A REPRESENTATION THEOREM FOR UNIVERSAL HORN CLASSES CATEGORICAL IN POWER

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Introduction

In what follows let $K$ be a universal Horn class categorical in (some suitably large, infinite) power. The author showed (in Givant [78], hereafter referred to as [G]) that either $K$ or a principal extension of $K$ (obtained by adding a new constant $c$ to the language of $K$ and a single equational axiom concerning $c$) must be definitionally equivalent to one of a class of standard examples of universal Horn classes categorical in power. In the present paper we describe some new examples of such classes (Section 1) and prove that $K$ itself must be definitionally equivalent to one (and essentially just one) of the enlarged class of examples (Section 4). In particular, the passage to a principal extension of $K$ is avoided. For instance, every variety (i.e. equationally defined class of algebras) categorical in power is polynomially definitionally equivalent to one of the following: (1) the variety of affine spaces—over the ring of $n \times n$ matrices with entries from a particular skew field—with a certain distinguished projection; (2) the variety of $n$th powers of "sets" (roughly, the $n$th powers of sets together with certain additional operations that allow us to characterize this construction equationally); (3) the variety of $n$th powers of "pointed sets".

These results were announced (in a different form) in Givant [75]. Soon afterwards there appeared in Palyutin [75] another representation theorem for universal Horn classes categorical in power. The connections between Palyutin's representation and the one given here are discussed in Section 5.

0. Preliminaries

In general we shall follow the notation of [G]. The letters $k, l, m, n$ denote finite cardinals, $\omega$ the first infinite cardinal, and $\kappa$ an arbitrary cardinal. Throughout this paper we shall assume that $m, n \geq 1$. The cardinality of a set $A$ is denoted by $|A|$. Capital German letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ denote algebraic structures (possibly with

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1 See the historical remarks at the end of Section 5.

2 The author is indebted to Professor Ralph McKenzie for reading a draft of this paper and suggesting several alterations which were subsequently adopted.
relations) and Roman letters \( A, B, C, \ldots \) the corresponding universes. Structures with operations only (no relations) are called \emph{algebras}. Given a structure \( \mathfrak{A} \) and a family \( \langle O_i : i \in I \rangle \) of operations on \( A \), we let \( \langle \mathfrak{A}, O_i \rangle_{i \in I} \) denote the structure obtained from \( \mathfrak{A} \) by adjoining the \( O_i \) as new fundamental operations. A structure is \emph{trivial} if its universe contains only one element. For notational convenience we shall construe "\( A \) as the set of \( n \)-tuples \( \langle x_1, \ldots, x_n \rangle \) where each \( x_i \in A \) (note that the indexing is from 1 to \( n \)). The corresponding \( n \)th (Cartesian) power of \( \mathfrak{A} \) is written "\( \mathfrak{A}^n \).

We use the letter \( K \) to denote a non-degenerate class of algebraic structures of the same similarity type \( \rho K \); here "non-degenerate" means there are non-trivial structures in \( K \). By a \emph{variety}, respectively \emph{quasivariety}, we understand a class of algebras which is defined by a set of equations, respectively universal Horn sentences.\(^3\) Given a class \( K \) of structures, we define three new classes, \( K_{\alpha}, K^\circ, \) and \( K^\ast \): \( K_{\alpha} \) is obtained by excluding from \( K \) all finite structures; \( K^\circ \) is obtained by excluding from \( K \) just those trivial structures which are not substructures of some non-trivial member of \( K \); \( K^\ast \) is obtained by adjoining to \( K \) all trivial structures of type \( \rho K \). Thus, \( K, K^\circ, \) and \( K^\ast \) differ only in their trivial structures. When \( K \) is a quasivariety, then so are \( K^\circ \) and \( K^\ast \), as is readily seen.

When \( K \) is a variety, then of course \( K = K^\ast \). \( K \) is \emph{categorical in power} provided it is categorical in some infinite power \( \kappa \geq |\rho K| \), i.e. \( K \) has a member of power \( \kappa \) and any two members of power \( \kappa \) are isomorphic.

We associate with each similarity type \( \rho \) a first-order language; its variables, in order, are \( \nu_1, \nu_2, \nu_3, \ldots \). \( \mathcal{I}(\rho) \) denotes the set of terms of this language, and \( \mathcal{I}(K) \) is an abbreviation for \( \mathcal{I}(\rho K) \). We use \( \tau, \sigma, \gamma \) to denote terms. The notation \( \tau(\nu_1, \ldots, \nu_k) \) conveys the fact that the variables occurring in \( \tau \) are among \( \nu_1, \ldots, \nu_k \) (but not all of these need occur); \( \tau(\sigma_1, \ldots, \sigma_k) \) is the term obtained by simultaneously substituting each term \( \sigma_i \) for \( \nu_i \) \((1 \leq i \leq k)\) in \( \tau \). We write \( \tau^{(\omega)} \) to denote the \( k \)-ary polynomial induced by the term \( \tau(\nu_1, \ldots, \nu_k) \) in a \( \rho \)-structure \( \mathfrak{A} \).

The above notations generalize in the obvious way to formulas. The phrase \( \tau \) is a \emph{term} of \( K \) means \( \tau \in \mathcal{I}(K) \), and the phrase \( K \) has \emph{constants} means that for some term \( \tau(\nu_1) \) the equation \( \tau(\nu_1) = \tau(\nu_2) \) holds in \( K \). A set \( T \subseteq \mathcal{I}(\rho) \) is \emph{closed under substitution} provided it contains all variables, and whenever \( \tau(\nu_1, \ldots, \nu_k), \sigma_1, \ldots, \sigma_k \in T \), then we also have \( \tau(\sigma_1, \ldots, \sigma_k) \in T \). Every set \( T \subseteq \mathcal{I}(\rho) \) is included in a smallest set \( T^* \) of terms which is closed under substitution.

We now come to a group of less familiar notions which will play an important conceptual role in the rest of this paper. Actually, the first of these notions seems to be new. A term \( \tau(\nu_1, \ldots, \nu_k) \) is \emph{idempotent over a term} \( \sigma \) in \( \mathfrak{A} \), or \( \sigma \)-\emph{idempotent in} \( \mathfrak{A} \) for short, provided the equation \( \tau(\sigma_1, \ldots, \sigma_k) = \sigma \) holds in \( \mathfrak{A} \). This is equivalent to saying that \( \tau^{(\omega)}(a, \ldots, a) = a \) for each \( a \) in the range of \( \sigma^{(\omega)} \). When

\(^3\) In Mal’cev [73] the term “variety” means a class of structures (possibly with relations) defined by a set of atomic formulas. A “quasi-variety” there is a class of structures defined by a set of “quasi-identities”, i.e. universal Horn sentences which have an atomic formula as one of the disjuncts.
\( \sigma \) is \( v_1 \) we obtain the usual general algebraic notion of idempotence (see, e.g., Urbanik [65]). \( \tau \) is \( \sigma \)-idempotent in \( K \) provided it is \( \sigma \)-idempotent in each \( \mathcal{A} \in K \).

Given a set \( T \subseteq \mathcal{I}(\rho) \) and a \( \rho \)-structure \( \mathcal{A} \) we take the polynomial reduct of \( \mathcal{A} \) to \( T \), written \( \mathcal{A}_T \), to be the structure \( \langle A, \tau^{(\mathcal{A})}\rangle_{\tau \in T} \); for \( K \) a class of \( \rho \)-structures we naturally set \( K_T = \{ \mathcal{A}_T : \mathcal{A} \in K \} \). Of course \( \mathcal{A}_T \) is a reduct of \( \mathcal{A}_{\mathcal{I}(\rho)} \) in the usual sense of the word. The \( \sigma \)-idempotent reduct of \( \mathcal{A} \) (respectively of \( K \)) is just \( \mathcal{A}_T \) (respectively \( K_T \)) where \( T \) is the set of \( \sigma \)-idempotent terms. When \( \sigma = v_1 \) we obtain the usual notion of an idempotent reduct.

We write \( K =_{\pi, \zeta} L \) to express the fact that \( =_{\pi, \zeta} \) is a polynomial (definitional) equivalence between two classes, \( K \) and \( L \), of algebras. Roughly speaking, for each \((k\text{-ary})\) operation symbol \( O \) in the language of \( K \), \( O^w \) is a term of \( \mathcal{I}(L) \) which defines \( O \cdot \cdots \cdot v_k \) in \( L \), and for each \((l\text{-ary})\) operation symbol \( Q \) in the language of \( L \), \( Q^\zeta \) is a term of \( \mathcal{I}(K) \) which defines \( Q \cdot v_1 \cdot \cdots \cdot v_l \) in \( K \) (see, e.g., [G], Section 1.3). The notation \( K = L \) means there is a polynomial equivalence between \( K \) and \( L \). Suppose \( K =_{\pi, \zeta} L \) and \( \tau(v_1, \ldots , v_k) \in \mathcal{I}(K) \), \( \sigma(v_1, \ldots , v_k) \in \mathcal{I}(L) \). We shall say that \( \tau \) and \( \sigma \) correspond under \( =_{\pi, \zeta} \) if \( \tau^{(\mathcal{A})} = \sigma^{(\mathcal{B})} \) whenever \( \mathcal{A} \in K \), \( \mathcal{B} \in L \), and \( \mathcal{A} =_{\pi, \zeta} \mathcal{B} \). This is obviously equivalent to saying, for example, that the equation \( \tau^w = \sigma \) holds in \( L \), where \( \tau^w \) is the translation of \( \tau \) induced by \( \pi \).

Two sets of terms, \( T \subseteq \mathcal{I}(K) \) and \( S \subseteq \mathcal{I}(L) \), are equivalent under \( =_{\pi, \zeta} \) provided that, under \( =_{\pi, \zeta} \), each \( \tau \in T \) corresponds to at least one \( \sigma \in S^* \), and for each \( \sigma' \in S \) there is at least one \( \tau' \in T^* \) corresponding to \( \sigma' \). The following theorem is easily verified.

**Theorem 0.1.** Suppose \( K =_{\pi, \zeta} L \), \( T \subseteq \mathcal{I}(K) \) and \( S \subseteq \mathcal{I}(L) \). If \( T \) and \( S \) are equivalent under \( =_{\pi, \zeta} \), then \( K_T =_{\pi, \zeta} L_S \), where \( =_{\pi, \zeta} \) is obtained from \( =_{\pi, \zeta} \) in the obvious way.

Of some importance is the special case when \( K = L \) and \( O^w = O^\zeta = O \cdot v_1, \ldots , v_k \) for each operation symbol \( O \) in the language of \( K \) (where \( k \) is the rank of \( O \)), so that \( \mathcal{A} =_{\pi, \zeta} \mathcal{A} \) for each \( \mathcal{A} \in K \). In this case we replace the reference to \( =_{\pi, \zeta} \) in the notions of correspondence and equivalence by one to \( K \). Thus \( \tau \) and \( \sigma \) correspond in \( K \) if \( \tau = \sigma \) holds in \( K \); \( T, S \subseteq \mathcal{I}(K) \) are equivalent in \( K \) if for each \( \tau \in T \) there is a \( \sigma \in S^* \) and for each \( \sigma' \in S \) there is a \( \tau' \in T^* \) such that \( \tau = \sigma \) and \( \tau' = \sigma' \) hold in \( K \). For this special case Theorem 0.1 says that if \( T, S \subseteq \mathcal{I}(K) \) are equivalent in \( K \), then \( K_T = K_S \).

Two algebras \( \mathcal{A} = (A, O_i)_{i \in I} \) and \( \mathcal{B} = (B, Q_j)_{j \in J} \) are variants of one another if they are the same except for their index sets, i.e. if there is a bijection \( f \) of \( I \) onto \( J \) such that

\[
\langle A, O_i \rangle_{i \in I} = \langle B, Q_{f(i)} \rangle_{i \in I}.
\]

This notion extends in the obvious way to structures and classes of structures. Two classes of algebras which are variants of one another are trivially polynomially equivalent.
1. Affine spaces derived from modules

Throughout this section let $\mathcal{R}$ be a ring with unit 1. We denote by $\mathcal{R}^{(n)}$ the ring of $n \times n$ matrices with entries from $\mathcal{R}$ (the universe of $\mathcal{R}$). We shall usually use $\mu, \nu, \lambda, \rho,$ and $\chi$ to denote matrices. Since $\mathcal{R}^{(n)}$ is also a ring with unit 1 (the identity matrix), we may construct, for each $m,$ the ring $(\mathcal{R}^{(n)})^{(m)}.$ Obviously this ring is isomorphic to $\mathcal{R}^{(n \cdot m)}$ in a canonical way. In practice we shall identify the two.

We conceive of an $\mathcal{R}$-module as an algebra

$$\mathcal{A} = \langle A, +^{(\mathcal{R})}, \alpha^{(\mathcal{R})} \rangle_{\alpha \in \mathcal{R}},$$

where $+^{(\mathcal{R})}$ is the usual binary operation of module addition and, for each $\alpha \in \mathcal{R},$ $\alpha^{(\mathcal{R})}$ is a unary operation, called scalar multiplication by $\alpha,$ which assigns to each $x \in A$ the (left-hand) product $\alpha^{(\mathcal{R})}x$ (or $\alpha x,$ as it is usually written). The class of all $\mathcal{R}$-modules, denoted by Mod ($\mathcal{R}$), is a variety. Indeed an equational axiom set can be obtained from the usual $\mathcal{R}$-module axioms (e.g., in Birkhoff–MacLane [67], pp. 71, 190) by replacing the additive identity and inverse laws with: $v_1 + 0v_1 = v_1,$ $0v_1 = Ov_2.$

For each $n$ and each $\mathcal{R}$-module $\mathcal{A}$ we construct an $\mathcal{R}^{(n)}$-module $\mathcal{W}^{(n)}$ in the usual way: the universe is the set $A$ of $n$-dimensional column vectors with entries from $A$ (for typographical convenience we write column vectors as rows: $x = (x_1, \ldots, x_n)$); addition of vectors is pointwise and scalar multiplication by $\mu \in \mathcal{R}^{(n)}$ is the usual operation of matrix multiplication of column vectors by $\mu.$ It is not difficult to show that each $\mathcal{R}^{(n)}$-module is isomorphic to an $\mathcal{R}^{(n)}$-module of the form $\mathcal{W}(n),$ where $\mathcal{A}$ is an $\mathcal{R}$-module.

Following Ostermann–Schmidt [66] we now define a class of affine spaces over $\mathcal{R}.$ Fix an $\mathcal{R}^{(n)}$-module $\mathcal{A}$ and let $\vdash^{(\mathcal{R})}$ be the ternary and $\check{\alpha}^{(\mathcal{R})}$ the binary operations determined by the formulas:

$$\vdash^{(\mathcal{R})}(x, y, z) = (x +^{(\mathcal{R})} y) -^{(\mathcal{R})} z \quad \text{for} \quad x, y, z \in A;$$

$$\check{\alpha}^{(\mathcal{R})}(x, y) = \alpha^{(\mathcal{R})} x +^{(\mathcal{R})} (1 - \alpha)^{(\mathcal{R})} y \quad \text{for} \quad x, y \in A \quad \text{and} \quad \alpha \in \mathcal{R}.$$

The affine space (over $\mathcal{R}$) derived from $\mathcal{A}$ is defined to be the algebra

$$\mathcal{A}^{(\text{aff})} = \langle A, \vdash^{(\mathcal{R})}, \check{\alpha}^{(\mathcal{R})} \rangle_{\alpha \in \mathcal{R}};$$

Aff ($\mathcal{R}$) denotes the class of all affine space over $\mathcal{R}.$ Ostermann–Schmidt [64], [66] showed that Aff ($\mathcal{R}$) is a variety and that, in case $\mathcal{R}$ is actually a field, this notion of affine space is equivalent to the usual one (given, for example, in Birkhoff–MacLane [67], p. 420; see also p. 425).

Let $\rho \in \mathcal{R}^{(n)}$ be an idempotent matrix, i.e. $\rho \rho = \rho.$ For each $\mathcal{R}^{(n)}$-module $\mathcal{B}$ the operation $\rho^{(\mathcal{B})}$ may be viewed as a projection of the whole space $\mathcal{B}$ onto the subset defined by the equation $\rho_1 v_1 = v_1.$ The affine space (over $\mathcal{R}^{(n)}$) derived from $\mathcal{B}$ with a distinguished projection $\rho$ is defined to be the algebra

$$\mathcal{B}^{(\text{aff})}_\rho = \langle \mathcal{B}, \vdash^{(\mathcal{R})}, \check{\alpha}^{(\mathcal{R})}, \rho^{(\mathcal{R})} \rangle_{\alpha \in \mathcal{R}^{(n)}} (= \langle \mathcal{A}^{(\text{aff})}, \rho^{(\mathcal{R})} \rangle).$$
We shall see that the class \( \text{Aff}(\mathfrak{M}(n), \rho) \) of all such spaces is a variety. As a special case of this construction, let \( 0 \leq k \leq n \) and take as \( \rho \) the matrix \( \delta_k \) for which the only non-zero entries are the \((i, i)\)th entries, with \( 1 \leq i \leq k \), these entries being 1. 
(When \( k = 0 \), \( \delta_k \) is just the zero matrix.) In this case we write \( \mathfrak{B}_k^{(\text{aff})} \) for \( \mathfrak{B}_k^{(\text{aff})} \) and \( \text{Aff}(\mathfrak{M}(n), k) \) for \( \text{Aff}(\mathfrak{M}(n), \delta_k) \). Notice that when \( \mathfrak{B} = \mathfrak{A}(n) \), the subset projected by \( \delta_k \) consists of all vectors of the form \( (x_1, \ldots, x_k, 0, \ldots, 0) \).

Again let \( \mathfrak{A} \) be a fixed \( \mathfrak{R} \)-module. Define \( \text{Mod}(\mathfrak{R}, \mathfrak{A}) \) to be the class of algebras

\[
\mathfrak{C} = (C, +^{(\mathfrak{A})}, \alpha^{(\mathfrak{A})}, a^{(\mathfrak{A})})_{a \in \mathfrak{R}, a \in \mathfrak{A}}
\]

such that \( (C, +^{(\mathfrak{A})}, \alpha^{(\mathfrak{A})})_{a \in \mathfrak{R}} \) is an \( \mathfrak{R} \)-module, the \( a^{(\mathfrak{A})} \) are constants in \( C \), and the mapping \( a \mapsto a^{(\mathfrak{A})} \) is an (isomorphic) embedding of \( \mathfrak{A} \) into \( (C, +^{(\mathfrak{A})}, \alpha^{(\mathfrak{A})})_{a \in \mathfrak{R}} \). \( \text{Mod}(\mathfrak{R}, \mathfrak{A}) \) is a quasivariety with an axiom set consisting of the axioms for \( \text{Mod}(\mathfrak{R}) \), together with the equations and inequalities which insure that the mappings \( a \mapsto a^{(\mathfrak{A})} \) are embeddings. \( \text{Aff}(\mathfrak{R}, \mathfrak{A}) \) is defined as the class of algebras

\[
\mathfrak{C}^{(\text{aff})} = (C, +^{(\mathfrak{A})}, \tilde{\alpha}^{(\mathfrak{A})}, \tilde{a}^{(\mathfrak{A})})_{a \in \mathfrak{R}, a \in \mathfrak{A}}
\]

where \( \mathfrak{C} \in \text{Mod}(\mathfrak{R}, \mathfrak{A}) \) and \( \tilde{a}^{(\mathfrak{A})} \) is the unary operation defined by the formula:

\[
\tilde{a}^{(\mathfrak{A})}x = x +^{(\mathfrak{A})} a^{(\mathfrak{A})} \quad \text{for} \quad x \in C.
\]

Finally, let \( \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), \rho) \) be the class of algebras

\[
\mathfrak{C}_\rho^{(\text{aff})} = (C, +^{(\mathfrak{A})}, \tilde{\alpha}^{(\mathfrak{A})}, \tilde{a}^{(\mathfrak{A})}, \rho^{(\mathfrak{A})})_{a \in \mathfrak{R}(n), a \in \mathfrak{A}}
\]

where \( \mathfrak{C} \in \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)) \). It is unnecessary to consider arbitrary \( \mathfrak{R}(n) \)-modules \( \mathfrak{B} \) instead of \( \mathfrak{A}(n) \) in this construction because when \( \mathfrak{B} = \mathfrak{A}(n) \), then \( \text{Mod}(\mathfrak{R}(n), \mathfrak{B}) \) is simply a variant of \( \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)) \) (see the preliminaries). When \( \rho \) is \( \delta_k \) we again write \( \mathfrak{C}_k^{(\text{aff})} \) and \( \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), k) \).

**Theorem 1.1.** Let \( \mathfrak{A} \) be an \( \mathfrak{R} \)-module and \( \rho \) an idempotent matrix in \( \mathfrak{R}(n) \).

(i) \( \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), \rho) \) is a quasivariety.

(ii) \( \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), \rho) \) is a variety iff \( |A| = 1 \), in which case it is a polynomial definitional extension of \( \text{Aff}(\mathfrak{R}(n), \rho) \).

(iii) Let \( T \) be the set of terms \( \alpha_1 v_1 + \cdots + \alpha_m v_m + a \) of \( \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)) \) such that \( \left( \sum_{i=1}^{m} \alpha_i \right) \rho = \rho \) and \( m \geq 1 \). Then

\[
\text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n))_T = \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), \rho).
\]

(iv) \( \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)) = \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), 0) \).

(v) \( \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n)) = \text{Aff}(\mathfrak{R}(n), \mathfrak{A}(n), n) \).

**Proof.** To see (i) set

\[
K = \{ (\mathfrak{C}_\rho^{(\text{aff})}, 0^{(\mathfrak{A})}) : \mathfrak{B} \in \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)) \text{ and } 0^{(\mathfrak{A})} \text{ is its zero vector} \}.
\]

\[
K \models \text{Mod}(\mathfrak{R}(n), \mathfrak{A}(n)).
\]
Indeed a polynomial definition of $K$ in $\text{Mod}(\mathfrak{M}(n), \mathfrak{A}(n))$ is inherent in the definition of the algebras in $K$. On the other hand, in $K$ we take $\mathfrak{T}(v_1, v_2, 0)$, $\mathfrak{A}(v_1, 0)$, and $\mathfrak{A}(0)$ as definitions of $v_1 + v_2$, $\lambda v_1$ for $\lambda \in F(n)$, and $a$ for $a \in \mathfrak{A}$, respectively.

If $\mathfrak{B} \in \text{Mod}(\mathfrak{M}(n), \mathfrak{A}(n))$ and $b \in B$, then the mapping $x \mapsto x + (0)\rho(0)(b)$ defines an isomorphism of $(\mathfrak{B}(\rho), 0(0))$ onto $(\mathfrak{B}(\rho), \rho(0)(b))$ (here the idempotence of $\rho$ is used). Thus

$$\text{if } \mathfrak{B} \in \text{Aff}(\mathfrak{M}(n), \mathfrak{A}(n), \rho) \text{ and } b \in B, \text{ then } (\mathfrak{B}, \rho(0)(b)) \in K. \quad (1.2)$$

Now let $\Gamma_0$ be the axiom set for $\text{Mod}(\mathfrak{M}(n), \mathfrak{A}(n))$ mentioned above and suppose $\Gamma_1$ is the axiom set for $K$ induced by $\Gamma_0$ and the polynomial equivalence of (1.1). Pick a variable $v_n$ not occurring in $\mathfrak{M}$ and let $\Gamma_2$ be the set obtained from $\Gamma_1$ by replacing each occurrence of $0$ with $\rho v_n$.

To obtain (ii) observe that $\Gamma_0$ (and hence $\Gamma_2$) is a set of equations if $|A| = 1$. On the other hand, if $|A| > 1$, then any member of $\text{Aff}(\mathfrak{M}(n), \mathfrak{A}(n), \rho)$ has a homomorphic image (obtained by identifying all constants) which is not in the class.

To prove (iii), assume that $\sum_{i=1}^{m} \alpha_i \rho = \rho$ and $a \in \mathfrak{A}$. Now in $\text{Mod}(\mathfrak{M}(n), \mathfrak{A}(n))$ the equation

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + a = \alpha_1 v_1 + \cdots + \alpha_m v_m + 0v_1 + a$$

is trivially valid, and, since $0 = (1 - \sum_{i=1}^{m} \alpha_i)\rho$, the term on the right-hand side of this equation clearly corresponds to a term obtained by repeated substitution from the terms $\mathfrak{T}(v_1, v_2, v_3)$, $\mathfrak{A}(v_1, v_2)$, $\ldots$, $\mathfrak{A}(v_1, v_2)$, $\rho(v_1)$, and $\mathfrak{A}(v_1)$. Thus we obtain (iii) from Theorem 0.1. Finally, (iv) follows from (1.1) and the proof of (v) is trivial.

Taking an $\mathfrak{A}$ with $|A| = 1$ in Theorem 1.1, we obtain the corresponding results for varieties:

**Theorem 1.2.** Let $\rho$ be an idempotent matrix in $R(n)$.

(i) $\text{Aff}(\mathfrak{M}(n), \rho)$ is a variety.

(ii) For $T$ the set of $\rho v_1$-idempotent terms of $\text{Mod}(\mathfrak{M}(n))$ we have

$$\text{Mod}(\mathfrak{M}(n))_T = \text{Aff}(\mathfrak{M}(n), \rho);$$

i.e. $\text{Aff}(\mathfrak{M}(n), \rho)$ is polynomially equivalent to the class of all $\rho v_1$-idempotent reducts of $\mathfrak{M}(n)$-modules.

(iii) $\text{Mod}(\mathfrak{M}(n)) = \text{Aff}(\mathfrak{M}(n), 0)$.

(iv) $\text{Aff}(\mathfrak{M}(n)) = \text{Aff}(\mathfrak{M}(n), n)$.

In the sequel we shall not bother, for the most part, to state separately such results for varieties, but shall simply refer to the corresponding results for quasivarieties.

From Theorem 1.1 (iv), (v) we see that $\text{Mod}(\mathfrak{M}(n), \mathfrak{A}(n))$ and $\text{Aff}(\mathfrak{M}(n), \mathfrak{A}(n))$ are essentially special cases of the $\text{Aff}(\mathfrak{M}(n), \mathfrak{A}(n), k)$ construction. In Theorem 1.3 below
this result is somewhat extended. Consider first a term $\sigma$ of the form $\mu v_1 + c$ where $\mu \in R^{(n)}$ is invertible and $c \in "A$. Let $\sigma^{-1}$ denote the term $\mu^{-1}v_1 - \mu^{-1}c$. For each $\mathbb{C} \in \text{Mod}(\mathbb{R}^{(n)}, \mathbb{U}^{(n)})$ set

$$\mathbb{C}^{(n)} = \langle C, +^{(n)}, \alpha^{(n)}, a^{(n)} \rangle_{a \in R^{(n)}, a \in "A}$$

where $+^{(n)}$ is the binary operation, $\alpha^{(n)}$ the unary operation, and $a^{(n)}$ the constant determined by the formulas:

$$+^{(n)}(x, y) = (x +^{(n)} y) -^{(n)} c^{(n)} \quad \text{for} \quad x, y \in C;$$

$$\alpha^{(n)}(x) = \sigma(a\sigma^{-1})^{(n)}(x) \quad \text{for} \quad x \in C \quad \text{and} \quad a \in R^{(n)};$$

$$a^{(n)} = \sigma^{(n)}(a) \quad \text{for} \quad a \in "A.$$

It is easy to check that the correspondence $x \mapsto \sigma^{(n)}(x)$ defines an isomorphism of $\mathbb{C}$ onto $\mathbb{C}^{(n)}$, so that $\mathbb{C}^{(n)}$ is in $\text{Mod}(\mathbb{R}^{(n)}, \mathbb{U}^{(n)})$. This also holds if we take $\sigma^{-1}$ instead of $\sigma$. Since $\mathbb{C}^{(n)} \cong \mathbb{C}$ we obtain

$$(*) \quad \text{Mod}(\mathbb{R}^{(n)}, \mathbb{U}^{(n)}) = \{\mathbb{C}^{(n)} : \mathbb{C} \in \text{Mod}(\mathbb{R}^{(n)}, \mathbb{U}^{(n)})\}.$$

**Theorem 1.3.** Let $\mathbb{F}$ be a skew field, $\mathbb{A}$ an $\mathbb{F}$-module, and $\rho$ an idempotent matrix in $F^{(n)}$ of rank $k$. Then

$$\text{Aff}(\mathbb{F}^{(n)}, \mathbb{A}^{(n)}, \rho) = \text{Aff}(\mathbb{F}^{(n)}, \mathbb{A}^{(n)}, k).$$

**Proof.** Since $\mathbb{F}$ is a skew field, there is an invertible matrix $\mu$ such that $\mu \rho \mu^{-1} = \delta_k$. We define $=_{\pi, \xi}$ by setting

$$+^\pi = +((v_1, v_2, v_3), \quad +^\xi = +((v_1, v_2, v_3),$$

$$\alpha^\pi = \mu \alpha \mu^{-1}(v_1, v_2), \quad \alpha^\xi = \mu^{-1} \alpha \mu(v_1, v_2),$$

$$\delta^\pi = \mu \delta(v_1), \quad \delta^\xi = \mu^{-1} \delta(v_1),$$

$$\rho^\pi = \rho(v_1), \quad \rho^\xi = \rho(v_1).$$

Taking $\sigma$ to be $\mu v_1 + 0$, it is straightforward to show that

$$(\mathbb{C}^{(n)})_{\rho} = _{\pi, \xi} \mathbb{C}^{(n)}$$

for each $\mathbb{C}$ in $\text{Mod}(\mathbb{R}^{(n)}, \mathbb{U}^{(n)})$. Since from $(*)$ it follows that

$$\text{Aff}(\mathbb{F}^{(n)}, \mathbb{A}^{(n)}, \rho) = \{(\mathbb{C}^{(n)})_{\rho} : \mathbb{C} \in \text{Mod}(\mathbb{F}^{(n)}, \mathbb{A}^{(n)})\},$$

this completes the proof.

For later use we prove:

**Theorem 1.4.** Let $\mathbb{A}$ be an $\mathbb{R}$-module. For each $c \in A$ there is a polynomial equivalence $=_{\pi, \xi}$ such that

(i) $\text{Mod}(\mathbb{R}, \mathbb{A}) = _{\pi, \xi} \text{Mod}(\mathbb{R}, \mathbb{A}),$

(ii) $0$ corresponds to $c$ under $=_{\pi, \xi}.$
Proof. Set
\[ +^* = + (v_1, v_2, c), \quad +^c = + (v_1, v_2, -c), \]
\[ \alpha^* = \bar{\alpha}(v_1, c), \quad \alpha^c = \bar{\alpha}(v_1, -c), \]
\[ a^* = \bar{\alpha}(c), \quad a^c = \bar{\alpha}(-c). \]

Taking \( \sigma \) to be \( v_1 + c \), we have \( \mathcal{C}^{(\sigma)} = \pi_{\cdot \cdot \cdot C} \mathcal{C} \) for each \( \mathcal{C} \) in \( \text{Mod} (\mathcal{H}, \mathcal{A}) \). Hence the theorem follows from (*)

2. The \( n \)th-power construction

We now recall a method of constructing new varieties and quasivarieties from old ones which preserves categoricity in power. Given an \( n \) and a non-empty set \( A \) we define on "\( A \) an \( n \)-ary operation \( d_n^A \) and \( n \) unary operations \( p_i, 1 \leq i \leq n \), by stipulating
\[ d_n^A ((x_1, \ldots, x_{1,n}), \ldots, (x_n, \ldots, x_{n,n})) = (x_{1,1}, \ldots, x_{n,n}), \]
\[ p_i^n ((x_1, \ldots, x_n)) = (x_i, \ldots, x_n). \]

For each structure \( \mathfrak{A} \) and class \( K \) of structures set
\[ K_n = \{ \mathfrak{B} : \mathfrak{B} \equiv \mathfrak{A}_n \text{ for some } \mathfrak{A} \in K \}. \]

The class \( K_n \) is closely related to the class of \( n \)th powers of members of \( K \). The diagonal operation \( d_n^A \) and the projections \( p_i^n \) allow one to express this \( n \)th power construction by means of equations. When no confusion can arise we will suppress "\( n \)" in the notations for the operation symbols, \( d_n \) and \( p_i^n \), of the language associated with \( K_n \), writing \( d \) and \( p_i \) instead. Assume that \( K \) is closed under isomorphisms. Then, as is known, \( K_n \) is a variety or quasivariety just in case \( K \) is. Moreover \( K_n \) is categorical in some infinite power just in case \( K \) is categorical in the same infinite power. A simple proof of these facts as well as relevant historical remarks can be found, e.g., in Section 2.2 of [G].

Lawvere observed that
\[ \text{Mod} (\mathfrak{H})_n = \text{Mod} (\mathfrak{H}^{(n)}). \]
(See Neumann [70], p. 8.) We generalize this somewhat in the next theorem.

Theorem 2.1. Let \( \mathfrak{F} \) be a skew field, \( \mathfrak{A} \) an \( \mathfrak{F} \)-module, and \( 0 \leq k \leq m \). Then
\[ \text{Aff} (\mathfrak{F}^{(m)}, \mathfrak{A}^{(m)}, k)_n = \text{Aff} (\mathfrak{F}^{(m-n)}, \mathfrak{A}^{(m-n)}, k). \]

Proof. By extending the definitions implicit in Lawvere's observation one easily obtains a polynomial equivalence \( \epsilon_{\cdot \cdot \cdot C} \) such that
\[ \text{Mod} (\mathfrak{F}^{(m)}, \mathfrak{A}^{(m)})_n = \epsilon_{\cdot \cdot \cdot C} \text{Mod} (\mathfrak{F}^{(m-n)}, \mathfrak{A}^{(m-n)}). \]
A representation theorem for universal Horn classes categorical in power

Now \( p_1(\delta_k v_1) \) clearly corresponds, under \( \equiv \), to a term \( p v_1 \) where \( p \) is an idempotent matrix of \( F^{(m \cdot n)} \). Let \( T \) be the set of all terms of Mod (\( \mathcal{G}(m) \), \( \mathcal{U}(m) \)) of the form \( \tau(v_1, \ldots, v_t) + d a_1 \cdots a_n \) where \( \tau \) is \( p_1(\delta_k v_1) \)-idempotent and \( a \in n^{(m \cdot A)} \); similarly let \( S \) be the set of all terms of Mod (\( \mathcal{G}(m \cdot n) \), \( \mathcal{U}(m \cdot n) \)) of the form \( \alpha_1 v_1 + \cdots + \alpha_t v_t + b \) where \( \alpha_1, \ldots, \alpha_t \in F^{(m \cdot n)} \), \( b \in A^{(m \cdot n)} \), and \( \sum_{i=1}^t \alpha_i \rho = \rho \). One readily shows that \( T \) and \( S \) are equivalent under \( \equiv \). Hence, by Theorem 0.1 and (2.1),

\[
[\text{Mod } \mathcal{G}(m) \mathcal{U}(m)]_T \cong [\text{Mod } \mathcal{G}(m \cdot n) \mathcal{U}(m \cdot n)]_S. \tag{2.2}
\]

Now it is not difficult to see that

\[
\text{Aff } \mathcal{G}(m) \mathcal{U}(m) = \text{Mod } \mathcal{G}(m) \mathcal{U}(m). \tag{2.3}
\]

Combining (2.2), (2.3) and Theorem 1.1 (iii) we obtain

\[
\text{Aff } \mathcal{G}(m) \mathcal{U}(m) = \text{Aff } \mathcal{G}(m \cdot n) \mathcal{U}(m \cdot n). \tag{2.4}
\]

The theorem follows at once from (2.4) and Theorem 1.3.

**Theorem 2.2.** Let \( \mathcal{A} \) be a skew field, \( \mathcal{A} \) an \( \mathcal{A} \)-module, and \( 0 \leq k \leq n \). Then \( \text{Aff } (\mathcal{G}(n) \mathcal{U}(n), k) \) is categorical in every infinite power \( > |F| + |A| \).

**Proof.** From the well-known categoricity of Mod (\( \mathcal{G}, \mathcal{U} \)) in all infinite powers \( > |F| + |A| \) we deduce the same result, first for Mod (\( \mathcal{G}, \mathcal{U} \), \( n \)), and then, by the generalized form of Lawvere's observation (2.1), for Mod (\( \mathcal{G}(n) \), \( \mathcal{U}(n) \)). Since \( \text{Aff } (\mathcal{G}(n) \mathcal{U}(n), k) \) is a polynomial reduct of the latter class, we obtain the theorem.

3. Algebras derived from groups

Throughout this section let \( \mathcal{G} = \langle G, \cdot, ^{-1}, 1 \rangle \) be a fixed group. By a \( \mathcal{G} \)-algebra we understand an algebra

\[
\mathcal{A} = \langle A, g^{(a)} \rangle_{g \in G}
\]

where each \( g^{(a)} \) is a unary operation and the correspondence \( \langle g, x \rangle \mapsto g^{(a)} x \) from \( G \times A \) to \( A \) defines an action of the group \( \mathcal{G} \) on \( A \) (in the sense of, e.g., Birkhoff-MacLane [67]). In particular, the set \( \{g^{(a)} : g \in G\} \) forms a group of permutations of \( A \) under the usual operation of functional composition. We shall occasionally allow the universe \( A \) and the operations \( g^{(a)} \) to be empty. The resulting \( \mathcal{A} \) is said to be empty and is denoted by \( \mathcal{D} \). \( \mathcal{D} \) is not a \( \mathcal{G} \)-algebra, but it proves convenient to consider it together with the \( \mathcal{G} \)-algebras.

We can turn \( \mathcal{G} \) itself into a \( \mathcal{G} \)-algebra

\[
\mathcal{G}^{*} = \langle G, g^{*} \rangle_{g \in G}
\]

by defining \( g^{*}(f) = g \cdot f \) for each \( f \in G \). Moreover, the disjoint union (in the
algebraic sense) of a non-empty family of (κ) isomorphic copies of \( G^* \) is again a \( G \)-algebra. We call such a union a (κ-) multiple of \( G^* \). Finally, the disjoint union of a (κ-) multiple of \( G^* \) with another \( G \)-algebra \( A \) is again a \( G \)-algebra and is called a (κ-) multiple of \( G^* \) over \( A \). We consider \( A \) itself to be a multiple of \( G^* \) over \( A \), namely the 0th multiple. A (non-zero) multiple of \( G^* \) over \( O \) is, of course, just a multiple of \( G^* \).

A fixed point of a \( G \)-algebra \( A \) is an \( a \in A \) such that \( g^{(a)} a = a \) for some \( g \neq 1 \). \( A \) is of type 2 if \( g^{(a)} \) has at most one fixed point for each \( g \neq 1 \). The set \( F^{(a)} \) of fixed points of \( A \) is readily seen to form a subuniverse of \( A \). Let \( \mathfrak{H}^{(a)} \) denote the corresponding subalgebra when \( F^{(a)} \neq \emptyset \) and \( \mathfrak{H} \) when \( F^{(a)} = \emptyset \).

**Theorem 3.1.** Every \( G \)-algebra \( A \) is a \( \kappa \)-multiple of \( G^* \) over \( \mathfrak{H}^{(a)} \) for a unique cardinal \( \kappa \).

This fact is not difficult to prove directly, but it can also be deduced from known facts about the orbits of a permutation group. We omit the details.

Denote the unique cardinal \( \kappa \) in Theorem 3.1 by \( m(A) \). Using Theorem 3.1 it is simple to show that \( m(A) \) and the isomorphism type of \( G^* \) completely determine the isomorphism type of \( A \):

**Theorem 3.2.** Let \( A, B \) be \( G \)-algebras. Then \( A \equiv B \) iff \( \mathfrak{H}^{(a)} = \mathfrak{H}^{(b)} \) and \( m(A) = m(B) \).

Suppose \( A \) is either a \( G \)-algebra or empty. Define \( \text{Per}(G, A) \) to be the class of algebras

\[ C = \langle C, g^{(a)}, a^{(a)} \rangle_{a \in G, a \in A} \]

satisfying the following conditions:

(i) the algebra \( C' = \langle C, g^{(a)} \rangle_{a \in G} \) is a \( G \)-algebra and the \( a^{(a)} \) are constants in \( C \);
(ii) \( F^{(a)} \subseteq \{ a^{(a)} : a \in A \} \);
(iii) if \( A \neq \mathfrak{H} \), then the mapping \( a \mapsto a^{(a)} \) defines an isomorphism of \( A \) into \( C' \).

The following simple facts about the class \( \text{Per}(G, A) \) are pointed out in 2.1.3 of [G]: it is a quasivariety if \( A \) is of type 2 or empty, and a variety iff \( |G| = 1 \) and \( |A| \leq 1 \); when \( |G| = |A| = 1 \) it is polynomially equivalent to the variety \( S_1 \) of "pointed sets" \( C = \langle C, c \rangle \), where \( c \in C \); when \( |G| = 1 \) and \( A = \emptyset \) it is polynomially equivalent to the variety \( S_0 \) of "sets" \( C = \langle C \rangle \) (more precisely, algebras without operations); its members are precisely the \( \text{Per}(G, A) \)-free algebras, and hence it is categorical in every infinite power \( > |G| + |A| \). Indeed it follows rather easily from Theorem 3.2 and conditions (i)-(iii) that \( \text{Per}(G, A) \) consists precisely of all the (non-empty) multiples of \( G^* \) over \( A \) with the elements of \( A \) adjoined as distinguished individual constants, and all isomorphic copies of such structures. (This is actually but a special case of a more general phenomenon which applies to every universally axiomatizable class of unary algebras which is categorical in power; see Theorem 2.4 of Givant [79] for details.)
4. The representation theorem

**Theorem 4.1.** I. A quasivariety $K$ is categorical in power iff $K^e$ is polynomially equivalent to one of the following classes:

(i) $\text{Aff}(\mathcal{F}^{(n)}, \mathcal{A}^{(n)}, k)$ for some skew field $\mathcal{F}$, $\mathcal{F}$-module $\mathcal{A}$, and $k, n$ with $0 \leq k \leq n$;

(ii) $\text{Per}(\mathcal{G}, \mathcal{B})_n$ for some group $\mathcal{G}$, some $\mathcal{B}$ either a $\mathcal{G}$-algebra of type 2 or empty, and some $n$.

II. A variety $K$ is categorical in power iff it is polynomially equivalent to one of the following classes:

(i) $\text{Aff}(\mathcal{F}^{(n)}, k)$ for some skew field $\mathcal{F}$ and $k, n$ with $0 \leq k \leq n$;

(ii) $(S_0)_n$ or $(S_1)_n$ for some $n$.

**Proof.** The easy direction of the theorem, that $K$ is categorical in power in case it, or $K^e$, is polynomially equivalent to one of the classes in I(i), (ii) or II(i), (ii) follows at once from Theorem 2.2 and the remarks in Sections 2 and 3.

In the other direction, assume that $K$ is a quasivariety categorical in power. If there is an equation strongly minimal in $K$, a situation which must occur if $K$ has a constant term (see 3.1.10 and 3.1.13 in [G])—then, as was shown in [G], $K^e$ is polynomially equivalent to one of the classes

$$\text{Mod}(\mathcal{F}, \mathcal{A})_n, \text{Aff}(\mathcal{F}, \mathcal{A})_n, \text{Per}(\mathcal{G}, \mathcal{B})_n,$$ (4.1)

for some $n$, where $\mathcal{F}$ is some skew field, $\mathcal{A}$ an $\mathcal{F}$-module, $\mathcal{G}$ a group, and $\mathcal{B}$ either a $\mathcal{G}$-algebra of type 2 or else empty. Hence, by Theorem 1.1 (iv), (v) and Theorem 2.1 we obtain the desired representation. Assume therefore that $K$ has no strongly minimal equation and, in particular, that $K$ has no constant terms.

In 3.1.10 of [G] an equation $e$ in the language of $K$ of the form $\sigma(v_1) = v_1$ is constructed with the following properties:

the equation $\sigma(\sigma) = \sigma$, i.e. $e(\sigma)$, holds in $K$; \hfill (4.2)

for every equation $\psi(v_1)$, one of $e \to \psi$ and $e \to \neg \psi$ holds in $K$. \hfill (4.3)

For each term $\gamma(v_1, \ldots, v_m)$ in $\mathcal{L}(K)$ there is another term $\overline{\gamma}(v_1)$ in $\mathcal{L}(K)$ of the form $\gamma(\tau_1(v_1), \ldots, \tau_m(v_1))$ such that $\overline{\gamma}(\gamma) = \gamma$ holds in $K$—see 3.1.9 of [G]. (Notice that, by (4.2), we may take $\overline{\sigma}$ to be $\sigma$.) Thus for each $C \in K$ and $x \in C$ we have $x$ in the range of $\gamma^{(C)}$ iff $x$ satisfies the equation $\overline{\gamma}(v_1) = v_1$ in $C$. From this, (4.2), and (4.3) we obtain

for each term $\gamma$ and each $C$ in $K$, the range of $\sigma^{(C)}$ is either disjoint from, or included in, the range of $\gamma^{(C)}$. \hfill (4.4)

Now set

$$K(e) = \{(C, c) : C \in K \text{ and } c \text{ satisfies } e \text{ in } C\}.$$
Choose a new individual constant symbol $c$ and, given a set $\Gamma$ of axioms for $K$, let $\Gamma \cup \{c(\varepsilon)\}$ be the corresponding set of axioms for $K(\varepsilon)$. Then by (4.2) and (4.3) we have:

for every term $\tau(\v_1, \ldots, \v_m) \in \mathcal{I}(K)$, if the equation $\tau(c, \ldots, c) = c$ holds in $K(\varepsilon)$, then $\tau(\sigma, \ldots, \sigma) = \sigma$ (4.5) holds in $K$.

By 3.1.17, 2.2.6, and 2.1.4 of [G], $K(\varepsilon)^o$ is polynomially equivalent to one of the classes in (4.1). Since $K(\varepsilon)^o$ has a constant term and $\text{Aff} \left( \mathcal{F}, \mathcal{A} \right)_n$ does not, $K(\varepsilon)^o$ must in fact be polynomially equivalent to either $\text{Mod} \left( \mathcal{F}, \mathcal{A} \right)_n$ or $\text{Per} \left( \mathcal{G}, \mathcal{B} \right)_n$ for some $n$, $\mathcal{F}$, $\mathcal{A}$, $\mathcal{G}$, and $\mathcal{B}$ as in (4.1). Thus the proof naturally splits into two cases.

Reformulating case one with the help of (2.1) in Theorem 2.1, we suppose that

$$K(\varepsilon)^o = \sigma_\varepsilon \text{Mod} \left( \mathcal{F}(n), \mathcal{A}(n) \right).$$

By Theorem 1.4 we may assume that $c$ corresponds to $0$ under $\equiv_{\pi, \varepsilon}$. Clearly, for each term $\tau(\v_1, \ldots, \v_1)$ of $K(\varepsilon)$ there is a unique term $\tau'(\v_1, \ldots, \v_1)$ of $\text{Mod} \left( \mathcal{F}(n), \mathcal{A}(n) \right)$ of the form $\mu_1\v_1 + \cdots + \mu_m\v_m + a$ (i.e. in normal form) such that $\tau$ corresponds to $\tau'$ under $\equiv_{\pi, \varepsilon}$. Thus, for example, $c^* = 0$. Set

$$\Lambda = \{\tau^*: \tau \in \mathcal{I}(K)\}.$$

Since $\mathcal{I}(K)$ is obviously equivalent to $\Lambda$ under $\equiv_{\pi, \varepsilon}$, it follows from Theorem 0.1 that

$$K(\varepsilon)_{\mathcal{I}(K)} = \text{Mod} \left( \mathcal{F}(n), \mathcal{A}(n) \right).$$

However, $K(\varepsilon)_{\mathcal{I}(K)}$ is plainly polynomially equivalent to $K^o$, so we arrive at

$$K^o = \text{Mod} \left( \mathcal{F}(n), \mathcal{A}(n) \right).$$

Thus the real task is to characterize $\Lambda$. This we do with the help of $\equiv_{\pi, \varepsilon}$ and several lemmas.

For each term $\tau(\v_1, \ldots, \v_m) \in \mathcal{I}(K)$, the equation $\tau(c, \ldots, c) = c$ holds in $K(\varepsilon)^o$ iff $\tau^* = \mu_1\v_1 + \cdots + \mu_m\v_m + 0$ for some $\mu_1, \ldots, \mu_m \in F(n)$. (4.8)

Indeed $\tau(c, \ldots, c) = c$ holds in $K(\varepsilon)^o$ iff $\tau^*(0, \ldots, 0) = 0$ holds in $\text{Mod} \left( \mathcal{F}(n), \mathcal{A}(n) \right)$, which in turn happens iff $\tau^*$ has the desired form.

Since $\sigma(c) = c$ (i.e. $\varepsilon(c)$) holds in $K(\varepsilon)^o$, we can apply (4.8) to find a $\rho \in F(n)$ for which $\sigma^* = \rho\v_1 + 0$. Using this and (4.8) we obtain the following translation of (4.5):

$$\text{if } \mu_1\v_1 + \cdots + \mu_m\v_m + 0 \in \Lambda, \text{ then } \left( \sum_{i=1}^{m} \mu_i \right)\rho = \rho.$$ (4.9)

Applying (4.9) to $\rho\v_1 + 0$ ( = $\sigma^*$), we see that

$$\rho\rho = \rho \quad \text{(i.e. } \rho \text{ is an idempotent matrix).}$$ (4.10)

(This also follows from (4.2).)
Our goal is to prove

\[ \Lambda = \{ \mu_1 v_1 + \cdots + \mu_m v_m + a : (\sum_{i=1}^{m} \mu_i) \rho = \rho \}. \]  

(4.11)

The most important step in establishing (4.11) is the extension of (4.9) to terms in \( \Lambda \) of the form \( \lambda v_1 + a \).

Fix a \( \lambda v_1 + a \in \Lambda \). Since \( \mathcal{E}^{(n)} \) is a regular ring, i.e. for every \( \chi \in F^{(n)} \) there is a \( \nu \in F^{(n)} \) satisfying \( \chi \nu = \chi \) (see, e.g., McCoy [64], p. 124), we can find a \( \nu \in F^{(n)} \) such that

\[ (\lambda \rho - \rho) \nu (\lambda \rho - \rho) = \rho - \lambda \rho. \]

From this, a simple computation yields

\[ \lambda \rho \nu \lambda \rho + \lambda \rho - \lambda \rho \nu \rho = \rho + \rho \nu \lambda \rho - \rho \nu \rho. \]  

(4.12)

Choose a term \( \tau(v_1, v_2) \in \mathcal{E}(K) \) for which

\[ \tau(v_1, c)^* = \nu v_1 + 0. \]  

(4.13)

Since \( \mathbf{v} \mathbf{0} = \mathbf{0} \) holds in Mod \((\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\), the equation \( \tau(c, c) = c \) must hold in \( K(e)^0 \). Thus, by (4.8) we can find \( \nu' \), \( \nu'' \in F^{(n)} \) for which

\[ \tau(v_1, v_2)^* = \nu' v_1 + \nu'' v_2 + 0. \]  

(4.14)

From (4.13) and (4.14) we see that \( \nu' v_1 + \nu'' 0 + 0 = \nu v_1 + 0 \) holds in Mod \((\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\). Hence \( \nu' = \nu \) and

\[ \tau(v_1, v_2)^* = \nu v_1 + \nu'' v_2 + 0. \]

Because \( \tau^* \in \Lambda \) we may apply (4.9) to obtain

\[ \nu \rho + \nu'' \rho = \rho. \]  

(4.15)

Set \( \mu = \nu \lambda \rho + \nu'' \rho \) and compute:

\[ \begin{aligned}
\lambda \rho \mu &= \lambda \rho (\nu \lambda \rho + \nu'' \rho) \quad \text{by definition of } \mu, \\
&= \lambda \rho (\nu \lambda \rho + \rho - \nu \rho) \quad \text{by (4.15),} \\
&= \lambda \rho \nu \lambda \rho + \lambda \rho \rho - \lambda \rho \nu \rho \quad \text{by distributivity,} \\
&= \rho + \rho \nu \lambda \rho - \rho \nu \rho \quad \text{by (4.10) and (4.12),} \\
&= \rho (\rho + \nu \lambda \rho - \nu \rho) \quad \text{by (4.10) and distributivity,} \\
&= \rho (\rho + \nu \lambda \rho + \nu'' \rho - \rho) \quad \text{by (4.15),} \\
&= \rho \mu \quad \text{by definition of } \mu.
\end{aligned} \]

This establishes

\[ \lambda \rho \mu = \rho \mu. \]  

(4.16)
Now fix an arbitrary $\mathbb{C}$ in $\text{Mod}(\mathbb{S}^{(n)}, \mathbb{X}^{(n)})$. Under $=_{\mathbb{C}}$, (4.4) translates as follows:

for every term $\chi v_1 + b$ in $\Lambda$, the range of $\rho^{(\mathbb{C})}$ is

either disjoint from, or included in, the range of $(\chi v_1 + b)^{(\mathbb{C})}$.  \hspace{1cm} (4.17)

Suppose $\gamma(v_1)$ is a term of $\mathbb{K}$ for which $\gamma^* = \lambda v_1 + a$. Then one can compute that $\sigma(\gamma(\sigma), \sigma)^*$ is just $\rho \mu v_1 + \rho \nu a$. Hence this latter term is in $\Lambda$. Clearly the range of $(\rho \mu v_1 + \rho \nu a)^{(\mathbb{C})}$ is included in the range of $\rho^{(\mathbb{C})}$. The reverse inclusion follows from (4.17), so we obtain:

$\rho^{(\mathbb{C})}$ and $(\rho \mu v_1 + \rho \nu a)^{(\mathbb{C})}$ have the same range. \hspace{1cm} (4.18)

Now consider $\mathbb{C}$ as the universe of a (left) $\mathbb{G}$-module. We can endow $\mathbb{C}$ with a right $\mathbb{G}$-module structure which has the additional property that $(\alpha x)\beta = \alpha(\beta x)$ for $\alpha, \beta \in F$ and $x \in \mathbb{C}$. (To do this, use the fact that a left $\mathbb{G}$-module is isomorphic to a direct sum of the 1-dimensional left $\mathbb{G}$-module derived from $\mathbb{G}$ itself; each summand can clearly be endowed with a right module structure over $\mathbb{G}$ which satisfies the desired additional property.) This enables us to treat $\rho^{(\mathbb{C})}$ and $\rho \mu^{(\mathbb{C})}$ as linear transformations of $\mathbb{C}$, viewed as a right $\mathbb{G}$-module. The range of $\rho \mu^{(\mathbb{C})}$ is certainly included in that of $\rho^{(\mathbb{C})}$. Therefore it is also included in the range of the affine transformation $(\rho \mu v_1 + \rho \nu a)^{(\mathbb{C})}$, by (4.18). A well-known vector space basis argument now shows that $\rho \mu^{(\mathbb{C})}$ and $(\rho \mu v_1 + \rho \nu a)^{(\mathbb{C})}$ must have the same range. Combining this with (4.18), we see that

$\rho^{(\mathbb{C})}$ and $\rho \mu^{(\mathbb{C})}$ have the same range. \hspace{1cm} (4.19)

By (4.16), $\lambda^{(\mathbb{C})}$ is the identity map on the range of $\rho \mu^{(\mathbb{C})}$. From (4.19) it follows that $\lambda^{(\mathbb{C})}$ is also the identity map on $\rho^{(\mathbb{C})}$. However, $\mathbb{C}$ was arbitrary, so $\lambda \rho = \rho$. We have established the desired extension of (4.9):

if $\lambda v_1 + a \in \Lambda$, then $\lambda \rho = \rho$. \hspace{1cm} (4.20)

To generalize this to arbitrary terms $\mu_1 v_1 + \cdots + \mu_m v_m + a$ in $\Lambda$, simply substitute $v_i$ for each $v_i$, $1 \leq i \leq m$, and set $\lambda = \sum_{i=1}^m \mu_i$. This gives us

$\Lambda \subseteq \left\{ \mu_1 v_1 + \cdots + \mu_m v_m + a : \left( \sum_{i=1}^m \mu_i \right)^{\rho} = \rho \right\}$. \hspace{1cm} (4.21)

To establish the reverse inclusion assume $(\sum_{i=1}^m \mu_i)^{\rho} = \rho$ and $a \in "\Lambda$. Choose a term $\tau(v_1, \ldots, v_m, v_{m+1})$ in $\mathbb{X}(\mathbb{K})$ for which

$\tau(v_1, \ldots, v_m, c)^* = \mu_1 v_1 + \cdots + \mu_m v_m + a$. \hspace{1cm} (4.22)

Let $\tau(v_1, \ldots, v_m, v_{m+1})^* = \lambda_1 v_1 + \cdots + \lambda_m v_m + \lambda_{m+1} v_{m+1} + b$. Then, by (4.22), the equation

$\lambda_1 v_1 + \cdots + \lambda_m v_m + \lambda_{m+1} 0 + b = \mu_1 v_1 + \cdots + \mu_m v_m + a$
must hold in \( \text{Mod} (\mathfrak{B}(n), \mathfrak{U}(n)) \) (because \( c^* = 0 \)). Therefore \( \lambda_i = \mu_i \) for \( 1 \leq i \leq m \) and \( b = a \), i.e.

\[
\tau^* = \mu_1 v_1 + \cdots + \mu_m v_m + \lambda_{m+1} v_{m+1} + a. \quad (4.23)
\]

Since \( \tau^* \in \Lambda \) by definition of \( \Lambda \), (4.23) and (4.21) lead to the conclusion \( [\Sigma_{i=1}^{m} \mu_i + \lambda_{m+1}] p = p \). But we have assumed that \( (\Sigma_{i=1}^{m} \mu_i) p = p \), so obviously \( \lambda_{m+1} p = 0 \). Combining this with (4.23) we get

\[
\tau(v_1, \ldots, v_m, \sigma(v_1))^* = \mu_1 v_1 + \cdots + \mu_m v_m + a.
\]

But \( \tau(v_1, \ldots, v_m, \sigma(v_1)) \in \mathfrak{I}(K) \), so \( \mu_1 v_1 + \cdots + \mu_m v_m + a \) is in \( \Lambda \) as was to be shown. This proves the formula (4.11).

In view of (4.11) and Theorem 1.1 (iii), \( \text{Mod} (\mathfrak{B}(n), \mathfrak{U}(n)) \Lambda \) is polynomially equivalent to \( \text{Aff} (\mathfrak{B}(n), \mathfrak{U}(n), \rho) \). Letting \( k \) be the rank of \( \rho \), we see from Theorem 1.3 that this latter class is, in turn, polynomially equivalent to \( \text{Aff} (\mathfrak{B}(n), \mathfrak{U}(n), k) \).

Combining these facts with (4.7), we obtain

\[
K = \text{Aff} (\mathfrak{B}(n), \mathfrak{U}(n), k),
\]

which proves \( I \) under the assumption (4.6).

For the other case of \( I \), suppose

\[
K(\varepsilon)^o = \sigma, \tau \text{ Per} (\mathfrak{B}, \mathfrak{B}). \quad (4.24)
\]

We shall show that this is incompatible with the assumption that \( K \infty \) has no strongly minimal equation. Let \( \tau(v_1, v_2) \in \mathfrak{I}(K) \) be such that

\[
\tau(v_1, c) \text{ corresponds to } p_1 v_1 \text{ under } =_{\pi, \zeta}. \quad (4.25)
\]

It is easily shown that \( \tau(v_1, v_2) \) corresponds under \( =_{\pi, \zeta} \) to a term of \( \text{Per} (\mathfrak{B}, \mathfrak{B})_n \) of the form

\[
d(p_1, g_1 w_1), \ldots, p_n, (g_n w_n), \quad (4.26)
\]

where \( 1 \leq j_1, \ldots, j_n \leq n \), each \( g_j \in G \), and each \( w_i \) is either \( v_1, v_2 \), or \( b \) for some \( b \in \mathfrak{B} \). (Indeed every term \( \gamma(v_1, v_2) \) of \( \text{Per} (\mathfrak{B}, \mathfrak{B})_n \) can be put into this form.) Let \( c \) correspond to \( c' \) under \( =_{\pi, \zeta} \), and for each \( i, 1 \leq i \leq n \), let \( w'_i = c' \) if \( w_i = v_2 \), and \( w'_i = w_i \) if \( w_i \neq v_2 \). Then from (4.25) and the correspondence of \( \tau \) to (4.26) it follows that the equation

\[
d(p_1, (g_1 w'_1), \ldots, p_n, (g_n w'_n)) = p_1 v_1
\]

holds in \( \text{Per} (\mathfrak{B}, \mathfrak{B})_n \). Applying \( p_i \) to both sides of this equation we see that

\[
p_i (g_i w'_i) = p_1 v_1
\]

holds in \( \text{Per} (\mathfrak{B}, \mathfrak{B})_n \) (use the laws \( p_id(v_1, \ldots, v_n) = p_i v_1 \) and \( pp_jv_1 = p_j v_1 \)). Since \( p_1 v_1 \) is not a constant term, this forces \( w'_i = v_1 \), and therefore \( w_i = v_1 \). Now \( i \) was an arbitrary integer between 1 and \( n \), so \( v_2 \) cannot occur in (4.26). This means \( \tau(v_1, v_2) \) is independent of \( v_2 \) in \( K \), i.e. \( \tau(v_1, v_2) = \tau(v_1, v_3) \) holds in \( K \). But then
\( \tau(v_1, v_1) = \tau(v_1, c) \) holds in \( K(\varepsilon)^e \), so that \( \tau(v_1, v_1) \) also corresponds to \( p_1 v_1 \) under \( \equiv_{\pi,e} \). Because \( p_1 v_1 = v_1 \) is strongly minimal in \( (\text{Per}(\mathcal{G}, \mathcal{B}), \omega) \), \( \tau(v_1, v_1) = v_1 \) must be strongly minimal in \( K(\varepsilon)^e \), and hence also in \( K^e \). But this contradicts our assumption on \( K \). The proof of I is now complete.

To establish the non-trivial direction of II, assume that \( K \) is a variety categorical in power. As before, \( K^e \) or \( K(\varepsilon)^e \) is polynomially equivalent to one of the classes in (4.1); moreover in this case we have \( |\mathcal{A}| = 1 \) and \( |G| = 1 \), \( |B| \leq 1 \), by the remarks after 3.1.17 in \([G]\). Thus we can use I as well as Theorem 1.1 (ii) and the remarks in Section 3 to conclude that \( K^e \) is polynomially equivalent to one of the classes \( \text{Aff}(\mathcal{G}^{(n)}(k), (S_0)_n), (S_1)_n \). As remarked in \([G]\) after 3.1.17, it follows at once that \( K^e = K \), so we have proved II.

As was pointed out, the key step in the above proof is the demonstration of (4.20). When \( K \) is a variety this becomes almost trivial. For assume that \( K(\varepsilon)^e \equiv_{\pi,e} \text{Mod}(\mathcal{G}^{(n)}(\pi)) \). Since every term of the latter class is 0-idempotent and since \( c \) must correspond to 0 under \( \equiv_{\pi,e} \) (0 is essentially the only constant), every term of \( K \) (and even of \( K(\varepsilon) \)) must be \( c \)-idempotent in \( K(\varepsilon) \). Hence, by (4.5), \( \tau(\sigma) = \sigma \) holds in \( K \) for every \( \tau(\sigma) \in \mathcal{T}(K) \). This is essentially just (4.20). Notice that we do not need Theorem 1.4.

In 3.1.19 of \([G]\) it is shown that a universal Horn class (in a similarity type admitting relations) which is categorical in power must in fact be a definitional extension, in a very simple way, of a quasivariety categorical in power. Thus Theorem 4.1 actually gives a representation of all universal Horn classes categorical in power.

A quasivariety \( K \) is free if every algebra in \( K \) is \( K \)-free; \( K \) is free in power \( \kappa \) if it has a member of power \( \kappa \) and all members of power \( \kappa \) are \( K \)-free.

**Proposition 4.2.** A. A quasivariety is free iff it is polynomially equivalent to one of classes in Theorem 4.1.1. (i), (ii) with \( n = 1 \). In particular, a variety is free iff it is polynomially equivalent to one of

\[
\text{Mod}(\mathcal{G}), \quad \text{Aff}(\mathcal{G}), \quad S_0, \quad S_1
\]

for some skew field \( \mathcal{G} \).

B. If a quasivariety \( K \) is free, but not categorical, in some regular, infinite power \( \kappa \), then \( K^e \) is free.

In fact, A is an immediate consequence of Theorem 4.1 and some simple facts about the free algebras in the classes of Theorem 4.1. To prove B assume that \( K \) is free, but not categorical, in power \( \kappa \), where \( \kappa \) is a regular, infinite cardinal. It is not difficult to show that \( K \) is polynomially equivalent to a quasivariety with similarity type \( \rho \kappa \kappa \) of power \( \leq \kappa \) (see 3.2.10 (1) and 1.4.1 of \([G]\) for a proof). Thus we may assume that \( |\rho \kappa \kappa| \leq \kappa \). It follows that \( K \) is \( \kappa \)-unique in the sense that, up to isomorphisms, there is exactly one algebra in \( K \) which is generated by a set of power \( \kappa \) and
by no set of smaller cardinality. Hence, by 3.1.11 of [G], K is categorical in all powers \(|\rho K| + \omega\), so Theorem 4.1 applies to K. Using the fact that K is free, but not categorical, in power \(\kappa\), it follows from Theorem 4.1 that \(K^\circ\) is polynomially equivalent to one of the classes in I (i), (ii) with \(n = 1\) (see, e.g., 2.2.1 (iv) of [G]). Thus \(K^\circ\) is free.

Both A and B are proved in [G] (B for arbitrary \(\kappa \geq \omega\)) using very different techniques. However, the method of proof used in [G] also establishes, for example, the following result (first stated on p. 48 of [G]) which is not a consequence of Theorem 4.1:

**Proposition 4.3.** If K is a quasivariety and every non-trivial member of K which is generated by a set of power \(\leq \omega_1\) is K-free, then K is free.

It is an open problem whether we can replace \(\omega_1\) by \(\omega\) in this theorem. A related open problem is whether, for a quasivariety K with \(|\rho K| > \omega\), the assumption that K is \(\kappa\)-unique for some infinite \(\kappa < |\rho K|\) implies that K is categorical in power \(>|\rho K|\).

When does a quasivariety categorical in power have two or more distinct representations? Put another way, when are the quasivarieties in Theorem 4.1.1. (i), (ii) polynomially equivalent? The answer is contained in the following theorem.

**Theorem 4.4.** For \(i = 1, 2\) let \(\mathcal{F}_i\) be a skew field, \(\mathcal{U}_i\) an \(\mathcal{F}_i\)-module, \(\mathcal{G}_i\) a group, \(\mathcal{B}_i\) either a \(\mathcal{G}_i\)-algebra of type 2 or empty, \(1 \leq n_i\), and \(0 \leq k_i \leq n_i\).

(i) \(\text{Aff} (\mathcal{F}_0, \mathcal{U}_0, k_0) = \text{Aff} (\mathcal{F}_1, \mathcal{U}_1, k_1)\) iff \(n_0 = n_1\), \(k_0 = k_1\), \(\mathcal{G}_0 = \mathcal{G}_1\), and \(\mathcal{U}_0\) is isomorphic to a variant of \(\mathcal{U}_1\).

(ii) \(\text{Per} (\mathcal{G}_0, \mathcal{B}_0, n_0) = \text{Per} (\mathcal{G}_1, \mathcal{B}_1, n_1)\) iff \(n_0 = n_1\), \(\mathcal{G}_0 = \mathcal{G}_1\), and either \(\mathcal{B}_0\) is isomorphic to a variant of \(\mathcal{B}_1\), or both are empty.

(iii) \(\text{Aff} (\mathcal{F}_0, \mathcal{U}_0, n_0, k_0) \neq \text{Per} (\mathcal{G}_1, \mathcal{B}_1, n_1)\).

We limit ourselves to a sketch of the proof of the non-trivial direction of (i). Let \(\pi_\mu\) be a polynomial equivalence between the two quasivarieties. Then each term \(\lambda(v_1, v_2)\) of the first class corresponds to a term \(\lambda(v_1, v_2)\) of the second class under \(\pi_\mu\). This induces a correspondence \(\mu \rightarrow \lambda\) of \(\mathcal{F}_0(n)\) to \(\mathcal{F}_1(n)\) which is in fact an isomorphism.

Take \(\mathcal{N}_i\) to be the \(\mathcal{F}_i(n)\)-module of dimension one with universe \(\pi F_i\). Because \(\mathcal{F}_0(n) = \mathcal{F}_1(n)\) and \(\mathcal{N}_i\) is, up to isomorphisms, the unique simple algebra in \(\text{Mod} (\mathcal{F}_i(n))\), we have \(\mathcal{N}_0\) isomorphic to a variant of \(\mathcal{N}_1\). Thus \(\mathcal{N}_0\) and \(\mathcal{N}_1\) have isomorphic endomorphism rings. But it is known that the ring of endomorphisms of \(\mathcal{N}_i\) is isomorphic to \(\mathcal{F}_i\) (see, e.g., Cohn [71], p. 2), so \(\mathcal{F}_0 = \mathcal{F}_1\).

5 The author is indebted to Professor George Bergman for calling these facts to his attention.
Now consider systems \( \langle \mu_i: 1 \leq i \leq m \rangle \) of elements in \( F_0^{(n_0)} \) satisfying the following conditions:

\[
\sum_{i=1}^{m} \mu_i = 1; \quad \mu_i \mu_i = \mu_i \neq 0, \quad \mu_i \mu_j = 0
\]

for \( 1 \leq i, j \leq m \) and \( i \neq j \). (4.27)

If \( C \) is the right \( F_0 \)-module of dimension \( n_0 \) with universe \( n_0 F_0 \), then \( F_0^{(n_0)} \) can be considered the ring of endomorphisms of \( C \), and each system satisfying (4.27) induces a decomposition of \( C \) into the direct sum of \( m \) non-trivial right \( F_0 \)-modules. By the uniqueness of dimension, \( n_0 \) is the largest value \( m \) can assume in (4.27). However, since \( F_0^{(n_0)} \approx F_1^{(n_1)} \), it is easily seen that there must be a system satisfying 4.27 with \( m = n_1 \). Hence \( n_1 \leq n_0 \). By symmetry, \( n_1 = n_0 \).

Set \( n = n_0 \). By passing to a variant of \( A_1 \) we may assume that \( A_0 = F_1 \). The dimension \( \kappa_i \) of \( A_i \) as an \( F_0 \)-module is characterizable as the maximum of the cardinalities of irredundant sets of generators of \( A_i^{(n)} \), i.e. sets \( X \) of generators such that, for each \( x \in X \), \( X \setminus \{x\} \) is no longer a set of generators of \( A_i^{(n)} \). Using the polynomial equivalence \( =_{\pi, \kappa} \) we can establish, as before, that \( A_0^{(n)} \) is isomorphic to a variant of \( A_1^{(n)} \). It follows that \( \kappa_0 = \kappa_1 \), and hence that \( A_0 \cong A_1 \).

Suppose that \( \delta_{k_0} \nu \) corresponds to \( v_1 + a \) under \( =_{\pi, \kappa} \). Then \( \delta_{k_0} \) and \( \nu \) have the same matrix rank. Clearly \( \nu \delta_{k_1} = \delta_{k_1} \) by Theorem 1.1 (iii), so \( k_0 \geq k_1 \) (i.e. rank \( \nu \geq \text{rank} \delta_{k_1} \)). A symmetric argument shows that \( k_1 \geq k_0 \).

The details of the above, as well as the proofs of (ii) and (iii) are left to the reader.

Csákány [75] was the first to investigate the polynomial equivalence of varieties of affine spaces (his work dates from 1973). He observed that for rings with unit, \( H \) and \( G \), we have \( \text{Aff} (H) \cong \text{Aff} (G) \) iff \( H \cong G \).

5. Palyutin's representation theorem

In Palyutin [75] a somewhat different representation of universal Horn classes categorical in power is given.\(^6\)\(^7\) (All references below are to the English translation, which we abbreviate by [P].) Palyutin describes his various "standard quasivarieties" categorical in power by giving a set of axioms for each. (He does not give a full description of the models.) We shall establish here the relevant

\(^6\) Palyutin appears to have been aware of some of the work of Baldwin, Lachlan, and the author, but it is not clear to what extent this influenced his own work; see the reference in [P], p. 86, to Baldwin–Lachlan [73] and in Palyutin [73], English translation, p. 913, to Baldwin–Lachlan–McKenzie [72], which is opposite the author's announcement, Givant [72].

\(^7\) Palyutin's description of varieties categorical in power (see [P], Lemma 9.1, p. 106) is incorrectly formulated. Condition 3) should read: if \( X = (n, k, \kappa, \Psi) \), then \( \kappa = k \). (Since Palyutin assumes \( \kappa \geq k \), his condition \( \kappa = 0 \) forces \( k = 0 \).) Consequently, the proof of his Corollary 2, p. 106, which is based on Lemma 9.1, 3), contains an error.
polynomial equivalences between Palyutin's standard examples and those listed in Theorem 4.1.1 (i), (ii) above. Only one of these equivalences, the first, presents serious difficulties. In the process of deriving the equivalences we shall also obtain full model descriptions of his standard examples as certain polynomial reducts, \( K^*_s \), of our standard examples \( K \) listed in Theorem 4.1.1 (i), (ii) (see (5.36), (5.43) and (5.49)). The use of \( K^*_s \) instead of \( K_s \) is necessitated by Palyutin's definition of a quasivariety as the class of models of a set of quasi-identities (see Footnote 3); thus his standard examples contain all trivial structures (see preliminaries). When referring to the similarity type of one of Palyutin's quasivarieties we shall use his notation for its type, or "characteristic".

We now fix a skew field \( \mathfrak{F} \), finite cardinals \( k \) and \( n \) with \( 0 < k < n \), a finite or infinite cardinal \( \kappa \geq k \), and a vector space \( \mathfrak{A} \) over \( \mathfrak{F} \) of dimension \( |\{ \alpha : k \leq \alpha < \kappa \}| \). For \( e = 0, 1 \) let \( M_e \) be the class of all structures of type \( \langle n, e, k, \kappa, \mathfrak{F} \rangle \) which model Palyutin's axioms \( C_1 - C_{26} \) and, in case \( e = 1 \), also \( C_{27} - C_{29} \) (see [P], pp. 97-98).

**Theorem 5.1.** \( M_e = \text{Aff} (\mathfrak{F}^{(n)}, \mathfrak{A}^{(n)}, n - (k + e)) \).

**Proof.** We begin by defining several \( n \times n \) matrices. In each case we specify only the non-zero entries.

- \( \lambda^* \) for \( \lambda \in F \), has \( \lambda \) in the \((n, n)\)th place. \( \quad (5.1) \)
- \( \varepsilon_{ij} \) has 1 in the \((i, j)\)th place. \( \quad (5.2) \)
- \( \rho^e \) has 1 in the \((j, j)\)th place for \( j = k + 1, \ldots, n - 1 \), and, in case \( e = 0 \), it also has 1 in the \((n, n)\)th place. \( \quad (5.3) \)
- \( \lambda^{(j)} \) for \( \lambda = (\lambda_{ij}) \in F^{(n)} \), has \( \lambda_{ij} \) in the \((i, j)\)th place for each \( i = 1, \ldots, n \). (Thus \( \lambda \) and \( \lambda^{(j)} \) have the same \( j \)th column). \( \quad (5.4) \)

In addition we stipulate:

- \( a^* \) for \( a \in A \), is the \( n \)-tuple of "\( A \) with \( a \) in the \( n \)th place (and 0 everywhere else). \( \quad (5.5) \)

\( ^8 \)It should be pointed out that \( C_{10} \) is incorrectly formulated. It should read: \( r_e (g_1(y, x)) = s(x, r_e (y), y) \). Moreover, it seems that Palyutin's axiomatization as it stands is not quite adequate. For example, I see no way of deriving an equation which relates the additive identity to \( p_0 \). Thus it would seem that an axiom such as \( p_0 (g_1(y, x), y) = y \) or, more simply, \( p_0 (x, y) = y \), must be added. Finally, Palyutin's axioms can be very much simplified. For example, in view of \( C_{15}, C_{16} \), one can take, in place of

\[ C_2: \quad s(g_1(z, x), s(g_1(z, y), g_1(z, u), z), z) = s(s(s(g_1(z, x), g_1(z, y), z), g_1(z, u), z), z), \]

the strong and simpler axiom

\[ s(x, s(y, u, z), z) = s(s(x, y, z), u, z). \]

Analogous remarks apply, e.g., to \( C_1 \) and \( C_{3} - C_{9} \) as well, while \( C_{27} \) is a trivial consequence of \( C_{29} \).
By Theorem 1.3 it suffices to prove
\[ M_e = \text{Aff}(\mathfrak{g}^{(n)}, \mathfrak{h}^{(n)}, \rho^e)^{\ast}. \]  
(5.6)

Set \( K = \text{Mod}(\mathfrak{g}^{(n)}, \mathfrak{h}^{(n)}) \) and let \( \langle a_\alpha : k \leq \alpha < \kappa \rangle \) be a basis for \( \mathfrak{h} \). We define a series of terms in \( \mathcal{S}(K) \).

\[
f = \left(1 + \sum_{j=1}^{n-1} e_{j,n}\right) v_1 - e_{1,n} v_2 - e_{2,n} v_3 - \cdots - e_{n-1,n} v_n. \]  
(5.7)

\[
g_1 = (1 - e_{n,n}) v_1 + e_{n,n} v_2. \]  
(5.8)

\[
g_j = (1 + e_{n,j-1} - e_{n,n}) v_1 + (e_{n,n} - e_{n,j-1}) v_2 \quad \text{for} \quad j = 2, \ldots, n. \]  
(5.9)

\[
s = e_{n,n} v_1 + e_{n,n} v_2 + [1 - (e_{n,n} + e_{n,n})] v_3. \]  
(5.10)

\[
p_\lambda = \lambda^* v_1 + (1 - \lambda^*) v_2 \quad \text{for} \quad \lambda \in F. \]  
(5.11)

\[
r_a = v_1 + a^* \quad \text{for} \quad k \leq \alpha < \kappa. \]  
(5.12)

\[
r_i = (1 - e_{n,i+1}) v_1 \quad \text{for} \quad 0 \leq i < k. \]  
(5.13)

\[
o = (1 - e_{n,n}) v_1. \]  
(5.14)

Let \( S_e \) be the set of all terms defined in (5.7)-(5.13) when \( e = 0 \) and (5.7)-(5.14) when \( e = 1 \). Further, let \( T_e \) be the set of terms \( v_1 + v_2 - v_3, \lambda v_1 + (1 - \lambda) v_2 \) for \( \lambda \in F^{(n)}, v_1 + a \) for \( a \in ^*A \), and \( \rho^e v_1 \). Thus \( K_{T_e} \) is nothing but \( \text{Aff}(\mathfrak{g}^{(n)}, \mathfrak{h}^{(n)}, \rho^e) \).

\[ K_{S_e} \simeq K_{T_e}. \]  
(5.15)

To prove this we shall use Theorem 0.1 in the case \( K = L \). That every term of \( S_e \) is equivalent in \( K \) to a term of \( T_e^* \) follows at once from (5.7)-(5.14), Theorem 1.1 (iii), and the definition of \( \rho^e \). To prove the other direction we define a series of terms in \( S_e^* \).

\[
t_j(v_1, v_2, v_3) = s(g_1(v_1, v_3), g_j(v_2, v_3), v_3) \quad \text{for} \quad j = 2, \ldots, n. \]  
(5.16)

\[
s_2(v_1, v_2, v_3) = g_1(f[v_3, p_{-1}(f_2, v_3), \ldots, p_{-1}(f_n, v_3)], s(v_1, v_2, v_3)). \]  
(5.17)

\[
s_m^m(v_1, v_2, v_{m+1}) = s_2(v_1, v_2, v_{m+1}) \quad \text{for} \quad m \geq 2. \]  
(5.18)

\[
s_{k+1}^m(v_1, \ldots, v_{k+1}, v_{m+1}) = s_2(s_k^m(v_1, \ldots, v_k, v_{m+1}), v_{k+1}, v_{m+1}) \quad \text{for} \quad m \geq 2 \quad \text{and} \quad k = 2, \ldots, m-1. \]  
(5.19)

\[
l_\lambda^{(j)}(v_1, v_2) = f(p_{\lambda_{j+1}}(g_j, v_2), p_{\lambda_{j+1}}(g_j, v_2), \ldots, p_{\lambda_{n-1}}(g_j, v_2)) \quad \text{when} \quad j = 1, \ldots, n-1. \]  
(5.20)

\[
l_\lambda^{(n)}(v_1, v_2) = f(p_{\lambda_{n-1}}(p_{\lambda_{n-1}}, \ldots, p_{\lambda_{1}})) \quad \text{for} \quad \lambda = (\lambda_{l_1}, \ldots, \lambda_{l_{i_1}}) \in F^{(n)} \quad \text{where} \quad \gamma_{l_i} = \lambda_{n_i} - \lambda_{l_i}. \]  
(5.21)

\[
l_\lambda(v_1, v_2) = s_m^m(l_\lambda^{(1)}, \ldots, l_\lambda^{(n)}(v_2)) \quad \text{for} \quad \lambda \in F^{(n)}. \]  
(5.22)
A representation theorem for universal Horn classes categorical in power

\[ x_b(v_1) = s^n_b(l_{e_{1,n}}(w_{b_1}; v_1), \ldots, l_{e_{n,n}}(w_{b_n}; v_1), v_1) \]

for \( b = (b_1, \ldots, b_n) \in \mathbb{A}^n. \) (5.23)

\[ y_i(v_1) = I_{e_{1,n}}(r_i; v_1) \quad \text{for} \quad i = 0, \ldots, k - 1. \] (5.24)

\[ z^e(v_1) = \begin{cases} v_1 & \text{if} \quad k = e = 0; \\ y_0(y_1(\cdots(y_{k-1}(v_1))\cdots)) & \text{if} \quad k > 0 \quad \text{and} \quad e = 0; \\ o & \text{if} \quad k = 0 \quad \text{and} \quad e = 1; \\ y_0(y_1(\cdots(y_{k-1}(o))\cdots)) & \text{if} \quad k > 0 \quad \text{and} \quad e = 1. \end{cases} \] (5.25)

On the basis of (5.7)–(5.14) it is easy to verify that the following equations hold in \( K. \)

\[ t_i = e_{n,i-1}v_1 + e_{n,i-1}v_2 + [1 - (e_{n,i-1} + e_{n,i-1})]v_3. \] (5.26)

\[ s_2 = v_1 + v_2 - v_3. \] (5.27)

\[ s'^n_k = v_1 + \cdots + v_k - (k - 1)v_{m+1} \quad \text{for} \quad k = 2, \ldots, m. \] (5.28)

\[ l^{(j)}_{\lambda} = \lambda^{(j)}v_1 + (1 - \lambda^{(j)})v_2 \quad \text{for} \quad \lambda \in F^{(n)}. \] (5.29)

\[ l_{\lambda} = \lambda v_1 + (1 - \lambda)v_2 \quad \text{for} \quad \lambda \in F^{(n)}. \] (5.30)

\[ w_a = v_1 + a^* \quad \text{for} \quad a \in \mathbb{A}. \] (5.31)

\[ x_b = v_1 + b \quad \text{for} \quad b \in \mathbb{A}. \] (5.32)

\[ y_i = (1 - e_{i+1,i+1})v_1 \quad \text{for} \quad i = 0, \ldots, k - 1. \] (5.33)

\[ z^e = p^e v_1. \] (5.34)

From (5.27), (5.30), (5.32) and (5.34) it follows that every term of \( T_e \) is equivalent in \( K \) to a term of \( S^*_e. \) In view of Theorem 0.1 this establishes (5.15).

We next show that

\[ K^*_e \subseteq M^*_e. \] (5.35)

The proof is a straightforward, but tedious, step-by-step verification that each of the axioms \( C_1 - C_{26} \) when \( e = 0, \) and \( C_1 - C_{29} \) when \( e = 1, \) holds in \( K^*_e. \) For example, \( C_{25} \) reads

\[ g_1(v_1, r_i(v_2)) = r_i(g_{i+2}(v_1, v_2)) \quad \text{for} \quad i = 0, \ldots, k - 1. \]

Thus we must check that

\[ (1 - e_{n,n})v_1 + e_{n,n}(1 - e_{n,i+1})v_2 = (1 - e_{n,i+1})[(1 + e_{n,i+1} - e_{n,n})v_1 + (e_{n,n} - e_{n,i+1})v_2] \]

\(^9\) See Footnote 8.
holds in $K$, and this reduces to verifying the simple matrix identities
\[ (1 - \varepsilon_{n,i+1})(1 + \varepsilon_{n,i+1} - \varepsilon_{n,n}) = 1 - \varepsilon_{n,n}, \]
\[ \varepsilon_{n,n}(1 - \varepsilon_{n,i+1}) = (1 - \varepsilon_{n,i+1})(\varepsilon_{n,n} - \varepsilon_{n,i+1}) = \varepsilon_{n,n} - \varepsilon_{n,i+1}. \]

We leave the proof that the other axioms hold in $K^*_s$ to the reader.

\[ M_e = K^*_s. \]  

(5.36)

This follows from (5.35), using (5.15) and the fact that $M_e$ is categorical in all powers $\kappa + \omega$ (see [P], pp. 98–99). Indeed if $\mathfrak{B} \in M_e$ has power $\kappa + \omega$ simply choose a $\mathfrak{C} \in K^*_s$ of power $|\mathfrak{B}|$. By (5.35) and the categoricity of $M_e$ we have $\mathfrak{B} \equiv \mathfrak{C}$. Since $K^*_s$ is closed under isomorphisms by (5.15) and Theorem 1.1 (i), we conclude that $\mathfrak{B} \in K^*_s$. If $\mathfrak{B}$ in $M_e$ is non-trivial and of power $\leq \kappa$, then $\mathfrak{B}$ is in $K^*_s$ by the above argument. Since $\mathfrak{B}$ is embeddable in $K^*_s$. We conclude again by (5.15) and Theorem 1.1(i) that $\mathfrak{B}$ is in $K^*_s$. Because $M_e$ and $K^*_s$ have the same trivial structures, this completes the proof of (5.36).

Together, (5.15) and (5.36) establish (5.6) and hence the theorem.

In the next theorem $\mathfrak{F}, \mathfrak{V}, \kappa, \text{and } n$ continue to be as above; but we now take $k = 1$, and therefore do not mention it explicitly. Let $N$ be the class of all structures of type $(n, \kappa, \mathfrak{F})$ which model Palyutin's axioms $B_1$–$B_{15}$.

**Theorem 5.2.** $N = \text{Mod} (\mathfrak{F}(n), \mathfrak{V}(n))^*.$

**Proof.** The proof is very similar to the proof of Theorem 5.1, so we limit ourselves to a sketch. By (2.1) in Theorem 2.1 it suffices to show
\[ N = \text{Mod} (\mathfrak{F}, \mathfrak{V})^*_n. \]  

(5.37)  

Set $K = \text{Mod} (\mathfrak{F}, \mathfrak{V})_n$ and let $\langle a_\alpha : 1 \leq \alpha < \kappa \rangle$ be a basis for $\mathfrak{V}$. Let $S$ be the set of terms of $\mathfrak{F}(K)$ defined as follows.

\[ g_i = p_i v_1 \text{ for } i = 1, \ldots, n. \]  

(5.38)  

\[ f = d v_1 \cdots v_n. \]  

(5.39)  

\[ +_p = p_1 v_1 + p_1 v_2. \]  

(5.40)  

\[ \lambda_p = \lambda(p_1 v_1) \text{ for each } \lambda \in F. \]  

(5.41)  

\[ c_0 = 0 \text{ and } c_\alpha = a_\alpha \text{ for } 1 \leq \alpha < \kappa. \]  

(5.42)

Proceeding as in Theorem 5.1, we prove (5.37) by showing

\[ K^*_s = K \text{ and } K^*_s = N. \]  

(5.43)

The proof of the first part of (5.43) is based on Theorem 0.1 and uses, among other things, the validity of the following two equations in $K$:

\[ v_1 + v_2 = f(g_1(v_1) + p g_1(v_2), \ldots, g_n(v_1) + p g_n(v_2)); \]  

(5.44)  

\[ \lambda v_1 = f(\lambda_p(g_1), \ldots, \lambda_p(g_n)). \]  

(5.45)
To obtain the second part of (5.43), just verify that axioms B₁–B₁₅ hold in \( K_5 \), so that \( K_5 \subseteq N \), and then imitate the proof of (5.36).

Fix a group \( G = \langle G, \cdot, ^{-1}, 1 \rangle \) and let \( \mathcal{S} \) be the set of all those sets \( E \) of subgroups (more correctly, subuniverses) of \( G \) which satisfy the following two conditions (given as a)–c) in [P], pp. 95–96):

(a) if \( H \in E \) and \( g \in G \sim H \), then \( gHg^{-1} \) (the conjugate subgroup of \( H \) by \( g \)) is in \( E \) and is different from \( H \);

(b) if \( H, H' \in E \) are distinct, then \( H \cap H' = \{1\} \).

Observe that \( \emptyset \in \mathcal{S} \). When \( |G| = 1 \), then \( \{G\} \) is the only other member of \( \mathcal{S} \). When \( |G| > 1 \), then no \( E \) in \( \mathcal{S} \) contains the trivial subgroup, by (a).

For each non-empty \( E \in \mathcal{S} \), define a \( \mathcal{O} \)-algebra \( \mathcal{O}(E) = \langle E, g^{(\mathcal{O}(E))}_{g \in G} \rangle \) by specifying \( g^{(\mathcal{O}(E))}(H) = gHg^{-1} \) for each \( H \in E \); define \( \mathcal{O}(\emptyset) \) to be \( \emptyset \).

**Theorem 5.3.** \( \mathcal{A} \) is isomorphic to \( \mathcal{S}(E) \) for some \( E \in \mathcal{S} \) iff \( \mathcal{A} \) is empty or:

(i) a \( \mathcal{O} \)-algebra of type 2 in which every element is a fixed point, when \( |G| > 1 \);

(ii) a trivial \( \mathcal{O} \)-algebra, when \( |G| = 1 \).

**Proof.** Suppose \( E, \in \mathcal{S} \), is non-empty. An \( H \in E \) is the fixed point of \( g^{(\mathcal{O}(E))} \) iff \( g \in H \), by (a). From this, using (a) and (b), one checks that \( \mathcal{O}(E) \) is as in (i) or (ii).

Now assume that \( \mathcal{A} \) is given as in (i) or (ii). For each \( a \in A \), let \( H_a \) be the subgroup of \( G \) leaving \( a \) fixed and set \( \mathcal{S}(\mathcal{A}) = \{H_a : a \in A\} \). Using the simple fact \( gH_a g^{-1} = H_{g(a)} \) one easily verifies that \( \mathcal{S}(\mathcal{A}) \in \mathcal{S} \) and that the correspondence \( a \mapsto H_a \) gives an isomorphism of \( \mathcal{A} \) onto \( \mathcal{O}(\mathcal{S}(\mathcal{A})) \). This proves Theorem 5.3.

**Theorem 5.4.** If \( E, E' \in \mathcal{S} \) are distinct, then \( \mathcal{O}(E) \neq \mathcal{O}(E') \).

Indeed, if \( \mathcal{O}(E) \cong \mathcal{O}(E') \), then \( E = E' \) because each \( H \) in \( E \) (or \( E' \)) is completely determined by the \( g \) for which \( g^{(\mathcal{O}(E))} \) leaves \( H \) fixed.

We can summarize Theorems 5.3 and 5.4 by saying that the correspondence \( E \mapsto \mathcal{O}(E) \) essentially maps \( \mathcal{S} \) one-one onto the set of isomorphism types of \( \mathcal{O}^{(\mathcal{A})} \) (see Section 3) where \( \mathcal{A} \) ranges over \( \mathcal{O} \)-algebras of type 2 and \( \mathcal{O} \); in case \( |G| = 1 \), it is necessary to adjoin the type of the trivial \( \mathcal{O} \)-algebra.

Now for each ordinal \( \xi \) let \( \mathcal{O}_\xi \) be the isomorphic image of \( \mathcal{O} \) with universe \( G \times \{\xi\} \) which is induced by the mapping \( g \mapsto (g, \xi) \). For each \( E \in \mathcal{S} \) and each finite or infinite cardinal \( \kappa \) set

\[
\mathcal{O}(E, \kappa) = \bigcup_{\xi < \kappa} \mathcal{O}_\xi \cup \mathcal{O}(E).
\]

**Theorem 5.5.** (i) For every \( \mathcal{A} \), either empty or a \( \mathcal{O} \)-algebra of type 2, we have \( \mathcal{A} \cong \mathcal{O}(E, \kappa) \) for some \( E \in \mathcal{S} \) and some \( \kappa \); in fact

\[
\mathcal{A} \cong \mathcal{O}(\mathcal{S}(\mathcal{A}), m(\mathcal{A})).
\]
(ii) If $G(E, \kappa) \cong G(E', \kappa')$, then $E = E'$ and $\kappa = \kappa'$, when $|G| > 1$; if $G(E, \kappa) \cong G(E, \kappa')$, then $\kappa = \kappa'$, when $|G| = 1$; $G(\emptyset, \kappa) \cong G(\{G\}, \kappa')$ iff $\kappa = \kappa' \geq \omega$ or $\kappa = \kappa' + 1 < \omega$, when $|G| = 1$.

**Proof.** (i) follows from Theorems 3.1 and 5.3, using $S^{(\infty)} \cong G(S(G^{(\infty)})�).

(ii) is a consequence of Theorems 3.2, 5.4, and the fact that, when $|G| = 1$ and $\mathfrak{A} = G(\{G\}, \kappa')$, we have $S^{(\infty)} = \emptyset$ and $m(\mathfrak{A}) = \kappa' + 1$.

Thus, except in the trivial case, the isomorphism type of $G$ and of each $G$-algebra of type 2 is completely determined by a unique $E \in S$ and a unique cardinal $\kappa$. This clarifies the relationship between $E, \kappa$ in Palyutin's description of his "standard quasivarieties of type A" and $\mathfrak{A}$ in our description of Per ($G, \mathfrak{A}$).

Now fix an $E \in S$, a finite or infinite cardinal $\kappa$, and an $n$. Let $P$ be the class of structures of type $(n, \kappa, G, E)$ which model Palyutin's axioms $A_1 - A_{15}$.

**Theorem 5.6.** $P = \text{Per} (G(G(E, \kappa)))_{\kappa}^n$.

**Proof.** The proof is analogous to the proofs of Theorems 5.1 and 5.2. Set $K = \text{Per} (G(G(E, \kappa)))_{\kappa}^n$ and $1_E = (1, \xi)$. Then $(1_E: \xi < \kappa)$ is a free basis for $\bigcup_{\xi < \kappa} G^*_\xi$. Let $S$ be the set of terms of $S(K)$ defined as follows:

\begin{align*}
    g_i &= \, p_i v_1 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad f = d v_1 \cdots v_n; \quad (5.46) \\
    h_p &= h(p, v_1) \quad \text{for each} \quad h \in G; \quad (5.47) \\
    c_\xi &= 1_\xi \quad \text{for} \quad \xi < \kappa \quad \text{and} \quad c_H = H \quad \text{for} \quad H \in E. \quad (5.48)
\end{align*}

Just as in Theorem 5.2 one now proves the theorem by showing

\[ K_s = K \quad \text{and} \quad K_s^* = P. \quad (5.49) \]

The details are left to the reader.

6. Historical remarks

The author first raised the problem of describing (up to polynomial equivalence) all quasivarieties categorical in power in 1971 while working on a related problem posed by Tarski (see [G], pp. 23–24, 38–39, and 44 for relevant historical details). Earlier, Burris [71] had obtained a complete description of all varieties of unary algebras categorical in power: they are polynomially equivalent to either $S_0$ or $S_1$. Later, Fajtlowicz [72] announced a description of all varieties which are categorical in every (finite and infinite) power: again, they must be polynomially equivalent to $S_0$ or $S_1$. The author announced his description of the free quasivarieties (i.e. quasivarieties $K$ in which all members are $K$-free) in Givant [72], and his description, up to a principal extension, of the quasivarieties categorical in power in Givant [73].
A representation theorem for universal Horn classes categorical in power

A simple consequence of this description is the theorem that a quasivariety $K$ which is categorical in some infinite power $\geq |\rho K|$ must be categorical in every infinite power greater than the cardinality of its smallest non-trivial member. This result was obtained independently by Palyutin [73] using different methods. The special case when $|\rho K| = \omega$ had already been established in Abakumov–Palyutin–Taitslin-Shishmarev [72] and, independently, in Baldwin-Lachlan [73]. (Of course the well-known results of Morley and Shelah concerning elementary classes were obtained even earlier.)

The standard examples in Givant [73] of quasivarieties categorical in power all have the property that there is a strongly minimal formula for the infinite models. The first example of a universal Horn class categorical in power which does not have this property was constructed in Palyutin [73]. One can show that Palyutin's example is definitionally equivalent to $\text{Aff}(\mathbb{Q}_2^{(2)})$, where $\mathbb{Q}_2$ is the field of integers modulo 2. In general, it is not hard to see that there will be a strongly minimal formula for the infinite members of $\text{Aff}(\mathbb{Q}^{(n)}, \mathbb{A}^{(n)}, k)$ iff $k = 0$ or $k = 1$; in this case $\delta_1 v_1 = v_1$ is one such formula.

As was mentioned in the introduction, the author's full description of quasivarieties categorical in power, avoiding the passage to a principal extension, was announced (for varieties) in Givant [75] (submitted October 29, 1974), and Palyutin's description in Palyutin [75] (submitted January 20, 1975).

References