{i,j\in V_+, i<j} b_i b_j + \sum_{i,j\in V_-, i<j} b_i b_j\Big).$$

Since for any set X

$$\sum_{i,j\in X, i<j} b_i b_j = \frac{1}{2}\Big(\Big(\sum_{i\in X} b_i\Big)^2 - \sum_{i\in X} b_i^2\Big),$$

using (10) we obtain

$$h_G(t, b) = -k(k+1) + (1+t)\Big(k^2 + (k+1)^2 - \sum_{i=1}^{n} b_i^2\Big)\Big/2. \tag{12}$$

Setting here $2k+1 = \sum_{i=1}^{n} |b_i|$ in the second term, we obtain the first 3 terms of (11).

If $E_b(G) \neq \emptyset$, then the right hand side of the equality (12) obtains additional negative summand

$$-t \sum_{(i,j)\in E_b(G)} |b_i|\,|b_j|.$$

Now we are done. □

We note that a pure n-vector $b = (b_1, \ldots, b_n)$ which determines a pure $(2k+1)$-gonal inequality exists if and only if $2k+1 \leqslant n$. Hence we have 2 cases: $2k+1 \leqslant n$, and $2k+1 > n$ when there is no pure vector b.

In the first case when $k \leqslant \lceil (n-1)/2 \rceil$ for a pure vector b we have $n_b = 2k+1$, $K(b) = K_{k,k+1}$, and the equality (11) takes the form

$$h_G(t, b) = k^2 t - k - t\,|E_b(G)|. \tag{13}$$

Since $x(x-1) \geqslant 0$ for all integral x, Lemma 8 has obvious

Corollary 9. *Let b determine a $(2k+1)$-gonal inequality. Then*

$$\max_b h_G(t, b) \leqslant k^2 t - k, \tag{14}$$

with equality if and only if G contains the complete bipartite graph $K_{k,k+1}$.

Recall that $d^t(G)$ is $(2k+1)$-gonal if $h_G(t, b) \leqslant 0$ for all b determining a $(2k+1)$-gonal inequality. It is easy to verify the following.

Corollary 10. *Let $n \geqslant 2m+1$. Set $d_m(G) \equiv m^3 d^t(G)$, where $t = 1/m + 1/m^3$. Then:*

(a) *$d_m(G)$ is $(2k+1)$-gonal for all $k < m$.*

(b) *$d_m(G)$ is $(2m+1)$-gonal except when $G = K_{m,m+1}$, and in this case only the pure $(2m+1)$-gonal inequality is violated.*

Proof of P2. For a graph G on n vertices, consider integral distance $d_m(G)$. By Corollary 10, $d_m(G)$ satisfies all $(2m+1)$-gonal inequalities if and only if G does contain $K_{m,m+1}$ as an induced subgraph. Hence P2 and P4 are equivalent. □

Proof of P3. Let $k = \lceil (n-1)/2 \rceil$. For $s = k, k-1, \ldots, 1$, we define $t = 1/s + 1/s^3$ and consider the distance $d_s(G)$ defined in Corollary 10. Let m be the largest integer such that G contains $K_{m,m+1}$ as an induced subgraph. Consider the answers given to question P3. If $s > m$, G does not contain a $K_{s,s+1}$, so by Corollary 10 either the answer to P3 is d_s hypermetric, or d_s is not hypermetric but violates a $(2p+1)$-gonal inequality for $p > s$. When $s = m$, G contains a $K_{s,s+1}$. By Corollary 10 the answer to P3 must be that d_s is not hypermetric and the minimum hypermetric inequality violated is $(2s+1)$-gonal. The answer to P3 gives us the value m that answers P4′. Therefore P3 is also NP-hard. □

The problems in this seciton are related to some computational problems on integer quadratic forms. Let $Q = (q_{ij})$ be an integer symmetric $(n-1) \times (n-1)$ matrix and let c be an integer $(n-1)$-vector. Define the quadratic form

$$g(x) = cx - x^{\mathrm{T}} Q x. \tag{15}$$

It is known that the minimum of (15) (and hence testing whether $g(x) \leqslant 0$) over *binary* vectors $x \in \{0, 1\}^{n-1}$ can be found in polynomial time (see e.g., [3, Ch. 7, Corollary 7.4]). Now (1) is equivalent to $g(x) \leqslant 0$ for all *integer* vectors x, where

$$q_{ij} = \begin{cases} d_{in} & 1 \leqslant i = j \leqslant n-1, \\ (d_{in} + d_{jn} - d_{ij})/2 & 1 \leqslant i, j \leqslant n-1 \end{cases}$$

and $c_i = d_{in}$ for $1 \leq i \leq n-1$. Indeed, if (1) is violated for some integer vector b, then setting $x_i = b_i$ for $1 \leq i \leq n-1$ gives an integer vector x for which $g(x) > 0$. Conversely, from an integer vector x for which $g(x)$ is positive we can construct a vector b generating a violated hypermetric inequality for d.

If Problem P1 is NP-hard, it would imply that minimizing (15) over the integers is NP-hard. In a similar way, the complexity results for Problems P2 and P3 can be interpreted in terms of the NP-hardness of corresponding computational problems for quadratic forms over the integers.

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References

[1] D. Avis and M. Deza, The cut cone, l^1-embeddability, complexity, and multicommodity flows, Networks 21 (1991) 595–617.

[2] S.S. Ryškov and E. P. Baranovskii, C-types of n-dimensional lattices and 5-dimensional primitive parallelohedra (with application to the theory of coverings), Trudy Matematicheskogo Institut imeni V.A. Steklova 137 (1976, No4) = Proceedings of the Steklov Institute of Mathematics (1978, No4).

[3] G. L. Nemhauser and L.A. Wolsey, Integer and Combinatorial Optimization (Wiley, New York, 1988).

[4] M. Deza, V.P. Grishukhin and M. Laurent, The hypermetric cone and L-polytopes, Rep. R 286 of IASI, Rome, 1990; as: The hypermetric cone is polyhedral, Combinatorica, to appear.

[5] M. Deza, V.P. Grishukhin and M. Laurent, Extreme hypermetrics and L-polytopes, in: Sets, Graphs, and Numbers, Budapest (Hungary), Colloq. Math. Soc. János Bolyai 60 (1991) .

[6] A. Brønstead, Convex Polytopes (Springer, Berlin, 1983).

[7] P. Terwilliger and M. Deza, The classification of finite connected hypermetric spaces, Graphs Combin. 3 (1987) 293–298.

[8] M.R. Garey and D.S. Johnson, Computers and Intractability (Freeman, New York, 1979).