Homogenization of a non-stationary Stokes flow in porous medium including a layer

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Abstract
We study the behavior of the solution to the non-stationary Stokes equations in a porous medium with characteristic size of the pores $\varepsilon$ and containing a thin fissure $\{0 \leq x_n \leq \eta\}$ of width $\eta$. The limit when $\varepsilon$ and $\eta$ tend to zero gives the homogenized behavior of the flow, which depends on the comparison between $\varepsilon$ and $\eta$.

Keywords: Homogenization; Stokes; Fissure; Porous medium

1. Introduction
In this paper we consider the non-stationary Stokes equations in a periodic porous medium containing a thin fissure. Our goal is to find the homogenized equations corresponding to limit when $\varepsilon$, the size of the pores, and $\eta$, the width of the fissure both tend to zero. The preliminary results on such a problem were studied by Bourgeat, Marusic-Paloka and Mikelić [4–6], in which they considered the Laplace’s equation and a stationary Stokes system. How to get the homogenized equations of Stokes or Navier–Stokes equations in porous medium was researched by several authors, see [1–3,7–9,11].

The results we obtain here correspond to three characteristic situations as in [6]. The first one is $\eta < \kappa \varepsilon^{2/3}$, in which case we conclude that the homogenized fluid flow is governed by the Darcy’s law in the whole domain, i.e. in this case, the inter-layer does not give any contribution.

The second characteristic situation appears when $\eta \sim \kappa \varepsilon^{2/3}$, in which case the convergence proofs in the previous situation break down. We find that the limit problem is now given by the classical Darcy’s law in the porous medium. But on the surface $\{x_n = 0\}$, an additional bilinear form is generated by the tangential derivative of the pressure, which satisfies the Reynold’s law. The homogenized pressure is continuous while on the surface $\{x_n = 0\}$ the normal component of the homogenized velocity has a jump given by the measure concentrated thereon.

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Finally, in the third characteristic case \( \eta > \kappa \varepsilon^{\frac{2}{3}} \), we prove that the effective velocity is a Dirac measure concentrated on \( \{x_n = 0\} \), which implies that the fissure is dominant and the tangential flow at \( \{x_n = 0\} \) is dominant. Meanwhile, in the other part of the porous medium the effective velocity is equal to zero.

Due to the non-stationary case, we cannot hope to get the strong convergence for the pressure as in [6], which brings more difficulty when we discuss the convergence process. We use a trick of “lift” operator (see [7,11]) to overcome this difficulty in this paper. We should mention here that since our \textit{a priori} estimates for the velocity in the fissure are different from [6], the convergence results in the fissure when \( \eta < \kappa \varepsilon^{\frac{2}{3}} \) are also different.

2. Definition of the problem

2.1. Notations

In this paper, we use the same symbols as Bourgeat [4]. For completeness, we repeat them as following (see Fig. 1). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and

\[
\Omega_+ = \Omega \cap \{x_n > 0\}; \quad \Omega_- = \Omega \cap \{x_n < 0\}; \quad \Sigma = \Omega \cap \{x_n = 0\}.
\]

For some \( \eta_0 > 0 \) we define the domain

\[
D = \Omega_- \cup (\Omega_+ + \eta_0 e_n) \cup (\Sigma \times [0, \eta_0]),
\]

where \( e_n = (0, 0, \ldots, 0, 1) \).

Let \( \varepsilon > 0, 0 < \eta < \eta_0 \) be two small parameters. With \( \Omega \) we associate a microstructure through the periodic cell \( Y = (0, 1)^n, n \geq 2 \), made of two complementary parts: the solid part \( A \), which is closed and strictly contained in \( Y \) with a smooth boundary \( \partial A \), and the fluid part \( Y^* = Y \setminus A \).

Defining \( A_k = A + k, k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \). We also denote

\[
A^- = \bigcup_{k \in \mathbb{Z}_n^+} A^k, \quad A^+ = \bigcup_{k \in \mathbb{Z}_n^-} A^k,
\]

all the solid parts in \( \mathbb{R}^n \), where \( \mathbb{Z}_n^- = \{k \mid k \in \mathbb{Z}^n, k_n < 0\} \) and \( \mathbb{Z}_n^+ = \{k \mid k \in \mathbb{Z}^n, k_n > 0\} \). It is obvious that \( \mathcal{E}^* = (\mathbb{R}^n \setminus (A^- \cup A^+)) \cap \Omega \) is the fluid part in \( \Omega \).

Following Bourgeat [4] we make the following assumptions on \( \mathcal{Y}^*, \mathcal{E}^*, A \) and \( A^* = A^+ \cup A^- \):

(a) \( \mathcal{Y}^* \) is an open connected set of strictly positive measure, with a locally Lipschitz boundary;
(b) \( A \) has strictly positive measure in \( \bar{Y} \);
(c) \( \mathcal{E}^* \) and the interior of \( A^* \) are open sets with boundaries of class \( C^{0,1} \) and are locally located on one side of their boundaries. Moreover \( \mathcal{E}^* \) is connected.

We also define

\[
A_e^- = \varepsilon A^-, \quad A_{e\eta}^+ = \eta e_n + \varepsilon A^+, \quad \mathcal{S}_{e\eta} = \partial (A_e^- \cup A_{e\eta}^+),
\]
We denote by
$$
A_\epsilon \eta = A_\epsilon^+ \cup A_\epsilon^- \quad \text{the solid part of the domain } D,
$$
$$
D_\epsilon \eta = D \setminus A_\epsilon \eta \quad \text{the fluid part of the domain } D \text{ (including the fissure)},
$$
$$
I_\eta = \Sigma \times (0, \eta) \quad \text{the fissure in } D,
$$
$$
\Omega_\epsilon \eta = D_\epsilon \eta \setminus I_\eta \quad \text{the fluid part of the porous medium}
$$
and
$$
\Omega_\epsilon^+ = D_\epsilon \eta \cap \{ x_n > \eta \}, \quad \Omega_\epsilon^- = D_\epsilon \eta \cap \{ x_n < 0 \}, \quad \Gamma_\eta = \partial \Sigma \times (0, \eta).
$$
Finally we define
$$
D^+ = D \cap \{ x_n > 0 \}, \quad D^- = \Omega^-.
$$

2.2. \((\epsilon, \eta)\)-Problem

In this paper, we consider the following problem

$$
\begin{align*}
\frac{\partial u_\epsilon \eta}{\partial t} - \Delta u_\epsilon \eta + \nabla p_\epsilon \eta &= f \quad \text{in } D_\epsilon \eta \times (0, T), \\
\text{div } u_\epsilon \eta &= 0 \quad \text{in } D_\epsilon \eta \times (0, T), \\
u_\epsilon \eta &= 0 \quad \text{on } \partial D_\epsilon \eta \text{ for all } t \in (0, T),
\end{align*}
$$

(1)

where \( u_\epsilon \eta, p_\epsilon \eta, f \) stand for the velocity, the pressure of the fluid and the external force term, respectively. We assume that

$$
f \in C^1(\bar{D} \times [0, T]).
$$

The initial value of the velocity is given by

$$
u_\epsilon \eta(0, x) = u_\epsilon \eta^0 \quad \text{for all } x \in D_\epsilon \eta.
$$

In the sequel, we always assume that

$$
\| u_\epsilon \eta^0 \|_{L^2(D_\epsilon \eta)} \leq C(\epsilon + \eta^{2}),
$$

where \( C \) is a positive constant which is independent of \( \epsilon \) and \( \eta \).

For any fixed \( \epsilon \) and \( \eta \), under the assumptions on \( f(x, t) \) and \( u_\epsilon \eta^0(x) \), a classical result shows that there exists at least one weak solution pair \( \{ u_\epsilon \eta, p_\epsilon \eta \} \in L^2(0, T; H^1_0(D_\epsilon \eta)) \times L^2(0, T; H^{-1}(D_\epsilon \eta)) \), where \( p_\epsilon \eta \) is uniquely defined up to a constant (see the third chapter in [10]). We are interested in the behavior of \( u_\epsilon \eta \) and \( p_\epsilon \eta \) when \( \epsilon, \eta \) tend to zero.

3. A priori estimates

We start with a priori estimates for the velocity. Before stating our theorem, we first introduce the following lemma (see Bourgeat [4]):

\textbf{Lemma 3.1.} Let \( \varphi \in L^2(0, T; H^1(D_\epsilon \eta))^n \) and \( \varphi = 0 \) on \( S_\epsilon \eta \) for all \( t \in (0, T) \). Then

$$
\begin{align*}
(i) \quad \| \varphi \|_{L^2(\Omega_\epsilon \eta \times (0, T))}^2 &\leq C \epsilon \| \nabla \varphi \|_{L^2(\Omega_\epsilon \eta \times (0, T))}^2, \\
(ii) \quad \| \varphi \|_{L^2(I_\eta \times (0, T))}^2 &\leq C \sqrt{\eta} \sqrt{\epsilon + \eta} \| \nabla \varphi \|_{L^2(D_\epsilon \eta \times (0, T))}.
\end{align*}
$$

(2)

\textbf{Proof.} Inequality (2(ii)) is the standard Poincaré’s inequality in porous medium (see Tartar [9]). We only need to prove the second part.

It is obvious that

$$
\| \varphi \|_{L^2(I_\eta \times (0, T))} \leq C \eta \| \nabla \varphi \|_{L^2(I_\eta \times (0, T))}.
$$
Next, if we choose a point \( x_1 \in A_{\varepsilon\eta} \), which is close to the point \( x \in I_\eta \), then we have
\[
\varphi(x) - \varphi(x_1) = \nabla \varphi(\xi)(x - x_1) \leq (\varepsilon + \eta)|\nabla \varphi|.
\]
\( \varphi(x_1) = 0 \) since \( x_1 \in A_{\varepsilon\eta} \). Thus,
\[
\| \varphi(x) \|_{L^2(I_\eta \times (0, T))} \leq (\varepsilon + \eta) \| \nabla \varphi \|_{L^2(I_\eta \times (0, T))} \leq C(\varepsilon + \eta) \| \nabla \varphi \|_{L^2(D_{\varepsilon\eta} \times (0, T))}.
\]
Multiplying above two inequalities we obtain (2(ii)).

**Theorem 3.1.** Let \( u^{\varepsilon\eta} \) be a solution for (1). Then
\[
\| \nabla u^{\varepsilon\eta} \|_{L^2(D_{\varepsilon\eta} \times (0, T))} \leq C(\varepsilon + \eta^\frac{3}{2}) \tag{3}
\]
\[
\| u^{\varepsilon\eta} \|_{L^2(\Omega_{\varepsilon\eta} \times (0, T))} \leq C(\varepsilon^2 + \varepsilon\eta^\frac{3}{2}) \tag{4}
\]
\[
\| u^{\varepsilon\eta} \|_{L^2(I_\eta \times (0, T))} \leq C(\eta^\frac{3}{2} \varepsilon^\frac{1}{4} + \eta^\frac{5}{2}) \tag{5}
\]

**Proof.** By multiplying Eq. (1) with \( u^{\varepsilon\eta} \) and integrating over \( D_{\varepsilon\eta} \times (0, T) \), we obtain
\[
\int_0^T \int_{D_{\varepsilon\eta}} \frac{\partial u^{\varepsilon\eta}}{\partial t} \cdot u^{\varepsilon\eta} \, dx \, dt - \int_0^T \int_{D_{\varepsilon\eta}} \Delta u^{\varepsilon\eta} \cdot u^{\varepsilon\eta} \, dx \, dt + \int_0^T \int_{D_{\varepsilon\eta}} \nabla p^{\varepsilon\eta} \cdot u^{\varepsilon\eta} \, dx \, dt = \int_0^T \int_{D_{\varepsilon\eta}} f \cdot u^{\varepsilon\eta} \, dx \, dt.
\]
It is obvious that
\[
\int_0^T \int_{D_{\varepsilon\eta}} \nabla p^{\varepsilon\eta} \cdot u^{\varepsilon\eta} \, dx \, dt = 0,
\]
since \( u^{\varepsilon\eta} \) is divergence-free.

Integrating by parts, we get the energy equality
\[
\frac{1}{2} \int_{D_{\varepsilon\eta}} |u^{\varepsilon\eta}|^2 \, dx + \frac{T}{2} \int_{D_{\varepsilon\eta}} |\nabla u^{\varepsilon\eta}|^2 \, dx \, dt = \frac{1}{2} \int_{D_{\varepsilon\eta}} |u_0^{\varepsilon\eta}|^2 \, dx + \frac{T}{2} \int_{D_{\varepsilon\eta}} f \cdot u^{\varepsilon\eta} \, dx \, dt.
\]
Using (2) we easily get
\[
\int_0^T \int_{D_{\varepsilon\eta}} f \cdot u^{\varepsilon\eta} \, dx \, dt \leq \int_0^T \int_{\Omega_{\varepsilon\eta}} f \cdot u^{\varepsilon\eta} \, dx \, dt + \int_0^T \int_{I_\eta} f \cdot u^{\varepsilon\eta} \, dx \, dt \leq C(\eta^\frac{1}{2} \varepsilon^\frac{3}{2} + \eta^\frac{5}{2}) \| \nabla u^{\varepsilon\eta} \|_{L^2(D_{\varepsilon\eta} \times (0, T))}.
\]
Inserting above inequality into the energy equality we obtain
\[
\int_{D_{\varepsilon\eta}} \frac{1}{2} |u^{\varepsilon\eta}|^2 \, dx + \frac{T}{2} \int_{D_{\varepsilon\eta}} |\nabla u^{\varepsilon\eta}|^2 \, dx \, dt \leq \int_{D_{\varepsilon\eta}} |u_0^{\varepsilon\eta}|^2 \, dx + C(\varepsilon + \eta^\frac{5}{2}) \| \nabla u^{\varepsilon\eta} \|_{L^2(D_{\varepsilon\eta} \times (0, T))}.
\]
From the hypotheses on the initial data we get
\[
\begin{align*}
\left\| \nabla u^{\varepsilon \eta} \right\|_{L^2(D_{\varepsilon \eta} \times (0,T))} & \leq C(\varepsilon + \eta^3), \\
\left\| u^{\varepsilon \eta} \right\|_{L^\infty(0,T;L^2(D_{\varepsilon \eta}))} & \leq C(\varepsilon + \eta^3).
\end{align*}
\]
Using (2) once more we obtain
\[
\left\| u^{\varepsilon \eta} \right\|_{L^2(I_{\varepsilon \eta} \times (0,T))} \leq C(\varepsilon^2 + \varepsilon \eta^3),
\]
and since it is defined up to a constant we take
\[
\left\| u^{\varepsilon \eta} \right\|_{L^2(I_{\varepsilon \eta} \times (0,T))} \leq C(\varepsilon^{\frac{3}{2}} \eta^2 + \varepsilon^2).
\]

In the next step we estimate the pressure. Contrary to the velocity estimate, \textit{a priori} estimate for pressure requires extending the pressure to the whole domain \(D\). For this purpose, we introduce (see Bourgeat [5])

**Lemma 3.2.** Let \( \varphi \in L^2(I_{\eta} \times (0, T)) \) be such that \( \int_0^T \int_{I_{\eta}} \varphi \, dx \, d\tau = 0 \). Then
\[
\left\| \varphi \right\|_{L^2(I_{\eta} \times (0,T))} \leq \frac{C}{\eta} \left\| \nabla \varphi \right\|_{L^2(0,T;H^{-1}(I_{\eta}))}.
\] (6)

We denote the extended pressure again by \( p^{\varepsilon \eta} \) and since it is defined up to a constant we take \( p^{\varepsilon \eta} \) such that \( \int_0^T \int_D p^{\varepsilon \eta} \, dx = 0 \). For such extended pressure we obtain the following result:

**Theorem 3.2.** Let \( p^{\varepsilon \eta} \) be defined above, \( c^{\varepsilon \eta} = \frac{1}{|I_{\eta}|} \int_0^T \int_{I_{\eta}} p^{\varepsilon \eta} \, dx \, dt \). Then
\[
\left\| p^{\varepsilon \eta} \right\|_{L^2(D \times (0,T))} \leq C \left( 1 + \frac{\eta^3}{\varepsilon} \right),
\]
\[
\left\| p^{\varepsilon \eta} - c^{\varepsilon \eta} \right\|_{L^2(I_{\eta} \times (0,T))} \leq C \left( \sqrt{\eta} + \frac{\varepsilon}{\eta} \right),
\] (7)
where \( C \) is a positive constant and independent of \( \varepsilon, \eta \).

**Proof.** We first show that
\[
\left\| \nabla p^{\varepsilon \eta} \right\|_{L^2(0,T;H^{-1}(D))} \leq C \left( 1 + \frac{\eta^3}{\varepsilon} \right).
\]
In fact, for any \( \varphi \in L^2(0,T;H^1(D)) \) with \( \left\| \varphi \right\|_{L^2(0,T;H^1(D))} \leq 1 \), we have
\[
\left\| \nabla p^{\varepsilon \eta} \right\|_{L^2(0,T;H^{-1}(D))} = \sup_{\varphi \in L^2(0,T;H^1(D))} \left\| \int_0^T \nabla p^{\varepsilon \eta} : \nabla \varphi \, dx \, dt \right\|
\]
\[
\leq \sup_{\varphi \in L^2(0,T;H^1(D))} \left\| \int_0^T f \cdot R_{\varepsilon} \varphi \, dx \, dt \right\| + \sup_{\varphi \in L^2(0,T;H^1(D))} \left\| \int_0^T \int_{D_{\varepsilon \eta}} \Delta u^{\varepsilon \eta} : R_{\varepsilon} \varphi \, dx \, dt \right\|
\]
\[
\leq C \left( 1 + \frac{\eta^3}{\varepsilon} \right).
\]
Then, the first inequality in (7) follows by use of Nečas inequality (see Temam [10]) in \( D \). On the other hand, let \( \varphi \in L^2(0,T;H^1(I_{\eta})) \). Then
\[
\left\| \int_0^T \int_{I_{\eta}} p^{\varepsilon \eta} \, dx \, dt \right\| = \left\| \int_0^T \int_{I_{\eta}} \nabla u^{\varepsilon \eta} : \nabla \varphi \, dx \, dt + \int_0^T f \varphi \, dx \, dt \right\|
\]
\[
\leq \left[ C(\varepsilon + \eta^3) + \eta \sqrt{\varepsilon + \eta} \right] \left\| \varphi \right\|_{L^2(0,T;H^1(I_{\eta}))}.
\]
So that
\[ \| \nabla p^{\varepsilon \eta} \|_{L^2(0,T;H^{-1}(I_0))} \leq C(\varepsilon + \eta^{\frac{3}{2}}). \]

Using Lemma 3.2, we obtain the second inequality in (7).

4. Convergence results

Let \( w^k, r^k \) \((k = 1, 2, \ldots, n)\) be the solutions of the auxiliary problem (see [4,9,11])

\[
\begin{align*}
-\Delta w^k + \nabla r^k &= e_k, & \text{div } w^k &= 0 \quad \text{in } Y^*, \\
w^k &= 0 \quad \text{on } \partial A, & \{ w^k, r^k \} \text{ is } Y^*-\text{periodic},
\end{align*}
\]

and let \( K \) be the permeability tensor defined by

\[ [K]_{ij} = \int_{Y^*} (w^j)_j \, dx. \]

Since \textit{a priori} estimates are different in the porous part \( \Omega_{\varepsilon \eta} \) and in the fissure \( I_\eta \), we need to introduce the auxiliary velocity

\[
u^{\varepsilon \eta} = \begin{cases} u^{\varepsilon \eta} & \text{in } \Omega_{\varepsilon \eta} \times (0, T), \\ 0 & \text{in } I_\eta \times (0, T). \end{cases}
\]

So that from Theorem 3.1, we have

\[
\| \nu^{\varepsilon \eta} \|_{L^2(D \times (0,T))} = \| u^{\varepsilon \eta} \|_{L^2(\Omega_{\varepsilon \eta} \times (0,T))} \leq C(\varepsilon^2 + \varepsilon \eta^{\frac{3}{2}}).
\]

In order to study the behavior of \( u^{\varepsilon \eta}, p^{\varepsilon \eta} \) in the fissure, we rewrite our equations in the unit cylinder \( I_1 = \Sigma \times (0,1) \) by introducing the change of variable \( z = \frac{x_n}{\eta} \). We define the new functions

\[
U^{\varepsilon \eta}(t, x', z) = u^{\varepsilon \eta}(t, x', \eta z), \quad P^{\varepsilon \eta}(t, x', z) = p^{\varepsilon \eta}(t, x', \eta z) - c^{\varepsilon \eta}
\]

and

\[
\tilde{U}^{\varepsilon \eta} = (U_1^{\varepsilon \eta}, U_2^{\varepsilon \eta}, \ldots, U_n^{\varepsilon \eta})
\]

with

\[
c^{\varepsilon \eta} = \frac{1}{|I_\eta|} \int_{I_\eta} p^{\varepsilon \eta} \, dx,
\]

where \( x' = (x_1, x_2, \ldots, x_{n-1}), x = (x', x_n) \).

Using the functions \( U^{\varepsilon \eta}, P^{\varepsilon \eta} \) we rewrite the problem (1) in a new form

\[
\begin{cases}
\frac{\partial U^{\varepsilon \eta}}{\partial t} - \left( \Delta U^{\varepsilon \eta} + \frac{1}{\eta^2} \frac{\partial^2 U^{\varepsilon \eta}}{\partial z^2} \right) + \sum_{i=1}^{n-1} \frac{\partial P^{\varepsilon \eta}}{\partial x_i} e_i + \frac{1}{\eta} \frac{\partial P^{\varepsilon \eta}}{\partial z} e_n = f(t, x', \eta z), \\
\text{div}_{x'} U^{\varepsilon \eta} + \frac{1}{\eta} \frac{\partial U_n^{\varepsilon \eta}}{\partial z} = 0 \quad \text{in } I_1 \times (0,T), \\
u^{\varepsilon \eta} = 0 \quad \text{on } \partial \Sigma \times (0,1) \text{ for all } t \in (0,T),
\end{cases}
\]

where \( \{ e_i \}_{i=1}^n \) is the orthonormal basis of \( R^n \).
From $U^{\varepsilon\eta}(t, x', z) = u^{\varepsilon\eta}(t, x', \eta z)$, we find that $U^{\varepsilon\eta}(0, x', z) = u_0^{\varepsilon\eta}(x', \eta z)$. Using the hypotheses on the initial data, it is easy to deduce that

$$
\left\| U^{\varepsilon\eta}(0, x', z) \right\|_{L^2(I_1)} = \left( \frac{1}{\eta} \int_{I_1} |u^{\varepsilon\eta}(0, x', z)|^2 \, dx \, dz \right)^{\frac{1}{2}} \leq C \left( \frac{1}{\sqrt{\eta}} (\varepsilon + \eta^2) \right) \leq C. 
$$

With all the preparations above, we state our convergence result.

**Theorem 4.1.**

(i) Let $\eta \leq \kappa \varepsilon^\frac{3}{2}$. $v^{\eta \varepsilon}$ is defined by (10). Then

$$
\frac{1}{\varepsilon^2} v^{\varepsilon\eta} \rightharpoonup v^0 \text{ weakly in } L^2(D \times (0, T)), \\
p^{\varepsilon\eta} \rightharpoonup p^0 \text{ weakly in } L^2(D \times (0, T)),
$$

where $\{v^0, p^0\}$ is a solution pair of the Darcy’s flow system

$$
\begin{cases}
  v^0 = K(f - \nabla p^0), & \text{div } v^0 = 0 \text{ in } D \times (0, T), \\
  v^0 \cdot n = 0 & \text{on } \partial D \text{ for all } t \in (0, T).
\end{cases}
$$

(ii) Let $\eta \geq \kappa \varepsilon^\frac{3}{2}$ and $\{U^{\varepsilon\eta}, P^{\varepsilon\eta}\}$ be a solution pair of (12). Then there exists $U^0 \in L^2(I_1 \times (0, T))$, such that $U_n^0 = 0$, $P^0 \in L^2(I_1 \times (0, T))$, up to a subsequence,

$$
\frac{1}{\eta^2} U^{\varepsilon\eta} \rightharpoonup U^0 \text{ weakly in } L^2(I_1 \times (0, T)), \qquad P^{\varepsilon\eta} \rightharpoonup P^0 \text{ weakly in } L^2(I_1 \times (0, T)).
$$

Furthermore, let $U^0 \in L^2(I_1 \times (0, T))$ and $P^0 \in L^2(I_1 \times (0, T))$ be defined by (13). Then

$$
\tilde{U}^0(t, x', z) = \frac{1}{2}(z - z') (f(t, x', 0) - \partial_\alpha P^0(t, x')), 
$$

in the sense of distribution, where $\tilde{U}^0 = \{U_1^0, U_2^0, ..., U_{n-1}^0\}$, $U^0(0) = (\tilde{U}^0, 0)$, $\tilde{f} = \{f_1, f_2, ..., f_{n-1}\}$, $\tilde{f} = (\tilde{f}, f_n)$, $\partial_\alpha = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{n-1}})$.

Especially, if $\eta > \kappa \varepsilon^\frac{3}{2}$, then

$$
\frac{1}{\eta^3} U^{\varepsilon\eta} \rightharpoonup^* U^0 \delta_\Sigma \text{ weak in } L^2(0, T; \mathcal{M}(D)),
$$

where $U^0 \in L^2(\Sigma \times (0, T))$, with $U_n^0 = 0$, is given by

$$
\tilde{U}^0 = \frac{1}{12}(\tilde{f}(t, x', 0) - \partial_\alpha P^0(t, x')), \qquad U^0 = (\tilde{U}^0, 0),
$$

and $P^0$ is the solution of the Reynold’s equation on $\Sigma \times (0, T)$

$$
\begin{cases}
  \text{div}_x \tilde{U}^0 = \text{div}_x \left( \tilde{f}(t, x', 0) - \partial_\alpha P^0(t, x') \right) = 0 & \text{on } \Sigma \times (0, T), \\
  \tilde{U}^0 \cdot n_\alpha = (\tilde{f}(t, x', 0) - \partial_\alpha P^0(t, x')) \cdot n_\alpha & \text{on } \partial D \text{ for all } t \in (0, T),
\end{cases}
$$

where $n_\alpha$ is the outward normal on $\Sigma$.

(iii) Let $\eta = \kappa \varepsilon^\frac{3}{2}$, then there is a Darcy’s velocity $v^0$, a Reynold’s velocity $U^0$ and a pressure field $p^0$ such that

$$
\frac{1}{\varepsilon^2} u^{\varepsilon\eta} \rightharpoonup^* v^0 + \kappa^3 U^0 \delta_\Sigma \text{ weak in } L^2(0, T; \mathcal{M}(D)), \\
p^{\varepsilon\eta} \rightharpoonup p^0 \text{ weakly in } L^2(D \times (0, T)),
$$

where $\delta_\Sigma$ is the Dirac measure concentrated on $\Sigma$ and $\mathcal{M}(D)$ is the space of Radon measure on $D$. 
The proof is based on the idea from Bourgeat [4]. Using Lemma 4.1. Furthermore, let (ii) the velocities \( v^0 \) and \( U^0 \) are linked with \( p^0 \) through the Darcy’s law

\[
v^0 = K(f - \nabla p^0) \quad \text{in } D,
\]

and the Reynold’s law

\[
\widetilde{U}^0 = \frac{1}{12} (\tilde{f}(t, x', 0) - \partial_u P^0(t, x', 0)) \quad \text{on } \Sigma.
\]

The pressure field \( P^0 \in V_S = \{ u \in L^2(D \times (0, T)); u(t, x', 0) \in L^2(D \times (0, T)) \} \) is the solution of the variational problem

\[
\int_0^T \int_D K (f - \nabla p^0) \cdot \nabla \varphi + \frac{\kappa^3}{12} \int_0^T \int_\Sigma (\tilde{f}(t, x', 0) - \partial_u P^0(t, x', 0)) \cdot \partial_u \varphi(t, x', 0) = 0,
\]

for every \( \varphi \in V_S \). Moreover, there exists a constant \( C \) such that

\[
p^0(t, x', 0) = p^0(t, x') + C \quad \text{for all } (t, x') \in (0, T) \times \Sigma.
\]

The proof of Theorem 4.1 will be divided in several steps and which is included by some lemmas.

### 4.1. Problem in the porous part \((\eta \leq \kappa \varepsilon^2)\)

In this part, we give the proof of Theorem 4.1 in the case of \( \eta \leq \kappa \varepsilon^2 \).

**Lemma 4.1.** Let \( \eta \leq \kappa \varepsilon^2 \) and let \( \{v^{\varepsilon n}, p^{\varepsilon n}\} \) be solution for (1), \( \{v^{\eta}\} \) is defined by (10). Then

(i) There exist subsequences of \( \{v^{\varepsilon n}\} \), \( \{p^{\varepsilon n}\} \) (still denoted by the same symbols), and functions \( v^0 \in L^2(D \times (0, T)), p^0 \in L^2(D \times (0, T)) \) such that

\[
\frac{1}{\varepsilon^2} v^{\varepsilon n} \rightharpoonup v^0 \quad \text{weakly in } L^2(D \times (0, T)),
\]

\[
p^{\varepsilon n} \rightharpoonup p^0 \quad \text{weakly in } L^2(D \times (0, T)).
\]

(ii) Furthermore, let \( v^0 \in L^2(D \times (0, T)), p^0 \in L^2(D \times (0, T)) \) be given by (14). Then

\[
v^0 = K(f - \nabla p^0) \quad \text{in } D'.
\]

**Proof.** The proof is based on the idea from Bourgeat [4]. Using a priori estimates (7) and (11) we find functions \( v^0 \in L^2(D \times (0, T)), p^0 \in L^2(D \times (0, T)) \) and extract the subsequence \( \{v^{\varepsilon n}\}, \{p^{\varepsilon n}\} \) such that (14) holds and let \( \{u^k, r^k\} \) be the solution of the auxiliary problem. We set

\[
w^{k}_{\varepsilon n} = \begin{cases} w^k(\frac{x}{\varepsilon}) & \text{in } \Omega_{\varepsilon n}^-; \\
\kappa (\frac{x}{\varepsilon}) & \text{in } \Omega_{\varepsilon n}^+; \\
r^{k}_{\varepsilon n} = \begin{cases} r^k(\frac{x}{\varepsilon}) & \text{in } \Omega_{\varepsilon n}^-; \\
r^k(\frac{x}{\varepsilon}) & \text{in } \Omega_{\varepsilon n}^+; \\
\end{cases}
\end{cases}
\]

where \( u^k, r^k \) are extended by periodicity to \( R^n \). Then

\[
\begin{align*}
-\varepsilon^2 \Delta u^{k}_{\varepsilon n} + \varepsilon \nabla r^{k}_{\varepsilon n} &= e_k & \text{in } \Omega_{\varepsilon n}, \\
\text{div } u^{k}_{\varepsilon n} &= 0 & \text{in } \Omega_{\varepsilon n}.
\end{align*}
\]

Let us consider the convergence in \( D^+ \times (0, T) \). Let \( \varphi \in C^\infty(0, T; C_0^\infty(D^+)) \). Multiplying the first equation in (1) by \( u^{k}_{\varepsilon n} \varphi \) and integrating over \( D \times (0, T) \),

\[
\int_0^T \int_{D^+} \frac{\partial u^{k}_{\varepsilon n}}{\partial t} \cdot u^{k}_{\varepsilon n} \varphi \, dx \, dt - \int_0^T \int_{D^+} \Delta u^{k}_{\varepsilon n} \cdot u^{k}_{\varepsilon n} \varphi \, dx \, dt + \int_0^T \int_{D^+} \nabla p^{k}_{\varepsilon n} \cdot u^{k}_{\varepsilon n} \varphi \, dx \, dt = \int_0^T \int_{D^+} f \cdot u^{k}_{\varepsilon n} \varphi \, dx \, dt.
\]
Now, we discuss the convergence of each term in (17). Firstly, we have

$$\int_0^T \int_{D^+} \frac{\partial u^{\varepsilon \eta}}{\partial t} \cdot w^{k}_{\varepsilon \eta} \varphi \, dx \, dt = \int_0^T \int_{D^+} u^{\varepsilon \eta}_0 \cdot w^{k}_{\varepsilon \eta} \varphi(x, 0) \, dx - \int_0^T \int_{D^+} u^{\varepsilon \eta} \cdot w^{k}_{\varepsilon \eta} \frac{\partial \varphi}{\partial t} \, dx \, dt = I_1 + I_2.$$ 

Using the assumption on the initial data, we have

$$|I_1| \leq \left\| u^{\varepsilon \eta}_0 \right\|_{L^\infty(0, T; L^2(D^+))} \left\| w^{k}_{\varepsilon \eta} \right\|_{L^\infty(0, T; L^2(D^+))} \left\| \varphi(x, 0) \right\|_{L^1(0, T; L^\infty(D^+))}$$

$$\leq C(\varepsilon + \eta^2) \to 0, \quad \text{as } \varepsilon \text{ tends to zero.}$$

Furthermore, using a priori estimate (4) we get

$$|I_2| \leq \left\| u^{\varepsilon \eta} \right\|_{L^2(D^+ \times (0, T))} \left\| w^{k}_{\varepsilon \eta} \right\|_{L^\infty(0, T; L^2(D^+))} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(0, T; L^\infty(D^+))}$$

$$\leq C(\varepsilon^2 + \varepsilon \eta^3) \to 0, \quad \text{as } \varepsilon \text{ tends to zero.}$$

Next, we consider the convergence of second term in (17), after integrating by parts, we have

$$- \int_0^T \int_{D^+} \Delta u^{\varepsilon \eta} \cdot w^{k}_{\varepsilon \eta} \varphi \, dx \, dt = \int_0^T \int_{D^+} \nabla u^{\varepsilon \eta} : \nabla w^{k}_{\varepsilon \eta} \varphi \, dx \, dt + \int_0^T \int_{D^+} \nabla u^{\varepsilon \eta} : (w^{k}_{\varepsilon \eta} \otimes \nabla \varphi) \, dx \, dt.$$ 

Using a priori estimate (3) we get

$$\int_0^T \int_{D^+} \nabla u^{\varepsilon \eta} : (w^{k}_{\varepsilon \eta} \otimes \nabla \varphi) \, dx \, dt \leq \left\| \nabla u^{\varepsilon \eta} \right\|_{L^2(D^+ \times (0, T))} \left\| w^{k}_{\varepsilon \eta} \right\|_{L^\infty(0, T; L^2(D^+))} \left\| \nabla \varphi \right\|_{L^2(0, T; L^\infty(D^+))}$$

$$\leq C(\varepsilon + \eta^2) \to 0, \quad \text{as } \varepsilon \text{ tends to zero.}$$

Integrating by parts, we have

$$\int_0^T \int_{D^+} \nabla u^{\varepsilon \eta} \cdot \nabla w^{k}_{\varepsilon \eta} \varphi \, dx \, dt = - \int_0^T \int_{D^+} u^{\varepsilon \eta} \cdot \Delta w^{k}_{\varepsilon \eta} \varphi \, dx \, dt - \int_0^T \int_{D^+} u^{\varepsilon \eta} \otimes \nabla \varphi : \nabla w^{k}_{\varepsilon \eta} \, dx \, dt = I_1 + I_2.$$ 

Using a priori estimate (4) once again, we get

$$|I_2| \leq \left\| u^{\varepsilon \eta} \right\|_{L^2(D^+ \times (0, T))} \left\| \nabla w^{k}_{\varepsilon \eta} \right\|_{L^\infty(0, T; L^2(D^+))} \left\| \nabla \varphi \right\|_{L^2(0, T; L^\infty(D^+))}$$

$$\leq C(\varepsilon^2 + \varepsilon \eta^3) \left\| \nabla w^{k}_{\varepsilon \eta} \right\|_{L^\infty(0, T; L^2(D^+))} \leq C(\varepsilon + \eta^2) \to 0, \quad \text{as } \varepsilon \text{ tends to zero.}$$

By use of the auxiliary problem we obtain

$$I_1 = \varepsilon^{-2} \int_0^T \int_{D^+} u^{\varepsilon \eta} \cdot (e_k - \varepsilon \nabla r^{k}_{\varepsilon \eta}) \varphi \, dx \, dt = \varepsilon^{-2} \int_0^T \int_{D^+} u^{\varepsilon \eta} \cdot e_k \varphi \, dx \, dt - \varepsilon^{-1} \int_0^T \int_{D^+} u^{\varepsilon \eta} \cdot \nabla r^{k}_{\varepsilon \eta} \varphi \, dx \, dt$$

$$= I_{11} + I_{12}.$$ 

Taking the limit as $\varepsilon \to 0$, using (14), we obtain

$$I_{11} \to \int_0^T \int_{D^+} v^0_k \varphi \, dx \, dt.$$ 

Integrating by parts once again and using (4), we have
It is easy to verify our claim since

\[ I_{12} = \varepsilon^{-1} \int_0^T \int_{D^+} \varphi \cdot \nabla \psi_k \ dx \ dt \leq \varepsilon^{-1} \left\| \varphi \right\|_{L^2(D^+ \times (0,T))} \left\| \nabla \psi_k \right\|_{L^\infty(0,T;L^2(D^+))} \left\| \nabla \varphi \right\|_{L^2(0,T;L^\infty(D^+))} \]

\[ \leq C(\varepsilon + \eta^2) \to 0, \quad \text{as } \varepsilon \text{ tends to zero.} \]

Consequently, we have

\[ - \int_0^T \Delta u^\varepsilon \cdot w^k \varphi \ dx \ dt \to \int_0^T \int_{D^+} \psi_0 \varphi \ dx \ dt, \quad \text{as } \varepsilon \text{ tends to zero.} \quad (18) \]

It is difficult to obtain the convergence of \( \int_0^T \int_{D^+} \nabla p^\varepsilon \cdot w^k \varphi \ dx \ dt \) since \( \nabla p^\varepsilon \) and \( w^k \) are both convergent weakly in their own spaces. To this end, we introduce the "lift" operator (see [7,11]).

**Lemma 4.2.** Let \( \Omega \) be an arbitrary bounded domain with Lipschitz boundary \( \partial \Omega \) and \( 1 < q < +\infty \). Then for any \( f \in L^q_0(\Omega) \), there exists a function \( u \in [W^{1,q}_0(\Omega)]^n \), satisfying \( f = \text{div} \ u \). Furthermore, there exists a constant \( C = C(\varepsilon, \Omega) \), such that

\[ \left\| u \right\|_{W^{1,q}_0(\Omega)^n} \leq C \left\| f \right\|_{L^q_0(\Omega)}. \]

Let \( V^k_\varepsilon = u^k_\varepsilon - \frac{1}{|D^+|} \int_{D^+} u^k_\varepsilon \ dx \). For \( q = 2 \), it is obvious that \( V^k_\varepsilon \to 0 \) in \( L^2_0(D^+) \). By Lemma 4.2, we can construct \( \{ \tilde{\psi}^k_\varepsilon \} \subseteq [W^{1,2}_0(D^+)]^n \) such that \( \text{div} \tilde{\psi}^k_\varepsilon = V^k_\varepsilon \). We also have the following estimate

\[ \left\| \tilde{\psi}^k_\varepsilon \right\|_{W^{1,2}_0(D^+)^n} \leq C \left\| V^k_\varepsilon \right\|_{L^2_0(D^+)}. \]

Since \( \{ \tilde{\psi}^k_\varepsilon \} \) is bounded in \( [W^{1,2}_0(D^+)]^n \) and \( V^k_\varepsilon \to 0 \) in \( L^2_0(D^+) \), up to a subsequence, we can assume that \( \tilde{\psi}^k_\varepsilon \to \tilde{\psi} \) in \( [W^{1,2}_0(D^+)]^n \), \( \text{div} \tilde{\psi}^k_\varepsilon = \tilde{\psi} \).

Let \( \psi^k_\varepsilon = \tilde{\psi}^k_\varepsilon - \tilde{\psi}^k - \frac{1}{|D^+|} \int_{D^+} (\tilde{\psi}^k_{\varepsilon,0} - \tilde{\psi}^k) \ dx \). It is obvious that \( \text{div} \psi^k_\varepsilon = V^k_\varepsilon \) and \( \psi^k_\varepsilon \to 0 \) in \( [W^{1,2}_0(D^+)]^n \).

The Rellich theorem implies that \( \psi^k_\varepsilon \to 0 \) in \( L^2_0(D^+) \).

We claim that \( I_1 \) and \( I_2 \) both converge to 0. In fact, from the definition of \( \psi^k_\varepsilon \), we notice that it equals to zero out of \( D^+ \). On the other hand, we have an estimate on \( \nabla p^\varepsilon \) as in (7), therefore,

\[ |I_1| \leq C \left( \left\| \psi^k_\varepsilon \right\|_{L^2_0(D^+)^n} + \varepsilon \left\| \nabla \psi^k_\varepsilon \right\|_{L^2_0(D^+)^n} \right). \]

For the same reason we get

\[ |I_2| \leq C \left( \left\| \psi^k_\varepsilon \right\|_{L^2_0(D^+)^n} + \varepsilon \left\| \nabla \psi^k_\varepsilon \right\|_{L^2_0(D^+)^n} \right). \]

It is easy to verify our claim since \( \psi^k_\varepsilon \to 0 \) in \( L^2_0(D) \). Consequently, we have

\[ \int_0^T \int_{D^+} p^\varepsilon w^k_\varepsilon \cdot \nabla \varphi \ dx \ dt - \int_0^T \int_{D^+} p^\varepsilon \nabla \varphi \left( \frac{1}{|D^+|} \int_{D^+} w^k_\varepsilon \ dx \right) \ dx \ dt \to 0, \quad \text{as } \varepsilon \to 0. \]
That is,
\[
\int_0^T \int_{D^+} p^{\varepsilon \eta} u_{\varepsilon \eta}^k \cdot \nabla \varphi \, dx \, dt \to \int_0^T \int_D p^0 \nabla \varphi \cdot K_k \, dx \, dt = - \int_0^T \int_D \nabla p^0 \cdot K_k \varphi \, dx \, dt.
\]  
(19)
Using the hypothesis on the external force, we obtain
\[
\int_0^T \int_{D^+} f \cdot u_{\varepsilon \eta}^k \varphi \, dx \, dt \to \int_0^T \int_D f \varphi \, dx \, dt,
\]  
(20)
as \varepsilon \to 0.

Using (18)–(20), we get
\[
\int_0^T \int_{D^+} v_0^\varepsilon \varphi \, dx \, dt \to \int_0^T \int_D (f - \nabla p^0) \varphi \, dx \, dt.
\]

Since \( \varphi \) is arbitrary, we obtain
\[
v_0^\varepsilon = \mathcal{K}(f - \nabla p^0) \quad \text{in} \quad \mathcal{D}'.
\]

In order to consider the convergence process in \( D^- \times (0, T) \), we only need to choose \( \varphi \in C^\infty(0, T; C_0^\infty(D^-)) \). Multiplying the first equation in (1) by \( u_{\varepsilon \eta}^k \) and integrating over \( D \times (0, T) \), using the same method as above, we can get the same result. \( \Box \)

4.2. Problem in the fissure \((\eta \geq \kappa \varepsilon^2)\)

Before studying the homogenized equation satisfied in the fissure, we should get some new \textit{a priori} estimates for Eqs. (12).

*Lemma 4.3.* Let \( U^{\varepsilon \eta}, \ P^{\varepsilon \eta} \) be solutions of (12). Then
\[
\| U^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\eta^2, \quad \| \frac{\partial U^{\varepsilon \eta}}{\partial z} \|_{L^2(I_1 \times (0, T))} \leq C\eta^2,
\]
\[
\| \partial_\eta U^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\eta, \quad \| P^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C.
\]  
(21)

*Proof.* Using (2) and (5) we obtain by simple change of variables
\[
\| U^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} = \left( \int_0^T \int_{I_1} |U^{\varepsilon \eta}|^2 \, dx \, dz \, dt \right)^{\frac{1}{2}} \leq \left( \frac{1}{\eta} \int_0^T \int_{I_1} |u^{\varepsilon \eta}|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\eta}} \| u^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))}
\]
\[
\leq C(\varepsilon^\frac{1}{2} \eta^\frac{3}{2} + \eta^2) \leq C\eta^2,
\]
\[
\| \frac{\partial U^{\varepsilon \eta}}{\partial z} \|_{L^2(I_1 \times (0, T))} = \frac{1}{\sqrt{\eta}} \| \eta \partial_\eta u^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\sqrt{\eta}(\varepsilon + \eta^2) \leq C\eta^2,
\]
\[
\| \partial_\eta U^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} = \frac{1}{\sqrt{\eta}} \| \partial_\eta u^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\frac{1}{\sqrt{\eta}}(\varepsilon + \eta^2) \leq C\eta,
\]
\[
\| P^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} = \frac{1}{\sqrt{\eta}} \| P^{\varepsilon \eta} - C^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\sqrt{\eta}(\sqrt{\eta} + \frac{\varepsilon}{\eta}) \leq C,
\]
\[
\| U^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T)} \leq \limsup_{\varepsilon \to 0^{+}} \left( \frac{1}{\eta} \int_{I_1} |u^{\varepsilon \eta}|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\eta}} \| u^{\varepsilon \eta} \|_{L^2(I_1 \times (0, T))} \leq C\frac{1}{\sqrt{\eta}}(\varepsilon + \eta^2) \leq C\eta,
\]
and (21) is proved. \( \Box \)
Furthermore using estimates (21) and the compactness we prove the following lemma:

**Lemma 4.4.** Let \( \eta \geq \kappa \varepsilon^{\frac{5}{2}} \) and \( \{U^{\varepsilon \eta}, P^{\varepsilon \eta}\} \) be solutions of (12). Then

(i) There exists \( U^0 \in L^2(I_1 \times (0, T)) \), such that \( U^0_n = 0 \), \( P^0 \in L^2(I_1 \times (0, T)) \), up to a subsequence,

\[
\frac{1}{\eta^2} U^{\varepsilon \eta} \rightharpoonup U^0 \quad \text{weakly in } L^2(I_1 \times (0, T)),
\]

\[
P^{\varepsilon \eta} \rightharpoonup P^0 \quad \text{weakly in } L^2(I_1 \times (0, T)).
\]

(ii) Furthermore, let \( U^0 \in L^2(I_1 \times (0, T)) \) and \( P^0 \in L^2(I_1 \times (0, T)) \) be defined by (22). Then

\[
\tilde{U}^0(t, x', z) = \frac{1}{2}(z - z_2^2)(\tilde{f}(t, x', 0) - \partial_\alpha P^0(t, x')),
\]

in the sense of distribution, where \( U^0 = (\tilde{U}^0, 0) \).

**Proof.** Using Lemma 4.3 and standard compactness arguments we find \( U^0, P^0 \) such that (22) holds. Moreover

\[
1 \eta \int_0^T \int_{I_1} \left( \text{div} \tilde{U}^{\varepsilon \eta} + \frac{1}{\eta} \frac{\partial U^{\varepsilon \eta}}{\partial z} \right) \varphi \, dx' \, dz \, dt = -\frac{1}{\eta} \int_0^T \int_{I_1} \tilde{U}^{\varepsilon \eta} \cdot \partial_\alpha \varphi \, dx' \, dz \, dt - \frac{1}{\eta^2} \int_0^T \int_{I_1} U^{\varepsilon \eta} \frac{\partial \varphi}{\partial z} \, dx' \, dz \, dt.
\]

Taking the limit as \( \varepsilon, \eta \to 0 \) we find

\[
\int_0^T \int_{I_1} U^0_n \frac{\partial \varphi}{\partial z} \, dx' \, dz \, dt = 0.
\]

Hence

\[
U^0_n = U^0_n(t, x').
\]

Since \( U^0_n, \frac{\partial U^0_n}{\partial z} \in L^2(I_1 \times (0, T)) \), the traces \( U^0_n(t, x', 0), U^0_n(t, x', 1) \) are well defined in \( L^2(\Sigma) \) for all \( t \in (0, T) \). Using the trace theorem in the porous medium (see Bourgeat [5]) and (3) we obtain

\[
\frac{1}{\eta^2} \left\| U^0_n(t, x', 0) \right\|_{L^2(\Sigma \times (0, T))} \leq C \frac{\sqrt{\varepsilon}}{\eta^2} \left\| \nabla u^{\varepsilon \eta} \right\|_{L^2(D_{\eta^2} \times (0, T))} \leq C \frac{\sqrt{\varepsilon}}{\eta^2} (\varepsilon + \eta^2)
\]

\[
\leq C \left( \frac{\varepsilon^2}{\eta^2} + \sqrt{\frac{\varepsilon}{\eta}} \right) \to 0, \quad \text{as } \varepsilon, \eta \to 0,
\]

which implies that

\[
U^0_n(t, x', 0) = 0,
\]

and analogously

\[
U^0_n(t, x', 1) = 0.
\]

We conclude that \( U^0_n(t, x') = 0 \) since \( U^0_n(t, x') \) is independent of \( z \) and (i) is proved.
Now we turn to prove (ii). Using (22) we obtain for any $\varphi \in C_0^\infty((0, T); C_0^\infty(I_1))$

\[
-\int_0^T \int_{I_1} U^{\varepsilon \eta} \cdot \frac{\partial \varphi}{\partial t} + \int_0^T \int_{I_1} \partial_a U^{\varepsilon \eta} \cdot \partial_a \varphi + \frac{1}{\eta^2} \int_0^T \int_{I_1} \frac{\partial U^{\varepsilon \eta}}{\partial z} \cdot \frac{\partial \varphi}{\partial z} \, dx \, dz dt + \int_{I_1} (U_0^{\varepsilon \eta} \cdot \varphi) \big|_{t=0} \, dx \, dz + \int_0^T \int_{I_1} f(t, x', \eta z) \cdot \varphi + f_{x'} \cdot \varphi + \frac{1}{\eta} \int_0^T \int_{I_1} P^{\varepsilon \eta} \, dz dt = 0. \tag{23}
\]

Multiplying by $\eta$ and taking the limits as $\varepsilon, \eta \to 0$ we have

\[
\int_0^T \int_{I_1} p^0 \frac{\partial \varphi}{\partial z} \, dx' \, dz' dt = 0,
\]

which implies that

\[
p^0(t, x', z) = p^0(t, x'),
\]

since we have the following convergence results to each term in (23).

Using Lemma 4.3, for the first term in (23), we have

\[
-\eta \int_0^T \int_{I_1} U^{\varepsilon \eta} \cdot \frac{\partial \varphi}{\partial t} \, dx' \, dz \, dt \leq \eta \left\| U^{\varepsilon \eta} \right\|_{L^2(I_1 \times (0, T))} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(I_1 \times (0, T))} \leq C \eta^3 \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

For the second term in (23), using Lemma 4.3, we obtain

\[
\eta \int_0^T \int_{I_1} \partial_a U^{\varepsilon \eta} \cdot \partial_a \varphi \, dx' \, dz \, dt \leq \eta \left\| \partial_a U^{\varepsilon \eta} \right\|_{L^2(I_1 \times (0, T))} \left\| \partial_a \varphi \right\|_{L^2(I_1 \times (0, T))} \leq C \eta^2 \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

For the third term in (23), using Lemma 4.3 once more, we have

\[
\frac{1}{\eta} \int_0^T \int_{I_1} \frac{\partial U^{\varepsilon \eta}}{\partial z} \cdot \frac{\partial \varphi}{\partial z} \, dx' \, dz \, dt \leq \frac{\eta}{\eta^2} \left\| \frac{\partial U^{\varepsilon \eta}}{\partial z} \right\|_{L^2(I_1 \times (0, T))} \left\| \frac{\partial \varphi}{\partial z} \right\|_{L^2(I_1 \times (0, T))} \leq C \eta \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

Using the assumption on the external force, we have

\[
\eta \int_0^T \int_{I_1} f(t, x', \eta z) \cdot \varphi \, dx' \, dz \, dt \leq C \eta \left\| f(t, x', \eta z) \right\|_{L^\infty(I_1 \times (0, T))} \left\| \varphi \right\|_{L^\infty(I_1 \times (0, T))} \leq C \eta \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

It is obvious that

\[
\eta \int_{I_1} \left( U^{\varepsilon \eta} \cdot \varphi \right) \big|_{t=0} \, dx' \, dz \leq C \eta \left\| U^{\varepsilon \eta}(0, x', z) \right\|_{L^2(I_1)} \left\| \varphi(0, x) \right\|_{L^2(I_1)} \leq C \eta^2 \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

Finally, for the sixth term in (23), using Lemma 4.3, we obtain

\[
\eta \int_0^T \int_{I_1} P^{\varepsilon \eta} \text{div}_{x'} \varphi \, dx' \, dz \, dt \leq C \eta \left\| P^{\varepsilon \eta} \right\|_{L^2(I_1 \times (0, T))} \left\| \text{div}_{x'} \varphi \right\|_{L^2(I_1 \times (0, T))} \leq C \eta \to 0, \quad \text{as } \varepsilon, \eta \to 0.
\]

Since $P^0(t, x') \in L^2(I_1 \times (0, T))$ and using Fubini's theorem we have
\[ \int_0^T \int_{I_1} |P^0(t, x')|^2 \, dx' \, dz + \int_0^T \int_{\Sigma} \left| P^0(t, x') \right|^2 \, dx' \, dz = \int_0^T \int_{I_1} f(t, x', \eta z) \cdot \varphi \, dx' \, dz + \int_0^T \int_{\Sigma} P^0(t, x') \cdot \text{div}_x' \varphi \, dx' \, dz, \]

which gives \( P^0(t, x') \in L^2(\Sigma \times (0, T)) \).

Choosing in (23) a smooth function \( \varphi_n \) such that \( \varphi_n = 0 \) we get

\[ \int_0^T \int_{I_1} \partial_\alpha U_{\varepsilon \eta} : \partial_\alpha \varphi \, dx' \, dz + \frac{1}{\eta^2} \int_0^T \int_{I_1} \partial \varphi \cdot \partial \varphi \, dx' \, dz + \int_0^T \int_{I_1} \partial U_{\varepsilon \eta} \cdot \partial \varphi \, dx' \, dz + \int_0^T \int_{I_1} f(t, x', \eta z) \cdot \varphi \, dx' \, dz + \int_0^T \int_{\Sigma} P_{\varepsilon \eta} \cdot \text{div}_x' \varphi \, dx' \, dz. \]  

(24)

Taking the limits as \( \varepsilon, \eta \to 0 \) in (24), using (22) and Lemma 4.3, we obtain

\[ \int_0^T \int_{I_1} \partial_\alpha U_{\varepsilon \eta} : \partial_\alpha \varphi \, dx' \, dz \leq \left\| \partial_\alpha U_{\varepsilon \eta} \right\|_{L^2(I_1 \times (0, T))} \left\| \partial_\alpha \varphi \right\|_{L^2(I_1 \times (0, T))} \leq C \eta \to 0, \quad \text{as} \ \varepsilon, \eta \to 0, \]

\[ \frac{1}{\eta^2} \int_0^T \int_{I_1} \partial \varphi \cdot \partial \varphi \, dx' \, dz \to \int_0^T \int_{I_1} \partial U^0 \cdot \partial \varphi \, dx' \, dz, \quad \text{as} \ \varepsilon, \eta \to 0, \]

\[ \int_0^T \int_{I_1} f(t, x', \eta z) \cdot \varphi \, dx' \, dz \to \int_0^T \int_{I_1} f(t, x', 0) \cdot \varphi \, dx' \, dz, \quad \text{as} \ \varepsilon, \eta \to 0, \]

\[ \int_0^T \int_{I_1} P_{\varepsilon \eta} \cdot \text{div}_x' \varphi \, dx' \, dz \to \int_0^T \int_{I_1} P^0(t, x') \cdot \text{div}_x' \varphi \, dx' \, dz, \quad \text{as} \ \varepsilon, \eta \to 0. \]

Combining all above information, we obtain

\[ \int_0^T \int_{I_1} \partial U^0 \cdot \partial \varphi = \int_0^T \int_{I_1} f(t, x', 0) \cdot \varphi + \int_0^T \int_{I_1} P^0(t, x') \cdot \text{div}_x' \varphi. \]

(25)

Note that \( U^0_n = 0 \), (25) can be rewritten in the form

\[ \left\{ \begin{array}{l} -\partial_z^2 \bar{U}_n = \bar{f}(t, x', 0) - \partial_a \bar{P}^0(t, x'), \\
U^0_n(t, x', 0) = U^0(t, x', 1) = 0. \end{array} \right. \]

(26)

It is an ODE with order 2 and its solution is given by

\[ \bar{U}_n(t, x', z) = \frac{1}{2}(z - z^2)(\bar{f}(t, x', 0) - \partial_a \bar{P}^0(t, x')). \]

(ii) is proved. \( \square \)

4.3. Effects of coupling (\( \eta = \kappa \varepsilon^{\frac{2}{3}} \))

The conclusion of the previous two sections is that for any sequence of solutions \( \{v^{\varepsilon \eta}, p^{\varepsilon \eta}\} \) with \( \eta \leq \kappa \varepsilon^{\frac{2}{3}} \) or \( \{U^{\varepsilon \eta}, P^{\varepsilon \eta}\} \) with \( \eta \geq \kappa \varepsilon^{\frac{2}{3}} \) and \( \varepsilon, \eta \to 0 \), we can extract subsequences \( \{v^{\varepsilon \eta}\}, \{p^{\varepsilon \eta}\}, \{U^{\varepsilon \eta}\}, \{P^{\varepsilon \eta}\} \) and find functions
\( v^0 \in L^2(D \times (0, T)) \), \( p^0 \in L^2(D \times (0, T)) \), \( U^0 \in L^2(I_1 \times (0, T)) \), \( P^0 \in L^2(0, T; L^2(\Sigma)) \) such that

\[
\frac{1}{\varepsilon^2} v^{\varepsilon \eta} \rightharpoonup v^0 \quad \text{weakly in } L^2(D \times (0, T)),
\]

\( p^{\varepsilon \eta} \rightharpoonup p^0 \quad \text{weakly in } L^2(D \times (0, T)),
\]

\[
\frac{1}{\eta^2} U^{\varepsilon \eta} \rightharpoonup U^0 \quad \text{weakly in } L^2(I_1 \times (0, T)), \quad U^0 = (\tilde{U}^0, 0),
\]

\( P^{\varepsilon \eta} \rightharpoonup P^0 \quad \text{weakly in } L^2(\Sigma \times (0, T)). \quad (27)
\]

Moreover, those limit functions \( v^0, p^0, U^0, P^0 \) necessarily satisfy the equations

\[
v^0 = \mathcal{K}(f - \nabla p^0) \quad \text{in } D \times (0, T),
\]

\[
v^0 \cdot \hat{n} = 0 \quad \text{on } \partial D \text{ for all } t \in (0, T),
\]

\[
\tilde{U}^0 = \frac{1}{2} (z - z^2)(\tilde{f}(t, x', 0) - \partial_x P^0) \quad \text{on } \Sigma \times (0, T),
\]

\[
\tilde{U}^0 \cdot \hat{n} = 0 \quad \text{on } \partial \Sigma \text{ for all } t \in (0, T). \quad (28)
\]

We are going to find the connection between the functions \( p^0 \) and \( P^0 \), i.e. to find the coupling effects between the solution in the porous medium and in the fissure.

**Lemma 4.5.** Let \( \eta = \kappa \varepsilon^2 \) and let \( \{p^{\varepsilon \eta}\} \subset L^2(D \times (0, T)) \), \( p^0 \in L^2(D \times (0, T)) \), \( P^0 \in L^2(0, T; L^2(\Sigma)) \) be such that (27) and (28) hold. Then

\[
\int_0^T \int_D \mathcal{K}(f - \nabla p^0) \cdot \nabla \varphi + \frac{\kappa^3}{12} \int_0^T \int_\Sigma (\tilde{f}(t, x', 0) - \partial_x P^0(t, x')) \cdot \partial_\alpha \varphi(t, x', 0) = 0,
\]

for every \( \varphi \in V_\Sigma = \{\varphi \in L^2(0, T; H^1(D)), \varphi(t, x', 0) \in L^2(0, T; H^1(\Sigma))\}. \quad (29)

**Proof.** Let \( \varphi \in V_\Sigma \). We use the divergence-free of \( u^{\varepsilon \eta} \) to deduce that

\[
0 = \int_0^T \int_D u^{\varepsilon \eta} \cdot \nabla \varphi = \int_0^T \int_D v^{\varepsilon \eta} \cdot \nabla \varphi + \frac{\kappa^3}{12} \int_0^T \int_{I_1} U^{\varepsilon \eta} \cdot \nabla \varphi(t, x', \eta z).
\]

Taking the limit as \( \varepsilon, \eta \to 0 \) and using (27), we obtain

\[
\int_0^T \int_D v^0 \cdot \nabla \varphi + \kappa^3 \int_0^T \int_{I_1} \tilde{U}^0 \cdot \partial_\alpha \varphi(t, x', 0) = 0.
\]

Finally (28) implies (29).

Hence, we prove the second part in Theorem 4.1. \( \square \)

In fact, we can also prove that the relationship between \( p(t, x', 0) \) and \( P^0(t, x') \) is given by the following lemma:

**Lemma 4.6.** Let \( \eta = \kappa \varepsilon^2 \) and \( p(t, x', 0), P^0(t, x') \) be the limit functions from (27). Then there exists \( C \in R \) such that

\[
p^0(t, x', 0) = P^0(t, x') + C. \quad (30)
\]

The proof can be seen in [4] with slightly change. The main difficulty we meet there is also the convergence of the pressure term, and it can be treated just the same as above, so we omit it for short.

Now, we only need to prove the special case in the third part in Theorem 4.1 to complete the whole proof. We have
Lemma 4.7. Let $\eta > \kappa \varepsilon^{\frac{2}{3}}$. Then
\[
1 \eta^{3} u^{\eta} \rightharpoonup^{*} U^{0} \delta \Sigma \quad \text{weakly in } L^{2}(0, T; \mathcal{M}(D)),
\]
where
\[
\tilde{U}^{0}(t, x') = \frac{1}{12} \left( \tilde{f}(t, x', 0) - \partial_{a} P^{0}(t, x') \right),
\]
and $P^{0} \in L^{2}(0, T; H^{1}(\Sigma)/R)$ is the unique solution of the following problem
\[
\begin{cases}
\Delta x_{'} P^{0} = \text{div} x_{'} \tilde{f}(t, x', 0) & \text{in } \Sigma \text{ for any } t \in (0, T), \\
(\tilde{f}(t, x', 0) - \partial_{a} P^{0}(t, x')) \cdot n_{a} = 0 & \text{on } \partial \Sigma \text{ for any } t \in (0, T).
\end{cases}
\]  
(31)

**Proof.** We choose the test function $\varphi \in C^{\infty}_{0}(D \times (0, T))$, then we have
\[
0 = \int_{0}^{T} \int_{D} \text{div } u^{\eta} \varphi = - \int_{0}^{T} \int_{D} v^{\eta} \cdot \nabla \varphi - \eta \int_{0}^{T} \int_{I_{1}} U^{\eta} \cdot \nabla \varphi(t, x', \eta z),
\]
therefore
\[
\int_{0}^{T} \int_{I_{1}} \frac{U^{\eta}_{\varphi}}{\eta^{2}} \cdot \partial_{a} \varphi(t, x', \eta z) = - \frac{1}{\eta^{2}} \int_{0}^{T} \int_{D} v^{\eta} \cdot \nabla \varphi - \frac{1}{\eta^{2}} \int_{0}^{T} \int_{I_{1}} U^{\eta}_{\varphi} \partial_{a} \varphi(t, x', \eta z).
\]  
(32)

Using (11), we get
\[
\frac{1}{\eta^{3}} \| v^{\eta} \|_{L^{2}(D \times (0, T))} \leq \frac{C}{\eta^{3}} (\varepsilon^{2} + \varepsilon \eta^{2}) \to 0, \quad \text{as } \varepsilon, \eta \to 0 \text{ tend to zero.} \tag{33}
\]
Passing to the limit in (32), using (13), (33) and $U^{0}_{\varphi} = 0$, we obtain
\[
\int_{0}^{T} \int_{I_{1}} \tilde{U}^{0} \cdot \partial_{a} \varphi(t, x', 0) = 0.
\]
Using Lemma 4.4, we obtain
\[
\frac{1}{12} \int_{0}^{T} \int_{\Sigma} (\tilde{f}(t, x', 0) - \partial_{a} P^{0}) \cdot \partial_{a} \varphi(t, x', 0) = 0,
\]
thus $P^{0}$ is the unique solution of (31).

Finally, choosing $\varphi \in C^{\infty}_{0}(D \times (0, T))$, we get
\[
\frac{1}{\eta^{3}} \int_{0}^{T} \int_{D} u^{\eta} \cdot \varphi = \frac{1}{\eta^{3}} \int_{0}^{T} \int_{D} v^{\eta} \cdot \varphi + \frac{1}{\eta^{2}} \int_{0}^{T} \int_{I_{1}} U^{\eta} \cdot \varphi(t, x', \eta z).
\]
Using (13), we obtain
\[
\frac{1}{\eta^{3}} \int_{0}^{T} \int_{D} u^{\eta} \cdot \varphi \to \int_{0}^{T} \int_{I_{1}} \tilde{U}^{0} \cdot \tilde{\varphi}(t, x', 0) = \int_{0}^{T} \int_{\Sigma} \tilde{U}^{0} \cdot \tilde{\varphi}(t, x', 0) = \langle U^{0} \delta \Sigma, \varphi \rangle_{L^{2}(0, T; \mathcal{M}(D)), C_{0}(D \times (0, T))},
\]
where $\tilde{\varphi} = (\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n-1})$, $\varphi = (\tilde{\varphi}, \varphi_{n})$, which implies that
\[
\frac{1}{\eta^{3}} u^{\eta} \rightharpoonup^{*} U^{0} \delta \Sigma \quad \text{weak* in } L^{2}(0, T; \mathcal{M}(D)).
\]
Hence, we prove the main theorem in this paper. \(\square\)
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