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Differential Geometry of Kaehler Submanifolds

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INTRODUCTION

Theory of complex manifolds is one of the most interesting objects in differential geometry. Since a Kaehler structure consists of a complex structure and a Riemann metric, there can be two points of view in the study of Kaehler manifolds:

(A) analytic point of view (i.e., with emphasis on complex structure)

(B) differential geometric point of view (i.e., with emphasis on Riemann metric).

For example: Let $P_2(C)$ be a 2-dimensional complex projective space with Fubini–Study metric of constant holomorphic sectional curvature and z_0, z_1, z_2 be a homogeneous coordinate system of $P_2(C)$. Let $P_1(C) = \{(z_0, z_1, z_2) \in P_2(C) \mid z_2 = 0\}$ and $Q_1(C) = \{(z_0, z_1, z_2) \in P_2(C) \mid z_0^2 + z_1^2 + z_2^2 = 0\}$. Then $P_1(C)$ and $Q_1(C)$ are complex analytically equivalent (i.e., equivalent from the viewpoint (A)), but they are not equivalent from the viewpoint (B) with respect to the induced Kaehler structures.

In general, "if a complex manifold M admits a Kaehler metric satisfying..., then M is complex analytically equivalent to..." is a result from the viewpoint (A), and "if a Kaehler manifold M satisfies..., then M is complex analytically isometric to..." is a result from the viewpoint (B).

In this note, we arrange results on complex submanifolds mainly from the viewpoint (B). By a *Kaehler submanifold* we mean a complex submanifold with the induced Kaehler structure.

1. Kaehler Submanifolds of a Complex Space Form

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. There are three types of complex space forms: *elliptic*, *hyperbolic*, or *flat* according as the holomorphic sectional curvature is positive, negative, or zero.

Let $P_m(C)$ be an m-dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. Then $P_m(C)$ is a complete and simply connected elliptic complex space form.

Complex Euclidean space C^m endowed with the usual Hermitian metric is a complete and simply-connected flat complex space form.

Let D_m be the open unit ball in C^m endowed with the natural complex structure and the Bergman metric of constant holomorphic sectional curvature -1. Then D_m is a complete and simply-connected hyperbolic complex space form.

Any m-dimensional complex space form is (after multiplying the metric by a suitable constant) locally complex analytically isometric to $P_m(C)$, C^m, or D_m, according as the holomorphic sectional curvature is positive, zero, or negative.

Let $\tilde{M}_{n+p}(\tilde{c})$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$ and let M_n be an n-dimensional Kaehler submanifold of $\tilde{M}$ (i.e., complex submanifold with the induced Kaehler structure).

Let J (resp. $\tilde{J}$) be the complex structure of M (resp. $\tilde{M}$) and let g (resp. $\tilde{g}$) be the Kaehler metric of M (resp. $\tilde{M}$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

and it satisfies

$$\begin{aligned} \sigma(X, Y) &= \sigma(Y, X) \\ \sigma(JX, Y) &= \sigma(X, JY) = \tilde{J}\sigma(X, Y). \end{aligned} \tag{1.1}$$

We choose a local field of orthonormal frames $e_1, \ldots, e_n$, $e_{1^*} = \tilde{J}e_1, \ldots, e_{n^*} = \tilde{J}e_n$, $e_{\bar{1}}, \ldots, e_{\bar{p}}$, $e_{\bar{1}^*} = \tilde{J}e_{\bar{1}}, \ldots, e_{\bar{p}^*} = \tilde{J}e_{\bar{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1,\ldots, e_n$, $e_{1^*},\ldots, e_{n^*}$ are tangent to M.[1] If we set

$$g(A_\lambda X, Y) = \tilde{g}(\sigma(X, Y), e_\lambda)$$

or

$$\sigma(X, Y) = \sum g(A_\lambda X, Y)\, e_\lambda\,,$$

then $A_{\tilde{1}},\ldots, A_{\tilde{p}}$, $A_{\tilde{1}^*},\ldots, A_{\tilde{p}^*}$ are local fields of symmetric linear transformations. We can easily see from (1.1) that

$$A_{\alpha^*} = JA_\alpha \qquad \text{and} \qquad JA_\alpha = -A_\alpha J$$

so that, in particular,

$$\operatorname{tr} A_\alpha = \operatorname{tr} A_{\alpha^*} = 0.$$

This implies that M is a minimal submanifold of $\tilde{M}$.

With respect to the frame field of $\tilde{M}$ chosen above, let

$$\omega^1,\ldots, \omega^n, \omega^{1^*},\ldots, \omega^{n^*}, \omega^{\tilde{1}},\ldots, \omega^{\tilde{p}}, \omega^{\tilde{1}^*},\ldots, \omega^{\tilde{p}^*}$$

be the field of dual frames. Then the Kaehler metrics can be expressed locally as

$$g = \sum \omega^i \otimes \omega^i, \qquad \tilde{g} = \sum \omega^I \otimes \omega^I.$$

The structure equations of $\tilde{M}$ are given by

$$d\omega^I = -\sum \omega_J{}^I \wedge \omega^J \tag{1.2}$$

$$\omega_J{}^I + \omega_I{}^J = 0, \tag{1.3}$$

$$\omega_b^a = \omega_{b^*}^{a^*}, \qquad \omega_\beta^\alpha = \omega_{\beta^*}^{\alpha^*}, \qquad \omega_\beta^a = \omega_{\beta^*}^{a^*}$$

$$\omega_{b^*}^a = \omega_{a^*}^b, \qquad \omega_{\beta^*}^\alpha = \omega_{\alpha^*}^\beta, \qquad \omega_{\beta^*}^a = \omega_{a^*}^\beta$$

$$d\omega_J{}^I = -\sum \omega_K{}^I \wedge \omega_J{}^K + \tilde{\Omega}_J{}^I \tag{1.4}$$

$$\tilde{\Omega}_J{}^I = \frac{1}{2} \sum \tilde{R}^I_{JKL}\omega^K \wedge \omega^L$$

$$\tilde{R}^I_{JKL} = \frac{\tilde{c}}{4}(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK} + J_{IK}J_{JL} - J_{IL}J_{JK} + 2J_{IJ}J_{KL}), \tag{1.5}$$

[1] We use the following convention on the range of indices unless otherwise stated:

$$\begin{aligned} A, B, C, D &= 1,\ldots, n, \tilde{1},\ldots, \tilde{p} \\ I, J, K, L &= 1,\ldots, n, 1^*,\ldots, n^*, \tilde{1},\ldots, \tilde{p}, \tilde{1}^*,\ldots, \tilde{p}^* \\ a, b, c, d &= 1,\ldots, n \\ i, j, k, l &= 1,\ldots, n, 1^*,\ldots, n^* \\ \alpha, \beta &= \tilde{1},\ldots, \tilde{p} \\ \lambda, \mu &= \tilde{1},\ldots, \tilde{p}, \tilde{1}^*,\ldots, \tilde{p}^*. \end{aligned}$$

where $\tilde{J} = \sum \tilde{J}_{IJ} e_I \otimes \omega^J$ so that

$$(\tilde{J}_{IJ}) = \left(\begin{array}{cc|cc} 0 & -I_n & & \\ I_n & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -I_p \\ \multicolumn{2}{c|}{0} & I_p & 0 \end{array}\right),$$

I_s being the identity matrix of degree s.

Restricting these forms to M, we have the structure equations of the immersion:

$$\omega^\lambda = 0 \tag{1.6}$$

$$d\omega^i = -\sum \omega_j^{\ i} \wedge \omega^j \tag{1.7}$$

$$d\omega_j^{\ i} = -\sum \omega_k^{\ i} \wedge \omega_j^{\ k} + \Omega_j^{\ i} \tag{1.8}$$

$$\Omega_j^{\ i} = \frac{1}{2} \sum R^i_{jkl} \omega^k \wedge \omega^l$$

$$\Omega_j^{\ i} = \tilde{\Omega}_j^{\ i} - \sum \omega_\lambda^{\ i} \wedge \omega_j^{\ \lambda} \qquad \text{(the equation of Gauss).} \tag{1.9}$$

From (1.2) and (1.6) we have $\sum \omega_i^{\ \lambda} \wedge \omega^i = 0$. By Cartan's lemma we may write

$$\omega_i^{\ \lambda} = \sum h^\lambda_{ij} \omega^j, \qquad h^\lambda_{ij} = h^\lambda_{ji} . \tag{1.10}$$

We can easily see

$$h^\lambda_{ij} = g(A_\lambda e_i , e_j) \quad \text{or} \quad \sigma(e_i , e_j) = \sum h^\lambda_{ij} e_\lambda \tag{1.11}$$

$$\text{or} \quad \sigma = \sum h^\lambda_{ij} \omega^i \otimes \omega^j \otimes e_\lambda .$$

We sometimes write $A_\lambda = (h^\lambda_{ij})$ instead of (1.11).

Let $A_\alpha' = (h^\alpha_{ab})$ and $A''_\alpha = (h^\alpha_{ab*})$. Then we can easily see that

$$A_\alpha = \begin{pmatrix} A_\alpha' & A''_\alpha \\ A''_\alpha & -A_\alpha' \end{pmatrix} \quad \text{and} \quad A_{\alpha*} = \begin{pmatrix} -A''_\alpha & A_\alpha' \\ A_\alpha' & A''_\alpha \end{pmatrix}.$$

The equation of Gauss is written as

$$R^i_{jkl} = \sum (h^\lambda_{ik} h^\lambda_{jl} - h^\lambda_{il} h^\lambda_{jk})$$
$$+ \frac{\tilde{c}}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl}). \tag{1.9$'$}$$

Let R be the curvature tensor field of M so that

$$R^i_{jkl} = g(R(e_k, e_l)\, e_j, e_i).$$

Then the equation of Gauss is written as

$$\begin{aligned} g(R(X, Y)\,Z, W) &= \tilde{g}(\sigma(X, W), \sigma(Y, Z)) - \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \\ &\quad + \frac{\tilde{c}}{4}\{g(X, W)\,g(Y, Z) - g(X, Z)\,g(Y, W) \\ &\quad + g(JX, W)\,g(JY, Z) - g(JX, Z)\,g(JY, W) + 2g(X, JY)\,g(JZ, W)\} \end{aligned} \tag{1.12}$$

or

$$\begin{aligned} g(R(X, Y)\,Z, W) &= \sum \{g(A_\lambda X, W)\,g(A_\lambda Y, Z) - g(A_\lambda X, Z)\,g(A_\lambda Y, W)\} \\ &\quad + \frac{\tilde{c}}{4}\{g(X, W)\,g(Y, Z) - g(X, Z)\,g(Y, W) \\ &\quad + g(JX, W)\,g(JY, Z) - g(JX, Z)\,g(JY, W) + 2g(X, JY)\,g(JZ, W)\}. \end{aligned} \tag{1.12$'$}$$

Let S be the Ricci tensor of M. Then we have

$$S(X, Y) = \frac{n+1}{2}\tilde{c}g(X, Y) - 2\sum g(A_\alpha X, A_\alpha Y). \tag{1.13}$$

Let ρ be the scalar curvature of M. Then we have

$$\rho = n(n+1)\tilde{c} - \|\sigma\|^2, \tag{1.14}$$

where $\|\sigma\|$ is the length of the second fundamental form σ of the immersion so that

$$\begin{aligned} \|\sigma\|^2 &= \sum \operatorname{tr} A_\lambda{}^2 = 2\sum \operatorname{tr} A_\alpha{}^2 \\ &= \sum h^\lambda_{ij} h^\lambda_{ij} = 2\sum h^\alpha_{ij} h^\alpha_{ij}. \end{aligned} \tag{1.15}$$

We can see from (1.12) or (1.12)$'$ that the sectional curvature K of M determined by orthonormal vectors X and Y is given by

$$\begin{aligned} K(X, Y) &= \frac{\tilde{c}}{4}\{1 + 3g(X, JY)^2\} + \tilde{g}(\sigma(X, X), \sigma(Y, Y)) - \|\sigma(X, Y)\|^2 \\ &= \frac{\tilde{c}}{4}\{1 + 3g(X, JY)^2\} + \sum \{g(A_\lambda X, X)\,g(A_\lambda Y, Y) - g(A_\lambda X, Y)^2\}. \end{aligned} \tag{1.16}$$

In particular, the holomorphic sectional curvature H of M determined by a unit vector X is given by

$$H(X) = \tilde{c} - 2\,\|\sigma(X, X)\|^2 = \tilde{c} - 2 \sum g(A_\lambda X, X)^2. \tag{1.17}$$

As an immediate consequence of (1.13), (1.14), and (1.17) we have

PROPOSITION 1.1. *Let M_n be an n-dimensional Kaehler submanifold of $\tilde{M}_{n+p}(\tilde{c})$. Then*

(a) $S - ((n+1)/2)\,\tilde{c}g$ *is negative semi-definite*

(b) $\rho \leqslant n(n+1)\tilde{c}$

(c) $H \leqslant \tilde{c}$.

1′. COMPLEX VERSION OF §1

We present a brief summary of complex version of the results given in §1.

Let $T_x(M)$ be the tangent space to M at x and $T_x{}^c(M)$ its complexification. Let $T_x^{1,0}(M) = \{X - \sqrt{-1}\, JX \mid X \in T_x(M)\}$ and $T_x^{0,1}(M) = \{X + \sqrt{-1}\, JX \mid X \in T_x(M)\}$. Then

$$T_x{}^c(M) = T_x^{1,0}(M) + T_x^{0,1}(M).$$

The similar results hold for $\tilde{M}$. Let

$$\xi_A = \tfrac{1}{2}(e_A - \sqrt{-1}\, e_{A*}), \qquad \xi_{\bar{A}} = \tfrac{1}{2}(e_A + \sqrt{-1}\, e_{A*})$$
$$\theta^A = \omega^A + \sqrt{-1}\, \omega^{A*}, \qquad \theta^{\bar{A}} = \omega^A - \sqrt{-1}\, \omega^{A*}.$$

Then ξ_A's (resp. $\xi_{\bar{A}}$'s) form a complex basis of $T_x^{1,0}(\tilde{M})$ (resp. $T_x^{0,1}(\tilde{M})$) at each point x, and ξ_a's (resp. $\xi_{\bar{a}}$'s) form a complex basis of $T_x^{1,0}(M)$ (resp. $T_x^{0,1}(M)$) at each point x. The complex structure J of M defines a linear isomorphism of $T_x{}^c(M)$ at each point x, which we denote by the same letter J. With respect to the basis $\xi_1, \ldots, \xi_n, \xi_{\bar{1}}, \ldots, \xi_{\bar{n}}$ of $T_x{}^c(M)$, J is represented by the matrix

$$\begin{pmatrix} \sqrt{-1}\, I_n & 0 \\ 0 & -\sqrt{-1}\, I_n \end{pmatrix}.$$

The similar results hold for $\tilde{J}$.

Restricting θ_A's to M, we have

$$\theta^\alpha = 0. \tag{1′.6}$$

The Kaehler metrics g and $\tilde{g}$ are given, respectively, by

$$g = \sum \theta^a \otimes \theta^{\bar{a}} \qquad \text{and} \qquad \tilde{g} = \sum \theta^A \otimes \theta^{\bar{A}}.$$

Let

$$\theta_B{}^A = \omega_B{}^A + \sqrt{-1}\,\omega_B^{A*}, \qquad \theta_{\bar{B}}{}^{\bar{A}} = \omega_B{}^A - \sqrt{-1}\,\omega_B^{A*}$$
$$\tilde{\Phi}_B{}^A = \tilde{\Omega}_B{}^A + \sqrt{-1}\,\tilde{\Omega}_B^{A*}, \qquad \tilde{\Phi}_{\bar{B}}{}^{\bar{A}} = \tilde{\Omega}_B{}^A - \sqrt{-1}\,\tilde{\Omega}_B^{A*}$$
$$\Phi_b{}^a = \Omega_b{}^a + \sqrt{-1}\,\Omega_b^{a*}, \qquad \Phi_{\bar{b}}{}^{\bar{a}} = \Omega_b{}^a - \sqrt{-1}\,\Omega_b^{a*}.$$

Then we have

$$d\theta^A = -\sum \theta_B{}^A \wedge \theta^B \tag{1'.2}$$

$$\theta_B{}^A + \theta_{\bar{A}}{}^{\bar{B}} = 0 \tag{1'.3}$$

$$d\theta_B{}^A = -\sum \theta_C{}^A \wedge \theta_B{}^C + \tilde{\Phi}_B{}^A \tag{1'.4}$$

$$\tilde{\Phi}_B{}^A = \sum \tilde{K}^A_{BC\bar{D}}\theta^C \wedge \theta^{\bar{D}}$$

$$\tilde{K}^A_{BC\bar{D}} = \frac{1}{2}\{\tilde{R}^A_{BCD} + \tilde{R}^A_{B^*CD^*} + \sqrt{-1}\,(\tilde{R}^A_{BCD^*} - \tilde{R}^A_{B^*CD})\}$$

$$\tilde{K}^A_{BC\bar{D}} = \frac{\tilde{c}}{4}(\delta_{AC}\delta_{BD} + \delta_{AB}\delta_{CD}) \tag{1'.5}$$

or

$$\tilde{\Phi}_B{}^A = \frac{\tilde{c}}{4}\left(\theta^A \wedge \theta^{\bar{B}} + \delta_{AB}\sum \theta^C \wedge \theta^{\bar{C}}\right). \tag{1'.5)'$$

Restricting these forms to M, we have

$$\theta^\alpha = 0 \tag{1'.6}$$

$$d\theta^a = -\sum \theta_b{}^a \wedge \theta^b \tag{1'.7}$$

$$d\theta_b{}^a = -\sum \theta_c{}^a \wedge \theta_b{}^c + \Phi_b{}^a \tag{1'.8}$$

$$\Phi_b{}^a = \sum K^a_{bc\bar{d}}\theta^c \wedge \theta^{\bar{d}}$$

$$\Phi_b{}^a = \tilde{\Phi}_b{}^a - \sum \theta_\alpha{}^a \wedge \theta_b{}^\alpha = \tilde{\Phi}_b{}^a + \sum \theta_{\bar{a}}{}^{\bar{\alpha}} \wedge \theta_b{}^\alpha. \tag{1'.9}$$

From (1′.2) and (1′.6) we have $\sum \theta_a{}^\alpha \wedge \theta^a = 0$ and $\sum \theta_{\bar{a}}{}^{\bar{\alpha}} \wedge \theta^{\bar{a}} = 0$. By Cartan's lemma we may write

$$\begin{aligned} \theta_a{}^\alpha &= \sum k^\alpha_{ab}\theta^b, \qquad k^\alpha_{ab} = k^\alpha_{ba} \\ \theta_{\bar{a}}{}^{\bar{\alpha}} &= \sum k^{\bar{\alpha}}_{\bar{a}\bar{b}}\theta^{\bar{b}}, \qquad k^{\bar{\alpha}}_{\bar{a}\bar{b}} = k^{\bar{\alpha}}_{\bar{b}\bar{a}}. \end{aligned} \tag{1'.10}$$

We can easily see

$$k^{\alpha}_{ab} = h^{\alpha}_{ab} - \sqrt{-1}\, h^{\alpha}_{ab*}$$
$$k^{\bar\alpha}_{\bar a\bar b} = h^{\alpha}_{ab} + \sqrt{-1}\, h^{\alpha}_{ab*}\,, \tag{1$'$.18}$$

and

$$\sigma = \sum (k^{\alpha}_{ab}\theta^a \otimes \theta^b \otimes \xi_\alpha + k^{\bar\alpha}_{\bar a\bar b}\theta^{\bar a} \otimes \theta^{\bar b} \otimes \xi_{\bar\alpha}). \tag{1$'$.11}$$

The equation of Gauss is written as

$$K^a_{bc\bar d} = \frac{\tilde c}{4}(\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd}) - \sum k^{\alpha}_{bc}k^{\bar\alpha}_{\bar a\bar d}\,. \tag{1$'$.9$'$}$$

Moreover $\| \sigma \|$ is given by

$$\| \sigma \|^2 = 4 \sum k^{\alpha}_{ab}k^{\bar\alpha}_{\bar a\bar b}\,. \tag{1$'$.15}$$

Let $\iota : T_x(M) \to T^{1,0}_x(M)$ and $\bar\iota : T_x(M) \to T^{0,1}_x(M)$ be isomorphisms defined by

$$\iota(X) = \tfrac{1}{2}(X - \sqrt{-1}\, JX)$$
$$\bar\iota(X) = \tfrac{1}{2}(X + \sqrt{-1}\, JX).$$

Then, for each A_α, there corresponds a unique linear mapping $B_\alpha : T_x{}^c(M) \to T_x{}^c(M)$ satisfying $\bar\iota \circ A_\alpha = B_\alpha \circ \iota$ and $\iota \circ A_\alpha = B_\alpha \circ \bar\iota$. In fact, let $X = \sum X^a e_a + \sum X^{a^*} e_{a^*} \in T_x(M)$. Then

$$\iota(X) = \tfrac{1}{2}(X - \sqrt{-1}\, JX) = \sum (X^a + \sqrt{-1}\, X^{a^*})\, \xi_a\,,$$
$$\bar\iota(X) = \tfrac{1}{2}(X + \sqrt{-1}\, JX) = \sum (X^a - \sqrt{-1}\, X^{a^*})\, \xi_{\bar a}$$

and

$$A_\alpha X = \sum h^{\alpha}_{ak}X^k e_a + \sum h^{\alpha}_{a^*k}X^k e_{a^*}\,.$$

Hence we have

$$\iota(A_\alpha X) = \sum (h^{\alpha}_{ab} + \sqrt{-1}\, h^{\alpha}_{ab*})(X^b - \sqrt{-1}\, X^{b^*})\, \xi_a$$
$$\bar\iota(A_\alpha X) = \sum (h^{\alpha}_{ab} - \sqrt{-1}\, h^{\alpha}_{ab*})(X^b + \sqrt{-1}\, X^{b^*})\, \xi_{\bar a}\,.$$

Therefore $B_\alpha : T_x^c(M) \to T_x^c(M)$ is defined to be a linear mapping represented by the matrix

$$\begin{pmatrix} 0 & A_\alpha' + \sqrt{-1}\,A_\alpha'' \\ A_\alpha' - \sqrt{-1}\,A_\alpha'' & 0 \end{pmatrix} = \begin{pmatrix} 0 & h_{ab}^\alpha + \sqrt{-1}\,h_{ab*}^\alpha \\ h_{ab}^\alpha - \sqrt{-1}\,h_{ab*}^\alpha & 0 \end{pmatrix}$$

with respect to the complex basis $\xi_1, \dots, \xi_n, \xi_{\bar{1}}, \dots, \xi_{\bar{n}}$ of $T_x^c(M)$. We sometimes write

$$B_\alpha = \begin{pmatrix} 0 & A_\alpha' + \sqrt{-1}\,A_\alpha'' \\ A_\alpha' - \sqrt{-1}\,A_\alpha'' & 0 \end{pmatrix}. \tag{1'.18)'}$$

Let $B_{\alpha*} = JB_\alpha$. Then $B_{\alpha*} : T_x^c(M) \to T_x^c(M)$ is a linear mapping represented by the matrix

$$\begin{pmatrix} \sqrt{-1}\,I_n & 0 \\ 0 & -\sqrt{-1}\,I_n \end{pmatrix} \begin{pmatrix} 0 & A_\alpha' + \sqrt{-1}\,A_\alpha'' \\ A_\alpha' - \sqrt{-1}\,A_\alpha'' & 0 \end{pmatrix}.$$

We can easily see

$$k_{ab}^\alpha = g(B_\alpha \xi_a, \xi_b) \tag{1'.11)'}$$

$$\| \sigma \|^2 = 2 \operatorname{tr} \sum B_\alpha^2. \tag{1'.15)'}$$

Let $X = \sum X^a e_a + \sum X^{a^*} e_{a^*} \in T_x(M)$. Then we have

$$g(A_\alpha X, X) = \sum h_{ab}^\alpha (X^a X^b - X^{a^*} X^{b^*}) + 2 \sum h_{ab*}^\alpha X^a X^{b^*}$$

$$g(A_{\alpha*} X, X) = -\sum h_{ab*}^\alpha (X^a X^b - X^{a^*} X^{b^*}) + 2 \sum h_{ab}^\alpha X^a X^{b^*}$$

$$\begin{aligned} g(B_\alpha \iota(X), \iota(X)) = {} & \sum h_{ab}^\alpha (X^a X^b - X^{a^*} X^{b^*}) + 2 \sum h_{ab*}^\alpha X^a X^{b^*} \\ & - \sqrt{-1} \left\{ \sum h_{ab*}^\alpha (X^a X^b - X^{a^*} X^{b^*}) - 2 \sum h_{ab}^\alpha X^a X^{b^*} \right\}, \end{aligned}$$

which imply

$$g(A_\alpha X, X)^2 + g(A_{\alpha*} X, X)^2 = | g(B_\alpha \iota(X), \iota(X)) |^2.$$

Therefore the holomorphic sectional curvature H of M determined by a unit vector X is given by

$$H(X) = \tilde{c} - 2 \sum | g(B_\alpha \iota(X), \iota(X)) |^2. \tag{1'.17}$$

2. EXAMPLES

The following result is well-known.

PROPOSITION 2.1. *Let M_n be an n-dimensional Kaehler submanifold of $\tilde{M}_{n+p}(\tilde{c})$. Then M is totally geodesic if and only if M satisfies one of the following conditions:*

(a) $\mathrm{Min}\{\tilde{c}, \tilde{c}/4\} \leqslant K \leqslant \mathrm{Max}\{\tilde{c}, \tilde{c}/4\}$ $(n \geqslant 2)$

(b) $H = \tilde{c}$

(c) $S = ((n+1)/2)\, \tilde{c} g$

(d) $\rho = n(n+1)\tilde{c}$.

Let $P_{n+1}(C)$ be an $(n+1)$-dimensional complex projective space with the Fubini–Study metric of constant holomorphic sectional curvature $\tilde{c}(>0)$ and let $z_0, z_1, \ldots, z_{n+1}$ be a homogeneous coordinate system of $P_{n+1}(C)$. Let

$$Q_n(C) = \left\{(z_0, z_1, \ldots, z_{n+1}) \in P_{n+1}(C) \,\middle|\, \sum z_i^2 = 0\right\}.$$

Then $Q_n(C)$ is complex analytically isometric to the Hermitian symmetric space $SO(n+2)/SO(2) \times SO(n)$ and the following result is well-known.

PROPOSITION 2.2. *With respect to the induced Kaehler structure, $Q_n(C)$ satisfies the following:*

(a) $0 \leqslant K \leqslant \tilde{c}$ $(n \geqslant 2)$
$K = \tilde{c}/2$ $(n = 1)$

(b) $\tilde{c}/2 \leqslant H \leqslant \tilde{c}$ $(n \geqslant 2)$
$H = \tilde{c}/2$ $(n = 1)$

(c) $S = (n/2)\, \tilde{c} g$

(d) $\rho = n^2 \tilde{c}$.

We have another example due to E. Calabi, which is a little more complicated than the preceding ones:

PROPOSITION 2.3. ([5]) *An n-dimensional complex projective space of constant holomorphic sectional curvature c can be imbedded as a Kaehler* *submanifold into an* $\{\binom{n+\nu}{\nu}-1\}$*-dimensional complex projective space of constant holomorphic sectional curvature* νc.

For $n = 1$ and $\nu = 2$, this is nothing but $Q_1(C)$ in $P_2(C)$.

Remark. The imbedding in Proposition 2.3 is given by all homogeneous monomials of degree ν in homogeneous coordinates: Let $z_0, \ldots, z_n$ be homogeneous coordinates in $P_n(C)$. Then the imbedding is given by

$$(z_0, \ldots, z_n) \to \left(z_0^{\ \nu}, \sqrt{\nu}\, z_0^{\nu-1} z_1, \ldots, \sqrt{\frac{\nu!}{\alpha_0! \cdots \alpha_n!}}\, z_0^{\alpha_0} \cdots z_n^{\alpha_n}, \ldots, z_n^{\ \nu}\right),$$

$$\text{where} \quad \sum_{i=0}^{n} \alpha_i = \nu.$$

3. Second Fundamental Form

If we define h_{ijk}^{λ} by

$$\sum h_{ijk}^{\lambda}\omega^k = dh_{ij}^{\lambda} - \sum h_{ik}^{\lambda}\omega_j^{\ k} - \sum h_{kj}^{\lambda}\omega_i^{\ k} + \sum h_{ij}^{\mu}\omega_{\mu}^{\ \lambda}, \tag{3.1}$$

then from (1.4), (1.5), (1.6), (1.7), and (1.10) we have

$$\begin{aligned} 0 = \tilde{\Omega}_i^{\ \lambda} &= d\omega_i^{\ \lambda} + \sum \omega_k^{\ \lambda} \wedge \omega_i^{\ k} + \sum \omega_{\mu}^{\ \lambda} \wedge \omega_i^{\ \mu} \\ &= \sum \left(dh_{ij}^{\lambda} - \sum h_{ik}^{\lambda}\omega_j^{\ k} - \sum h_{kj}^{\lambda}\omega_i^{\ k} + \sum h_{ij}^{\mu}\omega_{\mu}^{\ \lambda}\right) \wedge \omega^j \\ &= \sum h_{ijk}^{\lambda}\omega^k \wedge \omega^j. \end{aligned}$$

This implies that

$$h_{ijk}^{\lambda} \textit{ is symmetric with respect to } i, j, \textit{ and } k. \tag{3.2}$$

Moreover we can see

$$h_{a^*b^*k}^{\lambda} = -h_{abk}^{\lambda}, \qquad h_{ab^*k}^{\lambda} = h_{a^*bk}^{\lambda}. \tag{3.3}$$

We have the corresponding complex version. If we define k_{abc}^{α} and $k_{ab\bar{c}}^{\alpha}$ by

$$\sum k_{abc}^{\alpha}\theta^c + \sum k_{ab\bar{c}}^{\alpha}\theta^{\bar{c}} = dk_{ab}^{\alpha} - \sum k_{ac}^{\alpha}\theta_b^{\ c} - \sum k_{cb}^{\alpha}\theta_a^{\ c} + \sum k_{ab}^{\beta}\theta_{\beta}^{\ \alpha}, \tag{3.1$'$}$$

then from (1'.4), (1'.5), (1'.6), (1'.7), and (1'.10) we have

$$0 = \tilde{\Phi}_a{}^\alpha = d\theta_a{}^\alpha + \sum \theta_c{}^\alpha \wedge \theta_a{}^c + \sum \theta_\beta{}^\alpha \wedge \theta_a{}^\beta$$

$$= \sum \left(dk^\alpha_{ab} - \sum k^\alpha_{ac}\theta_b{}^c - \sum k^\alpha_{cb}\theta_a{}^c + \sum k^\beta_{ab}\theta_\beta{}^\alpha\right) \wedge \theta^b$$

$$= \sum (k^\alpha_{abc}\theta^c + k^\alpha_{ab\bar{c}}\theta^{\bar{c}}) \wedge \theta^b.$$

This implies that

$$k^\alpha_{abc} \text{ is symmetric with respect to } a, b, \text{ and } c \tag{3.2)'$$

$$k^\alpha_{ab\bar{c}} = 0 \qquad \text{or} \qquad k^{\bar{\alpha}}_{\bar{a}\bar{b}c} = 0. \tag{3.2)''$$

Let ∇' be the covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle). Then we have

$$(\nabla'_{e_k}\sigma)(e_i, e_j) = \sum h^\lambda_{ijk} e_\lambda \qquad \text{or} \qquad \nabla'\sigma = \sum h^\lambda_{ijk}\omega^i \otimes \omega^j \otimes \omega^k \otimes e_\lambda. \tag{3.4}$$

The second fundamental form σ of the immersion satisfies a differential equation:

PROPOSITION 3.1. ([19, 23]) *Let M_n be an n-dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$. Then*

$$\frac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 - \sum (\operatorname{tr} A_\lambda A_\mu)^2 + \frac{n+2}{2}\tilde{c}\|\sigma\|^2$$

or

$$\frac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 - 8 \operatorname{tr}\left(\sum A_\alpha{}^2\right)^2 - \sum (\operatorname{tr} A_\lambda A_\mu)^2 + \frac{n+2}{2}\tilde{c}\|\sigma\|^2,$$

where Δ denotes the Laplacian.

Proof. Since M is a minimal submanifold of $\tilde{M}$, the following holds ([9]):

$$\frac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 - \sum (\operatorname{tr} A_\lambda A_\mu)^2$$
$$+ \sum (4\tilde{R}^\lambda_{\mu ij}h^\lambda_{jk}h^\mu_{ik} - \tilde{R}^\lambda_{k\mu k}h^\lambda_{ij}h^\mu_{ij} + 2\tilde{R}^i_{jkj}h^\lambda_{il}h^\lambda_{kl} + 2\tilde{R}^i_{jkl}h^\lambda_{il}h^\lambda_{jk}).$$

Since $\tilde{M}$ is a complex space form of constant holomorphic sectional curvature $\tilde{c}$, the last term of the right hand side of the above equation is equal to $((n+2)/2)\tilde{c}\,\|\sigma\|^2$.

Moreover we have

$$
\begin{aligned}
&\sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 \\
&= -2\sum \operatorname{tr}\{A_\lambda{}^2 A_\mu{}^2 - (A_\lambda A_\mu)^2\} \\
&= -2\Big[\sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_\beta{}^2 - (A_\alpha A_\beta)^2\} + 2\sum \operatorname{tr}\{A_\alpha{}^2 A_{\alpha^*}^2 - (A_\alpha A_{\alpha^*})^2\} \\
&\qquad + 2\sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_{\beta^*}^2 - (A_\alpha A_{\beta^*})^2\} + \sum_{\alpha\neq\beta} \operatorname{tr}\{A_{\alpha^*}^2 A_{\beta^*}^2 - (A_{\alpha^*} A_{\beta^*})^2\}\Big] \\
&= -4\Big[\sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_\beta{}^2 - (A_\alpha A_\beta)^2\} + \sum \operatorname{tr}\{A_\alpha{}^2 A_{\alpha^*}^2 - (A_\alpha A_{\alpha^*})^2\} \\
&\qquad + \sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_{\beta^*}^2 - (A_\alpha A_{\beta^*})^2\}\Big] \\
&= -4\Big[\sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_\beta{}^2 - (A_\alpha A_\beta)^2\} + 2\sum \operatorname{tr} A_\alpha{}^4 + \sum_{\alpha\neq\beta} \operatorname{tr}\{A_\alpha{}^2 A_\beta{}^2 + (A_\alpha A_\beta)^2\}\Big] \\
&= -8\Big[\sum_{\alpha\neq\beta} \operatorname{tr} A_\alpha{}^2 A_\beta{}^2 + \sum \operatorname{tr} A_\alpha{}^4\Big] = -8\sum \operatorname{tr} A_\alpha{}^2 A_\beta{}^2 = -8 \operatorname{tr}\Big(\sum A_\alpha{}^2\Big)^2.
\end{aligned}
$$

(Q.E.D.)

A complex version of Proposition 3.1 is as follows.

PROPOSITION 3.1′. *Let M_n be an n-dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$. Then*

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - 8\operatorname{tr}\Big(\sum B_\alpha{}^2\Big)^2 - \sum(\operatorname{tr} B_\lambda B_\mu)^2 + \frac{n+2}{2}\tilde{c}\,\|\sigma\|^2.$$

Proof. Since

$$B_\alpha{}^2 = \begin{pmatrix} A_\alpha'^2 + A_\alpha''^2 - \sqrt{-1}\,(A_\alpha' A_\alpha'' - A_\alpha'' A_\alpha') & 0 \\ 0 & A_\alpha'^2 + A_\alpha''^2 + \sqrt{-1}\,(A_\alpha' A_\alpha'' - A_\alpha'' A_\alpha') \end{pmatrix},$$

we have

$$
\begin{aligned}
\operatorname{tr}\Big(\sum A_\alpha{}^2\Big)^2 &= 2\operatorname{tr}\Big[\Big\{\sum(A_\alpha'^2 + A_\alpha''^2)\Big\}^2 - \Big\{\sum(A_\alpha' A_\alpha'' - A_\alpha'' A_\alpha')\Big\}^2\Big] \\
&= \operatorname{tr}\Big(\sum B_\alpha{}^2\Big)^2.
\end{aligned}
$$

On the other hand, since

$$B_\alpha B_\beta = \begin{pmatrix} A_\alpha' A_\beta' + A_\alpha'' A_\beta'' - \sqrt{-1}(A_\alpha' A_\beta'' - A_\alpha'' A_\beta') & 0 \\ 0 & A_\alpha' A_\beta' + A_\alpha'' A_\beta'' + \sqrt{-1}(A_\alpha' A_\beta'' - A_\alpha'' A_\beta') \end{pmatrix}$$

and

$$B_\alpha B_{\beta*} = \begin{pmatrix} -A_\alpha' A_\beta'' + A_\alpha'' A_\beta' - \sqrt{-1}(A_\alpha' A_\beta' + A_\alpha'' A_\beta'') & 0 \\ 0 & -A_\alpha' A_\beta'' + A_\alpha'' A_\beta' + \sqrt{-1}(A_\alpha' A_\beta' + A_\alpha'' A_\beta'') \end{pmatrix},$$

we have

$$\begin{aligned} \sum (\mathrm{tr}\, A_\lambda A_\mu)^2 &= 2 \left\{ \sum (\mathrm{tr}\, A_\alpha A_\beta)^2 + \sum (\mathrm{tr}\, A_\alpha A_{\beta*})^2 \right\} \\ &= 8 \left[\sum \{\mathrm{tr}(A_\alpha' A_\beta' + A_\alpha'' A_\beta'')\}^2 + \sum \{\mathrm{tr}(A_\alpha' A_\beta'' - A_\alpha'' A_\beta')\}^2 \right] \\ &= 2 \left\{ \sum (\mathrm{tr}\, B_\alpha B_\beta)^2 + \sum (\mathrm{tr}\, B_\alpha B_{\beta*})^2 \right\} = \sum (\mathrm{tr}\, B_\lambda B_\mu)^2. \end{aligned}$$

Therefore Proposition 3.1′ follows from Proposition 3.1. (Q.E.D.)

COROLLARY 3.2. ([23]) *Let M be a complex curve in $\tilde{M}_{1+p}(\tilde{c})$. Then*

$$\frac{1}{2}\Delta \|\sigma\|^2 = \frac{3}{2}\|\sigma\|^2 (\tilde{c} - \|\sigma\|^2) + \|\nabla'\sigma\|^2.$$

Proof. Setting

$$A_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ b_\alpha & -a_\alpha \end{pmatrix} \qquad \text{so that} \quad A_{\alpha*} = JA_\alpha = \begin{pmatrix} -b_\alpha & a_\alpha \\ a_\alpha & b_\alpha \end{pmatrix}$$

we can easily see that

$$\mathrm{tr}\left(\sum A_\alpha^2\right)^2 = 2\left\{\sum (a_\alpha^2 + b_\alpha^2)\right\}^2 = \frac{1}{2}\left(\sum \mathrm{tr}\, A_\alpha^2\right)^2 = \frac{1}{8}\|\sigma\|^4$$

and

$$\begin{aligned} \sum (\mathrm{tr}\, A_\lambda A_\mu)^2 &= 2 \left\{ \sum (\mathrm{tr}\, A_\alpha A_\beta)^2 + \sum (\mathrm{tr}\, A_\alpha A_{\beta*})^2 \right\} \\ &= 8 \sum \{(a_\alpha a_\beta + b_\alpha b_\beta)^2 + (-a_\alpha b_\beta + a_\beta b_\alpha)^2\} \\ &= 8 \sum (a_\alpha^2 + b_\alpha^2)(a_\beta^2 + b_\beta^2) = 2 \sum \mathrm{tr}\, A_\alpha^2\, \mathrm{tr}\, A_\beta^2 \\ &= 2\left(\sum \mathrm{tr}\, A_\alpha^2\right)^2 = \frac{1}{2}\|\sigma\|^4. \end{aligned}$$

These, together with Proposition 3.1, imply

$$\frac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 + \frac{3}{2}\|\sigma\|^2(\tilde{c} - \|\sigma\|^2). \qquad \text{Q.E.D.}$$

COROLLARY 3.3. ([23]) *Let M be a complex curve in $\tilde{M}_2(\tilde{c})$. If $\|\sigma\| \neq 0$, or equivalently $K \neq \tilde{c}$, everywhere on M, then*

$$\Delta \log \|\sigma\|^2 = 3(\tilde{c} - \|\sigma\|^2).$$

Proof. We can choose locally an orthonormal frame field $\{e, Je\}$ with respect to which $A_{\tilde{1}} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ and $A_{\tilde{1}^*} = JA_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$. A straightforward computation yields that

$$\|\nabla'\sigma\|^2 = 8\|da\|^2.$$

On the other hand, since $\|\sigma\|^2 = 4a^2$ and $da^2 = 2ada$, we have

$$\|da\|^2 = \frac{1}{16}\frac{\|d\|\sigma\|^2\|^2}{\|\sigma\|^2}$$

provided that $\|\sigma\| \neq 0$. Hence, from Corollary 3.2, we have

$$\frac{\Delta\|\sigma\|^2}{\|\sigma\|^2} = 3(\tilde{c} - \|\sigma\|^2) + \left\|\frac{d\|\sigma\|^2}{\|\sigma\|^2}\right\|^2.$$

This, together with the fact that

$$\Delta \log \|\sigma\|^2 = \frac{\Delta\|\sigma\|^2}{\|\sigma\|^2} - \left\|\frac{d\|\sigma\|^2}{\|\sigma\|^2}\right\|^2,$$

implies $\Delta \log \|\sigma\|^2 = 3(\tilde{c} - \|\sigma\|^2)$. (Q.E.D.)

We prepare the following result for later use.

LEMMA 3.4. (1) $\mathrm{tr}(\sum A_\alpha{}^2)^2$ *is a geometric invariant, that is, it does not depend on the choice of* $e_{\tilde{1}}, \ldots, e_{\tilde{p}}$.

(2) $\sum(\mathrm{tr}\, A_\lambda A_\mu)^2$ *is a geometric invariant, and for a suitable choice of* $e_{\tilde{1}}, \ldots, e_{\tilde{p}}$, $\mathrm{tr}\, A_\lambda A_\mu = 0$ *for* $\lambda \neq \mu$.

(3) $\sum(\mathrm{tr}\, A_\lambda A_\mu)^2 \leqslant \frac{1}{2}\|\sigma\|^4$.

Proof. (1) is clear from (1.13).

(2) Let $\Lambda = (\mathrm{tr}\, A_\lambda A_\mu)$. Then Λ is a symmetric $(2p, 2p)$-matrix

and it is covariant for the change of $e_{\tilde{1}},\ldots, e_{\tilde{p}}, e_{\tilde{1}^*},\ldots, e_{\tilde{p}^*}$. In other words, let $\Lambda' = (\mathrm{tr}\, A_\lambda' A_\mu')$ be the corresponding matrix with respect to $e_1',\ldots, e_{\tilde{p}}', e_{\tilde{1}^*}',\ldots, e_{\tilde{p}^*}'$ and let $U = (U_{\lambda\mu})$ be the real representation of the unitary matrix given by $e_\mu' = \sum e_\lambda U_{\lambda\mu}$. Then $\Lambda' = {}^tU\Lambda U$. Since

$$\sum (\mathrm{tr}\, A_\lambda A_\mu)^2 = \mathrm{tr}\, \Lambda^2 = \mathrm{tr}({}^tU\Lambda U)^2 = \mathrm{tr}\, \Lambda'^2 = \sum (\mathrm{tr}\, A_\lambda' A_\mu')^2,$$

$\sum (\mathrm{tr}\, A_\lambda A_\mu)^2$ is a geometric invariant.

Moreover $\Lambda' = {}^tU\Lambda U$ implies that Λ can be diagonalized for a suitable choice of $e_{\tilde{1}},\ldots, e_{\tilde{p}}, e_{\tilde{1}^*},\ldots, e_{\tilde{p}^*}$, that is,

$${}^tU\Lambda U = \begin{pmatrix} \mathrm{tr}\, A_{\tilde{1}}'^2 & & & & & 0 \\ & \ddots & & & & \\ & & \mathrm{tr}\, A_{\tilde{p}}'^2 & & & \\ & & & \mathrm{tr}\, A_{\tilde{1}}'^2 & & \\ & & & & \ddots & \\ 0 & & & & & \mathrm{tr}\, A_{\tilde{p}}'^2 \end{pmatrix}$$

for some U. Therefore we have

$$\sum (\mathrm{tr}\, A_\lambda A_\mu)^2 = 2 \sum (\mathrm{tr}\, A_\alpha'^2)^2.$$

(3) From (2) we have

$$\sum (\mathrm{tr}\, A_\lambda A_\mu)^2 = 2 \sum (\mathrm{tr}\, A_\alpha'^2)^2 \leqslant 2 \left(\sum \mathrm{tr}\, A_\alpha'^2\right)^2 = \frac{1}{2} \| \sigma \|^4. \quad \text{(Q.E.D.)}$$

4. Complex Space Forms Immersed in Complex Space Forms

Let an n-dimensional complex space form $M_n(c)$ be immersed as a Kaehler submanifold in an $(n+p)$-dimensional complex space form $\tilde{M}_{n+p}(\tilde{c})$.

First we note that $c \leqslant \tilde{c}$. If $c = \tilde{c}$ then M is totally geodesic in $\tilde{M}$ (Proposition 2.1(b)). From now on we may therefore assume that $c \neq \tilde{c}$.

Since $H = c$ so that $S = ((n+1)/2)\, cg$, from (1.13) we have

$$\sum A_\alpha^2 = \frac{n+1}{4} (\tilde{c} - c) I, \tag{4.1}$$

where I denotes the identity transformation. Moreover since $\rho = n(n+1)c$, from (1.14) we have

$$\|\sigma\|^2 = n(n+1)(\tilde{c} - c). \tag{4.2}$$

From (1.17) we have

$$\|\sigma(X, X)\|^2 = \frac{\tilde{c} - c}{2} \tag{4.3}$$

for every unit vector X.

On the other hand, it is known that every anti-holomorphic sectional curvature of $M_n(c)$ is $c/4$. Therefore

$$K(X, Y) = K(X, JY) = \frac{c}{4}$$

provided that X, Y and JY are orthonormal. This, together with (1.1) and (1.16), implies

$$\|\sigma(X, Y)\|^2 = \frac{\tilde{c} - c}{4} \tag{4.4}$$

for orthonormal X, Y, and JY.

Let $e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}$ be local fields of orthonormal vectors on M as in §1. Then we have the following.

LEMMA 4.1 ([29]). *The $n(n+1)$ local fields of vectors $\sigma(e_a, e_b)$, $\tilde{J}\sigma(e_a, e_b)$, $1 \leqslant a \leqslant b \leqslant n$, are orthogonal.*

This, together with (4.3) and (4.4), implies that $\sigma(e_a, e_b)$, $\tilde{J}\sigma(e_a, e_b)$, $1 \leqslant a \leqslant b \leqslant n$, are linearly independent at each point. Therefore we have the following.

THEOREM 4.2 ([29]). *If $M_n(c)$ is a Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$ and if $p < n(n+1)/2$, then M is totally geodesic in $\tilde{M}$.*

Proposition 2.3 shows that the dimensional restriction in Theorem 4.2

cannot be improved. Theorem 4.2 gives a complete solution for the case $p < n(n+1)/2$.

We consider the case $p \geqslant n(n+1)/2$.

We choose local fields of orthonormal vectors $e_{\tilde{1}}, \dots, e_{\tilde{p}}, e_{\tilde{1}^*}, \dots, e_{\tilde{p}^*}$ normal to M in such a way that

$$e_{\tilde{a}} = \frac{\sqrt{2}}{\sqrt{\tilde{c} - c}}\,\sigma(e_a, e_a)$$

$$e_{\widetilde{(a,b)}} = \frac{2}{\sqrt{\tilde{c} - c}}\,\sigma(e_a, e_b),$$

where

$$(a, b) = \operatorname{Min}\{a, b\} + \frac{|a - b|\,(2n + 1 - |a - b|)}{2} \qquad \text{for} \quad a \neq b.$$

Since

$$\sigma(e_i, e_j) = \sum g(A_\lambda e_i, e_j)\, e_\lambda = \sum h^\lambda_{ij} e_\lambda,$$

we can see the following:

$$
A_{\tilde{a}} = (h^{\tilde{a}}_{ij}) =
\begin{array}{c}
\\ \\ a \\ \\ \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & a & & & & \end{array} \\
\left(\begin{array}{ccc|ccc}
0 & \vdots & & & & \\
 & 0 \;\vdots & & & & \\
\cdots & \circledast & \cdots & & 0 & \\
 & \vdots\; 0 & & & & \\
 & \vdots & 0 & & & \\
\hline
 & & & 0 & \vdots & \\
 & & & & 0\;\vdots & \\
 & 0 & & \cdots & -\circledast & \cdots \\
 & & & & \vdots\; 0 & \\
 & & & & \vdots & 0
\end{array}\right) \\
\begin{array}{ccccccc} & & & & n+a & & \end{array}
\end{array}
\begin{array}{c}
\\ \\ \\ \\ \\ \\ \\ n+a \\ \\ \\
\end{array}
\qquad (4.5)
$$

$$A_{\widetilde{(a,b)}} = (h_{ij}^{\widetilde{(a,b)}}) = \begin{array}{c|cccc|cccc|c}
 & & a & b & & & & & & \\
\hline
a & \cdots & \cdots & * & \cdots & & & & & \\
b & \cdots & * & \cdots & \cdots & & & 0 & & \\
\hline
 & & & & & \cdots & \cdots & -* & \cdots & n+a \\
 & & 0 & & & \cdots & -* & \cdots & \cdots & n+b \\
\hline
 & & & & & & n+a & n+b & &
\end{array}$$

$$A_\alpha = (h_{ij}^\alpha) = 0 \qquad \text{for} \quad \alpha > \widetilde{\frac{n(n+1)}{2}},$$

where

$$\circledast = \frac{\sqrt{\tilde{c}-c}}{\sqrt{2}} \qquad \text{and} \qquad * = \frac{\sqrt{\tilde{c}-c}}{2}.$$

It is easily seen that (4.5) is equivalent to

$$\begin{aligned}
&\omega_a^{\tilde{a}} = \frac{\sqrt{\tilde{c}-c}}{\sqrt{2}}\,\omega^a, \qquad \omega_{a*}^{\tilde{a}} = -\frac{\sqrt{\tilde{c}-c}}{\sqrt{2}}\,\omega^{a*} \\
&\omega_b^{\tilde{a}} = \omega_{b*}^{\tilde{a}} = 0 \qquad (b \neq a) \\
&\omega_a^{\widetilde{(a,b)}} = \frac{\sqrt{\tilde{c}-c}}{2}\,\omega^b, \qquad \omega_{a*}^{\widetilde{(a,b)}} = -\frac{\sqrt{\tilde{c}-c}}{2}\,\omega^{b*} \\
&\omega_c^{\widetilde{(a,b)}} = \omega_{c*}^{\widetilde{(a,b)}} = 0 \qquad (c \neq a,\ c \neq b) \\
&\omega_a^\alpha = \omega_{a*}^\alpha = 0 \qquad \left(\alpha > \widetilde{\frac{n(n+1)}{2}}\right).
\end{aligned} \tag{4.5)'}$$

From (4.5) we have

$$\sum (\operatorname{tr} A_\lambda A_\mu)^2 = 2 \sum (\operatorname{tr} A_\alpha{}^2)^2 = n(n+1)(\tilde{c}-c)^2. \tag{4.6}$$

Therefore, from Proposition 3.1, (4.1), (4.2), and (4.6), we have

$$\| \nabla' \sigma \|^2 = n(n+1)(n+2)(\tilde{c}-c)\left(\frac{\tilde{c}}{2} - c\right). \tag{4.7}$$

In consideration of Theorem 4.2, immediately from (4.7) we obtain the following.

THEOREM 4.3 ([27]). *Let $M_n(c)$ be a Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$. If $\tilde{c} > 0$, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c \leqslant \tilde{c}/2$.*

THEOREM 4.4 ([27]). *Let $M_n(c)$ be a Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$. If the second fundamental form of the immersion is parallel, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \tilde{c}/2$, the latter case arising only when $\tilde{c} > 0$. Moreover the immersion is rigid.*

We must prove the last assertion in Theorem 4.4 (the rigidity of the immersion).

From (3.1), (3.2), (3.3), (3.4) and (4.5) or (4.5)′ we have the following.

LEMMA 4.5 ([28]). *The following three conditions are mutually equivalent*:

(i)
$$\nabla' \sigma = 0$$

(ii)
$$h^\lambda_{ijk} = 0 \qquad \textit{for all} \quad \lambda, i, j, \textit{and } k$$

(iii)
$$\omega^{\tilde{a}}_{\tilde{a}^*} = 2\omega^a_{a^*}, \qquad \omega^{\tilde{a}}_{\tilde{b}} = 0, \qquad \omega^{\tilde{a}}_{\tilde{b}^*} = 0$$

$$\omega^{\widetilde{(a,b)}}_{\tilde{a}} = \sqrt{2}\, \omega^b_a, \qquad \omega^{\widetilde{(a,b)}}_{\tilde{a}^*} = \sqrt{2}\, \omega^b_{a^*}$$

$$\omega^{\tilde{a}}_{\widetilde{(b,c)}} = 0, \qquad \omega^{\tilde{a}}_{\widetilde{(b,c)}^*} = 0$$

$$\omega^{\widetilde{(a,b)}}_{\widetilde{(a,b)}^*} = \omega^a_{a^*} + \omega^b_{b^*}$$

$$\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} = \omega^b_c, \qquad \omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}^*} = \omega^b_{c^*}$$

$$\omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}} = 0, \qquad \omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}^*} = 0,$$

where a, b, c, and d are distinct, and

$$\omega_{\beta}^{\alpha} = \omega_{\beta^*}^{\alpha} = 0$$

if at least one index is greater than $\widetilde{n(n+1)/2}$.

According to the fundamental theorem of submanifolds, condition (iii) in Lemma 4.5 implies the rigidity of the immersion. Therefore the last assertion in Theorem 4.4 (the rigidity of the immersion) follows from Lemma 4.5.

It is well-known that the only complex curves of constant curvature immersed in $\tilde{M}_2(\tilde{c})$ are $M_1(\tilde{c})$ and $M_1(\tilde{c}/2)$, the latter case arising only when $\tilde{c} > 0$. Now we prove the following generalization.

THEOREM 4.6 ([28]). *Let $M_n(c)$ be a Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$. If $p = n(n+1)/2$, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \tilde{c}/2$, the latter case arising only when $\tilde{c} > 0$. Moreover the immersion is rigid.*

Proof. In consideration of Theorem 4.4 it suffices to show that $\nabla'\sigma = 0$. We give here a complex version of the original proof.[2]

From $(1'.9)'$ we have

$$\sum k_{bc}^{\alpha} k_{\bar{a}\bar{d}}^{\bar{\alpha}} = \frac{\tilde{c} - c}{4} (\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd}). \tag{4.8}$$

Let $H = (H_{\beta}{}^{\alpha})$ be the (p, p)-matrix, $p = n(n+1)/2$, defined by

$$H_{a}{}^{\alpha} = k_{aa}^{\alpha}$$
$$H_{\beta}{}^{\alpha} = k_{bc}^{\alpha} \qquad \text{for} \quad \beta = (\widetilde{b, c}).$$

Then (4.8) is equivalent to

$${}^t\bar{H}H = \left(\begin{array}{c|c} 2kI_n & 0 \\ \hline 0 & kI_{n(n-1)/2} \end{array}\right), \tag{4.8$'$}$$

[2] This complex version was suggested by T. Takahashi.

where $k = (\tilde{c} - c)/4$. Applying the covariant differentiation to the both sides of (4.8) or (4.8)′ and using (3.2)″, we obtain

$$\sum k^{\alpha}_{bce} k^{\bar{\alpha}}_{\bar{a}d} = 0 \qquad \text{or} \qquad {}^{t}\bar{H}\nabla'_{\xi_e} H = 0.$$

Since H is nonsingular provided that $c \neq \tilde{c}$, we can deduce

$$k^{\alpha}_{bce} = 0 \qquad \text{or} \qquad \nabla'_{\xi_e} H = 0.$$

This, combined with (3.2)″, implies that k^{α}_{ab} is parallel. (Q.E.D.)

Let

$$K_N = -\sum \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2 = 8 \operatorname{tr}\left(\sum A_\alpha{}^2\right)^2.$$

Then K_N is a geometric invariant (cf. Lemma 3.4) and is called the *scalar normal curvature* of the immersion. We have the following generalization of the result of Chen and Ludden ([7]).

COROLLARY 4.7. *Let M be a complex curve immersed in $\tilde{M}_2(\tilde{c})$. If K_N is constant, then M is either $M_1(\tilde{c})$ or $M_1(\tilde{c}/2)$, the latter case arising only when $\tilde{c} > 0$. Moreover the immersion is rigid.*

Proof. Since $\dim M = 1$, by the same argument as in the proof of Corollary 3.2, we obtain

$$K_N = 8 \operatorname{tr} A_{\bar{1}}^4 = \| \sigma \|^4,$$

which, together with the fact that

$$\| \sigma \|^2 = 2\tilde{c} - \rho = 2(\tilde{c} - K),$$

yields

$$K_N = 4(\tilde{c} - K)^2.$$

This implies that K_N is constant if and only if K is constant. Hence Corollary 4.7 is an immediate consequence of Theorem 4.6. (Q.E.D.)

The following results were proved implicitly by Calabi [6], which give a complete solution in the case of dimension one.

THEOREM 4.8 ([6, 16]). *Let $M_1(c)$ be a Kaehler submanifold immersed* *in $\tilde{M}_m(\tilde{c})$. If M cannot be immersed in any proper totally geodesic submanifold of $\tilde{M}$, then $c = \tilde{c}/m$ and the immersion is rigid.*

COROLLARY 4.9. *Let $M_1(c)$ be a Kaehler submanifold immersed in $\tilde{M}_m(\tilde{c})$. If $\tilde{c} \leqslant 0$, then $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$).*

Most results in this section can be considered as partial generalizations of the following.

THEOREM 4.10 ([5]). *Let $M_n(c)$ and $\tilde{M}_m(\tilde{c})$ be complete and simply-connected complex space forms. Then M can be imbedded in $\tilde{M}$ as a Kaehler submanifold if and only if*

(i) $\tilde{c} = \nu c$ *for some positive integer* ν

and

(ii) $m \geqslant \binom{n+\nu}{\nu} - 1$.

COROLLARY 4.11. *Let $M_n(c)$ and $\tilde{M}_m(\tilde{c})$ be complete and simply-connected complex space forms. If $\tilde{c} \leqslant 0$ and if M is imbedded in $\tilde{M}$ as a Kaehler submanifold, then M is totally geodesic in $\tilde{M}$.*

We conjecture that all the global assumptions in Theorem 4.10 and Corollary 4.11 can be removed.

5. KAEHLER SUBMANIFOLDS OF CONSTANT SCALAR CURVATURE IMMERSED IN COMPLEX SPACE FORMS

We consider Kaehler submanifolds, which are Einstein or more generally of constant scalar curvature, immersed in complex space forms.

The following result is well-known.

THEOREM 5.1 ([8, 30]). *Let M_n be a Kaehler hypersurface immersed in $\tilde{M}_{n+1}(\tilde{c})$. If $n \geqslant 2$ and if M is Einstein, then either M is totally geodesic in $\tilde{M}$ or $S = (n/2)\,\tilde{c}g$, the latter case arising only when $\tilde{c} > 0$. Moreover the immersion is rigid.*

Proof. Since M is Einstein so that $S = (\rho/2n)g$, from (1.13) and (1.14) we have

$$A_1^2 = \frac{1}{4n}\{n(n+1)\tilde{c} - \rho\}I = \frac{1}{4n}\|\sigma\|^2 I. \tag{5.1}$$

Therefore, for a suitable choice of $e_1, \ldots, e_n$, we can assume

$$A_{\tilde{1}} = (h^{\tilde{1}}_{ij}) = \begin{bmatrix} \lambda & & & & & \\ & \ddots & & & 0 & \\ & & \lambda & & & \\ & & & -\lambda & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda \end{bmatrix}, \tag{5.2}$$

where $\lambda = 1/(2\sqrt{n}) \| \sigma \|$.

If $\| \sigma \| = 0$, then M is totally geodesic in $\tilde{M}$. From now on we therefore assume that $\| \sigma \| \neq 0$.

From (3.1), (3.2), (3.3), and (5.2) we can deduce

$$h^{\mathbf{r}}_{ijk} = h^{\tilde{1}^*}_{ijk} = 0 \qquad \textit{for all} \quad i, j, \textit{ and } k \tag{5.3}$$

$$\omega^{\tilde{1}}_{\tilde{1}^*} = 2\omega^a_{a^*}. \tag{5.4}$$

It is clear that (5.3) is equivalent to

$$\nabla' \sigma = 0. \tag{5.3$'$}$$

On the other hand, since $\| \sigma \|$ is constant, from Proposition 3.1 and (5.1) we have

$$\| \nabla' \sigma \|^2 = \frac{n+2}{2n} \| \sigma \|^2 (\| \sigma \|^2 - n\tilde{c}).$$

This, together with (5.3)$'$, implies that $\| \sigma \|^2 = n\tilde{c}$ holds only when $\tilde{c} > 0$, or equivalently $S = (n/2)\,\tilde{c}g$ holds only when $\tilde{c} > 0$. Moreover the rigidity of the immersion follows from (5.4). (Q.E.D.)

A partial generalization of Theorem 5.1 is given in §6 (Corollary 6.2). The following result is useful.

LEMMA 5.2 ([4]). *If a complete Kaehler manifold satisfies*

(i) $K > 0$ *or* $c/2 < H \leqslant c$

(ii) ρ *is constant,*

then H *is constant.*

Immediately from Theorem 4.2 and Lemma 5.2 we have the following.

THEOREM 5.3. *Let M_n be a complete Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$, $\tilde{c} > 0$. If*

(i) $K > 0$

(ii) *ρ is constant*

(iii) $p < n(n+1)/2$,

then M is totally geodesic in $\tilde{M}$.

The following result follows immediately from Theorem 4.3 and Lemma 5.2.

THEOREM 5.4 ([25]). *Let M_n be a complete Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$, $\tilde{c} > 0$. If $H > \tilde{c}/2$ and if ρ is constant, then M is totally geodesic in $\tilde{M}$.*

6. POSITIVELY CURVED KAEHLER SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

We consider some Kaehler submanifolds of an elliptic complex space form. Without loss of generality we may assume that the ambiant manifold is an $(n+p)$-dimensional complex projective space $P_{n+p}(C)$ of constant holomorphic sectional curvature 1. Let M_n be an n-dimensional Kaehler submanifold of $P_{n+p}(C)$. In this situation we have had the following natural conjectures:

(I) If $H > 1/2$, then M is totally geodesic in $P_{n+p}(C)$.

(II) If $K > 0$ and if $p < n(n+1)/2$, then M is totally geodesic in $P_{n+p}(C)$.

(III) If $S > (n/2)g$, then M is totally geodesic in $P_{n+p}(C)$.

(IV) If $\rho > n^2$, then M is totally geodesic in $P_{n+p}(C)$.

Under a suitable topological restriction (for example, M is complete), these conjectures seem to be true. We have a complete solution for Conjecture (III), but only partial solutions for others.

The following result gives a complete solution for Conjecture (III):

THEOREM 6.1 ([26]). *Let M_n be an n-dimensional complete Kaehler* *submanifold immersed in* $P_{n+p}(C)$. *If every Ricci curvature of* M *is greater than* $n/2$, *then* M *is totally geodesic in* $P_{n+p}(C)$.

Proof. First we note that, by a theorem of Myers ([17]), M is compact. Since $S - (n/2)g$ is positive definite, we can see from (1.13) that $I - 4\sum A_\alpha^2$ is positive definite. This implies

$$\|\sigma\|^2 < n. \tag{6.1}$$

Moreover, since A_α's are symmetric linear transformations, $\sum A_\alpha^2$ is positive semi-definite. Since $\sum A_\alpha^2$ and $I - 4\sum A_\alpha^2$ can be transformed simultaneously by an orthogonal matrix into diagonal forms at each point of M, $(\sum A_\alpha^2)(I - 4\sum A_\alpha^2)$ is positive semi-definite. Hence we have

$$8 \operatorname{tr}\left(\sum A_\alpha^2\right)^2 \leqslant \|\sigma\|^2. \tag{6.2}$$

From Proposition 3.1, Lemma 3.4(3), (6.1), and (6.2) we have

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \frac{1}{2}\|\sigma\|^2(n - \|\sigma\|^2) \geqslant 0. \tag{6.3}$$

Hence, by a well-known theorem of E. Hopf, $\|\sigma\|^2$ is a constant so that $\Delta\|\sigma\|^2 = 0$. This, together with (6.1) and (6.3), implies $\|\sigma\| = 0$. Therefore M is totally geodesic. (Q.E.D.)

As an immediate consequence of Theorem 6.1, we have a partial generalization of Theorem 5.1:

COROLLARY 6.2. *Let* M_n *be an* n*-dimensional compact Kaehler submanifold immersed in* $P_{n+p}(C)$. *If* $n \geqslant 2$ *and if* M *is Einstein, then either* M *is totally geodesic in* $P_{n+p}(C)$ *or* $S \leqslant (n/2)g$.

In the case of $n = 1$, Theorem 6.1 gives the best possible solution for Conjecture (I), that is, we have the following.

COROLLARY 6.3 ([23, 32]). *Let* M *be a complete complex curve immersed in* $P_{1+p}(C)$. *If* $K > 1/2$ *everywhere on* M, *then* M *is a totally geodesic curve (i.e., a complex projective line) in* $P_{1+p}(C)$.

This can also be proved directly from Corollary 3.2.

The following two theorems give characterizations of a complex quadric $Q_1(C)$.

THEOREM 6.4 ([18]). *Let M be a compact complex curve immersed in $P_2(C)$. If $K \leqslant 1/2$ or $1/2 \leqslant K < 1$ everywhere on M, then M is complex analytically isometric to $Q_1(C)$, and hence $K = 1/2$. Moreover the immersion is rigid.*

Proof. Since
$$\| \sigma \|^2 = 2 - \rho = 2(1 - K),$$
$K \leqslant 1/2$ or $1/2 \leqslant K < 1$ is equivalent to $1 \leqslant \| \sigma \|^2$ or $0 < \| \sigma \|^2 \leqslant 1$. Therefore Corollary 3.3 and a well-known theorem of E. Hopf imply that $\| \sigma \|^2$ is constant so that $\| \sigma \|^2 = 1$. Hence $K = 1/2$, which implies that M is complex analytically isometric to $Q_1(C)$. The rigidity of the immersion follows from Theorem 4.6. (Q.E.D.)

THEOREM 6.5 ([23, 32]). *Let M be a complete complex curve immersed in $P_{1+p}(C)$. If $1/2 \leqslant K < 1$ everywhere on M, then M is complex analytically isometric to $Q_1(C)$ and hence $K = 1/2$. Moreover the immersion is rigid.*

Proof. Since $\| \sigma \|^2 = 2(1 - K)$, $1/2 \leqslant K < 1$ is equivalent to $0 < \| \sigma \|^2 \leqslant 1$. Therefore Corollary 3.2 and a well-known theorem of E. Hopf imply that $\| \sigma \|^2 = 1$ and $\nabla' \sigma = 0$. Hence $K = 1/2$ so that M is complex analytically isometric to $Q_1(C)$. The rigidity of the immersion follows from Theorem 4.4. (Q.E.D.)

For an imbedded (or non-singular) curve, there is the following result.

THEOREM 6.6 ([16]). *Let M be a complete nonsingular complex curve in $P_m(C)$. If $1/k < K \leqslant 1/(k-1)$ (resp. $1/k \leqslant K < 1/(k-1)$) everywhere on M for some integer k, $1 < k \leqslant m$, then $K = 1/(k-1)$ (resp. $K = 1/k$) and the imbedding is rigid.*

We can obtain the following two results from Theorem 6.1, which are partial solutions for Conjectures (I) and (II), respectively.

THEOREM 6.7 ([26]). *Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $H > \delta$, then M is totally geodesic in $P_{n+p}(C)$, where*
$$\delta = \begin{cases} \dfrac{3n-1}{3n+1} & (n \leqslant 5) \\[2mm] \dfrac{2n-3}{2n-2} & (n > 5). \end{cases}$$

Proof. This is an immediate consequence of Theorem 6.1 and the following lemma.

LEMMA ([3]). *If $\delta < H \leqslant 1$, then every Ricci curvature of M is greater than μ, where*

$$\mu = \begin{cases} \dfrac{(3n+1)\delta - (n-1)}{4} & (n \leqslant 5) \\ (n-1)\delta - \dfrac{n-3}{2} & (n > 5). \end{cases}$$

(Q.E.D.)

THEOREM 6.8 ([26]). *Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $n \geqslant 2$ and if $K > \delta$, then M is totally geodesic in $P_{n+p}(C)$, where*

$$\delta = \begin{cases} \dfrac{5}{23} & (n = 5) \\ \dfrac{5n - 2 - \sqrt{9n^2 + 60n + 4}}{8(n-5)} & (n \neq 5). \end{cases}$$

Proof. We have the following

LEMMA ([2]). *If $n \geqslant 2$ and if $\delta < K \leqslant 1$, then*

$$\frac{\delta(8\delta + 1)}{1 - \delta} < H.$$

On the other hand, let x be an arbitrary point of M and X be an arbitrary unit vector in $T_x(M)$. If $e_1 = X, e_2, \ldots, e_n, Je_1, \ldots, Je_n$ is an orthonormal basis of $T_x(M)$, then

$$S(X, X) = H(X) + \sum_{a=2}^{n} \{K(X, e_a) + K(X, Je_a)\}.$$

Hence, by Lemma, $K > \delta$ implies

$$S(X, X) > \frac{\delta(8\delta + 1)}{1 - \delta} + 2(n-1)\delta.$$

We can see that if

$$\delta = \begin{cases} \dfrac{5}{23} & (n = 5) \\ \dfrac{5n - 2 - \sqrt{9n^2 + 60n + 4}}{8(n - 5)} & (n \neq 5), \end{cases}$$

then

$$S(X, X) > \frac{n}{2}.$$

This, combined with Theorem 6.1, completes the proof. (Q.E.D.)

We have some more partial solutions for Conjecture (I).

THEOREM 6.9 ([25]). *Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If*

$$H > 1 - \frac{n + 2}{2(n + 2p)},$$

then M is totally geodesic in $P_{n+p}(C)$.

Proof. First we note that by a theorem of Tsukamoto ([34]) M is compact. From (1.17) we can see that if $H > 1 - \delta$, then the square of every eigenvalue of A_λ must be smaller than $\delta/2$. Therefore we have

$$\operatorname{tr} A_\lambda{}^2 A_\mu{}^2 \leqslant \frac{\delta}{2} \operatorname{tr} A_\lambda{}^2 \tag{6.4}$$

for all λ and μ. From (6.4) we have

$$\operatorname{tr}\left(\sum A_\alpha{}^2\right)^2 = \sum \operatorname{tr} A_\alpha{}^2 A_\beta{}^2 \leqslant \frac{p\delta}{2} \sum \operatorname{tr} A_\alpha{}^2 = \frac{p\delta}{4} \| \sigma \|^2. \tag{6.5}$$

On the other hand, from Lemma 3.4(2) we have

$$\begin{aligned} \sum (\operatorname{tr} A_\lambda A_\mu)^2 &= \operatorname{tr} A^2 = \operatorname{tr}({}^tUAU)^2 = 2 \sum (\operatorname{tr} A_\alpha'^2)^2 \\ &\leqslant 4n \sum \operatorname{tr} A_\alpha'^4, \end{aligned}$$

where we use the general fact that a symmetric $(2n, 2n)$-matrix A satisfies $(\operatorname{tr} A^2)^2 \leqslant 2n \operatorname{tr} A^4$. This, together with (6.4), implies

$$\sum (\operatorname{tr} A_\lambda A_\mu)^2 \leqslant n\delta \| \sigma \|^2. \tag{6.6}$$

Proposition 3.1, (6.5), and (6.6) imply

$$\frac{1}{2}\Delta \| \sigma \|^2 \geqslant \left\{ \frac{n+2}{2} - (n+2p)\delta \right\} \| \sigma \|^2.$$

Therefore if $\delta = (n+2)/2(n+2p)$, then, by a well-known theorem of E. Hopf, $\| \sigma \|^2$ is constant so that ρ is constant. Moreover we have

$$H > 1 - \delta = 1 - \frac{n+2}{2(n+2p)} \geqslant \frac{1}{2}.$$

Therefore Theorem 5.4 implies that M is totally geodesic. (Q.E.D.)

For hypersurfaces, Theorem 6.9 gives the best solution for Conjecture (I).

COROLLARY 6.10. *Let M_n be a complete Kaehler hypersurface immersed in $P_{n+1}(C)$. If $H > 1/2$, then M is totally geodesic in $P_{n+1}(C)$.*

The following theorem gives a partial solution for Conjecture (II).

THEOREM 6.11 ([24]). *Let M_n be a complete Kaehler hypersurface immersed in $P_{n+1}(C)$. If $n \geqslant 4$ and if $K > 0$, then M is totally geodesic in $P_{n+1}(C)$.*

Proof. We prove in Proposition 6.12 that M is compact.

At each point x of M, we can choose an orthonormal basis $e_1, \ldots, e_n$, $Je_1, \ldots, Je_n$ of $T_x(M)$ with respect to which the matrix of $A_{\tilde{1}}$ is of the form

$$\begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \lambda_n & & & & \\ & & & -\lambda_1 & & & \\ & 0 & & & \ddots & & \\ & & & & & -\lambda_n \end{bmatrix}$$

so that

$$\| \sigma \|^2 = 2 \operatorname{tr} A_{\tilde{1}}^{\,2} = 4 \sum \lambda_a^{\,2}.$$

From (1.16) we have, for $a \neq b$,

$$K\left(\frac{e_a + e_b}{\sqrt{2}}, \frac{Je_a - Je_b}{\sqrt{2}}\right) = \frac{1}{4} - \frac{\lambda_a^2 + \lambda_b^2}{2}.$$

Since $K > 0$ we have

$$\lambda_a^2 + \lambda_b^2 < \frac{1}{2} \qquad \text{for} \quad a \neq b. \tag{6.7}$$

From (6.7) we have

$$\lambda_a^4 + \lambda_a^2\lambda_b^2 \leqslant \frac{1}{2}\lambda_a^2$$

and hence

$$(n-1)\sum \lambda_a^4 + \sum_{a\neq b} \lambda_a^2\lambda_b^2 \leqslant \frac{n-1}{2}\sum \lambda_a^2,$$

or

$$(n-2)\sum \lambda_a^4 + \left(\sum \lambda_a^2\right) \leqslant \frac{n-1}{2}\sum \lambda_a^2.$$

Therefore we have

$$(n-2)\operatorname{tr} A_{\bar{1}}^4 + \frac{1}{2}(\operatorname{tr} A_{\bar{1}}^2)^2 \leqslant \frac{n-1}{2}\operatorname{tr} A_{\bar{1}}^2,$$

that is,

$$(n-2)\operatorname{tr} A_{\bar{1}}^4 + \frac{1}{8}\|\sigma\|^4 \leqslant \frac{n-1}{4}\|\sigma\|^2. \tag{6.8}$$

This, together with Proposition 3.1, implies

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \frac{n-4}{2(n-2)}\|\sigma\|^2(n - \|\sigma\|^2).$$

On the other hand, we can see from (6.7) that

$$\sum \lambda_a^2 < \frac{n}{4}, \qquad \text{i.e.,} \qquad \|\sigma\|^2 < n.$$

Therefore we have $\Delta\|\sigma\|^2 \geqslant 0$. Hence, by a well-known theorem of E. Hopf, $\|\sigma\|^2$ is constant so that ρ is constant. Therefore Theorem 5.3 implies that M is totally geodesic in $P_{n+1}(C)$. (Q.E.D.)

PROPOSITION 6.12 ([23]). *Let M be a complete Kaehler hypersurface* *immersed in $P_{n+1}(C)$. If $H \geqslant \delta > (1-n)/2$ for some constant δ, then M is compact.*

Proof. Let $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ be an orthonormal basis of $T_x(M)$ as in the proof of Theorem 6.11. For $X = \sum X^a e_a + \sum X^{a^*} Je_a$, we have from (1.13) that

$$S(X, X) = \frac{n+1}{2} g(X, X) - 2 \sum \lambda_a^2 (X^a X^a + X^{a^*} X^{a^*}).$$

On the other hand, (1.17) implies

$$H(e_a) = 1 - 2\lambda_a^2.$$

Since $H \geqslant \delta$, we have $2\lambda_a^2 \leqslant 1 - \delta$ so that

$$S(X, X) \geqslant \left(\frac{n-1}{2} + \delta\right) g(X, X).$$

This, together with a theorem of Myers ([17]), implies that M is compact. (Q.E.D.)

The following result gives a partial solution for Conjecture (IV).

THEOREM 6.13 ([32]). *Let M_n be an n-dimensional compact Kaehler submanifold immersed in $P_{n+p}(C)$. If $\rho > n(n+1) - (n+2)/3$ everywhere on M, then M is totally geodesic in $P_{n+p}(C)$.*

Proof. First we can prove

$$2 \operatorname{tr} A_\alpha^2 A_\beta^2 \leqslant (\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2). \tag{6.9}$$

In fact, since A's are of the form

$$\begin{pmatrix} B & C \\ C & -B \end{pmatrix},$$

we may assume that

$$A_\alpha^2 = \begin{bmatrix} \lambda_1^2 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_n^2 & & & & \\ & & & \lambda_1^2 & & & \\ & & & & \ddots & & \\ 0 & & & & & & \lambda_n^2 \end{bmatrix} \quad \text{and} \quad A_\beta^2 = \left(\begin{array}{c|c} a_{ab} & * \\ \hline * & a_{ab} \end{array}\right)$$

with $a_{aa} \geqslant 0$. Therefore

$$(\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2) - 2 \operatorname{tr} A_\alpha^2 A_\beta^2 = 4 \left\{ \left(\sum \lambda_a^2\right)\left(\sum a_{aa}\right) - \left(\sum \lambda_a^2 a_{aa}\right) \right\} \geqslant 0.$$

From (6.9) we have

$$\begin{aligned} \operatorname{tr} \left(\sum A_\alpha^2\right)^2 &= \sum \operatorname{tr} A_\alpha^2 A_\beta^2 \leqslant \frac{1}{2} \sum (\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2) = \frac{1}{2}\left(\sum \operatorname{tr} A_\alpha^2\right)^2 \\ &= \frac{1}{8} \| \sigma \|^4. \end{aligned} \tag{6.10}$$

Since $\rho > n(n+1) - (n+2)/3$ so that $\| \sigma \|^2 < (n+2)/3$ everywhere on M, Proposition 3.1, Lemma 3.4(3), and (6.10) imply that

$$\Delta \| \sigma \|^2 \geqslant \| \sigma \|^2 (n + 2 - 3 \| \sigma \|^2) \geqslant 0$$

holds everywhere on M. Hence, by a well-known theorem of E. Hopf, $\| \sigma \|^2$ is constant so that $\| \sigma \| = 0$. (Q.E.D.)

Let

$$\nu(x) = \dim_C\{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_x(M)\}$$

and

$$\nu = \operatorname*{Min}_{x \in M} \nu(x).$$

The following result is a partial solution for Conjecture (I) and Conjecture (II).

THEOREM 6.14 ([1]). *Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $2\nu \geqslant n$ and if $K > 0$ or $H > 1/2$, then M is totally geodesic in $P_{n+p}(C)$.*

Outline of Proof. The set

$$M' = \{x \in M \mid \nu(x) = \nu\}$$

is an open subset of M.

Let $\mathscr{D}$ be a distribution on M' defined by

$$x \to \{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_x(M)\}.$$

Then $\mathscr{D}$ is differentiable and involutive. Every integral manifold of $\mathscr{D}$

is a ν-dimensional complete Kaehler submanifold of M of constant holomorphic sectional curvature 1.

Let D_1 and D_2 be two integral manifolds of $\mathscr{D}$. Since $2\nu \geqslant n$ and $K > 0$ or $H > 1/2$, a theorem of Goldberg–Kobayashi ([10]) implies $D_1 \cap D_2 \neq \varnothing$. This is possible only when $\nu = n$, that is, M is totally geodesic.

7. Algebraic Manifolds

By a well-known theorem of Chow, a compact complex manifold is an algebraic manifold if it admits a complex analytic *imbedding* as a submanifold of a complex projective space of some dimension. In this section we consider some differential geometric properties of algebraic manifolds.

First we prove the following theorem of Gauss–Bonnet type.

Theorem 7.1 ([22]). *Let M_n be an n-dimensional compact Kaehler submanifold imbedded in $P_{n+p}(C)$. If M is a complete intersection of p non-singular hypersurfaces of degree $a_1, \ldots, a_p$ in $P_{n+p}(C)$, then*

$$\int_M \rho * 1 = n \left(n + p + 1 - \sum a_\alpha\right)\left(\prod a_\alpha\right) \frac{(4\pi)^n}{n!},$$

where $*1$ denotes the volume element of M.

Proof. Let $\tilde{h}$ be the generator of $H^2(P_{n+p}(C), Z)$ corresponding to the divisor class of a hyperplane $P_{n+p-1}(C)$. Then the first Chern class $c_1(P_{n+p}(C))$ of $P_{n+p}(C)$ is given by

$$c_1(P_{n+p}(C)) = (n + p + 1)\tilde{h}. \tag{7.1}$$

Let j: $M \to P_{n+p}(C)$ be the imbedding and let h be the image of $\tilde{h}$ under the homomorphism j^*: $H^2(P_{n+p}(C), Z) \to H^2(M, Z)$. Then the first Chern class $c_1(M)$ of M is given by

$$c_1(M) = \left(n + p + 1 - \sum a_\alpha\right) h. \tag{7.2}$$

Let Φ be the fundamental 2-form of M, that is, a closed 2-form defined by

$$\Phi(X, Y) = \tfrac{1}{2}\, g(JX, Y).$$

Let γ be the Ricci 2-form of M, that is, a closed 2-form defined by

$$\gamma(X, Y) = \frac{1}{4\pi} S(JX, Y).$$

Then the first Chern class $c_1(M)$ of M is represented by γ. We denote $[\Phi]$ and $[\gamma]$ to be the cohomology classes represented by Φ and γ, respectively, so that $c_1(M) = [\gamma]$.

Let $\tilde{\Phi}$ be the fundamental 2-form of $P_{n+p}(C)$. Since the Ricci tensor $\tilde{S}$ of $P_{n+p}(C)$ is given by

$$\tilde{S} = \frac{n+p+1}{2}\tilde{g},$$

the Ricci 2-form $\tilde{\gamma}$ of $P_{n+p}(C)$ satisfies

$$\tilde{\gamma} = \frac{n+p+1}{4\pi}\tilde{\Phi}.$$

Therefore we have

$$c_1(P_{n+p}(C)) = \frac{n+p+1}{4\pi}[\tilde{\Phi}]. \tag{7.3}$$

Since $\Phi = j * \tilde{\Phi}$, (7.1), (7.2) and (7.3) imply

$$c_1(M) = \frac{n+p+1-\sum a_\alpha}{4\pi}[\Phi].$$

Hence there exists a 1-form η which satisfies

$$\gamma = \frac{n+p+1-\sum a_\alpha}{4\pi}\Phi + d\eta. \tag{7.4}$$

Let Λ be the operator of interior product by Φ. Applying Λ to the both sides of (7.4) we have

$$\frac{\rho}{4\pi} = \frac{n(n+p+1-\sum a_\alpha)}{4\pi} + \Lambda\, d\eta$$

since $\Lambda\Phi = *(\Phi \wedge *\Phi) = n$ and $\Lambda\gamma = *(\Phi \wedge *\gamma) = \rho/4\pi$.

Let δ be the codifferential operator and let μ be the operator defined by $\mu\alpha = (r-s)\sqrt{-1}\,\alpha$, where α is a form of type (r, s). Then, using the well-known identity $d\Lambda - \Lambda d = \delta\mu - \mu\delta$, we have $\Lambda d\eta = -\delta\mu\eta$ since $d\Lambda\eta = \mu\delta\eta = 0$. Therefore we have

$$\frac{\rho}{4\pi} = \frac{n(n+p+1-\sum a_\alpha)}{4\pi} - \delta\mu\eta.$$

Integrating the both sides of the equation on M, we obtain

$$\int_M \rho * 1 = n\left(n+p+1-\sum a_\alpha\right)\int_M * 1. \tag{7.5}$$

Let $P_p(C)$ be a p-dimensional linear subspace of $P_{n+p}(C)$, and let ν be the number of points in $M \cap P_p(C)$. Then the dimension theorem for algebraic manifolds states that ν does not depend on the choice of $P_p(C)$ if $P_p(C)$ is in general position. By a theorem of Wirtinger ([35]), the volume of M is given by

$$\int_M * 1 = \nu \frac{(4\pi)^n}{n!}.$$

On the other hand, since M is a complete intersection, we have ([11])

$$\nu = \prod a_\alpha .$$

Therefore we obtain

$$\int_M * 1 = \left(\prod a_\alpha\right)\frac{(4\pi)^n}{n!},$$

which, combined with (7.5), completes the proof. (Q.E.D.)

Remark. Theorem 7.1 implies that the integral of the scalar curvature depends only on the degree of M. But the scalar curvature itself depends wholly on the equations defining M. For example: Let

$$M = \{(z_0, \dots, z_{n+1}) \in P_{n+1}(C) \mid z_0{}^2 + \cdots + z_n{}^2 + a z_{n+1}^2 = 0\},$$

where $z_0, \dots, z_{n+1}$ are homogeneous coordinates in $P_{n+1}(C)$. Then we have (cf. Theorem 7.4)

$$n^2 + 1 - a^2 \leqslant \rho \leqslant n(n+1) - \frac{n}{a} \qquad (a \geqslant 1)$$

$$n(n+1) - \frac{n}{a} \leqslant \rho \leqslant n^2 + 1 - a^2 \qquad (0 < a \leqslant 1).$$

Immediately from Theorem 7.1, we have the following result which is a partial solution for Conjecture (IV) in §6.

THEOREM 7.2 ([22]). *Let M_n be an n-dimensional compact Kaehler submanifold imbedded in $P_{n+p}(C)$. If M is a complete intersection and if $\rho > n^2$ everywhere on M, then M is totally geodesic in $P_{n+p}(C)$.*

Proof. Since $\rho > n^2$ everywhere on M, from Theorem 7.1 or (7.5) we have

$$n^2 \int_M *1 < n\left(n + p + 1 - \sum a_\alpha\right) \int_M *1,$$

which implies $\sum a_\alpha < p + 1$, that is, $a_1 = \cdots = a_p = 1$. Therefore M is a linear subspace of $P_{n+p}(C)$. (Q.E.D.)

The following result is also an immediate consequence of Theorem 7.1 or (7.5).

THEOREM 7.3. *Let M be an algebraic curve. If M is a complete intersection and if $K > 0$, then M is either a line or a complex quadric.*

Since an algebraic manifold is represented as the set of zeros of some homogeneous polynomials in homogeneous coordinates, the following problem arises naturally.

Problem. Find formulae of calculating some differential geometric invariants (curvature, Ricci curvature, scalar curvature, etc.) for algebraic manifolds.

There are few results in this direction. For a hypersurface we can compute the scalar curvature. In fact, we have the following.

THEOREM 7.4 ([23]) *Let M be a compact Kaehler hypersurface of $P_{n+1}(C)$ defined by a homogeneous equation $F(z_0, \ldots, z_{n+1}) = 0$. Then*

$$\rho = n(n+1) - \left(\sum z_i \bar{z}_i\right)\left\{\frac{\operatorname{tr} A\bar{A}}{{}^t\bar{\mathfrak{U}}\mathfrak{U}} - 2\frac{{}^t(\bar{A}\mathfrak{U})(A\bar{\mathfrak{U}})}{({}^t\bar{\mathfrak{U}}\mathfrak{U})^2} + \frac{({}^t\bar{\mathfrak{U}}A\bar{\mathfrak{U}})({}^t\mathfrak{U}\bar{A}\mathfrak{U})}{({}^t\bar{\mathfrak{U}}\mathfrak{U})^3}\right\},$$

where $\mathfrak{U} = (\partial F/\partial z_i)$ and $A = (\partial^2 F/\partial z_i \partial z_j)$, $i, j = 0, 1, \ldots, n+1$.

Proof. Let $w_1, \ldots, w_{n+1}$ and $x_1, \ldots, x_n$ be local coordinate systems in $P_{n+1}(C)$ and M, respectively, with respect to which M is represented locally by

$$w_A = f_A(x_1, \ldots, x_n),$$

where $A = 1, 2, \ldots, n+1$.

Let $g = \sum g_{a\bar{b}}\, dx_a \otimes d\bar{x}_b$ and $(g^{a\bar{b}})$ be the inverse matrix of $(g_{a\bar{b}})$, and let $\tilde{g} = \sum \tilde{g}_{A\bar{B}}\, dw_A \otimes d\bar{w}_B$ and $(\tilde{g}^{A\bar{B}})$ be the inverse matrix of $(\tilde{g}_{A\bar{B}})$. Then we can see that

$$\sum \tilde{g}^{A\bar{B}} \frac{\partial \bar{F}}{\partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial \bar{w}_B} \frac{\partial}{\partial w_A}$$

is a vector normal to M. Let ξ be the unit vector in this direction and let

$$k_{ab} = \tilde{g}\left(\sigma\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}\right), \xi\right).$$

Then we have the following relation for some α:

$$\alpha k_{ab} = -\sum \frac{\partial^2 F}{\partial z_i\, \partial z_j}\left(\frac{\partial z_i}{\partial w_A} + z_i \varphi_A\right)\left(\frac{\partial z_j}{\partial w_B} + z_j \varphi_B\right) \frac{\partial f_A}{\partial x_a} \frac{\partial f_B}{\partial x_b},$$

where

$$\varphi_A = -\frac{\partial}{\partial w_A} \log \sum z_i \bar{z}_i \qquad \text{and} \qquad \alpha\bar{\alpha} = \left(\sum z_i \bar{z}_i\right)\left(\sum \frac{\partial F}{\partial z_i} \frac{\partial \bar{F}}{\partial \bar{z}_i}\right).$$

Therefore, by a straightforward calculation, we have

$$\begin{aligned} \|\sigma\|^2 &= 4 \sum g^{a\bar{c}} g^{b\bar{d}} k_{ab} \bar{k}_{cd} \\ &= \left(\sum z_i \bar{z}_i\right)\left\{\frac{\operatorname{tr} A\bar{A}}{{}^t\bar{\mathfrak{U}}\mathfrak{U}} - 2\,\frac{{}^t(\bar{A}\mathfrak{U})(A\bar{\mathfrak{U}})}{({}^t\bar{\mathfrak{U}}\mathfrak{U})^2} + \frac{({}^t\bar{\mathfrak{U}}A\bar{\mathfrak{U}})({}^t\mathfrak{U}\bar{A}\mathfrak{U})}{({}^t\bar{\mathfrak{U}}\mathfrak{U})^3}\right\}. \end{aligned}$$

Q.E.D.

The following result gives a partial solution for Conjecture (II) in §6 (cf. Theorem 6.11).

THEOREM 7.5 ([23]). *Let M be a complete Kaehler hypersurface imbedded in $P_{n+1}(C)$. If $n \geqslant 2$ and if $K > 0$, then M is totally geodesic in $P_{n+1}(C)$.*

Proof. By the same argument as in the proof of Theorem 6.11, we can deduce $\|\sigma\|^2 < n$, that is, $\rho > n^2$. Since M is compact (cf. Proposition 6.12) so that it is an algebraic hypersurface in $P_{n+1}(C)$, Theorem 7.2 implies that M is totally geodesic. (Q.E.D.)

Remark. On account of $Q_1(C)$ in $P_2(C)$, the assumption $n \geqslant 2$ in Theorem 7.5 cannot be removed.

The following result gives a generalization of Theorem 5.1 in the case of algebraic hypersurfaces.

THEOREM 7.6 ([12]). *Let M be a compact Kaehler hypersurface imbedded in $P_{n+1}(C)$. If ρ is constant, then either M is totally geodesic in $P_{n+1}(C)$ or M is complex analytically isometric to $Q_n(C)$ in $P_{n+1}(C)$.*

Proof. Let Φ and γ be the fundamental 2-form and the Ricci 2-form of M, respectively. By the same argument as in the proof of Theorem 7.1, we can deduce that

$$[\gamma] = k[\Phi]$$

holds for some constant k.

On the other hand, since ρ is constant, by the harmonic integral theory we can see that γ is a harmonic form. Hence

$$\gamma = k\Phi$$

holds so that M is Einstein provided that $n \geqslant 2$. This, combined with Theorem 5.1 yields Theorem 7.6 for $n \geqslant 2$.

For $n = 1$, Theorem 7.6 reduces to a special case of Theorem 4.6. (Q.E.D.)

Using the vanishing theorem of Kodaira, Kobayashi and Ochiai proved the following result which gives among others a partial solution for Conjecture (I) and Conjecture (II) in §6.

THEOREM 7.7 ([14, 15]). *Let M_n be an n-dimensional complete intersection submanifold of $P_{n+p}(C)$. If $n \geqslant 2$ and if M admits a Kaehler metric of positive holomorphic bisectional curvature,*[3] *then M is a linear subspace of $P_{n+p}(C)$.*

Theorem 7.7 is a result from the viewpoint (A) in the Introduction.

[3] If $K > 0$ or $c/2 < H \leqslant c$, then the holomorphic bisectional curvature is positive.

8. Problems

1. Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $H > 1/2$, is M totally geodesic in $P_{n+p}(C)$? [Some partial solutions are given in Theorem 5.4, Corollary 6.3, Theorem 6.7, Theorem 6.9, Corollary 6.10 and Theorem 6.14.]

2. Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $K > 0$ and if $p < n(n+1)/2$, is M totally geodesic in $P_{n+p}(C)$? [Some partial solutions are given in Theorem 6.8, Theorem 6.11, and Theorem 7.5.]

3. Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$, is M totally geodesic in $P_{n+p}(C)$? [Some partial solutions are given in Theorem 6.13 and Theorem 7.2.]

4. If $M_n(c)$ is a Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$ and if $\tilde{c} \leqslant 0$, is M totally geodesic in M? [Some partial solutions are given in Theorem 4.2, Theorem 4.4, Theorem 4.6, Corollary 4.9, and Corollary 4.11.]

5. Let $M_n(c)$ be a Kaehler submanifold immersed in $\tilde{M}_m(\tilde{c})$. If $\tilde{c} > 0$ and if the immersion is full (i.e., M cannot be immersed in any proper totally geodesic submanifold of $\tilde{M}$), does the following hold? (i) $\tilde{c} = \nu c$ and (ii) $m = \binom{n+\nu}{\nu} - 1$ for some positive integer ν. [Some partial solutions are given in Theorem 4.6, Theorem 4.8, and Theorem 4.10.]

6. Let M be a Kaehler hypersurface immersed in $P_{n+1}(C)$. If ρ is constant, is M either totally geodesic or locally complex analytically isometric to $Q_n(C)$? [Generalization of Theorem 7.6.]

7. Let M_n be an n-dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$, $\tilde{c} > 0$. If M is irreducible (or Einstein) and if the second fundamental form is parallel, is M one of the following? $M_n(\tilde{c})$, $M_n(\tilde{c}/2)$, or locally $Q_n(C)$. [Generalization of Theorem 4.4.]

8. Let M_n be an n-dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$, $\tilde{c} \leqslant 0$. If M is Einstein, is M totally geodesic? [Generalization of Corollary 4.11 or Theorem 5.1.]

9. Can the assumption "M is Einstein" in Problems 7 and 8 be replaced by "ρ is constant"?

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