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Entropy Transmission in C^*-Dynamical Systems

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Three entropies of a state in C^*-dynamical systems are introduced and their relations and dynamical properties are studied. The entropy (information) transmission under a channel between two dynamical systems is considered. We find a condition under which our entropy becomes a dynamical invariant between two systems.

INTRODUCTION

The concept of state in a physical system is a powerful weapon to study the dynamical behavior of that system. The entropy of a state is a measure of uncertainty or randomness of a dynamical system. After Shannon, the concept of entropy plays an essential role in information theory as well as in statistical mechanics. In these theories, not only entropy but also its transmission is important. The mechanism of transmission is expressed by a so-called channel between input sources and output receivers.

LASER physics has been much developed and its usefulness to information theory has been recently realized. Many trials [5, 6, 8–10, 15] have been made to find a mathematical basis of quantum information theory. In a previous paper [15], we introduced several ergodic channels and studied their dynamical properties.

In this paper, we mainly study the entropy transmission in quantum systems within C^*-algebraic framework.

In Section 1, we introduce three different entropies of a state in C^*-dynamical systems. We then study their relations and properties.

In Section 2, the entropy transmission through a channel is investigated. We consider under what conditions on a channel the entropy (information) can be transferred from an input system to an output system without any loss. In other words, we find conditions under which the entropy of a state is a *dynamical invariant*.

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1. Entropies in C^*-Dynamical Systems

Since Von Neumann introduced a quantum mechanical entropy around 1932, many physicists have applied it in several dynamical systems and studied its general properties [11, 13, 14, 22].

This entropy is defined as follows: Let $\mathscr{H}$ be a separable Hilbert space with the inner product $\langle\ ,\ \rangle$ (in this paper we assume the separability of $\mathscr{H}$ for brevity) and $T(\mathscr{H})_{+,1}$ be the set of all positive trace class operators ρ with $\operatorname{tr}\rho = 1$. Then the Von Neumann entropy of ρ is defined by

$$S(\rho) = -\operatorname{tr}\rho\log\rho.$$

In the same vein, the relative entropy between two states ρ and σ is defined by

$$S(\rho \mid \sigma) = \operatorname{tr}\rho(\log\rho - \log\sigma).$$

These entropies play an important role in studying quantum dynamical systems.

By several physical reasons, we had better extend the above entropies to those of C^*-dynamical systems. The program of this extension has been almost made for the relative entropy by Lindblad, Umegaki, Araki and Uhlmann [2, 3, 12, 19, 20]. The generalized relative entropy has turned out to be useful to study quantum systems [1, 7, 16, 22]. In this section we proceed with this program for entropy.

Let $\mathscr{A}$ be a C^*-algebra with unity I, $\mathfrak{S}$ be the set of all states on $\mathscr{A}$ and $\alpha(R)$ be a strongly continuous one-parameter group of automorphisms of $\mathscr{A}$. We call this triple $(\mathscr{A}, \mathfrak{S}, \alpha(R))$ a C^*-dynamical system. It is known that most physical systems can be described by this triple.

For each state $\phi \in \mathfrak{S}$, there exists a unique (up to unitary equivalence) cyclic (GNS) representation $\{\mathscr{H}_\phi, \pi_\phi, \Omega_\phi\}$ such that $\phi(A) = \langle \Omega_\phi, \pi_\phi(A)\,\Omega_\phi\rangle$ for any A in $\mathscr{A}$.

Let us here introduce the following special subsets of $\mathfrak{S}$ for the sequel discussion: (i) $I(\alpha)$, the set of all α-invariant states (i.e., $\phi\circ\alpha_t = \phi$ for every $t \in R$), and (ii) $K(\alpha)$, the set of all KMS states with respect to α_t at a certain inverse temperature $\beta = 1$ (i.e., $\phi \in K(\alpha)$ if for any pair A, B in $\mathscr{A}$, there exists a bounded function $F_{A,B}(z)$ of complex number $z \in C$ continuous on and holomorphic in the strip $0 \leqslant \operatorname{Im} z \leqslant 1$ with boundary values $F_{A,B}(t) = \phi(\alpha_t(A)B)$ and $F_{A,B}(t+i) = \phi(B\alpha_t(A))$).

It is known that a KMS state w.r.t. α_t is automatically α-invariant and would be one of the most appropriate states to describe thermodynamic equilibrium.

We denote by ex $\mathscr{S}$ the set of all extreme points of a weak*-compact convex subset $\mathscr{S}$ of $\mathfrak{S}$ (e.g., $\mathscr{S} = I(\alpha), K(\alpha)$).

Now, for each $\phi \in I(\alpha)$, there exists a unique one-parameter unitary group $u^{\phi}(R)$ such that $\pi_{\phi}(\alpha_t(A)) = u_t^{\phi} \pi_{\phi}(A) u_{-t}^{\phi}$ and $u_t^{\phi} \Omega_{\phi} = \Omega_{\phi}$ hold for any $A \in \mathscr{A}$ and all $t \in R$. Let E_{ϕ} be the projection from the cyclic Hilbert space $\mathscr{H}_{\phi}$ to the set of all u^{ϕ}-invariant vectors of $\mathscr{H}_{\phi}$. The dynamical system $(\mathscr{A}, \alpha(R))$ is said to be G-abelian on $\phi \in I(\alpha)$ if $E_{\phi} \pi_{\phi}(\mathscr{A})'' E_{\phi}$ is an abelian Von Neumann algebra.

It is known [18] that for any state $\phi \in \mathfrak{S}$, there always exists (but not necessary unique) a probability measure μ on $\mathfrak{S}$ such that $\phi = \int \omega \, d\mu$ (i.e., ϕ is the barycenter of μ). Further there exists a unique σ-weakly continuous map θ_{μ} of $L^{\infty}(\mathfrak{S}, \mu)$ into $\pi_{\phi}(\mathscr{A})'$ such as $\theta_{\mu}(1) = I$ and $\langle \Omega_{\phi}, \theta_{\mu}(f) \pi_{\phi}(A) \Omega_{\phi} \rangle = \int f(\omega) \, \omega(A) \, d\mu$ for any $A \in \mathscr{A}$ and $f \in L^{\infty}(\mathfrak{S}, \mu)$. This measure μ is said to be orthogonal if $\theta_{\mu}(1_S) \theta(1 - 1_S) = 0$ for the characteristic function 1_S of any Borel subset S of $\mathfrak{S}$. When the measure μ is orthogonal, θ_{μ} is a *-isomorphism and the set $\mathscr{C}_{\mu} = \{\theta_{\mu}(f) \mid f \in L^{\infty}(\mathfrak{S}, \mu)\}$ is an abelian Von Neumann subalgebra of $\pi_{\phi}(\mathscr{A})'$. Moreover, for a weak*-compact convex subset $\mathscr{S}$ of $\mathfrak{S}$ and $\phi \in \mathscr{S}$, there exists a maximal (in the sense of Bishop–deLeeuw) measure μ on $\mathscr{S}$ such that ϕ is the barycenter of μ and μ is pseudosupported by ex $\mathscr{S}$ in the sense that $\mu(\mathscr{B}) = 1$ for every Baire subset $\mathscr{B}$ of $\mathscr{S}$ with $\mathscr{B} \supset \operatorname{ex} \mathscr{S}$. In this case, we write

$$\phi = \int_{(\operatorname{ex} \mathscr{S})} \omega \, d\mu.$$

This measure μ is not always unique. However, it is shown [18, p. 241], that for every $\phi \in \mathfrak{S}$ and every abelian Von Neumann algebra $\mathscr{C}$ of $\pi_{\phi}(\mathscr{A})'$, there exists a unique orthogonal measure μ such as $\mathscr{C}_{\mu} = \mathscr{C}$. In particular, (i) if $\mathscr{C}$ is the center $\mathscr{Z}_{\phi} = \pi_{\phi}(\mathscr{A})'' \cap \pi_{\phi}(\mathscr{A})'$, then the unique orthogonal measure μ w.r.t. $\mathscr{Z}_{\phi}$ is pseudosupported by the set of all factor states and it is called the central measure of ϕ, and (ii) if the dynamical system $(\mathscr{A}, \alpha(R))$ is G-abelian on ϕ in $I(\alpha)$ and $\mathscr{C}$ is the set $\pi_{\phi}(\mathscr{A})' \cap u^{\phi}(R)'$, then the orthogonal measure μ w.r.t. $\mathscr{C}$ is unique and μ is pseudosupported by ex $I(\alpha)$.

We now define the entropy of a state in our dynamical system as follows: For each state ϕ in a weak*-compact convex subset $\mathscr{S}$ of $\mathfrak{S}$, as discussed above, there exists a probability measure μ pseudosupported by ex $\mathscr{S}$. If this measure μ is atomic in the sense that ϕ is expressed by $\phi = \sum_n \lambda_n \phi_n$, $\phi_n \in \operatorname{ex} \mathscr{S}$, $\sum_n \lambda_n = 1$, $\lambda_n \geqslant 0$, then the entropy of ϕ with respect to $\mathscr{S}$ is given by

$$S^{\mathscr{S}}(\phi) = \inf \left\{ -\sum_n \lambda_n \log \lambda_n \right\},$$

where the infimum is taken for all possible discrete decompositions of ϕ because the measure μ is not always unique. If μ is not atomic, then $S^{\mathscr{S}}(\phi)$ is defined to be infinite. Naturally, $S^{\mathscr{S}}(\phi) \geqslant 0$ and $=0$ iff $\phi \in \operatorname{ex} \mathscr{S}$.

According to this definition, the entropy of ϕ does depend on the set $\mathscr{S}$ chosen. In this paper we consider three cases $\mathscr{S} = \mathfrak{S}$, $I(\alpha)$, $K(\alpha)$. Therefore we have three entropies $S^{\mathfrak{S}}(\phi)$ $(= S(\phi)$ shortly), $S^{I(\alpha)}(\phi)$ and $S^{K(\alpha)}(\phi)$, which are different in general even for a state $\phi \in K(\alpha)$. The physical meaning of, for example, $S^{I(\alpha)}(\phi)$ is the measure of uncertainty of $\phi \in I(\alpha)$ among all α-invariant states.

In this section, we study the properties of these entropies and their relations.

We expect that the randomness of a real physical system should be finite. Hence we extract the states with $S^{\mathscr{S}}(\phi) < \infty$ from the set $\mathscr{S}$; $\mathscr{S}_{\text{real}}$ $(= \mathscr{S}_r) = \{\phi \in S \mid S^{\mathscr{S}}(\phi) < \infty\}$. Further we denote by $\mathscr{S}_d$ the set of all states in $\mathscr{S}$ with a discrete decomposition into ex $\mathscr{S}$. Since the weak*-closure of convex-hull of ex $\mathscr{S}$ is $\mathscr{S}$ itself, we have

Proposition 1.1. $\bar{\mathscr{S}}_r^{w^*} = \mathscr{S}$.

We next show that our entropy is indeed an extension of Von Neumann's. Let $\mathscr{A}$ be the C^*-algebra generated by the set $C(\mathscr{H})$ of all compact operators on $\mathscr{H}$ and the identity operator I. We denote this algebra by $\tilde{C}(\mathscr{H})$.

Theorem 1.2. *Let* $\mathscr{A} = \tilde{C}(\mathscr{H})$ *and* $\alpha_t(A) = u_t A u_{-t}$ *by a unitary operator* u_t, *then for any state* $\phi \in \mathfrak{S}$ *given by* $\phi(A) = \operatorname{tr} \rho A$ *for any* $A \in \mathscr{A}$ *with a density operator* $\rho \in T(\mathscr{H})_{+,1}$, *we have the following*:

(1) $S(\phi) = -\operatorname{tr} \rho \log \rho$;

(2) *if* ϕ *is a* α-*invariant faithful state and every eigenvalue of* ρ *is non-degenerate, then* $S^{I(\alpha)}(\phi) = S(\phi)$;

(3) *if* ϕ *is a* α-*KMS state, then* $S^{K(\alpha)}(\phi) = 0$.

Proof. (1) Let $\rho = \sum_k \lambda_k \rho_k$ be a decomposition of ρ into extremal (pure) states ρ_k (i.e., $\rho_k^2 = \rho_k$ for each k). It is well known [22] that $-\sum_k \lambda_k \log \lambda_k$ attains to the minimum value when λ_k is the eigenvalue of ρ (the eigenvalue of multiplicity n is repeated precisely n times) and ρ_k is the one-dimensional projection from $\mathscr{H}$ to the subspace generated by a pairwise orthonormal eigenvector Φ_k associated with $\lambda_k : \rho_k = |\Phi_k\rangle\langle\Phi_k|$ (Dirac's notation). Hence, $S(\phi) = -\operatorname{tr} \rho \log \rho$.

(2) Since ϕ is α-invariant, the equality $[u_t, \rho] = 0$ holds for all $t \in R$. From the assumptions on the state ϕ and the eigenvalue of ρ, we have $[u_t, \rho_k] = 0$ for each $\rho_k = |\Phi_k\rangle\langle\Phi_k|$, where Φ_k is the eigenvector of ρ. Thus ρ_k is α-invariant for every k, by which we obtain $S(\phi) \geqslant S^{I(\alpha)}(\phi)$.

Let $\phi = \sum_k \lambda_k \phi_k$ be an ergodic decomposition of ϕ. Then there exists

$\rho_k \in T(\mathscr{H})_{+,1}$ such that $\phi_k(A) = \operatorname{tr} \rho_k A$ for any $A \in \mathscr{A}$. Suppose that ρ_k is not pure, then we have an extremal decomposition of ρ_k into pure states such as $\rho_k = \sum_n \mu_n^k \sigma_n^k$. Since ρ_k is α-invariant, the state σ_n^k becomes α-invariant as discussed before. This contradicts the fact that ρ_k is an ergodic state. Therefore, ρ_k is a pure state. This deduces the converse inequality: $S(\phi) \leqslant S^{I(\alpha)}(\phi)$.

(3) It is known [4] that there exists a unique KMS state for a given constant β and $\alpha(R)$ when $\mathscr{A} = \tilde{C}(\mathscr{H})$. Thus $S^{K(\alpha)}(\phi) = 0$. Q.E.D.

We now go back to the general discussion. We first show some relations among $S(\phi)$, $S^{I(\alpha)}(\phi)$ and $S^{K(\alpha)}(\phi)$.

THEOREM 1.3. *For any KMS state $\phi \in K(\alpha)$, we have*

(1) $S^{I(\alpha)}(\phi) \geqslant S^{K(\alpha)}(\phi)$,

(2) $S(\phi) \geqslant S^{K(\alpha)}(\phi)$,

(3) *if our dynamical system $(\mathscr{A}, \alpha(R))$ is G-abelian on ϕ, then* $S(\phi) \geqslant S^{I(\alpha)}(\phi) \geqslant S^{K(\alpha)}(\phi)$,

(4) *if our dynamical system $(\mathscr{A}, \alpha(R))$ is η-abelian on ϕ (i.e., $\lim_{T\to\infty} (1/T) \int dt\ \phi(C^*[\alpha_t(A), B]\, C) = 0$ for any A, B, C in $\mathscr{A}$), then* $S^{I(\alpha)}(\phi) = S^{K(\alpha)}(\phi)$.

Proof. It is enough for us to prove these statements when every state ψ is in $\mathscr{S}_d$ with $\mathscr{S} = \mathfrak{S}$, $I(\alpha)$, $K(\alpha)$ because otherwise $S^{\mathscr{S}}(\psi) = \infty$. This remark is always valid in all theorems of this paper.

As is mentioned before, the central decomposition of $\phi \in K(\alpha)_d$ is unique and orthogonal; $\phi = \sum_n \lambda_n \phi_n$ with $\phi_n \in \operatorname{ex} K(\alpha)$. Since $\operatorname{ex} K(\alpha) \subset I(\alpha)$, each ϕ_n can be further decomposed into ergodic states in $I(\alpha)$; $\phi_n = \sum_k \mu_k^n \psi_k$ with $\psi_k \in \operatorname{ex} I(\alpha)$. We therefore obtain

$$S^{I(\alpha)}(\phi) = \inf \left\{ -\sum_{k,n} \lambda_n \mu_k^n \log \lambda_n \mu_k^n \mid \{\mu_k^n\} \right\}$$

$$= \sum_n \lambda_n S^{I(\alpha)}(\phi_n) + S^{K(\alpha)}(\phi) \geqslant S^{K(\alpha)}(\phi).$$

This is the inequality of (1). The statement (2) is similarly proved.

When the system $(\mathscr{A}, \alpha(R))$ is G-abelian on ϕ, the ergodic decomposition of ϕ is unique and orthogonal [4]. Hence the inequality of (3) follows by similar argument as (1).

When the system $(\mathscr{A}, \alpha(R))$ is η-abelian on ϕ, the set inclusion $\operatorname{ex} K(\alpha) \subset \operatorname{ex} I(\alpha)$ holds [4], which concludes the equality (4). Q.E.D.

In the remainder of this section, we shall discuss some fundamental properties of our entropies.

Let us consider two states ϕ and ψ in $\mathscr{S}$ with the following central decompositions: $\phi = \sum_{n\in N} a_n \phi_n$ and $\psi = \sum_{m\in M} b_m \psi_m$, where N and M are the index sets. Since the central measure is unique and orthogonal, the central decomposition of new state $\omega = \lambda\phi + (1-\lambda)\psi$, $\lambda \in (0, 1)$, is given by $\omega = \sum_{n\in N} \lambda a_n \phi_n + \sum_{m\in M} (1-\lambda) b_m \psi_m$. There may exist common factor states in two parts of the above decomposition. Hence we should express the state ω as

$$\omega = \sum_{n\in I_1} \lambda a_n \phi_n + \sum_{k\in I_2} \{\lambda a_k + (1-\lambda) b_k\} \phi_k + \sum_{m\in I_3} (1-\lambda) b_m \psi_m,$$

where the index sets I_1, I_2, I_3 satisfy the relations $I_1 \cup I_2 = N$, $I_2 \cup I_3 = M$ and $I_k \cap I_j = \varnothing$ $(k \neq j)$.

In the case of $\mathscr{S} = \mathfrak{S}$, it is easily seen that

$$\begin{aligned} S(\omega) = &\sum_{n\in I_1} \lambda a_n S(\phi_n) + \sum_{k\in I_2} \{\lambda a_k + (1-\lambda) b_k\} S(\phi_k) \\ &+ \sum_{m\in I_3} (1-\lambda) b_m S(\psi_m) - \sum_{n\in I_1} \lambda a_n \log \lambda a_n \\ &- \sum_{k\in I_2} \{\lambda a_k + (1-\lambda) b_k\} \log\{\lambda a_k + (1-\lambda) b_k\} \\ &- \sum_{m\in I_3} (1-\lambda) b_m \log(1-\lambda) b_m. \end{aligned}$$

By the concavity of $-x \log x$, we obtain

$$\begin{aligned} S(\omega) \geqslant &\sum_{n\in N} \lambda a_n S(\phi_n) + \sum_{m\in M} (1-\lambda) b_m S(\psi_m) \\ &- \sum_{n\in N} \lambda a_n \log a_n - \sum_{m\in M} (1-\lambda) b_m \log b_m \\ = {}&\lambda \sum_{n\in N} \{a_n S(\phi_n) - a_n \log a_n\} \\ &+ (1-\lambda) \sum_{m\in M} \{b_m S(\psi_m) - b_m \log b_m\} \\ = {}&\lambda S(\phi) + (1-\lambda) S(\psi). \end{aligned}$$

When $\mathscr{S} = K(\alpha)$, $S^{K(\alpha)}(\phi_n) = S^{K(\alpha)}(\psi_n) = 0$. Hence, again by the concavity of $-x \log x$, we obtain the desired inequality:

$$S^{K(\alpha)}(\omega) \geqslant \lambda S^{K(\alpha)}(\phi) + (1-\lambda) S^{K(\alpha)}(\psi).$$

Finally, when $\mathscr{S} = I(\alpha)$, we have two cases: If ϕ is a KMS state w.r.t. α_t or, more generally, if the center $\mathscr{Z}_\phi = \pi_\phi(\mathscr{A})'' \cap \pi_\phi(\mathscr{A})'$ is pointwise invariant under the canonical extension $\tilde{\alpha}_t$ of α_t (i.e., $\tilde{\alpha}_t(Q) = u_t^\phi Q u_t^{\phi *}$, $Q \in \pi_\phi(\mathscr{A})''$ and $\tilde{\alpha}_t(Q) = Q$ for any Q in $\mathscr{Z}_\phi$), then the resulting factor states $\{\phi_n\}$ are α_t-invariant. By this fact, when ϕ and ψ satisfy one of the above conditions, we can prove the concavity

$$S^{I(\alpha)}(\omega) \geqslant \lambda S^{I(\alpha)}(\phi) + (1-\lambda)\, S^{I(\alpha)}(\psi)$$

by the same way as $S(\omega)$. The second case is when our dynamical system $(\mathscr{A}, \alpha(R))$ is G-abelian on both states ϕ and ψ. We can prove the above inequality by using the unique ergodic decomposition instead of the central decomposition.

Let us summarize these resuls as

THEOREM 1.4. *For two states ϕ and ψ in $\mathfrak{S}$, define $\omega = \lambda\phi + (1-\lambda)\psi$ for some $\lambda \in [0, 1]$. Then*

(1) $S(\omega) \geqslant \lambda S(\phi) + (1-\lambda) S(\psi)$,

(2) *if $\phi, \psi \in K(\alpha)$, then $S^{K(\alpha)}(\omega) \geqslant \lambda S^{K(\alpha)}(\phi) + (1-\lambda)\, S^{K(\alpha)}(\psi)$ and $S^{I(\alpha)} \geqslant \lambda S^{I(\alpha)}(\phi) + (1-\lambda)\, S^{I(\alpha)}(\psi)$,*

(3) *if $\phi, \psi \in I(\alpha)$ and if the centers $\mathscr{Z}_\phi$ and $\mathscr{Z}_\psi$ are pointwise invariant under $\tilde{\alpha}_t$ or the dynamical system $(\mathscr{A}, \alpha(R))$ is G-abelian on both ϕ and ψ, then $S^{I(\alpha)}(\omega) \geqslant \lambda S^{I(\alpha)}(\phi) + (1-\lambda)\, S^{I(\alpha)}(\psi)$.*

We next consider the "additivity" of our entropy $S^{\mathscr{S}}(\phi)$. Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras with unity and $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$ be the injective C^*-tensor product of $\mathscr{A}$ and $\mathscr{B}$ [18, p. 207]. For any $\phi \in \mathfrak{S}(\mathscr{A})$ and $\psi \in \mathfrak{S}(\mathscr{B})$, a state $\phi \otimes \psi \in \mathfrak{S}(\mathscr{C})$ is defined by $\phi \otimes \psi(A \otimes B) = \phi(A)\, \psi(B)$ for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The time evolution automorphism of $\mathscr{C}$ is given by $\gamma_t = \alpha_t \otimes \tau_t$ for all $t \in R$. Then the question to be studied is when an equality $S^{\mathscr{S}(\gamma)}(\phi \otimes \gamma) = S^{\mathscr{S}(\alpha)}(\phi) + S^{\mathscr{S}(\tau)}(\psi)$ is satisfied.

LEMMA 1.5. (1) *If $\phi \in \mathrm{ex}\, \mathfrak{S}(\mathscr{A})$ and $\psi \in \mathrm{ex}\, \mathfrak{S}(\mathscr{B})$, then $\phi \otimes \psi \in \mathrm{ex}\, \mathfrak{S}(\mathscr{C})$.*

(2) *If $\phi \in \mathrm{ex}\, K(\alpha)$ and $\psi \in \mathrm{ex}\, K(\tau)$, then $\phi \otimes \psi \in \mathrm{ex}\, K(\gamma)$.*

(3) *If $\phi \in \mathrm{ex}\, I(\alpha)$ and $\psi \in \mathrm{ex}\, I(\tau)$, then $\phi \otimes \psi \in \mathrm{ex}\, I(\gamma)$.*

Proof. Since $\pi_\phi(\mathscr{A})'' = B(\mathscr{H}_\phi)$ for a pure state ϕ, the (1) follows from the equalities $\pi_{\phi \otimes \psi}(\mathscr{C})'' = \pi_\phi(\mathscr{A})'' \otimes \pi_\psi(\mathscr{B})''$ and $B(\mathscr{H}_{\phi \otimes \psi}) = B(\mathscr{H}_\phi \otimes \mathscr{H}_\psi) = B(\mathscr{H}_\phi) \otimes B(\mathscr{H}_\psi)$ [18, p. 413]. The statement (2) is due to $\mathscr{Z}_{\phi \otimes \psi} = \mathscr{Z}_\phi \otimes \mathscr{Z}_\psi$ [18, Corollary 5.11]. Let us show (3): We first define three Von Neumann algebras $\mathfrak{N}_\phi = \{\pi_\phi(\mathscr{A}) \cup u^\phi(R)\}''$, $\mathfrak{N}_\psi = \{\pi_\psi(\mathscr{B}) \cup u^\psi(R)\}''$ and $\mathfrak{N}_\omega = \{\pi_\omega(\mathscr{C}) \cup u^\omega(R)\}''$, where $u(R)$ is a one-parameter unitary group associated to each state. The equality $\gamma_t = \alpha_t \otimes \tau_t$ implies $u^\omega(t) = u^\phi(t) \otimes u^\psi(t)$, hence $\mathfrak{N}_\omega = \mathfrak{N}_\phi \otimes \mathfrak{N}_\psi$. Since $\phi \in \mathrm{ex}\, I(\alpha)$ and $\psi \in \mathrm{ex}\, I(\tau)$, $\mathfrak{N}_\phi = B(\mathscr{H}_\phi)$ and

$\mathfrak{N}_{\psi}=B(\mathscr{H}_{\psi})$ hold. We thus have $\mathfrak{N}_{\omega}=B(\mathscr{H}_{\phi})\otimes B(\mathscr{H}_{\psi})=B(\mathscr{H}_{\omega})$, which concludes the result. Q.E.D.

Let a triple $(\mathscr{S}(\alpha), \mathscr{S}(\tau), \mathscr{S}(\gamma))$ be one of the following three: $(K(\alpha), K(\tau), K(\gamma))$, $(I(\alpha), I(\tau), I(\gamma))$ and $(\mathfrak{S}(\mathscr{A}), \mathfrak{S}(\mathscr{B}), \mathfrak{S}(\mathscr{C}))$.

We now suppose that every state in $\mathscr{S}(\gamma)$ is uniquely decomposed into extreme states in $\mathscr{S}(\gamma)$. Then any state in $\mathscr{S}(\alpha)$ or $\mathscr{S}(\tau)$ has a unique extremal decomposition. Indeed, if a state $\phi\in\mathscr{S}(\alpha)$ has two distinct extremal decompositions such as $\phi=\sum_n\lambda_n\phi_n$ and $\phi=\sum_n\mu_n\phi_n'$, then a state $\omega=\phi\otimes\psi$ with $\psi\in\operatorname{ex}\mathscr{S}(\tau)$ also has two distinct extremal decompositions due to Lemma 1.5. This contradicts the assumption. Under this assumption, let $\phi=\sum_n\lambda_n\phi_n$ and $\psi=\sum_m\mu_m\psi_m$ be extremal decompositions of $\phi\in\mathscr{S}(\alpha)$ and $\psi\in\mathscr{S}(\tau)$. By Lemma 1.5, we have a unique extremal decomposition of the state $\omega=\phi\otimes\psi$ such that $\omega=\sum_{n,m}\lambda_n\mu_m\phi_n\otimes\psi_m$. Therefore we obtain

$$\begin{aligned} S^{\mathscr{S}(\gamma)}(\omega) &= -\sum_{n,m}\lambda_n\mu_m\log\lambda_n\mu_m \\ &= -\sum_n\lambda_n\log\lambda_n-\sum_m\mu_m\log\mu_m \\ &= S^{\mathscr{S}(\alpha)}(\phi)+S^{\mathscr{S}(\tau)}(\psi). \end{aligned}$$

THEOREM 1.6. *If every state in $\mathscr{S}(\gamma)$ is uniquely decomposed into extremal states in $\mathscr{S}(\gamma)$, then for any states $\phi\in\mathscr{S}(\alpha)$ and $\psi\in\mathscr{S}(\tau)$, we have*

$$S^{\mathscr{S}(\gamma)}(\phi\otimes\psi)=S^{\mathscr{S}(\alpha)}(\phi)+S^{\mathscr{S}(\tau)}(\psi).$$

The condition of this theorem is indeed satisfied in the following cases:

(1) For the triple $(K(\alpha), K(\tau), K(\gamma))$, the condition is automatically satisfied.

(2) For the triple $(I(\alpha), I(\tau), I(\gamma))$, the dynamical system $(\mathscr{C}, \gamma(R))$ is G-abelian.

(3) For the triple $(\mathfrak{S}(\mathscr{A}), \mathfrak{S}(\mathscr{B}), \mathfrak{S}(\mathscr{C}))$, $\mathscr{C}$ is separable and $\pi_{\phi\otimes\psi}(\mathscr{C})'$ is abelian Von Neumann algebra. (In this case, it is known [4, p. 358] that the state $\phi\otimes\psi$ has a unique extremal decomposition.)

2. Entropy Transmission

It is important to consider the dynamical change of states in every physical system. One of most general descriptions of this state change is suggested in the communication theory of Shannon. In communication (information) theory, we have to consider two dynamical systems, namely, an input system described by a C^*-algebraic triple $(\mathscr{A}, \mathfrak{S}(\mathscr{A}), \alpha(R))$ and an output system by another C^*-algebraic triple $(\mathscr{B}, \mathfrak{S}(\mathscr{B}), \tau(R))$. An information (entropy) of the input system is sent to the output system through a channel. To investigate this transmission process is a central theme of information theory. By classical analogy [15, 17, 21], a quantum mechanical channel might be defined by a map Λ^* from $\mathfrak{S}(\mathscr{A})$ to $\mathfrak{S}(\mathscr{B})$ such that its dual map $\Lambda: \mathscr{B} \to \mathscr{A}$ is completely positive (i.e., for any positive $n \times n$-matrix $(Q_{i,j})$ with $Q_{i,j}$ in $\mathscr{B}$, the $n \times n$-matrix $(\Lambda Q_{i,j})$ is positive for all $n \in N$) with $\Lambda I_{\mathscr{B}} = I_{\mathscr{A}}$, where $I_{\mathscr{B}}$ and $I_{\mathscr{A}}$ are unities of $\mathscr{B}$ and $\mathscr{A}$, respectively (remark that we use the same notation I for every unity in the sequel discussion when no confusion occurs).

The channel provides us a mechanism of state-change appeared in several fields of quantum physics. We often meet the following channels.

EXAMPLE 1. Let X, Y be second countable compact Hausdorff spaces and $\mathscr{F}_X$, $\mathscr{F}_Y$ be their Borel fields respectively. We denote the sets of all regular probability measures by $P(X)$ on $(X, \mathscr{F}_X)$ and $P(Y)$ on $(Y, \mathscr{F}_Y)$. A map $\lambda: X \times \mathscr{F}_Y \to R^+$ satisfying that (i) $\lambda(x, \cdot) \in P(Y)$ for each fixed $x \in X$ and (ii) $\lambda(\cdot, Q)$ is a (continuous) measurable function on $(X, \mathscr{F}_X)$ for each fixed $Q \in \mathscr{F}_Y$ is a channel (we call it a "classical" channel). Then a probability measure $\mu \in P(X)$ is transferred to a probability measure $\mu' \in P(Y)$ such as $\mu'(Q) = \int_X \lambda(x, Q)\, \mu(dx)$ $(= \Lambda^*\mu(Q))$. In this case, we take $\mathscr{A}$ and $\mathscr{B}$ abelian C^*-algebras of all continuous functions on X and Y, respectively, and the mapping Λ is given by $(\Lambda f)(x) = \int_Y f(y)\, \lambda(x, dy)$ for any $f \in \mathscr{B}$.

EXAMPLE 2. When $\mathscr{A} = \mathscr{B} = \tilde{C}(\mathscr{H})$ and $\mathfrak{S}(\mathscr{A}) = \mathfrak{S}(\mathscr{B}) = \tilde{C}(\mathscr{H})^*_{+,1}$ $(\supset T(\mathscr{H})_{+,1})$ and $\{V_t \mid t \in R\}$ is a one-parameter isometric semigroup on $\mathscr{H}$, the time evolution of $\rho \in T(\mathscr{H})_{+,1}$ given by $\Lambda_t^*\rho = V_t \rho V_t^*$ is a channel for each $t \in R$.

EXAMPLE 3. Under the same conditions of Example 2, let $\sum_n q_n E_n$ be the spectral decomposition of an operator Q in $C(\mathscr{H})$. Then the so-called Von Neumann measurement $\Lambda^*\rho = \sum_n E_n \rho E_n$, $\rho \in T(\mathscr{H})_{+,1}$, is a channel.

EXAMPLE 4. Let σ be an automorphism of the injective C^*-tensor product $\mathscr{A} \otimes \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$. Furthermore, let $\mathscr{E}$ be a norm one projection from $\mathscr{A} \otimes \mathscr{B}$ to $\mathscr{A}$ $(= \mathscr{A} \otimes I)$. Then $\Lambda = \mathscr{E} \circ \sigma$ is a completely positive map and its dual map Λ^* defined by $\Lambda^*\psi(B) = \psi(\mathscr{E}(\sigma(B)))$, $\psi \in \mathfrak{S}(\mathscr{A})$, is a channel, where we identify B with $I \otimes B$.

EXAMPLE 5. Let $(X, \mathscr{F}_X)$, $P(X)$ be those given in Example 1 and $C(X)$ be the set of all continuous functions on X. Further, let $\mathscr{A}$ be a noncommutative C^*-algebra and $\mathfrak{S}(\mathscr{A})$ be its state space. Then a map Λ^* defined by $\Lambda^*\mu = \int_X \psi_x \mu(dx)$, $\mu \in P(X)$, $\psi_x \in \mathfrak{S}(\mathscr{A})$, is a channel from $P(X)$ to $\mathfrak{S}(\mathscr{A})$. This channel is often called a classical-quantum channel (c.q.-channel) [10]. Moreover, a quantum-classical (q.c.) channel $\Lambda^*: \mathfrak{S}(\mathscr{A}) \to P(X)$ can be constructed by $\Lambda^*\phi(\cdot) = \phi(M(\cdot))$, where $M: \mathscr{F}_X \to \mathscr{A}$ is a positive operator valued measure such as $M(\Delta) = \Lambda(1_\Delta)$ for any $\Delta \in \mathscr{F}_X$.

As in usual information theory, it is interesting to analyse ergodic properties of our quantum mechanical channels. This work has been partially done in [15], where we introduced the following ergodic channels:

(1) *A channel Λ^* is said to be stationary (write $\Lambda^* \in SC$) if $\Lambda \circ \tau_t = \alpha_t \circ \Lambda$ for all $t \in R$.*

(2) *A channel Λ^* is said to be ergodic ($\Lambda^* \in EC$) if $\Lambda^* \in SC$ and- $\Lambda^*(\mathrm{ex}\, I(\alpha)) \subset \mathrm{ex}\, I(\tau)$.*

(3) *A channel Λ^* is said to be KMS ($\Lambda^* \in KC$) if $\Lambda^* \in SC$ and $\Lambda^*(K(\alpha)) \subset K(\tau)$.*

Furthermore, we here introduce the following special channels for given weak*-compact convex subsets $\mathscr{S}(\mathscr{A}) \subset \mathfrak{S}(\mathscr{A})$ and $\mathscr{S}(\mathscr{B}) \subset \mathfrak{S}(\mathscr{B})$:

(4) *A channel Λ^* is said to be deterministic for $\mathscr{S}(\mathscr{A})$ ($\Lambda^* \in DC(\mathscr{S}(\mathscr{A}))$) if Λ^* is injective on $\mathscr{S}(\mathscr{A})$.*

(5) *A channel Λ^* is said to be perfect for a pair $\mathscr{S}(\mathscr{A})$ and $\mathscr{S}(\mathscr{B})$ ($\Lambda^* \in PC(\mathscr{S}(\mathscr{A}), \mathscr{S}(\mathscr{B}))$) if $\Lambda^*: \mathscr{S}(\mathscr{A}) \to \mathscr{S}(\mathscr{B})$ is bijective.*

If a channel is deterministic or perfect, then we can uniquely determine an input state by observing an output state. We hence expect that when such a channel is used, an information (entropy) of the input system would be equal to that obtained from the output system. One of our aims in this section is to rigorously prove this intuitive fact within our mathematical framework. More generally, the following question is pertinent to our investigation: *Under what conditions on a channel does our entropy become an invariant between two dynamical systems*?

By answering this question, it might be possible to characterize C^*-dynamical systems according to an invariant $S^{\mathscr{S}}(\phi)$ under a given channel Λ^*. This is, however, an open question.

LEMMA 2.1. *For a channel $\Lambda^*: \mathfrak{S}(\mathscr{A}) \to \mathfrak{S}(\mathscr{B})$, we have*

(1) *Λ^* is onto iff Λ iz injective,*

(2) *Λ^* is injective iff Λ is onto,*

(3) *Λ^* is bijective iff Λ is bijective.*

Proof. The statement (3) is an immediate consequence of (1) and (2), and (2) is similarly proved as (1). So we prove the statement (1).

Suppose that $\Lambda Q = \Lambda R$ for some Q, R in $\mathscr{B}$. Then $\Lambda^*\phi(Q) = \Lambda^*\phi(R)$ holds for every $\phi \in \mathfrak{S}(\mathscr{A})$. If Λ^* is onto, then the set $\{\Lambda^*\phi \mid \phi \in \mathfrak{S}(\mathscr{A})\}$ is identical to $\mathfrak{S}(\mathscr{B})$, which implies $Q = R$.

Conversely, suppose that Λ^* is not onto. Then there exists a state $\phi \in \mathfrak{S}(\mathscr{B})$ such that $\phi \notin \Lambda^*(\mathfrak{S}(\mathscr{A})) \subset \mathfrak{S}(\mathscr{B})$. Define a linear functional $\hat{B}$ for each $B \in \mathscr{B}$ by $\hat{B}(\phi) = \phi(B)$ for any $\phi \in \mathfrak{S}(\mathscr{B})$. There then exist B_1 and B_2 in $\mathscr{B}$ such that $\hat{B}_1 = \hat{B}_2$ on $\Lambda^*(\mathfrak{S}(\mathscr{A}))$ but $\hat{B}_1(\phi) \neq \hat{B}_2(\phi)$. Hence $\Lambda^*\psi(B_1) = \Lambda^*\psi(B_2)$ for any $\psi \in \mathfrak{S}(\mathscr{A})$, so $\Lambda(B_1) = \Lambda(B_2)$. It follows $B_1 = B_2$ because Λ is supposed to be injective. This contradicts $\hat{B}_1(\phi) \neq \hat{B}_2(\phi)$. Q.E.D.

Let us consider the question (stated before) concerning the invariance of entropy under a channel.

THEOREM 2.2. *If $\Lambda^* \in SC \cap DC(\mathfrak{S}(\mathscr{A}))$ and Λ is a *-homomorphism, then $S^{K(\alpha)}(\phi) = S^{K(\tau)}(\Lambda^*\phi)$ for any $\phi \in K(\alpha)$.*

Proof. As noted before, it is enough to consider the case $\phi \in K(\alpha)_d$. Then the state ϕ is written as $\phi = \sum_n \lambda_n \phi_n$ with $\phi_n \in \mathrm{ex}\, K(\alpha)$. From the conditions on Λ and Λ^*, the states $\Lambda^*\phi_n$ for all n satisfy the KMS condition w.r.t. τ_t and $\Lambda^*\phi_n \neq \Lambda^*\phi_m$ for $n \neq m$. Therefore we have only to show that $\Lambda^*\phi_n$ is extremal in $K(\tau)$ for every n, equivalently, show that $\Lambda^*\phi_n$ is a factor state. Put $\psi = \Lambda^*\phi$ and let $\{\mathscr{H}_\phi, \pi_\phi, \Phi\}$ and $\{\mathscr{H}_\psi, \pi_\psi, \Psi\}$ be the cyclic representations of ϕ and ψ, respectively. We here define an operator $\Lambda_{\phi,\psi}$ from the set $\pi_\psi(\mathscr{B})\Psi$ to $\mathscr{H}_\phi$ such as $\Lambda_{\phi,\psi}\pi_\psi(Q)\Psi = \pi_\phi(\Lambda Q)\Phi$ for any $Q \in \mathscr{B}$. We thus have

$$\begin{aligned}
&\langle \Lambda_{\phi,\psi}\pi_\psi(Q)\Psi, \Lambda_{\phi,\psi}\pi_\psi(Q)\Psi\rangle \\
&\quad = \langle \pi_\phi(\Lambda Q)\Phi, \pi_\phi(\Lambda Q)\Phi\rangle = \langle \Phi, \pi_\phi(\Lambda(Q)^* \Lambda(Q))\Phi\rangle \\
&\quad = \langle \Phi, \pi_\phi(\Lambda(Q^*Q))\Phi\rangle = \langle \pi_\psi(Q)\Psi, \pi_\psi(Q)\Psi\rangle,
\end{aligned}$$

where we used the condition that Λ is a *-homomorphism. Hence $\Lambda_{\phi,\psi}$ is an isometry. Since $\Lambda^* \in DC(\mathfrak{S}(\mathscr{A}))$, the map Λ is onto from $\mathscr{B}$ to $\mathscr{A}$ because of Lemma 2.1. Therefore

$$\{\pi_\phi(\Lambda\mathscr{B})\Phi\}^- = \{\pi_\phi(\mathscr{A})\Phi\}^- = \mathscr{H}_\phi.$$

It follows that $\Lambda_{\phi,\psi}$ can be extended to a unitary operator from $\mathscr{H}_\psi$ to $\mathscr{H}_\phi$. Moreover, for any B, Q in $\mathscr{B}$,

$$\begin{aligned}
&\Lambda_{\phi,\psi}\pi_\psi(B)\Lambda^*_{\phi,\psi}\pi_\phi(\Lambda Q)\Phi \\
&\quad = \Lambda_{\phi,\psi}\pi_\psi(B)\Lambda^*_{\phi,\psi}\Lambda_{\phi,\psi}\pi_\psi(Q)\Psi = \Lambda_{\phi,\psi}\pi_\psi(BQ)\Psi \\
&\quad = \pi_\phi(\Lambda(BQ))\Phi = \pi_\phi(\Lambda B)\pi_\phi(\Lambda Q)\Phi,
\end{aligned}$$

which implies that $\Lambda_{\phi,\psi}\pi_\psi(B)\Lambda^*_{\phi,\psi}=\pi_\phi(\Lambda B)$ for any B in $\mathscr{B}$. Thus it is easily seen that

$$\pi_\phi(\mathscr{A})''\cap\pi_\phi(\mathscr{A})'=\Lambda_{\phi,\psi}(\pi_\psi(\mathscr{B})''\cap\pi_\psi(\mathscr{B})')\Lambda^*_{\phi,\psi}$$

because the channel Λ^* is deterministic on $\mathfrak{S}(\mathscr{A})$. This concludes that if ϕ is a factor state on $\mathscr{A}$, then so is $\psi=\Lambda^*\phi$. Q.E.D.

We next consider the cases of $S^{I(\alpha)}(\phi)$ and $S(\phi)$.

THEOREM 2.3. (1) *If* $\Lambda^*\in SC\cap DC(\mathfrak{S}(\mathscr{A}))$ *and* Λ *is a *-homomorphism, then* $S^{I(\alpha)}(\phi)=S^{I(\tau)}(\Lambda^*\phi)$ *for any G-abelian* $\phi\in I(\alpha)$.

(2) *If* $\Lambda^*\in EC\cap PC(\operatorname{ex} I(\alpha),\operatorname{ex} I(\tau))$, *then* $S^{I(\alpha)}(\phi)=S^{I(\tau)}(\Lambda^*\phi)$ *for any* $\phi\in I(\alpha)$.

Proof. For any G-abelian $\phi\in I(\alpha)_d$, we have a unique ergodic decomposition $\phi=\sum_n\lambda_n\phi_n$. Since Λ is a *-homomorphism and $\Lambda^*\in SC$, it is an easy exercise to show that $(\mathscr{B},\tau(R))$ is G-abelian on $\Lambda^*\phi$. If we can prove that $\Lambda^*\phi$ $(=\psi)$ is ergodic for an ergodic state ϕ, then the conclusion (1) follows. The state ψ is τ-invariant because of $\phi\in I(\alpha)$ and $\Lambda^*\in SC$. The condition $\Lambda^*\in DC(\mathfrak{S}(\mathscr{A}))$ implies that an equality $u_t^\phi\Lambda_{\phi,\psi}=\Lambda_{\phi,\psi}u_t^\psi$ holds for all $t\in R$ in addition to a relation $\Lambda_{\phi,\psi}\pi_\psi(\mathscr{B})\Lambda^*_{\phi,\psi}=\pi_\phi(\mathscr{A})$ proved in Theorem 2.2. Let $\mathfrak{N}_\phi$ and $\mathfrak{N}_\psi$ be Von Neumann algebras generated by the sets $\{\pi_\phi(\mathscr{A}),u^\phi(R)\}$ and $\{\pi_\psi(\mathscr{B}),u^\psi(R)\}$, respectively. Then $\mathfrak{N}_\phi=\Lambda_{\phi,\psi}\mathfrak{N}_\psi\Lambda^*_{\phi,\psi}$ holds. Hence, if ϕ is an extremal α-invariant state on $\mathscr{A}$, then ψ is also an extremal τ-invariant state on $\mathscr{B}$.

Let us prove (2): For an ergodic decomposition $\phi=\sum_n\lambda_n\phi_n$, the decomposition $\Lambda^*\phi=\sum_n\lambda_n\Lambda^*\phi_n$ is ergodic because of $\Lambda^*\in EC\cap PC(\operatorname{ex} I(\alpha),\operatorname{ex} I(\tau))$. Thus we obtain

$$S^{I(\tau)}(\Lambda^*\phi)\leqslant\inf\left\{-\sum_n\lambda_n\log\lambda_n\right\}=S^{I(\alpha)}(\phi).$$

Now, let $\Lambda^*\phi=\sum_n\mu_n\psi_n$ be a certain ergodic decomposition of $\Lambda^*\phi$. Since Λ^* is perfect for $\operatorname{ex} I(\alpha)$, there exists a unique state ϕ_n in $\operatorname{ex} I(\alpha)$ for each ψ_n such that $\Lambda^*\phi_n=\psi_n$. Hence $\Lambda^*\phi=\Lambda^*\sum_n\mu_n\phi_n$, which means that $\phi=\sum_n\mu_n\phi_n$ is an ergodic decomposition of ϕ. Thus we also have $S^{I(\alpha)}(\phi)\leqslant S^{I(\tau)}(\Lambda^*\phi)$. This concludes the result. Q.E.D.

THEOREM 2.4. *If* $\Lambda^*\in PC(\mathfrak{S}(\mathscr{A}),\mathfrak{S}(\mathscr{B}))$, *then* $S(\phi)=S(\Lambda^*\phi)$ *for any* $\phi\in\mathfrak{S}(\mathscr{A})$.

Proof. For an extremal decomposition $\phi=\sum_n\lambda_n\phi_n$ of a state

$\phi \in \mathfrak{S}(\mathscr{A})_d$, we have a decomposition $\Lambda^*\phi = \sum_n \lambda_n \Lambda^*\phi_n$. Let us show that $\Lambda^*\phi_n$ is a pure state for each n. Suppose that $\Lambda^*\phi_n$ is not pure. Then there exist ψ and ω in $\mathfrak{S}(\mathscr{B})$ such that $\Lambda^*\phi_n = a\psi + (1-a)\omega$ with some $a \in (0, 1)$. Since Λ^* is perfect for $\mathfrak{S}(\mathscr{A})$, there exists a map Ξ such that $\Lambda^*\Xi = \Xi\Lambda^* =$ identity map. We thus have $\phi_n = \Xi\Lambda^*\phi_n = a\Xi\psi + (1-a)\Xi\omega$, which contradicts the purity of ϕ_n. Moreover, let $\Lambda^*\phi = \sum_n \mu_n \psi_n$ be an extremal decomposition of $\Lambda^*\phi$. Then it is readily shown that the decomposition $\phi = \sum_n \mu_n \Xi\phi_n$ is extremal in $\mathfrak{S}(\mathscr{A})$. These statements conclude the result. Q.E.D.

A channel provides us a rule describing relations between events (extremal states) of an input system and those of an output system. In the above theorems, we rigorously proved more or less intuitive facts expected to be formed under special channels. Moreover, the following converse problem might be interesting and is still open: Suppose that one of our entropies is a dynamical invariant under a certain channel. Then (1) how much can we say about this channel? and (2) under what conditions on dynamical systems can we determine the channel uniquely?

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