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Improved theory of generalized meteo-ballistic weighting factor functions and their use

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Abstract

It follows from the analysis of artillery fire errors that approximately two-thirds of the inaccuracy of indirect artillery fire is caused by inaccuracies in the determination of the meteo parameters included in fire error budget model. Trajectories calculated under non-standard conditions are considered to be perturbed. The tools utilized for the analysis of perturbed trajectories are weighting factor functions (WFFs) which are a special kind of sensitivity functions. WFFs are used for calculation of meteo ballistic elements μ_B (ballistic wind w_B, density ρ_B, virtual temperature τ_B, pressure p_B) as well. We have found that the existing theory of WFF calculation has several significant shortcomings. The aim of the article is to present a new, improved theory of generalized WFFs that eliminates the deficiencies found. Using this theory will improve methods for designing firing tables, fire control systems algorithms, and meteo message generation algorithms.

Keywords: Exterior ballistic; Non-standard projectile trajectory; Perturbation; Meteoconditions; Sensitivity function; (Generalized) weighting (factor) function (curve); (Generalized) effect function; Norm effect

1. Introduction

1.1. Motivation

It follows from the analysis of artillery fire errors, e.g. [1,2], that approximately two-thirds of the inaccuracy of indirect artillery fire is caused by inaccuracies in the determination of meteo parameters included in the error budget model [1]. Consequently, it is always important to pay close attention to the problems of including the actual meteo parameters in ballistic calculations [3]. The following meteo parameters μ are primarily utilized: Wind vector w, air pressure p, virtual temperature τ, and density ρ [2–6].

This paper deals only with problems relating to unguided projectiles without propulsion system for the sake of lucidity of the solved problems.

1.2. Weighting functions – basic information

The most important information about the influence of meteo parameters (and not only them) on the trajectory of an unguided projectile is included in the relevant weight or weighting functions [2,7–10].

List of notation

μ	met parameter (element)
$\mu(y)$	real or measured magnitude of met parameter μ in height y
$r(\mu)$	weighting factor function (curve, WFF)
Q_P, Q_{CP}	effect function
$\mu_{STD}(h)$	met parameter standard course with the height h
$\Delta\mu(y)$	absolute deviation of met element μ in height y
$\delta\mu(y)$	relative deviation of met element μ in height y
$\Delta\mu_B$	absolute ballistic deviation of ballistic element μ_B
$\delta\mu_B$	relative ballistic deviation of ballistic element μ_B

The basis for the derivation of the weighting functions is perturbation theory [11].

We are interested in the exercise of the perturbation theory in dynamical systems theory, primarily in the control theory of dynamical systems [12,13]. It is exercised especially in the exploration of stability and sensitivity [12,13]. The most

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widespread variant of the perturbation theory is the simplest one – the first-order perturbation theory. Its most important basis is the linearization of all requisite non-linear functions and equations [11–13].Unless otherwise specified, the following information refers to this theory.

A special subset of controlled systems is comprised of aerospace vehicles, i.e., aircrafts, space vehicles, rockets, space shuttles, guided missiles and spinning and non-spinning "unguided" projectiles with/without terminal guidance and Magnus rotors [4–6,14].

The state equations of non-linear dynamical systems have then the form

$$\boldsymbol{x}'(t)=\boldsymbol{X}[t,\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{d}(t),\boldsymbol{\alpha}(t)],\quad \boldsymbol{x}(0)=\boldsymbol{x}_0$$
$$\boldsymbol{y}(t)=\boldsymbol{Y}[t,\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{d}(t),\boldsymbol{\alpha}(t)] \tag{1}$$

where $\boldsymbol{x}$ is state variable vector, $\boldsymbol{u}$ is input control variable vector, $\boldsymbol{d}$ is input disturbance variable vector, $\boldsymbol{\alpha}$ is parameters vector, $\boldsymbol{y}$ is output variable vector.

The perturbation theory is used for transformation of these equations into their linearized form (first example) [5–9]. The linearized state equations have for example the form

$$\boldsymbol{x}'(t)\approx \boldsymbol{A}(t,\boldsymbol{\alpha}(t))\cdot\boldsymbol{x}(t)+\boldsymbol{B}(t,\boldsymbol{\alpha}(t))\cdot(\boldsymbol{u}(t)+\boldsymbol{d}(t)),\quad \boldsymbol{x}(0)=\boldsymbol{x}_0$$
$$\boldsymbol{y}(t)\approx \boldsymbol{C}(t,\boldsymbol{\alpha}(t))\cdot\boldsymbol{x}(t)+\boldsymbol{D}(t,\boldsymbol{\alpha}(t))\cdot(\boldsymbol{u}(t)+\boldsymbol{d}(t)) \tag{2}$$

For the analysis of dynamical systems, it is interesting to observe changes of the system properties while some parameters $\boldsymbol{\alpha}(t)$ change; the parameters are then often denoted as influence quantities. We speak of differential sensitivity analysis of the control system or of sensitivity of a system to parameter variations. The perturbation theory is used again for linearization of Eq. (2) relative to parameters $\boldsymbol{\alpha}$. We obtain a sensitivity model of the (linearized) dynamical system, for instance in the form [12,13,15,16]

$$\boldsymbol{\lambda}'_i(t,\boldsymbol{\alpha})\approx \boldsymbol{A}(t,\boldsymbol{\alpha})\cdot\boldsymbol{\lambda}_i(t,\boldsymbol{\alpha})+\frac{\partial \boldsymbol{A}}{\partial\alpha_i}\cdot\boldsymbol{x}(t,\boldsymbol{\alpha})+\frac{\partial \boldsymbol{B}}{\partial\alpha_i}\cdot(\boldsymbol{u}(t)+\mathbf{d}(t))$$
$$\boldsymbol{\lambda}_i(0,\boldsymbol{\alpha})=\boldsymbol{\lambda}_{0,i} \tag{3}$$
$$\boldsymbol{\eta}_i(t,\boldsymbol{\alpha})\approx \boldsymbol{C}(t,\boldsymbol{\alpha})\cdot\boldsymbol{\lambda}_i(t,\boldsymbol{\alpha})+\frac{\partial \boldsymbol{C}}{\partial\alpha_i}\cdot\boldsymbol{x}(t,\boldsymbol{\alpha})+\frac{\partial D}{\partial\alpha_i}\cdot(\boldsymbol{u}(t)+\mathbf{d}(t))$$

where

$$\boldsymbol{\lambda}_i\!: \boldsymbol{\lambda}_i(t,\boldsymbol{\alpha})=\frac{\partial \boldsymbol{x}}{\partial\alpha_i},\quad \boldsymbol{\eta}_i:\boldsymbol{\eta}_i(t,\boldsymbol{\alpha})=\frac{\partial \boldsymbol{y}}{\partial\alpha_i},\quad i=1,2,\ldots,n$$

are the absolute sensitivity functions. The absolute sensitivity functions of the output variables $\boldsymbol{\eta}_i$ are especially important for the practice. Non-dimensional Bode sensitivity functions are often used [12,13,15,16].

The perturbation theory is used in this second case for finding linearized relations between changes of system parameters $\Delta\boldsymbol{\alpha}$ and corresponding changes of the output variables $\Delta\boldsymbol{y}$, which are represented by the sensitivity functions $\boldsymbol{\eta}_i$ ($\boldsymbol{\eta}_i \approx \Delta\boldsymbol{y}/\Delta\alpha_i$, $i = 1, 2, \ldots, n$) and which can be expressed consecutively through the use of the corresponding transfer functions [12,13].

Standard test functions for the control variables $\boldsymbol{u}(t)$ and the disturbance variables $\boldsymbol{d}(t)$ are used for the analysis of properties of the systems that are described by Eqs. (1), (2) and (3). The unit impulse is usually used, and also the unit step, the function sine and/or cosine, etc. [12,13].

Such a procedure is not sufficient for analyses of movements of aerospace vehicles, so it is customary to use reference trajectories and maneuvers, respectively, which represent the typical maneuvers of a given type of aerospace vehicle [4–6,14].

Moreover, it is necessary to differentiate whether reference maneuvers are pursued under standard conditions or perturbed conditions.

Standard conditions are defined contractually and determine the standard/normal values of the parameters respectively $\boldsymbol{\alpha}_{\mathrm{STD}}(t)$, for instance, parameters of the standard atmosphere are considered. The reference maneuvers under standard conditions are utilized for the basic analysis of aerospace vehicle properties, Eqs. (1) or (2) are used withal ($\boldsymbol{d}(t) = 0$).

The reference maneuvers under perturbed conditions ($\boldsymbol{d}(t) \neq 0$, respectively $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_{\mathrm{STD}}(t) + \Delta\boldsymbol{\alpha}(t)$) serve for consequential analyses of stability or robustness of flight control; Eq. (3) are used together with Eqs. (1) or (2).

The reference maneuver under standard conditions in the exterior ballistics of unguided projectiles is represented just by the standard projectile trajectory, and the reference maneuver under perturbed conditions is identical to the relevant perturbed projectile trajectory.

As mentioned above, corresponding sets of transfer functions are referred to Eqs. (2) and (3); their equivalent in the time domain is the convolution operation represented by the convolution integral. Two functions f and g figure in the convolution integral. The functions f and g have a special significance in the control theory of dynamical systems. The function f represents a generalized input variable $z_m(t)$, $m = 1, 2, \ldots$ (respectively $u_j(t)$, $j = 1, 2, \ldots$ and $d_k(t)$, $k = 1, 2, \ldots$ and $\Delta\alpha_i(t)$, $i = 1, 2, \ldots$) and the function $g_{m,l}(t)$ is the weighting function that corresponds with the relevant transfer function. The integral value then corresponds to the system response y_l, $l = 1, 2, \ldots$, to the excitation by the input variable [12,13].

The weighting functions $g_{ml}(t)$ are impulse-response functions [12,13], i.e. responses of the dynamical system to the special excitation by impulse function $z_m(t) = z_{m0}\cdot\delta(t - t_p)$, where z_{m0} is the excitation amplitude and $\delta(t - t_p)$ is the Dirac delta function. The weighting function then has the form

$$g_{ml}(t-t_{\mathrm{p}})=(M_{mn}/z_{m0})\cdot\gamma_{ml}(t-t_{\mathrm{p}}) \tag{4}$$

where

t_{p} is the moment of the impulse occurrence,
$\gamma_{ml}(t - t_{\mathrm{p}})$ is the normed form of the weighting function and
M_{mn} is the relevant norm.

We have now presented all of the common information from the control theory of dynamical systems necessary to understand the importance of weighting functions in exterior ballistics.

The perturbation theory was already used – in a simple form – at the start of the 20th century in exterior ballistics, e.g. [17]. The equation system corresponding to Eq. (3) was first derived during the First World War. These problems are often presented with the name Theory of trajectory (differential) corrections. The starting points were the motion equations of a projectile as a mass point (3 DoF – degree of freedom), which is an analogy to the Eq. (1). The convolutory integral usage started in this period and therefore the usage of the needful weighting functions started too. Corresponding models were not published until after the war, starting in 1919, for instance [7].

We have adduced more information about the overall progress in perturbation theory utilization in exterior ballistics in our article [3]. Only complementary information will be introduce here.

The control theory started to form at the end of the1930s and was not developed in full until the 1950s, so the procedures introduced into exterior ballistics had been formed almost 30 to 40 years prior. It should be no surprise, then, that the weighting functions were introduced differently.

The weighting (factor) function (*WFF*) r_{ml} was introduced into the exterior ballistics as the normed step response function [2,3,9,10,18], followed by the response of the dynamical system to the special excitation by the step function $z_m(t) = z_{m0} \cdot H(t - t_p)$, where z_{m0} is the excitation amplitude and $H(t - t_p)$ is the Heaviside step function. Then, the non-normed weighting function has the form

$$R_{ml}(t - t_p) = \sigma_{ml}(N_{ml} / z_{m0}) \cdot r_{ml}(t - t_p) \tag{5}$$

where t_p is the moment of the leap/perturbation, $r_{ml}(t - t_p)$ is the normed weighting (factor) function (curve) WFF [2,3,7], N_{ml} is the relevant norm, $\sigma_{ml} = +1$ or -1 is the contractual sign – see sections 2.4, 2.5.4.

The non-normed weighting function or perturbation functions $R_{ml}(t - t_p)$ were named the effect functions (curves) – EFs originally [2,8–10,19].

It follows from the properties of the Dirac impulse function, the Heaviside step function and from system linearity that conversion relations among weighting functions are [13]

$$g_{ml}(t - t_p) = (-1) \cdot \frac{dR_{ml}}{dt_P} = -\left(\sigma_{ml} \cdot \left(\frac{N_{ml}}{z_{m0}}\right)\right) \cdot \frac{dr_{ml}}{dt_P} \tag{6}$$

We did not find an explanation for Eq. (5) and its links with Eq. (4) defined by Eq. (6) in an explicit form in the available literature, but all the authors implicitly assume its validity [2,7,9,10,18]. Without appreciating the validity of this equation, the relation between modern control theory and the traditional theory of exterior ballistics, including the relevant weighting functions (WFFs), is not clear.

In the initial period, the following WFFs were introduced for meteo parameters: r_{wx} for the range wind, r_{wz} for the cross wind and r_ρ for the air density e.g. [7]. In this period, it was still assumed that the drag coefficient $c_D(v_{air})$ depends only on the air speed v_{air}. It was not until the 1920s, especially in connection with the publication of the drag coefficient $c_D(M) = c_D(v_{air,\ true})$ by Dupuis law, the respect for the dependence of the drag coefficient on the *Mach number* and on so-called fictive or true air speed – *TAS* begin. Therefore, a WFF was introduced; we named it $r_{\tau/\rho}$, because it exists only in pairs with WFF r_ρ. In our article [3], we explained that there are other combinations of WFFs, see also [2,8,20], and we refer to the problem in this contribution as well. The achieved findings are published, for example, in [9,10,19].

Further development of this in the 1950s is documented, for example, in [8,20].

The development from the 1960s to the present can be considered paradoxical. Methods based on the theory of perturbations have been further developed and they are widely used in control theory, for example [4–6,12,13,16], whereas their use in exterior ballistics has declined. The status can be demonstrated by the content of important publications from this period.

No word about perturbation problems can be found in the key books [21,22]. McCoy [23] only pays attention to the problem of variable wind in two pages. Other authors, e.g. [24–26], clarify these problems through oversimplification and without a more detailed explanation of WFF problems. The book [2] deals with the problems of WFFs in the most detail, but a sensitivity model analogous to the Eq. (3) is not presented, unlike [27]. There are very few articles that deal with the given problems, for example [18,28–30] and our contributions [3,31,32].

The question arises as to why this development has occurred. We do not know the answer. We try only to present the following hypothesis, for which we will use the following proverb: *"They throw the baby out with the bathwater"*. What is the "water" and what is the "child"? The "*water*" is the numerical algorithms for quick calculations of perturbed trajectories, and the "*child"* is the WFFs.

Till the early 1960s the main endeavor of publications about perturbations of trajectories focused on finding the most effective algorithms to solve perturbed trajectories. This problem became uninteresting after the massive arrival of digital computers. As a result, perturbation theory was quickly abandoned, and it was forgotten that the possibility of calculating WFFs was also lost.

1.3. The main objectives of the contribution

The weighting factor functions (WFFs) are special representants of sensitivity functions – see Eq. (3) and (6), and should be primarily considered as a post-processing tool. They allow for the compression of useful information very effectively and also allow for the display of it in synoptic graphs. Our main goal is a return to the use of WFFs (sensitivity functions) in the exterior ballistics.

We expect from this to

- streamline the teaching of exterior ballistics as a result of increasing its lucidity,
- improve the suggestive power of the published outputs from research in problems of the sensitivity analysis of projectile trajectories.

The consequential aim is to contribute to the improvement of methods for making firing tables, algorithms of fire control systems, and methods for the preparation of documents for processing meteorological measurements and the subsequent generation of meteorological messages.

For the performance of the aims introduced, we present an improved theory of generalized meteo-ballistic weighting factor functions. The core of the theory is created by the publications of V. Cech [33,34]. Moreover, selected problems are finished in this contribution.

1.4. Perturbations versus correction of projectile trajectories

Trajectory perturbation follows logic (Eqs. (4) and (5)) – the primary change z_{m0} of any of the parameters/input variables leads to the trajectory perturbation, which is a change of the output variables vector $\Delta \boldsymbol{y}_{\mathrm{Pm}}$.

The theory of trajectory (differential) correction traditionally stems from a request [9,10,24,25,27,35–37] so the change of control variables $\Delta \boldsymbol{u}$, which also leads to the change of output variables $\Delta \boldsymbol{y}(\Delta \boldsymbol{u}) = \Delta \boldsymbol{y}_{\mathrm{C}}$, compensates for the effect of perturbation $\Delta \boldsymbol{y}_{\mathrm{P}}$, i.e., it is valid

$$\Delta \boldsymbol{y}_{\mathrm{C}} + \Delta \boldsymbol{y}_{\mathrm{Pm}} = 0 \tag{7}$$

so in the traditional notation and for the meteo-ballistic parameters [3] we will present the most frequent case for range correction ($\Delta y_{\mathrm{Cl}} = \Delta X$) [3]

$$\Delta X = Q_{x\mathrm{A}}(\mu|\mu_0)\cdot\frac{\Delta\mu_{\mathrm{B}}}{\Delta\mu_{\mathrm{BN}}} = Q_{x\mathrm{R}}(\mu|\mu_0)\cdot\frac{\delta\mu_{\mathrm{B}}}{\delta\mu_{\mathrm{BN}}} \tag{8}$$

where

$\Delta\mu_{\mathrm{B}} = \mu_{\mathrm{B}} - \mu_{\mathrm{STD}}$ – absolute ballistic deviation of ballistic element μ_{B}, see below,

$\delta\mu_{\mathrm{B}} = \Delta\mu_{\mathrm{B}}/\mu_{\mathrm{STD}}$ – relative ballistic deviation of ballistic element μ_{B}, see below,

$\Delta\mu_{\mathrm{BN}}$, $\delta\mu_{\mathrm{BN}}$ – constant norm values of the absolute and relative ballistic deviation that are presented in tabular firing tables,

$Q_{x\mathrm{A}}$, $Q_{x\mathrm{R}}$ – corresponding (unity) correction factors for range (x) that are presented in tabular firing tables. Indices A (absolute) or R (relative) inform us of the fact that what enters into the calculations is the absolute deviation $\Delta\mu_{\mathrm{B}}$ or the relative deviation $\delta\mu_{\mathrm{B}}$ of the ballistic elements. Next time – if no ambiguities will be possible – we will omit these indices.

The notation $Q(\mu|\mu_0)$ means [3] that the correction factor $Q(\mu)$ is calculated under the assumption that $Q(\mu_0)$ is used as a second correction factor and, for example, their common range correction $\Delta X_{\mathrm{com}} = \Delta X_{\mu} + \Delta X_{\mu 0}$.

It follows for the correction coefficient $Q_{x\mathrm{A}}$ from Eqs. (8) and (5)

$$Q_{x\mathrm{A}} \cong (-1)\cdot((\sigma_{ml}\cdot N_{ml})\,/\,\Delta\mu_{\mathrm{B0}})\cdot\Delta\mu_{\mathrm{BN}} \tag{9}$$

where $z_{m0} = \Delta\mu_{\mathrm{B0}}$ is the constant value by which the perturbation has been calculated. For example, $\Delta\mu_{\mathrm{B0}} = \pm 25$ m/s is recommended [37] for the range wind ($\Delta\mu_{\mathrm{B}} = w_x$). For the firing tables by NATO methodology [21,35,36] this is $\Delta\mu_{\mathrm{BN}} = 1$ knot = 0.514 444 m/s. For the firing tables by the Soviet methodology [2,24–27] it is $\Delta\mu_{\mathrm{BN}} = 10$ m/s. For the choice of $(\sigma_{ml}\cdot N_{ml})$, see sections 2.4 and 2.5.4.

Equations for the relative values of the correction coefficients $Q_{x\mathrm{R}}$ are determined by analogy, $z_{m0} = \delta\mu_{\mathrm{B0}}$ and $\delta\mu_{\mathrm{BN}}$. For example, this is recommended for the air density ρ [9,37] $\delta\mu_{\mathrm{B0}} = \delta\rho = \pm 0.1$ and $\delta\mu_{\mathrm{BN}} = 1\%$ (NATO), 10% (Soviet methodology).

Relations analogous to Eq. (8), Eq. (9) can be derived for the azimuth (unity) corrections ($\Delta y_{\mathrm{Cl}} = \Delta Z$, ($Q_{z\mathrm{A}}$, $Q_{z\mathrm{R}}$)), time of flight ($\Delta y_{\mathrm{Cl}} = \Delta t_{\mathrm{Pl}}$), etc. – see sections 2.2 and 2.3.

The linearization is understood in two different ways. The traditional way [4–6,11–13,16] is based on the Taylor series of the function at the working point, and then there is a numerical estimate of the partial derivatives of the $\boldsymbol{\eta}_i$ ($\boldsymbol{\eta}_i \approx \Delta \boldsymbol{y}/\Delta\alpha_i$, $i = 1, 2, \ldots, n$), which figured in Eq. (3) [8–10,19,24–27].

The second way is more general. Two linear approximations at the working point – from the left and from the right – are numerically estimated [35–37]. The aim is to extend the interval during which the linear approximation is sufficiently accurate. An alternative to this procedure is the use of second-order perturbation theory [8,11].

For more information on the linearization of functions, see [38].

1.5. Explicit versus implicit algorithm

What has been mentioned up to now relates to the explicit algorithm for calculations of the sensitivity functions and WFFs. A pre-requisite is to build Eqs. (2) and (3) in a form that corresponds to the analyzed model of the dynamical system [4–10,19,27].

The advantage of the explicit algorithm is the possibility to study the structure of the links in the state-space model (matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ and their derivatives). In fact, the real models are so complicated that they are confusing, and therefore several simpler, partial models are derived from them so that these new models will already be clear and appropriate for analysis [4–6].

As mentioned in [9], M. Garnier published in 1929 the WFF calculation method, which can be described as the implicit algorithm. Linearized Eqs. (2) and (3) do not have to be derived at all. The original non-linear system Eq. (1) is sufficient for the work. The basis of the algorithm is the definition of the partially perturbed functions – see section 2.2. The algorithm is extremely convenient for programming on a digital computer. In the following text we describe the essence of the method. The linearity can be assumed implicitly and will appear in the form of relations not used until post-processing. The disadvantage of the algorithm is that the structure of dependences in the perturbation model is not obvious.

2. Improved theory of generalized weighting factor functions

Pieces of work made by V. Cech form the core of the theory [33,34].

2.1. Ballistic atmosphere models

Atmospheric conditions have to be known in advance, at the time of planning the shot. This means that appropriate measurements have to be made in advance and consequently the results of these measurements need to be extrapolated in time and space, i.e. to the points the projectile will fly through. For data extrapolation it is necessary to choose hypotheses about their future changes. The methodology of measurements of the required magnitudes, along with the algorithms of their processing and extrapolation, form the core of the ballistic models of the atmosphere [2,9,39–41]. With respect to the aim of the article, we are going to define and describe five groups of models. The first group of models serves for practical calculations of firing data. The remaining four groups of models are used for theoretical analyses and tabular and graphical firing tables.

2.1.1. Current atmosphere models

Current atmospheric models have the following basic features. They are based on currently-measured data, from which the noise and relatively quick trend components are eliminated. The data are exported to the users in the form of meteorological messages. There are three subsets of them.

In the first group no concrete information about the weapon or the projectile planned for shooting is entered, i.e., the data are universally usable by any weapon system. This especially embodies METCMQ meteo messages according to NATO methodology [35,42,43] and METEO – 11 ("Meteo-average") according to Soviet methodology [2,3,24,26,28,41].

The second group is represented by meteo message METBKQ according to NATO methodology [35,40,43,44]. The data are modified by means of weighting factors – WFs, deduced from particular WFFs. The applied WFs [44] and WFFs are accurately valid only for a totally specific gun, projectile, charge and quadrant elevation. For other guns the data stated in METBKQ are valid only approximately.

The third group of models is represented by meteo message METGM [35,43,45,46].This modern method is based on complex modeling of the development of the meteorological situation and custom sending of the meteo messages [35].

2.1.2. Standard atmosphere models

In practice a number of general standard (normal, etalon) atmospheres are used. The most important is the International Standard Atmosphere (ISA) according to ISO 2533. In exterior ballistics a number of different standard atmospheres have also been used [8] – not only generally, but also special ones. For our purposes we will mention only two of them: Ventcel's atmosphere (also Artillery Normal Atmosphere – ANA) and ICAO standard atmosphere. Ventcel's atmosphere has been used for the majority of calculations of firing tables according to Soviet methodology [2,24–27,47]. ICAO standard atmosphere is being used for firing table calculations according to NATO methodology [4,12,21,23,37,40,43,48].

Figures about standard atmospheric parameters are indicated depending either on geometric altitude above mean sea level (MSL) H or geopotential altitude above MSL h_{geopot}. In the following text we will use only the altitude H.

This deals mainly with the following set of functions $\mu_{\mathrm{STD}}(H) = (\tau_{\mathrm{STD}}(H),\ p_{\mathrm{STD}}(H),\ \rho_{\mathrm{STD}}(H),\ a_{\mathrm{STD}}(H),\ g_{\mathrm{STD}}(H))$, namely: virtual temperature, air pressure, air density, speed of sound and gravity acceleration + $g_{\mathrm{N}} = 9.806\ 65\ \mathrm{m/s^2}$ – normal gravity acceleration.

2.1.3. Standard meteo–ballistic atmosphere model

In practice, a single model of meteo-ballistic atmosphere is used [2,40,41]. This model serves for evaluating measurements and processing METBKQ meteo messages according to NATO methodology [35,42,43] and METEO – 11 ("Meteo – average") according to Soviet methodology [2,3,24,26–28,41]. It is a selected standard atmosphere that is vertically shifted according to the relation

$$H = y_{\mathrm{Z}} = h - h_{\mathrm{MDP}} \tag{10}$$

where

h_{MDP} is the altitude above MSL of a meteorological station (Meteorological Datum Plane – MDP),
h is an altitude above MSL of the atmospheric layer measured [2,3,40,44],
y_{Z} is a superelevation of the atmospheric layer measured above MDP.

This transformation is illogical in principle and causes a number of complications when using the meteo messages METBKQ and METEO – 11 [2] in practice.

2.1.4. Standard firing table atmosphere models

In practice, a series of standard firing table atmosphere models is used. We are concerned especially about those models based on Ventcel's atmosphere (also ANA) and ICAO standard atmosphere.

The following relation is used for conversion of coordinates

$$H = h_{\mathrm{G}} + y \tag{11}$$

where

h_{G} is an altitude above MSL of the origin of a ballistic coordinate system (x, y, z),
y is the height of the projectile trajectory above the level (x,y,z), $y = 0$.

The altitude above MSL h_{G} can reach 10 000 m or more while shooting or bombing from an airplane.

Contractual – firing table values of the altitude h_{FT} ($h_{\mathrm{G}} = h_{\mathrm{FT}}$) are chosen for setting up the firing tables.

According to Soviet methodology [2,24–27,47], $h_{\mathrm{FT}} = 0$, 500, 1000, 1500, 2000, 2500, 3000 m above MSL. Firing tables set for $h_{\mathrm{FT}} > 0$ m are denoted as Mountain Firing Tables. This system has a convenient accuracy for approx. 95% of the continent's surface [2,33].

Swiss methodology used the implicit definition of firing table altitudes h_{FT} such as to define the table's standard densities $\rho_{\mathrm{STD}}(h_{\mathrm{FT}}) = 1208,\ 1150,\ 1100,\ \ldots,\ 900\ \mathrm{g/m^3}$ [9]. NATO methodology [35,36] presupposes only $h_{\mathrm{FT}} = 0$ m above MSL. So when shooting in the mountains, significant errors of shooting appear [10]. This system has a convenient accuracy for not more than approx. 50% of the continent's surface [33].

2.1.5. Perturbed firing table atmosphere models

For each standard atmosphere and each altitude h_G at least three perturbed table atmosphere models exist [8,9,20].

In the interest of maximal simplification of the computational algorithm, we will introduce a set of artificially-constructed relations in the model [34].

Perturbed magnitudes will be indicated by $\mu_P(H, t) = (\tau_P(H, t), p_P(H, t), \rho_P(H, t), a_P(H, t), g_P(H, t))$. These are generalized input step functions – Eq. (5). We will not consider the perturbation of the acceleration of gravity ($g_P(H, t) = g_{STD}(H)$).

We consider wind vector $\boldsymbol{w}_0 = (w_x, w_y, w_z)_0$ to be a disturbance input variables belonging to vector $\boldsymbol{d}$ (Eqs. (1), (2) and (3)) which are also perturbation constants (z_{m0}) – see Eqs. (5) and (9).

We will implement three perturbed virtual temperatures [34]

$$\tau_{Pi}(H,t) = (1+\delta\tau_{0i}\cdot\varepsilon_P(t))\cdot\tau_{STD}(H) + \Delta\tau_{0i}\cdot\varepsilon_P(t), \quad i = 1,2,3 \tag{12}$$

where

($\delta\tau_0$, $\Delta\tau_{0i}$) are perturbation constants (z_{m0}) – see Eqs. (5) and (9), $\varepsilon_P(t)$ is perturbation function. For the basic perturbation algorithm it is always $\varepsilon_P(t) = H(t - t_P)$ – see Eq. (5).

Perturbed hypsometric equation has the form [34]

$$p_{P1}(H,t) = p_{STD}(0)\cdot\exp\left[-\frac{H}{h_B(H,t)}\right] \tag{13}$$

where

$h_B(H,t)$ is perturbed pressure scale height [m]

$$\frac{1}{h_B(H,t)} = \left(\frac{g_N}{r_{DA}}\right)\cdot\frac{1}{\tau_{B,P}(H,t)} \tag{14}$$

where

r_{DA} is the gas constant of dry air [J/(kg·K)], $\tau_{B,P}(H,t)$ is the barometric average virtual temperature of the interval $\langle 0, H\rangle$ [K]

$$\frac{1}{\tau_{B,P}(H,t)} = \frac{1}{H}\int_0^H \left(\frac{g_P(H,t)}{g_N}\right)\cdot\frac{\mathrm{d}h}{\tau_{P1}(H,t)} \tag{15}$$

We will implement perturbed relative function of pressures [34]

$$P_{P1}(H,h_G,t) = \frac{p_{P1}(H,t)}{p_{P1}(h_G,t)} = \exp\left[-\left(\frac{H}{h_B(H,t)} - \frac{h_G}{h_B(h_G,t)}\right)\right] \tag{16}$$

and perturbed atmospheric pressure

$$p_p(H,t) = p_P(h_G,t)\cdot P_{P1}(H,h_G,t) \tag{17}$$

where

$$p_P(h_G,t) = (1+\delta p_0\cdot\varepsilon_P(t))\cdot p_{STD}(h_G,t) + \Delta p_0\cdot\varepsilon_P(t) \tag{18}$$

where

(δp_0, Δp_{0i}) are perturbation constants (z_{m0}) – see Eqs. (5) and (9).

Next we define the perturbed relative function of virtual temperatures [34]

$$T_{P2}(H,h_G,t) = \frac{\tau_{P2}(h_G,t)}{\tau_{P2}(H,t)} \tag{19}$$

and the perturbed relative function of air density [34]

$$Rho_{P1,2}(H,h_G,t) = \frac{\rho_{P1}(H,t)}{\rho_{P1}(h_G,t)} = P_{P1}(H,h_G,t)\cdot T_{P2}(H,h_G,t) \tag{20}$$

where

$$\rho_{P1}(h_G,t) = \frac{p_P(h_G,t)}{r_{DA}\cdot\tau_{P2}(h_G,t)} \tag{21}$$

then perturbed air density (the first output of the model) is defined by relation [34]

$$\rho_P(H,t) = \rho_P(h_G,t)\cdot Rho_{P1,2}(H,h_G,t) \tag{22}$$

where

$$\rho_P(h_G,t) = (1+\delta\rho_0\cdot\varepsilon_P(t))\cdot\rho_{P1}(h_G,t) + \Delta\rho_0\cdot\varepsilon_P(t) \tag{23}$$

where

($\delta\rho_0$, $\Delta\rho_{0i}$) are perturbation constants (z_{m0}) – see relations (5) and (9).

We define two perturbed speeds of sound [34]

$$a_{Pi}(H,t) = \sqrt{\kappa_{MA}\cdot r_{DA}\cdot\tau_{Pi}(H,t)}, \quad i = 2,3 \tag{24}$$

where

κ_{MA} is the adiabatic index of moist air. It approximately holds that [8] $\kappa_{MA} \approx \kappa_{DA}$, $\kappa_{DA}$is the adiabatic index of dry air.

The vector of the projectile towards the air (air speed) – see Eq. (5) – is

$$\boldsymbol{v}_{air} = \boldsymbol{v} - \boldsymbol{w}_0\cdot\varepsilon_P(t) = (v_{air,x}, v_{air,y}, v_{air,z}) \tag{25}$$

where

$\boldsymbol{v} = (v_x, v_y, v_z)$ is a vector of the projectile towards the Earth (ground speed), then we define two perturbed Mach numbers (second output of the model) [34]

$$M_{Pi}(H,t) = \frac{|\vec{v}_{air}|}{a_{Pi}(H,t)}, \quad i = 2,3 \tag{26}$$

For drag function $\varphi(M)$ as defined in [8] on page 95, and also [20], its perturbed form can be used [34]

$$\varphi_P(M_{P2}, M_{P3}) = M_{P2}^2(H,t)\cdot c_D(M_{P3}(H,t)) \tag{27}$$

where

$c_D(M_{P3}(H,t))$ is a drag coefficient [-], which is used in all models.

Table 1
Input data for perturbation models.

Option	1	2	3	4	etc.
$\delta\rho_0$	0	$\delta\rho_0$	0	0	
$\Delta\rho_0$	0	0	0	0	
δp_0	0	0	δp_0	0	
Δp_0	0	0	0	Δp_0	
$\delta\tau_{01}$	0	0	0	0	
$\delta\tau_{02}$	0	0	$\delta\tau_0$	0	
$\delta\tau_{03}$	0	$\delta\tau_0$	$\delta\tau_0$	0	
$\Delta\tau_{01}$	0	0	0	$\Delta\tau_0$	
$\Delta\tau_{02}$	0	0	0	$\Delta\tau_0$	
$\Delta\tau_{03}$	0	0	0	$\Delta\tau_0$	
w_{x0}	0	w_{x0}	w_{x0}	w_{x0}	
w_{y0}	0	0	0	0	
w_{z0}	0	w_{z0}	w_{z0}	w_{z0}	

It is appropriate to supplement the model with a perturbed ballistic coefficient [8,34].

The model can be included in projectile trajectory models with 3 DoF, 5 DoF or 6 DoF [21,23,27,45].

We get the (non-perturbed) standard table atmosphere model by setting all the perturbation constants as equal to zero – see Table 1 (Option 1).

In practice, basically three models of perturbed table atmosphere are used [8] – see Table 1 (Option 2, 3, 4).

The first model (Option 2) is the most widespread [8,10,19,20,29]. It is a prerequisite for firing table formations according to the NATO methodology [35–37] and generating meteo messages analogical to METBKQ [40,43,44]. For generating the WFF, r_ρ $\delta\rho_0 \neq 0$ and $\delta\tau_0 = w_{x0} = w_{z0} = 0$ is chosen. For generating the WFF $r_{\tau/\rho}$ is chosen $\delta\tau_0 \neq 0$ and $\delta\rho_0 = w_{x0} = w_{z0} = 0$. For generating the WFF r_{wx} $w_{x0} \neq 0$ and $\delta\rho_0 = \delta\tau_0 = w_{z0} = 0$ is chosen. For generating the WFF r_{wz} $w_{z0} \neq 0$ and $\delta\rho_0 = \delta\tau_0 = w_{x0} = 0$ is chosen. For research use, different combinations of nonzero values of $\delta\rho_0$, $\delta\tau_0$, w_{x0}, w_{z0} can be chosen.

Even in the 1940s and 1950s it was discovered [10] that this first model has a very inappropriate course of the WFF $r_{\tau/\rho}$. That is why an alternative solution has been sought. For example, in [20] the second model has been analyzed (Option 3). It has been discovered that using drag function $\varphi(M)$ – see Eq. (27) – instead of drag coefficient $c_D(M)$ brings significant improvement. The new WFF $r_{\tau/p}$ has a much more favorable course in the majority of cases in comparison with the original WFF $r_{\tau/\rho}$. Our brief comment can be found in [32]. For generating the WFF $r_{\tau/p}$ $\delta\tau_{01} = \delta\tau_{02} = \delta\tau_0 \neq 0$ and $\delta p_0 = w_{x0} = w_{z0} = 0$ is chosen. For generating the WFF r_p $\delta p_0 \neq 0$ and $\delta\tau_0 = \mathrm{w}_{x0} = \mathrm{w}_{z0} = 0$ is chosen.

The third model (Option 4) practically agrees with the original model set by P. Langevin [2,8,18,26–28]. It is used for setting up the firing tables according to the Soviet methodology and for meteo messages METEO – 11. Based on our previous research, while the corresponding WFF $r_{\tau/p0}$ has relatively the best features, nevertheless it does not comply under marginal conditions. The problem is not closed at all, so we plan to continue our research. For generating the WFF $r_{\tau/p0}$ $\tau_{01} = \Delta\tau_{02} = \Delta\tau_{03} = \Delta\tau_0 \neq 0$ and $\Delta p_0 = w_{x0} = w_{z0} = 0$ is chosen.

2.2. Garnier's algorithm of the weighting factor function calculation

The essence of Garnier's method (see section 1.5) resides in the cyclic calculation of the partially-perturbed trajectories [9,33]. It derives from the properties of the Heaviside step function $H(t - t_P)$ –Eq. (5).

The partially-perturbed trajectory is composed of two segments. The first segment ($H(t - t_P) = 0$ for $t < t_P$) represents the unperturbed trajectory. The second segment ($H(t - t_P) = 1$ for $t > t_P$) connects with the first segment and represents the perturbed trajectory.

The parameter (the variable) is the start time of the perturbation t_P, which is selected in steps from the interval $\langle 0, t_{end}\rangle$. In the time $t = t_{end}$, it is reached $(x, y, z)_{end}$, $(v_x, v_y, v_z)_{end}$, etc. The meaning of the time t_{end} will be explained in the section 2.3.

The standard (unperturbed) trajectory is a special case of the partially-perturbed trajectory – the whole trajectory is unperturbed ($t_P \geq t_{end} = t_{PI}$). The point of impact/burst (the index PI) arises for the standard trajectory in the time $t = t_{end} = t_{PI}$ – $(x, y, z)_{PI}$, $(v_x, v_y, v_z)_{PI}$, etc.

The (full) perturbed trajectory is also a special case of the partially-perturbed trajectory – the whole trajectory is perturbed. It is valid $t_p = 0$. In the time $t = t_{end,0}$, it is reached $(x, y, z)_{end,0}$, $(v_x, v_y, v_z)_{end,0}$, etc.

The numerical calculations generate the set of partially-perturbed trajectories that differ at chosen times t_{Pi}, $i = 0, 1, 2, \ldots$ The times $t_{Pi} \in \langle 0, t_{end}\rangle$ are selected densely to make it possible to consequently express the courses of the WFFs.

Now it is possible to calculate perturbations of the elements in the point of calculations termination — point of impact/burst ($t = t_{PI}$) for the perturbation of the meteo parameter μ

- range perturbation

$$\Delta X(\mu, t_P) = x_{end} - x_{PI} \tag{28}$$

- perturbation of height of impact/burst point

$$\Delta Y(\mu, t_P) = y_{end} - y_{PI} \tag{29}$$

- azimuth perturbation

$$\Delta Z(\mu, t_P) = z_{end} - z_{PI} \tag{30}$$

- time of flight perturbation

$$\Delta t(\mu, t_P) = t_{end} - t_{PI} \tag{31}$$

- perturbation of velocity horizontal component

$$\Delta v_x(\mu, t_P) = v_{x,end} - v_{x,PI} \tag{32}$$

etc.

These functions of times t_P and t_{PI} are the special cases of effect functions $R_{ml}(t - t_P)$ – Eq. (5).

From the practical point of view, the most significant effect functions (EFs) are of the range, height, azimuth and time of flight (Eqs. (28) to (31)). We will use the special notation $Q_P(\mu, t_P) = Q_P(t_P) = Q_P$ for these EFs.

2.3. Coordinate perturbation of the point of impact

There exist at least five ways to define the time t_{end}, in which the calculation of the partially-perturbed trajectory is finished, and so we have minimally five variants of perturbations of coordinates of the impact/burst point.

The first variant is the simplest from the view of numerical calculation. We choose contractually $t_{end} = t_{PI}$, i.e., the time of the calculation is always equal to the time of the projectile flight on the standard trajectory t_{PI}. Consequently, the isochronous perturbations are considered and $\Delta t_t(\mu, t_P) = 0$.

The time t_{end} is defined implicitly in the other four variants of perturbations.

In the second variant, the time t_{end} is defined by the condition $y(t_{end}) = y_{PI}$, consequently $\Delta Y_y(\mu, t_p) = 0$. The iso-height of impact/burst perturbations takes effect in this case. Corresponding correction factors to $\Delta X_y(\mu, t_P)$ and $\Delta t_y(\mu, t_P)$ for $t_P = 0$ are usually included in the firing tables – Eqs. (8) and (9) [18,26,35–37,47]. Trivial approximate formulas exist for the conversion of isochronal perturbations into iso-height [7,8,10,19,27].

In the third variant, the time t_{end} is defined by the condition $x(t_{end}) = x_{PI}$, consequently $\Delta X_x(\mu, t_P) = 0$. The iso-range of impact/burst perturbations takes effect in this case. Corresponding correction factors to $\Delta Y_x(\mu, t_P)$ and $\Delta t_x(\mu, t_P)$ for $t_P = 0$ are usually included in the firing tables – Eqs. (8) and (9) [18,26,35–37,47]. The perturbations $\Delta t_y(\mu, t_P)$ and $\Delta t_x(\mu, t_P)$ for $t_P = 0$ are transformed into corrections of a fuze setting. Trivial approximate formulas exist for the conversion of isochronal perturbations into iso-range.

In the fourth variant, the time t_{end} is defined by the condition $\varepsilon(t_{end}) = \varepsilon_{PI}$, consequently $\Delta\varepsilon_\varepsilon(\mu, t_p) = 0$, where ε_{PI} is the angle of impact point site. In this case, the perturbations $\Delta D_\varepsilon(\mu, t_P)$ are calculated of the slant range D of the impact point and perturbations of the time $\Delta t_\varepsilon(\mu, t_P)$. The iso-angle of site perturbations takes effect in this case.

In the fifth variant, the time t_{end} is defined by the condition $D(t_{end}) = D_{PI}$, consequently $\Delta D_D(\mu, t_P) = 0$, where D_{PI} is the slant range of impact point. In this case, the perturbations $\Delta\varepsilon_D(\mu, t_P)$ are calculated of the angle of impact point site and perturbations of the time $\Delta t_D(\mu, t_P)$. The iso-slant range perturbations take effect in this case.

The total perturbation $Q_{PS}(\Delta\mu)$ of parameters of the point of impact/burst – e.g. the range $\Delta X_P(\Delta\mu)$, Eqs. (5) to (9) – under known (measured) courses of the absolute deviations $\Delta\mu(t)$ from the standard values ($\Delta\mu(t) = \mu_{measured}(t) - \mu_{STD}(t)$) is given by the convolutory integral

$$\Delta X_P \approx Q_{PS}(\Delta\mu) \cong \int_0^{t_{PI}} \Delta\mu(t_P)\cdot\frac{dQ_P(\mu, t_P)}{dt_P}\cdot dt_P = (\sigma_Q\cdot N_Q)\cdot\left(\frac{\Delta\mu_B}{\Delta\mu_{B0}}\right) \approx -\Delta X \tag{33}$$

and

$$\Delta\mu_B \cong \int_0^{t_{PI}} \Delta\mu(t_P)\cdot\frac{dr_A(\mu, t_P)}{dt_P}\cdot dt_P \tag{34}$$

where

$(\sigma_Q\cdot N_Q)$ corresponds to Eq. (5), see sections 2.4 and 2.5.4,
$\Delta\mu_{B0}$ – see Eq. (9),
$\Delta\mu_B$ is the absolute ballistic deviation of the meteo ballistic element μ, $r_A(\mu, t_P)$ is the (normed) weighting factor function WFF, see Eqs. (5) and (6) [2,3,10]; the index A – see the commentary to the Eq. (8).

Analogous Eqs. to (33) and (34) are valid also for the known (measured) relative deviations $\delta\mu(t)$ ($\delta\mu(t) = \Delta\mu(t)/\mu_{STD}(t)$) [2,3] – see Eq. (8).

2.4. Weighting factor functions for flat-fire trajectories

Measured deviations are evaluated for the requirements of the flat-fire depending on the topographic range, i.e., on the coordinate x, and so ($\Delta\mu(x)$, $\delta\mu(x)$) is used [23,30]. As a consequence, Eqs. (33) and (34) must be modified.

We will use the function $t_P = F(x)$, which is valid for the unperturbed (standard) trajectory; then it will be $Q_P(\mu, x)$ and $r(\mu, x)$. Let us remind the reader that $dx = v_x\cdot dt_P$, thus Eq. (34) will have the form

$$\Delta\mu_B \cong \int_0^{x_{PI}} \Delta\mu(x)\cdot\frac{dr_A(\mu, x)}{dx}\cdot dx \tag{35}$$

We choose $(\sigma_Q\cdot N_Q) = Q_P(\mu, x)$ for $x = 0$ in all cases.

It is important to be aware that

$$\frac{dr_A(\mu, x)}{dx} = \frac{1}{v_x(t_P)}\cdot\frac{dr_A(\mu, t_P)}{dt_P} \tag{36}$$

is valid.

The WFFs for the range wind $r_{wx}(x)$ and for the cross wind $r_{wz}(x)$ are important only for the flat-fire from a practical point of view.

2.5. Weighting factor functions for common trajectories

2.5.1. Generalized two-branched effect function

For shooting at common trajectories, measured deviations ($\Delta\mu$, $\delta\mu$) are evaluated depending on coordinate y of the projectile trajectory, thus ($\Delta\mu(y)$, $\delta\mu(y)$) is used [7–10,17–27,33,45,49,50]. Therefore it is necessary to modify Eqs. (33) and (34) again.

The meteo message *de facto* determines ($\Delta\mu(y_{zi})$, $\delta\mu(y_{zi})$, $i = 0, 1, 2, \ldots$) [2,3,18,31–35,40–46]. As a consequence it is necessary to transform the data of the meteo message at first (discrete coordinates y_{zi}, see the paragraph 2.1.3) into data dependent on the coordinate y [2,24,26,34,35,40,41,43].

We will use the function $t_P = F(y)$ valid for standard trajectory. It is a one-to-two function. Such an essential failure will be eliminated by deliberating the particular dependence separately for the ascending branch (AB) $t_{P,AB} = t_{P1}(y) = F_{AB}(y)$ and separately for the descending branch (DB) $t_{P,DB} = t_{P2}(y) = F_{DB}(y)$. It holds that $t_{P1}(y) \le t_{P2}(y)$. In consequence of this, it is essential to separately consider traditional effect functions (EFs) for the ascending $Q_{P,AB}(\mu, y)$ and descending $Q_{P,DB}(\mu, y)$ branch. The

Table 2
Four groups of true projectile trajectories.

Trajectory	Angle of departure Θ_0	Point of impact/burst (x_{PI}, y_{PI})					
		Branch of the Trajectory	Angle of site ε_{PI}	Range R = x_{PI}	Height y_{PI}	y_{min}	y_{max}
1	$\Theta_0 > 0$	ascending	$\varepsilon_{PI} \in (\varepsilon_s, \theta_0\rangle$	$R \in \langle 0, x_s)$	$y_{PI} > 0$	0	y_{PI}
2		descending	$\varepsilon_{PI} \in (0, \varepsilon_s\rangle$	$R \in \langle x_s, x_{LP})$		0	y_s
3			$\varepsilon_{PI} \le 0$	$R \ge x_{LP}$	$y_{PI} \le 0$	y_{PI}	y_s
4	$\Theta_0 \le 0$		$\varepsilon_{PI} \le \theta_0 \le 0$	$R \ge 0$		y_{PI}	0

1. Indices: s– summit of the trajectory, LP – level point (point of fall), 2. x_{LP} – base of the trajectory, 3. x_s and ε_s – range and angle of site of the trajectory summit.

next step requires sorting the true projectile trajectories into four groups – Table 2.

By means of the traditional method it was possible to calculate WFFs $r(y)$ only in three special cases [10]: The trajectory has either only an ascending (1st trajectory) [10], or only a descending (4th trajectory) [10] branch, or the trajectory has both branches of equal height (3rd trajectory and $R = x_{LP}$) [2,7–10,18–20,24,26].

Our goal is to provide calculating WFFs $r(y)$ for each trajectory – Table 2. In order to do so, generalized effect functions (EFs) $Q_{CP}(t_P)$, defined for basic projectile trajectory, is introduced [33].

The basic trajectory [34] consists of true projectile trajectory and virtual sections that are chosen so that the height of the ascending branch will be the same as the descending branch. In virtual sections of the basic projectile trajectory all perturbations contractually equal zero.

The minimal basic trajectory [34] is a basic trajectory that has the same origin point or end point, i.e., point of impact, or both with a true trajectory – Table 2 and Fig. 1.

In accordance with the definition of the basic trajectory, the derivative of generalized effect functions [34] is valid

$$\frac{\mathrm{d}Q_{CPi}(t_P)}{\mathrm{d}t_P} = \begin{cases} \dfrac{\mathrm{d}Q_{Pi}}{\mathrm{d}t_P} & \text{if} \quad t_P \in \langle t_{Oi}, t_{PI,i}\rangle, \\ 0 & \text{otherwise,} \end{cases}$$

where

$i = 1, 2, 3, 4$ are indices of the true trajectories – Fig. 1, Table 2, t_P is the perturbation time for the basic trajectory.

The link between t_P and t_{Pi} is apparent from Fig. 1 and consequently we define generalized effect functions [34] – Fig. 1

$$Q_{CPi}(t_P) = \begin{cases} Q_{Pi}(0) & \text{if} \quad t_P < t_{Oi}, \\ Q_{Pi}(t_P) & \text{if} \quad t_P \in \langle t_{Oi}, t_{PI,i}\rangle, \\ 0 & \text{if} \quad t_P > t_{PI,i}. \end{cases} \tag{37}$$

Traditional effect functions $Q_{P,AB}(\mu, y)$ and $Q_{P,DB}(\mu, y)$ will be implemented by their generalized varieties $Q_{CP,AB}(\mu, y)$ and $Q_{CP,DB}(\mu, y)$ which have – unlike the traditional ones – the same height [34].

Next, by means of unifying $Q_{CP,AB}(\mu, y)$ and $Q_{CP,DB}(\mu, y)$ into one function we will create a generalized two-branched effect function (curve) $Q_{CP}(\mu, y) = Q_{CP}(y)$ – Fig. 2 [2,8–10,19, 20,33,34].

2.5.2. Generalized Garnier's effect function

Generalized Garnier's effect function $Q_{CG}(y)$ – Fig. 3 is calculated [34] according to the same definitional relation as a traditional Garnier's effect function $Q_G(y)$ [8,9,20,33,49] but differs in inputs ($Q_{CP,AB}(\mu, y)$, $Q_{CP,DB}(\mu, y)$) and ($Q_{P,AB}(\mu, y)$, $Q_{P,DB}(\mu, y)$) respectively

$$Q_{CG}(y) = Q_{CG}(\mu, y) = Q_{CP,AB}(\mu, y) - Q_{CP,DB}(\mu, y) = Q_{CP1} - Q_{CP2} \tag{38}$$

The value $Q_{CG}(\mu, y_0)$ represents the cumulative effect of all perturbations in heights $y \ge y_0$.

2.5.3. Generalized Bliss' effect function

Generalized Bliss' effect function $Q_{CB}(y)$ – Fig. 4 is counted [34] according to the same definitional relation as the traditional Bliss' effect function $Q_B(y)$ [2,7,8,10,18,19,24,28,33,43, 44,47], but differs in inputs $Q_{CG}(\mu, y)$ and $Q_P(\mu, y)$ respectively

$$Q_{CB}(y) = Q_{CB}(\mu, y) = Q_{CG}(\mu, 0) - Q_{CG}(\mu, y) \tag{39}$$

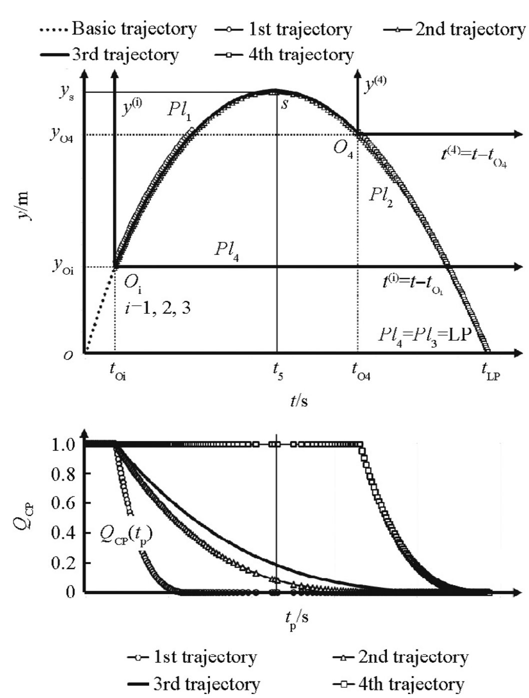

Fig. 1. Basic trajectory and its four subsets – true trajectories and their corresponding generalized effect functions $Q_{CP}(t_P)$ (illustrative example) [34], Table 2. O_i – origin of the i-th trajectory, PI_i – point of impact/burst of the i-th trajectory.

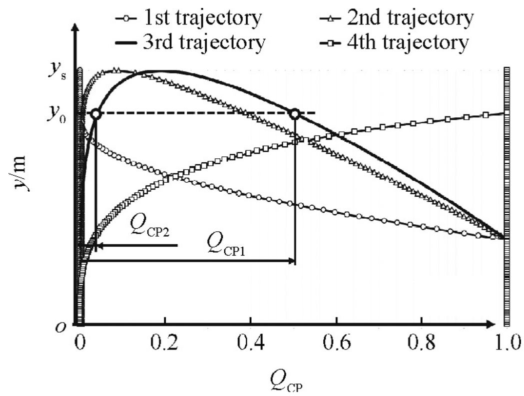

Fig. 2. Generalized two-branched effect functions (curves) $Q_{CP}(y)$ for four varieties of projectile trajectories (illustrative example follows up the Fig. 1) [34], Table 2.

Value $Q_{CB}(\mu, y_0)$ represents the cumulative effect of all perturbations in heights $y \le y_0$.

2.5.4. Generalized weighting factor functions

Generalized weighting factor functions WFFs are calculated by norming from generalized effect functions (curves).

For generalized Garnier's weighting factor functions WFFs it holds that [8,9,20,49]

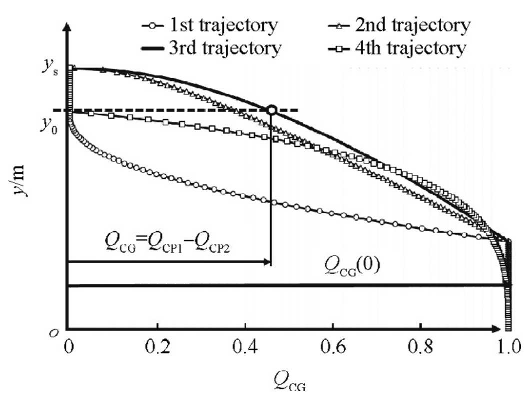

Fig. 3. Generalized Garnier's effect functions $Q_{CG}(y)$ (illustrative example follows up the Fig. 2) [34], Eq. (38).

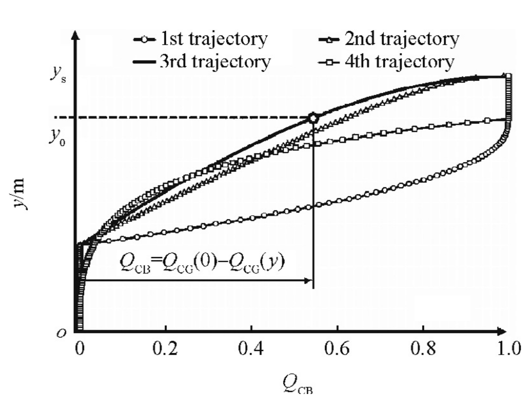

Fig. 4. Generalized Bliss' effect functions $Q_{CB}(y)$ (illustrative example follows up the Fig. 3) [33], Eq. (39).

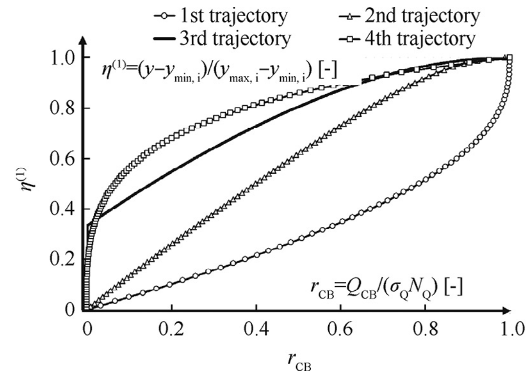

Fig. 5. Course of the generalized Bliss' WFF $sr_{CB}(\mu, \eta)$ (illustrative example follows up Fig. 4) [34], Eq. (41).

$$r_{CG}(\mu, y) = \frac{Q_{CG}(\mu, y)}{(\sigma_Q \cdot N_Q)} \tag{40}$$

where

$(\sigma_{ml} \cdot N_{ml}) = (\sigma_Q \cdot N_Q)$ – see the relations (5), (33), (42), (47), (48) and (49).

For generalized Bliss' weighting factor functions WFFs it holds that [2,3,7,8,10,18,19,23–28,31–34,41,44,50]

$$r_{CB}(\mu, y) = \frac{Q_{CB}(\mu, y)}{(\sigma_Q \cdot N_Q)} = r_{CG}(\mu, y_{\min}) - r_{CG}(\mu, y) \tag{41}$$

where

$$r_{CG}(\mu, y_{\min}) = \frac{Q_{CG}(\mu, y_{\min})}{(\sigma_Q \cdot N_Q)} = r_{CB}(\mu, y_{\max})$$

where

$y_{\min}$ – see Table 2,
$y_{\max}$ – see Table 2,
$Q_{CG}(\mu, y_{\min}) = Q_{CP}(\mu, y_{\min}) = Q_{CP}(\mu, t_P) = Q_{CP}(t_P)$ for $t_P = 0$ – see Figs. 1–3,
$r_{CG}(y_{\max}) = r_{CB}(y_{\min}) = 0$ – see Figs. 1–4.

Traditionally [2,3,7,8,10,18,19,23,24,26,28,30–34,49,50] are chosen (for $t_P = 0$)

$$\begin{aligned}(\sigma_Q \cdot N_Q) &= Q_{CG}(\mu, y_{\min}) = Q_{CP}(\mu, y_{\min}) \\ &= Q_{CP}(\mu, t_P) = Q_{CP}(t_P) \neq 0\end{aligned} \tag{42}$$

Hereafter, if the following condition is valid

$$|Q_{CP}(0)| = \max(|Q_{CG}(\mu, y)|) \neq 0 \tag{43}$$

then it always holds that

$$r_{CG}(y_{\min}) = r_{CB}(y_{\max}) = +1.$$

According to NATO and Soviet methodologies using Bliss' WFFs rB(μ, y) are presupposed, so we will limit our following analysis only to generalized Bliss' WFFs $r_{CB}(\mu, y)$ – Figs. 5 and 6. If interchange is not possible, we will no longer mention index "CB" in description of WFFs.

For the graphic presentation of the WFFs course the coordinate y is also normed. In Figs. 5 and 6 two of many possible varieties of norming are presented.

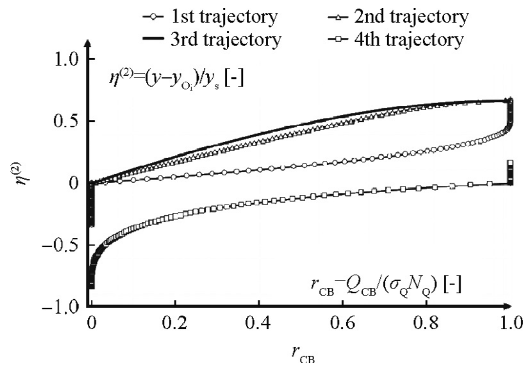

Fig. 6. The course of generalized Bliss' WFF $sr_{CB}(\mu, \eta)$ (illustrative example follows up Fig. 4) [34], Eq. (41).

Analogically for Eqs. (34) and (35), ($dy = v_y(t_P) \cdot t_P$) will apply [2,3,8,10,18,19,24,26,28,31–34]

$$\Delta\mu_B \cong \int_{y_{min}}^{y_{max}} \Delta\mu(y) \cdot \frac{dr_{A,CB}(\mu, y)}{dy} \cdot dy \tag{44}$$

It is important to realize that analogically, in Eq. (36) it holds that

$$\begin{aligned} \frac{dr_{A,CB}(\mu, y)}{dy} &= -\frac{dr_{A,CG}(\mu, y)}{dy} \\ &= \frac{1}{(\sigma_Q \cdot N_Q)} \left[\frac{1}{v_{y,DB}(t_{P,DB}(y))} \cdot \frac{dQ_{CP,DB}(\mu, t_{P,DB}(y))}{dt_P} \right. \\ &\quad \left. - \frac{1}{v_{y,AB}(t_{P,AB}(y))} \cdot \frac{dQ_{CP,AB}(\mu, t_{P,AB}(y))}{dt_P} \right] \end{aligned} \tag{45}$$

In the summit of the trajectory $(x, y)_s$ it applies that $v_y(y_s) = 0$, so it is necessary to use l'Hospital's rule for analyzing relations for trajectories from 2 to 4 [34] – Table 2.

When calculating WFF $r_{\tau/\rho}$ (section 2.1.5) Eq. (43) is not usually fulfilled and instead it usually applies that [3,10,20, 31–34]

$$|Q_{CP}(0)| < \max(|Q_{CG}(\mu, y)|) \tag{46}$$

The relation mentioned above is described as a "*norm effect*" [10]. In extreme cases it is possible that also $Q_{CP}(0) = 0$ [10,20]. In these cases it is essential, or more precisely, necessary to choose norm N_Q in a different way than according to the traditional method given by Eq. (42). Moreover it is necessary to add that the described complication can appear even while calculating other WFFs [20,33,34].

The authors of the book [10] chose the norm N_Q as the total variation of the function $Q_B(y)$ (in our case it is $Q_{CB}(\mu, y)$). If this norm is being used, the WFFs are indicated as "*normalized effect functions (curves)*". The introduced process is correct from the mathematical point of view, but it is not suitable in practice.

Based on the analyses of the problem, we propose the following norm [34]

$$N_Q = \max_y Q_{CG}(\mu, y) - \min_y Q_{CG}(\mu, y) \tag{47}$$

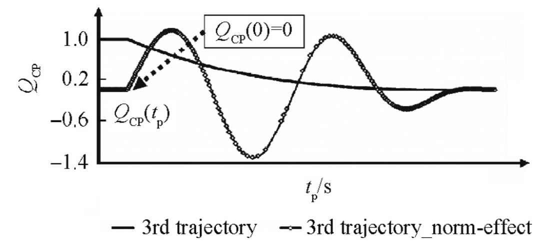

Fig. 7. Two generalized effect functions $Q_{CP}(t_P)$ (illustrative example follows up Fig. 1) [34]. Problem of the "norm-effect", Eqs. (46)–(49).

and at the same time

$$\sigma_Q = \text{sign}(Q_{CP}(0)) \tag{48}$$

if it applies that $Q_{CP}(t_P) \neq 0$ for $t_P = 0$ i.e. $Q_{CP}(0)$.

If $Q_{CP}(0) = 0$ then a number of varieties how to choose σ_Q exist. For example we can choose [34]

$$\sigma_Q = \text{sign}(Q_{CG}(\mu, y_{sup})) \tag{49}$$

where

$$|Q_{CG}(\mu, y_{sup})| = \max_y(|Q_{CG}(\mu, y)|)$$

In case that equality Eq. (43) holds, Eqs. (47) and (48) becomes consistent with the traditional Eq. (42).

Extreme variety ($Q_{CP}(0) = 0$) will be explained by means of Figs. 7–9.

3. Conclusion

This article presents our newly-conceived theory of generalized meteo-ballistic weighting factor functions – WFF as a special kind of sensitivity functions – Fig. 10. The limited extent of this contribution has allowed us only to indicate the applicational possibilities of the new theory.

In the publications that will follow we plan to especially apply the new theory to the problematics of calculating the reference height of the projectile trajectory Y_R [2,3,18,28,31, 33,34] that will mainly demand a detailed analysis of WFFs for the virtual temperature [20,32].

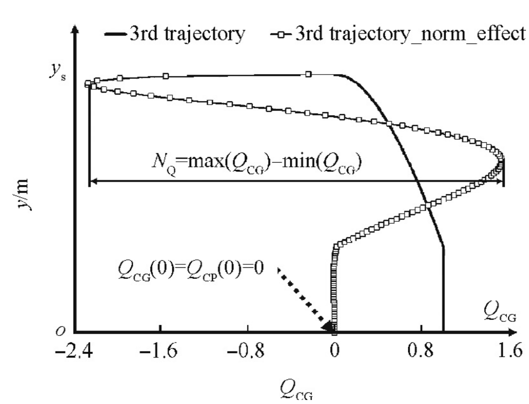

Fig. 8. Two generalized Garnier's effect functions $Q_{CG}(y)$ (illustrative example follows up Figs. 3 and 7) [34]. Problem of the "norm-effect", Eqs. (46)–(49).

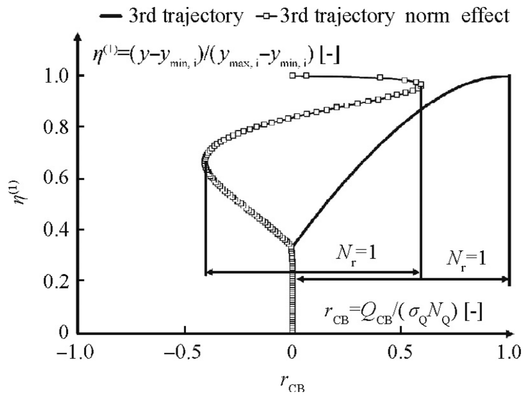

Fig. 9. Two generalized Bliss' WFFs $r_{CB}(\mu, \eta)$ (illustrative example follows up Figs. 5 and 8) [34]. Problem of the "norm-effect", Eqs. (46)–(49).

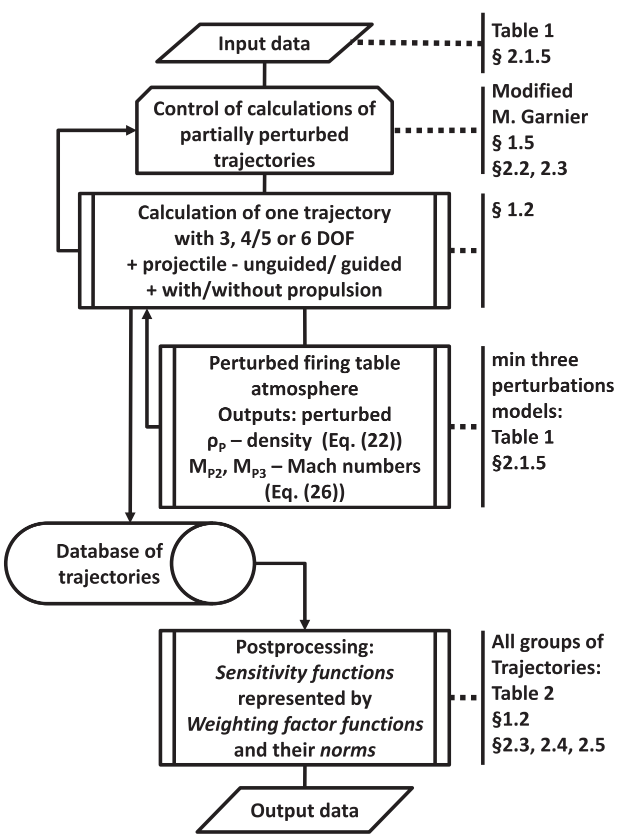

Fig. 10. The simplified flowchart of improved algorithm of the weighting factor functions and corresponding norms calculation.

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