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Harmonic superspaces from superstrings

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Abstract

We derive harmonic superspaces for $N = 2, 3, 4$ SYM theory in four dimensions from superstring theory. The pure spinors in ten dimensions are dimensionally reduced and yield the harmonic coordinates. Two anticommuting BRST charges implement Grassmann analyticity and harmonic analyticity. The string field theory action produces the action and field equations for $N = 3$ SYM theory in harmonic superspace.

1. Introduction

Pure spinors [1] in ten dimensions are complex commuting chiral spinorial ghosts $\lambda^{\hat\alpha}$ with $\hat\alpha = 1, \ldots, 16$ satisfying the ten nonlinear constraints

$$\lambda^{\hat\alpha}\gamma^{\hat m}_{\hat\alpha\hat\beta}\lambda^{\hat\beta} = 0 \qquad (1.1)$$

(hats denote 10-dimensional indices). They form the starting point for a new approach to the quantization of the superstring with coordinates $x^{\hat m}$, $\theta^{\hat\alpha}$ and $\lambda^{\hat\alpha}$ [2]. Due to these constraints on λ, the troublesome second class constraints of the superstring become effectively first class. One can relax these constraints and obtain a covariant formulation by introducing more ghosts as Lagrange multipliers [3]. The result is an $N = 2$ WZNW model [4]. The pure spinors in this covariant approach are real and the BRST charge maps $\theta^{\hat\alpha}$ into $\lambda^{\hat\alpha}$. In this Letter, though, we use complex constrained $\lambda^{\hat\alpha}$. Pure spinors also exist in other dimensions [1].

Harmonic superspace[1] was constructed to circumvent the no-go theorems for a full-fledged superspace description of N-extended supersymmetries (susy). The main idea is to let the R-symmetry group $U(N)$ (or $SU(N)$ for $N = 4$), which acts on the susy generators, become part of a coset approach. The generators

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[1] See [5] for a complete review of the subject and references. Two useful accounts of the subject can be found in [6] and in [7]. Projective harmonic superspace has been introduced in [8]. Harmonic superspace for hypermultiplets and with central charges was discussed in [9], with references to earlier work cited therein. The application to the AdS/CFT correspondence is studied in [10], and some developments of $N = 4$ harmonic superspace for SYM can be found in [11] and in [12].

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of $U(N)$ are divided into coset generators with coset coordinates u called harmonic variables, and subgroup generators. Superfields depend not only on x^m and half of the $\theta^\alpha_I, \bar\theta^{\dot\alpha I}$ (with $\alpha, \dot\alpha = 1, 2$ and $I = 1, \ldots, N$) but also on u's. For $N = 2, 3, 4$ the cosets most often used are

$$\frac{SU(2)}{U(1)}, \quad \frac{SU(3)}{U(1)\times U(1)}, \quad \frac{SU(4)}{S[U(2)\times U(2)]}, \tag{1.2}$$

respectively, although other choices are also possible [6].

In this Letter we present a derivation of four-dimensional harmonic superspaces from ten-dimensional pure spinors by using ordinary dimensional reduction in which we set the extra six coordinates to zero by hand. The spinors $\lambda^{\hat\alpha}$ decompose into λ^α_I and $\bar\lambda^{\dot\alpha I}$ where $I = 1, \ldots, 4$ is an $SU(4) \sim SO(6)$ index. The main idea is to factorize the pure spinors $\lambda^{\hat\alpha}$ into auxiliary variables λ^α_a and $\bar\lambda^{\dot\alpha}_a$ with $a = 1, 2$, and harmonic variables u^a_I and $\bar v^{aI}$. In this way we factorize the Lorenz group and the internal symmetry group $SU(4)$. Using this factorization, the pure spinor constraints turn into constraints on λ^α_a and $\bar\lambda^{\dot\alpha}_a$, and on u^a_I and $\bar v^{aI}$.

Contracting the operator $d_{z\hat\alpha}$ in the BRST charge [2]

$$Q = \oint dz\, \lambda^{\hat\alpha} d_{z\hat\alpha}, \tag{1.3}$$

with the harmonic coordinates leads to eight spinorial covariant derivatives

$$d^a_\alpha = u^a_I d^I_\alpha, \qquad \bar d^a_{\dot\alpha} = \bar v^{aI} \bar d_{\dot\alpha I}, \tag{1.4}$$

which satisfy the constraints

$$\{d^a_\alpha, d^b_\beta\} = \epsilon_{\alpha\beta}\{\bar d^a_{\dot\alpha}, \bar d^{\dot\alpha b}\}, \qquad \{d^a_\alpha, \bar d^b_{\dot\beta}\} = 0, \tag{1.5}$$

as a consequence of the constraints on u and $\bar v$, and in terms of which G (Grassmann) analyticity (dependence on half the θ's) of superfields is defined.

If one does not provide the information that d^a_α and $\bar d^a_{\dot\alpha}$ are linear in u^a_I and $\bar v^{aI}$, one looses information. We therefore construct a second BRST charge which only anticommutes with Q_H if d^a_α and $\bar d^a_{\dot\alpha}$ are factorized as in (1.4). It is constructed from the generators of $U(N)$ represented by the following differential operators[2]

$$d^a{}_{a'} = u^a_I \partial_{u^{a'}_I} - \bar u^I_{a'} \partial_{\bar u^I_a}. \tag{1.6}$$

Requiring that the vertex operators are annihilated by these BRST charges should yield the field equations of $N = 4$ harmonic superspace. In this Letter we work out the case of $N = 3$ and obtain by truncation the field equations of $N = 3$ SYM theory in harmonic superspace. We end by deducing an action for $N = 3$ SYM theory in harmonic superspace from the Chern–Simons action for string field theory [13].

The present analysis might provide a link between string theory with pure spinors and recent developments in twistor theory [14]. Another interesting aspect not covered in the present letter is deformed harmonic superspace [15]. It would be interesting to discover which kind of harmonic superspace one obtains for suitable Ramond–Ramond background fields [16].

In a future article we intend to extend these results to the $N = 4$ case and construct an action for $N = 4$ SYM theory [17]. In particular, this should give a conceptually simple derivation of the rather complicated measure. A similar analysis is pursued in [18].

2. The coordinates of $N = 4$, $N = 3$, and $N = 2$ harmonic superspace from pure spinors

We substitute the decomposition $\lambda^{\hat\alpha} = (\lambda^\alpha_I, \bar\lambda^{\dot\alpha I})$ into the pure spinor constraints, and use the representation of the matrices $\gamma^{\hat m}_{\hat\alpha\hat\beta}$ given in [19]. In this representation the Dirac matrices with $m = 0, 1, 2, 3$ are labelled by $\gamma^{\alpha\dot\beta}$ and those for $m = 4, \ldots, 9$ are labelled by $\gamma^{IJ} = -\gamma^{JI}$, and all matrix elements are expressed in terms of Kronecker delta's and the epsilon symbols $\epsilon^{\alpha\beta}, \epsilon^{\dot\alpha\dot\beta}$ and ϵ^{IJKL}. The pure spinor constraints decompose then into the following six plus four constraints

$$\lambda^\alpha_I \epsilon_{\alpha\beta} \lambda^\beta_J + \frac{1}{2}\epsilon_{IJKL} \bar\lambda^{\dot\alpha K} \epsilon_{\dot\alpha\dot\beta} \bar\lambda^{\dot\beta L} = 0,$$
$$\lambda^\alpha_I \bar\lambda^{\dot\alpha I} = 0. \tag{2.1}$$

The first relation corresponds to $m = 4, \ldots, 9$ while the second one corresponds to $m = 0, 1, 2, 3$. To solve these constraints we adopt the following ansatz

$$\lambda^\alpha_I = \lambda^\alpha_a u^a_I, \qquad \bar\lambda^{\dot\alpha J} = \bar\lambda^{\dot\alpha}_a \bar v^{aJ}, \tag{2.2}$$

[2] The R-symmetry group $SU(4)$ corresponds to the Lorentz generators in the extra dimensions. This suggests that the second BRST charge might be obtained by dimensional reduction of the BRST charge in ten dimensions, extended to include the ten-dimensional Lorentz generators.

where $a = 1, 2$. The new variables u_I^a and $\bar{v}^{aJ}$ are complex and commuting. They carry $GL(2, \mathbf{C})$ and $SU(4)$ indices. The spinors λ_a^α, $\bar{\lambda}_a^{\dot{\alpha}}$ are also complex and commuting, and carry a representation of $SL(2, \mathbf{C})$ and $GL(2, \mathbf{C})$. In this way, we separate the Lorentz group from the internal symmetry group $SU(4)$.

The decomposition in (2.2) is left invariant by the gauge transformations

$$u_I^a \to M^a{}_b u_I^b, \qquad \lambda_a^\alpha \to \lambda_b^\alpha \left(M^{-1}\right)^a{}_b,$$
$$\bar{v}^{aJ} \to \bar{M}^a{}_b \bar{v}^{bJ}, \qquad \bar{\lambda}_a^{\dot{\alpha}} \to \bar{\lambda}_b^{\dot{\alpha}} \left(\bar{M}^{-1}\right)^a{}_b, \tag{2.3}$$

where M and $\bar{M}$ are independent $GL(2, \mathbf{C})$ matrices. The factorization (2.2) plus the gauge invariance (2.3) yields 16 complex parameters. To reduce to the usual 11 independent complex parameters of pure spinors, we further impose the following two covariant constraints

$$u^a{}_I \bar{v}^{bI} = 0,$$
$$\lambda_a^\alpha \epsilon_{\alpha\beta} \epsilon^{ab} \lambda_b^\beta + \bar{\lambda}_a^{\dot{\alpha}} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{ab} \bar{\lambda}_b^{\dot{\beta}} = 0. \tag{2.4}$$

The first one imposes four complex conditions, while the second equation is a single invariant complex condition.

The first constraint in (2.4) and the gauge transformations in (2.3) reduce the 16 complex components of u_I^a and $\bar{v}^{aI}$ to 8 real parameters. This is the same number as the number of independent parameters of the coset

$$\frac{U(4)}{U(2) \times U(2)} = \frac{SU(4)}{S(U(2) \times U(2))}$$

used in [7] (see also [12] and [10]). The restriction of $U(2) \times U(2)$ to the subgroup $S(U(2) \times U(2))$ is due to second constraint of (2.4). The latter is preserved by the transformations M and $\bar{M}$ only after the identification $\det M = \det \bar{M}$.

To identify the $SU(4)$ of the coset space, we introduce new coordinates $u_I^{a,\dot{b}} = (u_I^{a,\dot{1}}, u_I^{a,\dot{2}})$ where

$$u_I^{a,\dot{1}} = u_I^a, \qquad u_I^{a,\dot{2}} = \epsilon^{ab} v_{bI}, \tag{2.5}$$

and $v_{bI} = (\bar{v}^{bI})^*$. The matrix $u_I^{(a,\dot{b})}$ is a $U(4)$ matrix because the harmonic variables u_I^a and $\bar{v}^{aI}$ satisfy the constraints (2.4) and they can be normalized as follows, using the gauge transformations (2.3),

$$u_I^a \bar{u}_b^I = \delta^a{}_b, \qquad \bar{v}^{aI} v_{bI} = \delta^a{}_b, \tag{2.6}$$

where $\bar{u}_b^I = (u_I^b)^*$.

To restrict $U(4)$ to $SU(4)$ we choose the gauge. Denoting this relation by $N_{IJ} = 0$, it is clear that $N_{IJ} \bar{v}^{aJ} = 0$ and $\epsilon^{IJKL} N_{KL} u_J^a = 0$ due to (2.4). This leaves the phase of $\det u_I^{a\dot{b}}$ undetermined. The gauge in (2.5) sets this phase to zero.

$$u_I^a \epsilon_{ab} u_J^b - \frac{1}{2} \epsilon_{IJKL} \bar{v}^{aK} \epsilon_{ab} \bar{v}^{bL} = 0. \tag{2.7}$$

This gauge choice is preserved by $S(U(2) \times U(2))$.

The normalizations (2.6) fix 4 real parameters for each $GL(2, \mathbf{C})$ in (2.3). The remaining 7 real parameters of $GL(2, \mathbf{C})$ (remaining after the identification $\det M = \det \bar{M}$), reproduce the subgroup $S(U(2) \times U(2))$. All equations are covariant under this subgroup. Thus the coordinates $u_I^A \equiv u_I^{a,\dot{a}}$, with $A = 1, \ldots, 4$, parametrize the coset $\frac{SU(4)}{S(U(2)\times U(2))}$.

Let us turn to $N = 3$ harmonic superspace. If we decompose the λ_I^α's and the $\bar{\lambda}^{\dot{\alpha}I}$'s into $N = 3$ vectors and $N = 3$ scalars we have $\lambda_I^\alpha = (\lambda_i^\alpha, \psi^\alpha)$ and $\bar{\lambda}^{\dot{\alpha}I} = (\bar{\lambda}^{\dot{\alpha}i}, \bar{\psi}^{\dot{\alpha}})$. In that basis, the pure spinor constraints in (2.1) become

$$\lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta + \epsilon_{ijk} \bar{\lambda}^{\dot{\alpha}k} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} = 0,$$
$$\lambda_i^\alpha \epsilon_{\alpha\beta} \psi^\beta + \epsilon_{ijk} \bar{\lambda}^{\dot{\alpha}j} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}k} = 0,$$
$$\lambda_i^\alpha \bar{\lambda}^{\dot{\alpha}i} + \psi^\alpha \bar{\psi}^{\dot{\alpha}} = 0. \tag{2.8}$$

The reduction to the $N = 3$ case is obtained by setting $\psi^\alpha = \bar{\psi}^{\dot{\alpha}} = 0$. Inserting this ansatz into the first two equations of (2.8), we obtain

$$\lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta = 0, \qquad \bar{\lambda}^{\dot{\alpha}j} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}k} = 0, \tag{2.9}$$

which is equivalent to requiring that all determinants of order 2 of the matrices λ_i^α and $\bar{\lambda}^{\dot{\alpha}i}$ vanish. It is well known (and easy to check) that if two of the 2×2 submatrices have vanishing determinant, so does the third. This implies (2.10). This means that the pure spinors can be factorized into

$$\lambda_i^\alpha = \lambda^\alpha u_i, \qquad \bar{\lambda}^{\dot{\alpha}i} = \bar{\lambda}^{\dot{\alpha}} \bar{v}^i, \tag{2.10}$$

and the equations (2.8) are solved by

$$\psi^\alpha = \bar{\psi}^{\dot{\alpha}} = 0, \qquad u_i \bar{v}^i = 0. \tag{2.11}$$

So for the $N = 3$ case no constraint is needed for λ^α and $\bar{\lambda}^{\dot{\alpha}}$. Notice that the two complex vectors u_i and $\bar{v}^i$ are defined up to a gauge transformation

$$u_i \to \rho u_i, \qquad \lambda^\alpha \to \rho^{-1} \lambda^\alpha,$$
$$\bar{v}^i \to \sigma \bar{v}^i, \qquad \bar{\lambda}^{\dot{\alpha}} \to \sigma^{-1} \bar{\lambda}^{\dot{\alpha}} \tag{2.12}$$

where $\rho, \sigma \in \mathbf{C}$. The two real parameters $|\rho|$ and $|\sigma|$ are used to impose the normalizations $u_i \bar{u}^i = 1$ and $v_i \bar{v}^i = 1$. If one also gauges away the overall phases of u_i and $\bar{v}^i$, the space of harmonic coordinates u_i and $\bar{v}^i$ is parametrized by six real parameters. This coincides with the number of free parameters of the coset $SU(3)/U(1) \times U(1)$. Indeed, we can construct 3×3 matrices

$$\left(u_i^1, u_i^2, u_i^3\right) = \left(u_i^{(1,0)}, u_i^{(0,-1)}, u_i^{(-1,1)}\right)$$

as follows:

$$u_i^1 \equiv u_i^{(1,0)} = u_i,$$
$$u_i^2 \equiv u_i^{(-1,1)} = \epsilon_{ijk} \bar{v}^j \bar{u}^k,$$
$$u_i^3 \equiv u_i^{(0,-1)} = v_i, \tag{2.13}$$

where $\bar{u}^i = (u_i)^*$ and $v_i = (\bar{v}^i)^*$. Fixing the phases of u_i^1 and u_i^3, the u_i^I form $SU(3)$ matrices which are coset representatives of $\frac{SU(3)}{U(1) \times U(1)}$. The $U(1) \times U(1)$ transformations generate the phases $\arg(\rho)$ and $\arg(\sigma)$. The notation $u_i^{(a,b)}$ indicates the $U(1) \times U(1)$ charges of the harmonic variables and they satisfy the hermiticity property

$$\overline{u_i^{(a,b)}} = u^{i(-a,-b)}.$$

We denote by u_I^i the inverse harmonics

$$u_I^i u_i^J = \delta_I{}^J, \qquad u_i^I u_I^j = \delta_i{}^j,$$
$$\det u = \epsilon^{ijk} u_i^1 u_j^2 u_k^3 = 1. \tag{2.14}$$

For later use we also list the components of the inverse matrix u_I^i:

$$u_1^i \equiv u^{i(-1,0)} = \overline{u_i^{(1,0)}} = \bar{u}^i,$$
$$u_2^i \equiv u^{i(1,-1)} = \epsilon^{ijk} v_j u_k,$$
$$u_3^i \equiv u^{i(0,1)} = \bar{v}^i. \tag{2.15}$$

Finally, we consider a further reduction to $N = 2$. We decompose the $N = 3$ pure spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha} i}$ into a vector of $N = 2$ and a singlet, $\lambda_i^\alpha = (\lambda_I^\alpha, \lambda_3^\alpha)$ and $\bar{\lambda}^{\dot{\alpha} i} = (\bar{\lambda}^{\dot{\alpha} I}, \bar{\lambda}^{\dot{\alpha} 3})$ where $I = 1, 2$. We set λ_3^α and $\bar{\lambda}_3^{\dot{\alpha}}$ to zero. The pure spinor equations (2.8) reduce then to

$$\lambda_I^\alpha \epsilon_{\alpha\beta} \lambda_J^\beta \epsilon^{IJ} = 0,$$
$$\bar{\lambda}^{\dot{\alpha} J} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta} K} \epsilon_{JK} = 0,$$
$$\lambda_I^\alpha \bar{\lambda}^{\dot{\alpha} I} = 0. \tag{2.16}$$

The first two equations imply that λ_I^α and $\bar{\lambda}^{\dot{\alpha} I}$ are factorized into $\lambda_I^\alpha = \lambda^\alpha u_I$ and $\bar{\lambda}^{\dot{\alpha} J} = \bar{\lambda}^{\dot{\alpha}} \bar{v}^J$ where $u_I \bar{v}^I = 0$. The vector $\bar{v}^I$ is proportional to $\epsilon^{IJ} u_J$. Hence without loss of generality one may write

$$\lambda_I^\alpha = \lambda^\alpha u_I, \qquad \bar{\lambda}^{\dot{\alpha} J} = \bar{\lambda}^{\dot{\alpha}} \epsilon^{IJ} u_I. \tag{2.17}$$

With this parametrization of the $N = 2$ case there are neither constraints on the λ's nor on the u's.

The vector u_I yields the usual parametrization of $N = 2$ harmonic superspace [5]. Namely, one introduces the $SU(2)$ matrix (u_I^+, u_I^-) where $u_I^+ = u_I$ and $u_I^- = (u^{+I})^*$ with $u_J^+ = \epsilon_{JK} u^{+K}$. The coset $SU(2)/U(1)$ is obtained by dividing by the subgroup $U(1)$ which generates the phases $u_I^\pm \to e^{\pm i\alpha} u_I^\pm$. In fact, Eqs. (2.17) are defined up to a rescaling of λ^α, $\bar{\lambda}^{\dot{\alpha}}$ and of u_I given by $u_I \to \rho u_I$, for $\rho \neq 0$. This yields the compact space $\mathbf{CP}^1$.

3. $N = 3$ harmonic superspace for SYM theory from superstrings

The field equation for $D = 4$, $N = 3$ SYM-theory in ordinary (not harmonic) superspace are given by [20]

$$\left\{\nabla_\alpha^i, \nabla_\beta^j\right\} = \epsilon_{\alpha\beta} \bar{W}^{ij}, \qquad \left\{\bar{\nabla}_{\dot{\alpha} i}, \bar{\nabla}_{\dot{\beta} j}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}} W_{ij},$$
$$\left\{\nabla_\alpha^i, \bar{\nabla}_{\dot{\beta} j}\right\} = \delta_j^i \nabla_{\alpha\dot{\beta}}. \tag{3.1}$$

The coordinates for this $N = 3$ superspace, $(x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i})$, are obtained by imposing the constraint $\theta_4^\alpha = \bar{\theta}^{\dot{\alpha} 4} = 0$. Since θ's transform into λ's under BRST transformations we also impose for consistency $\lambda_4^\alpha = \bar{\lambda}^{\dot{\alpha} 4} = 0$.

Using the decomposition of the $N = 3$ spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha} i}$ given in (2.10), and contracting the harmonic variables with the operators $d_{z\hat{\alpha}}$ in (1.3) yields two new spinorial operators

$$Q_G = \lambda^\alpha d_\alpha^1 + \bar{\lambda}^{\dot{\alpha}} \bar{d}_{3\dot{\alpha}},$$
$$d_\alpha^1 = u_i d_\alpha^i = u_i^1 d_\alpha^i = u_i^{(1,0)} d_\alpha^i,$$
$$\bar{d}_{3\dot{\alpha}} = \bar{v}^i \bar{d}_{\dot{\alpha} i} = u_3^i \bar{d}_{\dot{\alpha} i} = u^{i(0,1)} \bar{d}_{\dot{\alpha} i}. \tag{3.2}$$

The operator d_α^1 corresponds to $\xi_i D_\alpha^i$ and $\bar{d}_{3\dot{\alpha}}$ to $\eta^i \bar{D}_{\dot{\alpha} i}$ in [5].

Due to the constraints on the u's the operators d^1_α and $\bar d_{3\dot\alpha}$ satisfy the commutation relations

$$\{d^1_\alpha, d^1_\beta\} = 0, \qquad \{d^1_\alpha, \bar d_{3\dot\beta}\} = 0,$$
$$\{\bar d_{3\dot\alpha}, \bar d_{3\dot\beta}\} = 0. \tag{3.3}$$

To derive these relations one may use the dimensionally reduced relations

$$\{d^i_\alpha, d^j_\beta\} = \epsilon_{\alpha\beta}\Pi^{ij}, \qquad \{\bar d_{\dot\alpha i}, \bar d_{\dot\beta j}\} = \epsilon_{\dot\alpha\dot\beta}\bar\Pi_{ij},$$
$$\{d^i_\alpha, \bar d_{\dot\alpha j}\} = \delta^i_j \Pi_{\alpha\dot\beta}.$$

Hence, $Q_{\rm G}$ (where G stands for Grassmann) is nilpotent for any λ^α and $\bar\lambda^{\dot\alpha}$.

The BRST operator $Q_{\rm G}$ implements naturally the G-analyticity on the space of superfields $\Phi(x,\theta,\bar\theta,\lambda,\bar\lambda,u)$. A superfield with ghost number zero is given by $\Phi(x,\theta,\bar\theta,u)$ and G-analyticity means $Q_{\rm G}\Phi = 0$ which implies $D^1_\alpha\Phi = \bar D_{3\dot\alpha}\Phi = 0$ (since $\{d^1_\alpha, \Phi(x,\theta,\bar\theta,\lambda,\bar\lambda,u)\} = D^1_\alpha\Phi(x,\theta,\bar\theta,\lambda,\bar\lambda,u)$ and similarly for $\bar d_{3\dot\alpha}$). Such a superfield is called a G-analytic superfield in [5]. A generic superfield $\Phi(x,\theta,\bar\theta,\lambda,\bar\lambda,u)$ with ghost number one can be parametrized in terms of two u-dependent spinorial superfields A_α, $\bar A_{\dot\alpha}$ as follows:

$$\Phi^{(1)}(x,\theta,\bar\theta,\lambda,\bar\lambda,u) = \lambda^\alpha A_\alpha + \bar\lambda^{\dot\alpha}\bar A_{\dot\alpha}, \tag{3.4}$$

and $\{Q_{\rm G}, \Phi^{(1)}\} = 0$ implies the following constraints on these superfields

$$D^1_\alpha A_\beta + D^1_\beta A_\alpha = 0, \qquad \bar D_{3\dot\alpha}\bar A_{\dot\beta} + \bar D_{3\dot\beta}\bar A_{\dot\alpha} = 0,$$
$$D^1_\alpha \bar A_{\dot\beta} + \bar D_{3\dot\beta} A_\alpha = 0. \tag{3.5}$$

Assuming that A_α and $A_{\dot\alpha}$ factorize in the same way as $D^1_\alpha = u_i D^i_\alpha$ and $\bar D_{3\dot\alpha} = \bar v^i \bar D_{\dot\alpha i}$, so $A_\alpha = u_i A^i_\alpha$ and $A_{\dot\alpha} = \bar v^i A_{\dot\alpha i}$, the equations (3.5) reproduce (3.1). We stress that (3.5), unlike (3.1), do not put the theory on-shell; only the extra assumption of the factorization of A_α and $A_{\dot\alpha}$ puts the theory on-shell.

Gauge transformations are generated by a ghost-number zero scalar superfield $\Omega^{(0)}$. To lowest order in $\Phi^{(1)}$ they read $\delta\Phi^{(1)} = \{Q_{\rm G}, \Omega^{(0)}\}$ which yields $\delta A_\alpha = D_\alpha\Omega$ and $\delta A_{\dot\alpha} = \bar D_{\dot\alpha}\Omega$. Equations (3.5) are easily solved in $D = 4$; they imply that the superfields A_α and $\bar A_{\dot\alpha}$ are pure gauge. Hence the $Q_{\rm G}$-cohomology in the space of superfields with ghost number 1 vanishes.

To determine on which harmonic variables superfields depend, we construct a second BRST operator $Q_{\rm H}$ which is constructed from the $SU(3)$ generators

$$d^a{}_b = u^a_i\partial_{u^b_i} - u^i_b\partial_{u^i_a} = u^a_i p^i_b - u^i_b p^a_i, \tag{3.6}$$

where p^i_b can be represented by $\partial/\partial u^b_i$ and similarly for p^b_i. These generators split into three raising operators $d^1_2 = d^{(2,-1)}$, $d^2_3 = d^{(-1,2)}$, $d^1_3 = d^{(1,1)}$, three lowering operators $d^2_1 = d^{(-2,1)}$, $d^3_2 = d^{(1,-2)}$, $d^3_1 = d^{(-1,-1)}$, and two Cartan generators d^1_1 and d^2_2. The raising operators commute with $Q_{\rm G}$

$$[d^{(2,-1)}, d^1_\alpha] = [d^{(-1,2)}, d^1_\alpha] = [d^{(1,1)}, d^1_\alpha] = 0,$$
$$[d^{(2,-1)}, d_{3\dot\alpha}] = [d^{(-1,2)}, d_{3\dot\alpha}] = [d^{(1,1)}, d_{3\dot\alpha}]$$
$$= 0 \tag{3.7}$$

and form an algebra, in particular $[d^{(2,-1)}, d^{(-1,2)}] = d^{(1,1)}$. This suggests to construct a new nilpotent BRST operator $Q_{\rm H}$

$$Q_{\rm H} = \xi^3_1 d^1_3 + \xi^2_1 d^1_2 + \xi^3_2 d^2_3 - \beta^1_3\xi^2_1\xi^3_2, \tag{3.8}$$

where we introduced new pairs of anticommuting (anti)ghosts (ξ^3_1, β^1_3), (ξ^2_1, β^1_2), (ξ^3_2, β^2_3) with canonical anticommutation relations. It is convenient to use a notation in which the $U(1)\times U(1)$ weights are made explicit $\xi^3_1 \equiv \xi^{(-1,-1)}$, $\xi^2_1 \equiv \xi^{(-2,1)}$ and $\xi^3_2 = \xi^{(1,-2)}$.

Since $Q_{\rm H}$ and $Q_{\rm G}$ anticommute their sum $Q_{\rm tot}$ is obviously nilpotent. A generic superfield $\Phi^{(1)}$ with ghost number one can be decomposed into the following pieces

$$\Phi^{(1)} = \lambda^\alpha A^{(1,0)}_\alpha + \bar\lambda^{\dot\alpha}\bar A^{(0,1)}_{\dot\alpha}$$
$$+ \xi^3_1 A^{(1,1)} + \xi^2_1 A^{(2,-1)} + \xi^3_2 A^{(-1,2)}, \tag{3.9}$$

where $A^{(1,0)}_\alpha$, $\bar A^{(0,1)}_{\dot\alpha}$, $A^{(2,-1)}$, $A^{(-1,2)}$ and $A^{(1,1)}$ are harmonic superfields (superfields which depend on the variables u). The harmonic weights of the superfields follow from requiring that $\Phi^{(1)}$ has zero harmonic weight, just like the BRST charge $Q_{\rm tot}$. Note that $\Phi^{(1)}$ depends only upon the variables $x,\theta,\bar\theta,\lambda,\bar\lambda$'s and u's and not upon the conjugated momenta as a consequence of quantum mechanical rules. This forbids ghost-number one combinations of the form $\beta\xi\xi$, $\beta\xi\lambda, \ldots$.

The equations of motion for $N = 3$ SYM follow from the BRST-cohomology equations

$$\{Q_{\rm tot}, \Phi^{(1)}\} + \frac{1}{2}\{\Phi^{(1)}, \Phi^{(1)}\} = 0. \tag{3.10}$$

Decomposing the superfield $\Phi^{(1)}$ into $\Phi_{\rm H}^{(1)} + \Phi_{\rm G}^{(1)}$, where $\Phi_{\rm H}^{(1)}$ denotes the terms with ξ-ghosts and $\Phi_{\rm G}^{(1)}$ the terms with λ-ghosts, the Maurer–Cartan equations in (3.10) decompose as follows:

$$\{Q_{\rm G}, \Phi_{\rm G}^{(1)}\} + \frac{1}{2}\{\Phi_{\rm G}^{(1)}, \Phi_{\rm G}^{(1)}\} = 0, \tag{3.11}$$

$$\{Q_{\rm G}, \Phi_{\rm H}^{(1)}\} + \{Q_{\rm H}, \Phi_{\rm G}^{(1)}\} + \{\Phi_{\rm G}^{(1)}, \Phi_{\rm H}^{(1)}\} = 0, \tag{3.12}$$

$$\{Q_{\rm H}, \Phi_{\rm H}^{(1)}\} + \frac{1}{2}\{\Phi_{\rm H}^{(1)}, \Phi_{\rm H}^{(1)}\} = 0. \tag{3.13}$$

This system of equations is invariant under the infinitesimal gauge transformation

$$\Phi^{(1)} \to \Phi^{(1)} + \{Q_{\rm tot}, \Omega\} + \{\Phi^{(1)}, \Omega\}, \tag{3.14}$$

where Ω is a generic harmonic superfield with ghost number zero. According to the above decomposition of $\Phi^{(1)}$, one obtains

$$\delta\Phi_{\rm G}^{(1)} = \{Q_{\rm G}, \Omega\} + \{\Phi_{\rm G}^{(1)}, \Omega\},$$
$$\delta\Phi_{\rm H}^{(1)} = \{Q_{\rm H}, \Omega\} + \{\Phi_{\rm H}^{(1)}, \Omega\}.$$

To reduce the system of equations in (3.11)–(3.13) to the field equations of harmonic superspace, we use the fact that $Q_{\rm G}$ has no cohomology. This implies that Eq. (3.11) is solved by a pure gauge superfield

$$\Phi_{\rm G}^{(1)} = e^{-i\Delta}\left(Q_{\rm G} e^{i\Delta}\right),$$

where Δ is a ghost-number zero superfield known in the literature as the *bridge* (see, for example, [5]). Also the BRST cohomology of $Q_{\rm H}$ vanishes on the unconstrained superspace and therefore one can also solve the system (3.11)–(3.13) starting from the last equation.

In the harmonic superspace framework, one usually employs the bridge superfield $\Delta(x, \theta, \bar\theta, u)$ to bring the spinorial covariant derivatives to the 'pure gauge' form

$$\nabla_\alpha^{(1,0)} = e^{-i\Delta} d_\alpha^{(1,0)} e^{i\Delta},$$
$$\bar\nabla_{\dot\alpha}^{(0,1)} = e^{-i\Delta} \bar d_{\dot\alpha}^{(0,1)} e^{i\Delta}. \tag{3.15}$$

Here the bridge is seen as the most general solution of Eq. (3.11). By making a finite gauge transformation which sets $\Phi_{\rm G}^{(1)} = 0$, the gauge transformed $\Phi_{\rm H}^{(1)}$ is given by

$$e^{-i\Delta}\left(\Phi_{\rm H}^{(1)} + Q_{\rm H}\right)e^{i\Delta}$$
$$= \xi_1^3 V^{(1,1)} + \xi_1^2 V^{(2,-1)} + \xi_2^3 V^{(-1,2)}. \tag{3.16}$$

Eq. (3.12) becomes

$$D_\alpha^{(1,0)} V^{(2,-1)} = D_\alpha^{(1,0)} V^{(-1,2)} = D_\alpha^{(1,0)} V^{(1,1)} = 0,$$
$$\bar D_{\dot\alpha}^{(0,1)} V^{(2,-1)} = \bar D_{\dot\alpha}^{(0,1)} V^{(-1,2)} = \bar D_{\dot\alpha}^{(0,1)} V^{(1,1)}$$
$$= 0, \tag{3.17}$$

expressing the G-analyticity of the harmonic connections $V^{(1,1)}$, $V^{(2,-1)}$ and $V^{(-1,2)}$. The last equation (3.13) finally gives the SYM equations of motion of $N = 3$ harmonic superspace

$$D^{(2,-1)} V^{(-1,2)} - D^{(-1,2)} V^{(2,-1)}$$
$$+ \left[V^{(2,-1)}, V^{(-1,2)}\right] = V^{(1,1)},$$
$$D^{(2,-1)} V^{(1,1)} - D^{(1,1)} V^{(2,-1)}$$
$$+ \left[V^{(2,-1)}, V^{(1,1)}\right] = 0,$$
$$D^{(-1,2)} V^{(1,1)} - D^{(1,1)} V^{(-1,2)}$$
$$+ \left[V^{(-1,2)}, V^{(1,1)}\right] = 0, \tag{3.18}$$

where the harmonic derivatives $D^{(1,1)}$, $D^{(2,-1)}$ and $D^{(-1,2)}$ represent the action of $d^{(1,1)}$, $d^{(2,-1)}$ and $d^{(-1,2)}$ on u-dependent superfields. These are the field equations of $N = 3$ SYM harmonic superspace, see Eq. (12.57) in [5]. Eqs. (3.17), (3.18) are invariant under the gauge transformations

$$\delta V^{(2,-1)} = D^{(2,-1)}\omega + \left[V^{(2,-1)}, \omega\right],$$
$$\delta V^{(-1,2)} = D^{(-1,2)}\omega + \left[V^{(-1,2)}, \omega\right],$$
$$\delta V^{(1,1)} = D^{(1,1)}\omega + \left[V^{(1,1)}, \omega\right], \tag{3.19}$$

where the superfield ω satisfies

$$D_\alpha^{(1,0)}\omega = 0, \qquad \bar D_{\dot\alpha}^{(0,1)}\omega = 0. \tag{3.20}$$

4. The action and measure for $N = 3$ SYM theory

We start from the observation that the field equations (3.10) are of Chern–Simons form and can be derived from an action of the form

$$S_{\rm CS} = \int d\mu \left(\Phi^{(1)} Q_{\rm tot} \Phi^{(1)} + \frac{2}{3}\Phi^{(1)} \star \Phi^{(1)} \star \Phi^{(1)}\right) \tag{4.1}$$

where $\star$ denotes conventional matrix multiplication. The measure $d\mu$ has to be determined.

Instead of dimensionally reducing (4.1) we follow a different path. We have to define the integration measure for all zero modes in the theory. Since we are dealing with worldline models, the only contribution comes from the zero modes of x^μ, θ_i^α, $\bar\theta^{\dot\alpha i}$, λ_i^α, $\bar\lambda^{\dot\alpha i}$, u_i^I and ξ_1^3, ξ_1^2, ξ_2^3. The set of ghosts λ_i^α, $\bar\lambda^{\dot\alpha i}$ pertains to the BRST charge $Q_{\rm G}$ which implements the G-analyticity. Therefore, they implement kinematical constraints on the theory expressed by the equations:

$$[Q_{\rm G}, S_{N=3}] = 0, \qquad [Q_{\rm G}, d\mu_{\rm H}] = 0, \tag{4.2}$$

where $S_{N=3}$ is the off-shell $N = 3$ action and $d\mu_{\rm H}$ is the invariant measure in the space of the zero modes of x^μ, θ_i^α, $\bar\theta^{\dot\alpha i}$, u_i^I and ξ_1^3, ξ_1^2, ξ_2^3. In addition, $S_{N=3}$ has zero ghost number, while $d\mu_{\rm H}$ has ghost number three. Form [2] and [21] it is known that $d\mu_{\rm H} \in H^3(Q_{\rm H})$. This implies that $d\mu_{\rm H} = d\xi_1^3\, d\xi_1^2\, d\xi_2^3\, d\mu'$, where the measure $d\mu' = d\mu'(x^\mu, \theta_i^\alpha, \bar\theta^{\dot\alpha i}, u_i^I)$ has to be fixed by the G-analyticity (4.2).

First we consider the space formed by x^μ, θ_i^α, $\bar\theta^{\dot\alpha i}$. The conditions in (4.2) select the analytic subspace $(x_{\rm A}^m, \theta_\alpha^{(0,1)}, \theta_\alpha^{(1,-1)}, \bar\theta_{\dot\alpha}^{(1,0)}, \bar\theta_{\dot\alpha}^{(-1,1)})$, where $\theta^{(a,b)} = u^{i\,(a,b)}\theta_i$, and $x_{\rm A}^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + 2i\theta^{\alpha(-1,0)}\bar\theta^{\dot\alpha(1,0)} + 2i\theta^{\alpha(0,1)}\bar\theta^{\dot\alpha(0,-1)}$. Therefore, the only invariant measure is given by

$$d\mu' = d^4x_{\rm A}\, d^2\theta^{(0,1)}\, d^2\theta^{(1,-1)}\, d^2\bar\theta^{(1,0)} \times d^2\bar\theta^{(-1,1)}\, d\mu_u, \tag{4.3}$$

where $d\mu_u$ is the measure for the harmonic variables. In order to derive a $Q_{\rm G}$ invariant measure $d\mu_u$, we introduce the new variables (projective harmonic variables [22])

$$z_1 = \frac{u_1}{u_3}, \quad z_2 = \frac{u_2}{u_3}, \quad z_3 = \frac{v_1}{v_2}. \tag{4.4}$$

The measure $d\mu_u$ for the $N = 3$ harmonic space is the Haar measure for the coset $SU(3)/U(1) \times U(1)$. We compute it as follows. There are 18 real matrix elements for a general 3×3 complex matrix, which we denote by the column vectors (u_i, v_i, w_i) $(i = 1, 2, 3)$ and their complex conjugates. The orthormality relations yield 9 real constraints. They allow one to express w_i in terms of u_i and v_i as $w_i = \epsilon_{ijk}\bar u^j \bar v^k$ up to a phase which will drop out. One is left with the 6 real z variables in (4.4) and three angles. One of these angles is fixed by the requirement that the determinant be unity, while the two remaining angles parametrize the subgroup $U(1) \times U(1)$.

As suggested by this enumeration, we use the following set of 18 integration variables

$$z_1 = \frac{u_1}{u_3}, \quad z_2 = \frac{u_2}{u_3}, \quad z_3 = \frac{v_1}{v_3}, \quad z_4 = \frac{v_3}{v_2},$$
$$|u_3|^2, \quad \frac{\bar u^3}{u_3}, \quad |v_2|^2, \quad \frac{\bar v^2}{v_2},$$
$$w_1, \quad \bar w^1, \quad w_2, \quad \bar w^2, \quad |w_3|^2, \quad \frac{\bar w^3}{w_3} \tag{4.5}$$

and the conjugates of z's.

The integration over the delta functions which enforce the othornormality relations yields unity. Namely,

(1) $\int dw_1\, d\bar w^1\, \delta(w_i\bar u^i)$ yields $|u_1|^{-2}$,
(2) $\int dw_2\, d\bar w^2\, \delta(w_i\bar v^i)$ yields $|u_1|^2/|w_3|^2$,
(3) $\int d|w_3|^2\, \delta(|w|^2 - 1)$ yields $[1 + |w_1/w_3|^2 + |w_2/w_3|^2]$.

The product of these contributions is unity.

(4) $\int d|u_3|^2\, \delta(|u|^2 - 1)$ yields $|u_3|^{-2} = (1 + |z_1|^2 + |z_2|^2)$,
(5) $\int d|v_2|^2\, \delta(|v|^2 - 1)$ gives $|v_2|^{-2} = 1 + |z_3|^2 + |z_2 + z_1\bar z_3|^2$,
(6) $\int dz_4\, d\bar z^4\, \delta(u \cdot \bar v)$ cancels the previous two contributions because $u \cdot \bar v = u_3\bar v^2(z_1\bar z^3 + z_2 + \bar z^4)$.

One thus only obtains a contribution of the Jacobian. The evaluation of a 18×18 Jacobian may seems daunting, but most terms vanish, and one easily derives by hand

$$J = \left|u_3^2 u_3 \bar u^3 v_2^2 v_2 \bar v^2\right| = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_3|^2 + |z_2 + z_1\bar z^3|^2)^2}. \tag{4.6}$$

This yields Haar measure for $SU(3)/U(1) \times U(1)$ and agrees with [22] except for the power in the denominator

$$d\mu_u = \frac{\prod_{i=1}^3 dz_i\, d\bar z^i}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_3|^2 + |z_2 + z_1\bar z^3|^2)^2} \tag{4.7}$$

To conclude, the action for $N = 3$ SYM in harmonic superspace is given by

$$S = \int d\xi_1^3\, d\xi_1^2\, d\xi_2^3\, d^4x_A\, d^2\theta^{(0,1)} \times d^2\theta^{(1,-1)}\, d^2\bar{\theta}^{(1,0)}\, d^2\bar{\theta}^{(-1,1)}\, d\mu_u \times \left(\Phi_{\mathrm{H}}^{(1)} Q_{\mathrm{H}} \Phi_{\mathrm{H}}^{(1)} + \frac{2}{3} \Phi_{\mathrm{H}}^{(1)} \star \Phi_{\mathrm{H}}^{(1)} \star \Phi_{\mathrm{H}}^{(1)} \right), \tag{4.8}$$

where $\Phi_{\mathrm{H}}^{(1)}$ is given by the right hand side of (3.16) and S coincides with the action given in [5] after the integration over the ghost fields ξ's is performed.

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