Computers Math. Applic. Vol. 24, No. 1/2, pp. 17–24, 1992
Printed in Great Britain.

0097-4943/92 $5.00 + 0.00

THE TRANSIENT SOLUTION TO A CLASS OF MARKOVIAN QUEUES

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Abstract—In this paper, we study the transient behavior of Markovian queues such as $M/M/s$ queues and bulk-arrival $M/M/1$ queues. It will be shown that the transition probabilities of a birth-death process on the non-negative integers, governed by parameters $\{\lambda_n, \mu_n\}_{n=0}^{\infty}$ such that $\lambda_{n-1} = \lambda$ and $\mu_n = \mu$ for all $n \geq N$ and for some $N \geq 1$ with $\mu_0 = 0$, can be represented in terms of the busy period density of an $M/M/1$ queue having the arrival rate λ and the service rate μ and some exponential functions. The transition probabilities of a bulk-arrival $M/M/1$ queue can also be expressed in terms of its busy period density.

1. INTRODUCTION

In this paper, we study the transient behavior of Markovian queues such as $M/M/s$ queues and bulk-arrival $M/M/1$ queues.

Let $\{X(t),\, t \geq 0\}$ be an ergodic birth-death process on the non-negative integers $\{0, 1, 2, \dots\}$ governed by parameters $\{\lambda_n, \mu_n\}_{n=0}^{\infty}$ such that $\mu_0 = 0$. It is well known (see e.g., [1,2]) that, for such birth-death processes, the transition probabilities $P_{ij}(t) = \Pr[X(t) = j \mid X(0) = i]$ can be expressed as

$$P_{ij}(t) = \pi_j \int_0^{\infty} e^{-xt} Q_i(x)\, Q_j(x)\, d\psi(x), \tag{1}$$

where $\pi_n = \pi_{n-1}\, \lambda_{n-1}/\mu_n$, $n \geq 1$, with $\pi_0 = 1$, a set of polynomials $\{Q_i(x)\}_{i=0}^{\infty}$ is defined through the recursion relations

$$-xQ_i(x) = \mu_i Q_{i-1}(x) - (\lambda_i + \mu_i)\, Q_i(x) + \lambda_i Q_{i+1}(x), \qquad i \geq 0, \tag{2}$$

with $Q_{-1}(x) = 0$ and $Q_0(x) = 1$, and $\psi(x)$ is a positive spectral measure with respect to which $\{Q_i(x)\}_{i=0}^{\infty}$ constitutes an orthogonal system. Karlin and McGregor [3] showed that if there exists some N such that $\lambda_{n-1} = \lambda$ and $\mu_n = \mu$ for all $n \geq N$ then the measure $\psi(x)$ is obtainable so that its transition behavior is, in principle, completely determined through (1) and (2) (see also [2]). In particular, for the case $N = 1$ with $\rho = \lambda/\mu < 1$, i.e., an ergodic $M/M/1$ queue, $\psi(x)$ has the continuous density

$$\psi'(x) = \frac{\sqrt{4\lambda\mu - (\lambda + \mu - x)^2}}{2\pi\mu x}, \qquad |\lambda + \mu - x| < 2\sqrt{\lambda\mu}, \tag{3}$$

and has, in addition, a mass of amount $(1 - \rho)$ at $x = 0$. Also, for a bulk-arrival $M/M/1$ queue, some exact but complicated expressions for $P_{ij}(t)$ have been obtained in [4].

In the ergodic $M/M/1$ queue, on the other hand, Abate and Whitt [5] pointed out an interesting relation between $P_{i0}(t)$ and the busy period density $b(t)$ of the $M/M/1$ queue that

$$P_{i0}(t) = \int_0^t b^{(i)}(u)\, du - \rho \int_0^t b^{(i+1)}(u)\, du \tag{4}$$

This research is partially supported by General Scientific Research Grant (02680017) of the Ministry of Education, Science and Culture.

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and, due to the time reversibility $\rho^i P_{i0}(t) = P_{0i}(t)$,

$$P_{0i}(t) = \rho^i \int_0^t b^{(i)}(u)\, du - \rho^{i+1} \int_0^t b^{(i+1)}(u)\, du,$$

where $b^{(i+1)}(t) = b(t) * b^{(i)}(t)$, $i \geq 0$, with $b^{(0)}(t) = \delta(t)$, the delta function, and $*$ denotes the convolution operator. Moreover, Abate, Kijima and Whitt [6] obtained further relations such as

$$P_{i0}(t) - P_{i+1,0}(t) = \frac{b^{(i+1)}(t)}{\mu}, \qquad i \geq 0, \tag{5}$$

and useful inequalities between $P_{ij}(t)$. Evidently, such relations as (4) and (5) are very useful since they provide an insight for the better understanding of the transient behavior of the $M/M/1$ queue (for numerical purposes, see [7]). Note that these relations can not be derived from (1) through (3) by the first glance. Also, when N is large, it will be extremely difficult to determine the spectral measure $\psi(x)$ (see [8] for $M/M/s$ queues). Hence, it is of practical interest to relate the transition probabilities $P_{ij}(t)$ to known functions such as the busy period density by a different means. In this paper, we give a representation of $P_{ij}(t)$ for a birth-death process with the parameters as given above in terms of the busy period density of an $M/M/1$ queue having the arrival rate λ and the service rate μ and some other exponential functions. Some representations of $P_{i0}(t)$ for a bulk-arrival $M/M/1$ queue are also derived.

This paper is organized as follows. In the next section, we show that the results in (4) and (5) can be in fact derived directly by the spectral representation (1) for the $M/M/1$ case. For general N, however, this method seems not so useful and the development of such relations requires another means. The general case is treated in Section 3. The basic idea here is the decomposition of the sample path at the changing state N. Finally, in Section 4, we consider a bulk-arrival $M/M/1$ queue and show that the representations (4) for $i = 0$ and (5) hold true even for the bulk-arrival case.

2. THE $M/M/1$ QUEUE

Let $\hat{P}_{ij}(s) = \int_0^\infty e^{-st} P_{ij}(t)\, dt$ denote the Laplace transform of $P_{ij}(t)$ (throughout the paper, we denote the Laplace transform of a function $a(t)$ by $\hat{a}(s)$, i.e., using a hat, if it exists). Since $\psi(x)$ has a finite support [3], it follows from (1) that

$$\hat{P}_{ij}(s) = \pi_j \int_0^\infty Q_i(x)\, Q_j(x)\, \frac{d\psi(x)}{x+s}, \qquad \mathrm{Re}(s) > 0. \tag{6}$$

Denote the Stieltjes transform of the spectral measure by $B(s)$, i.e.,

$$B(s) = \int_0^\infty \frac{d\psi(x)}{x+s} = \hat{P}_{00}(s), \qquad \mathrm{Re}(s) > 0. \tag{7}$$

Due to the spatial homogeneity of $M/M/1$ queues, Karlin and McGregor [3] showed that

$$B(s) = -\frac{\lambda - \mu + s - \sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\mu s}. \tag{8}$$

But, the Laplace transform of the busy period density is given by

$$\hat{b}(s) = \frac{\lambda + \mu + s - \sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\lambda} \tag{9}$$

(see e.g., [9]). Hence, from (7) through (9), one has

$$\hat{P}_{00}(s) = \frac{1}{s} - \rho\, \frac{\hat{b}(s)}{s}. \tag{10}$$

Since $\hat{P}_{i0}(s) = \hat{b}^i(s)\,\hat{P}_{00}(s)$, it follows that

$$s\hat{P}_{i0}(s) = \hat{b}^i(s) - \rho\hat{b}^{i+1}(s), \tag{11}$$

which, in the real domain, leads to (4).

To prove (5), we note that

$$\hat{P}_{00}(s) = \frac{1}{\mu}\frac{\hat{b}(s)}{1-\hat{b}(s)}. \tag{12}$$

This follows at once, since, from (9),

$$\hat{b}(s) = \frac{\mu}{s+\lambda+\mu-\lambda\hat{b}(s)} \tag{13}$$

so that

$$\frac{s\hat{b}(s)}{\mu} = (1-\hat{b}(s))\,(1-\rho\hat{b}(s)). \tag{14}$$

Since $\hat{P}_{i0}(s) = \hat{b}^i(s)\,\hat{P}_{00}(s)$, and then since $\hat{P}_{00}(s) - \hat{P}_{10}(s) = \frac{\hat{b}(s)}{\mu}$ from (12), one has

$$\hat{P}_{i0}(s) - \hat{P}_{i+1,0}(s) = \frac{\hat{b}^{i+1}(s)}{\mu}, \qquad i \geq 0. \tag{15}$$

Hence, (5) holds in the real domain.

3. REPRESENTATION OF $P_{ij}(t)$

Let N be a positive integer and let $\{X(t)\}$ be a birth-death process defined in Section 1 with parameters such that $\lambda_{n-1} = \lambda$ and $\mu_n = \mu$ for all $n \geq N$ with $\rho = \frac{\lambda}{\mu} < 1$. Also, let $\mathbf{A}$ be the $N \times N$ tri-diagonal matrix defined by

$$\mathbf{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{N-2} & -\lambda_{N-2}-\mu_{N-2} & \lambda_{N-2} \\ & & & \mu_{N-1} & -\lambda-\mu_{N-1} \end{pmatrix}. \tag{16}$$

Note that the last row-sum in $\mathbf{A}$ is strictly negative. Hence the generator $\mathbf{A}$ governs a lossy birth-death process restricted on $\{0,\ldots,N-1\}$ (see e.g., [10]). Let $g_{ij}(t)$ be the transition probabilities of $X(t)$ restricted on the states $\{0,\ldots,N-1\}$, i.e.,

$$g_{ij}(t) = \Pr[X(t) = j, \qquad 0 \leq X(u) \leq N-1; \quad 0 \leq u \leq t \mid X(0) = i].$$

It is readily seen that $\mathbf{G}(t) = (g_{ij}(t)) = \exp\{\mathbf{A}t\}$.

Let $(-\gamma_n)$, $0 \leq n \leq N-1$, be the eigenvalues of the lossy generator $\mathbf{A}$. It is well known that all the eigenvalues are distinct, real and strictly negative [11]. Let $\mathbf{x}_n = (x_{n0},\ldots,x_{n,N-1})'$ and $\mathbf{y}_n = (y_{n0},\ldots,y_{n,N-1})'$ ($'$ denotes the transpose) be the right and left eigenvectors of $\mathbf{A}$ associated with $(-\gamma_n)$, $0 \leq n \leq N-1$, respectively. It is assumed that they are normalized to be $\mathbf{x}_n'\mathbf{y}_n = 1$. The system $\{\mathbf{x}_n, \mathbf{y}_n\}$ is biorthogonal, i.e., $\mathbf{x}_i'\mathbf{y}_j = 0$ for $i \neq j$. Hence, the spectral decomposition of $\mathbf{A}$ yields

$$g_{ij}(t) = \sum_{n=0}^{N-1} e^{-\gamma_n t} x_{ni}\, y_{nj}, \qquad 0 \leq i,j \leq N-1. \tag{17}$$

It should be noted that $\lim_{t\to\infty} g_{ij}(t) = 0$, since $\gamma_n > 0$, $0 \leq n \leq N-1$. Also, due to the time reversibility (see e.g., [10]), $\pi_i g_{ij}(t) = \pi_j g_{ji}(t)$ for $0 \leq i,j \leq N-1$.

Let $f_i(t)$, $0 \le i \le N-1$, be the first-passage-time density of $\{X(t)\}$ from state i to state N. It is easy to see that $f_i(t) = \lambda g_{i,N-1}(t)$. In particular,

$$\hat{f}_0(s) = \lambda \hat{g}_{0,N-1}(s) = \prod_{n=0}^{N-1} \frac{\gamma_n}{s+\gamma_n}$$

(see [12]). Let $\{T_n\}$ be a sequence of random variables distributed by the density function $f_{N-1}(t)$ and let B_n denote the n^{th} busy period of an $M/M/1$ queue having the arrival rate λ and the service rate μ. In the above birth-death setting, it is the first-passage-time density of $\{X(t)\}$ from state n to state $(n-1)$ for all $n \ge N$. Consider then an alternating renewal process $\{M(t), t \ge 0\}$ having up-times T_n and down-times B_n, i.e., $M(t)$ counts the number of transitions from up to down in $\{T_1, B_1, T_2, B_2, \cdots\}$ until time t. Let $m(t) = \frac{d}{dt} E[M(t)]$ be the renewal density of the alternating renewal process. The differentiability of $E[M(t)]$ follows easily. The Laplace transform of $m(t)$ is given by

$$\hat{m}(s) = \frac{\hat{f}_{N-1}(s)}{1 - \hat{f}_{N-1}(s)\,\hat{b}(s)}. \tag{18}$$

The following lemmas are of independent interest.

LEMMA 1. *In the birth-death process under consideration, the transition probability $P_{N-1,N-1}(t)$ is characterized by the renewal density of the alternating renewal process $\{M(t)\}$ as*

$$P_{N-1,N-1}(t) = \frac{1}{\lambda}\, m(t). \tag{19}$$

PROOF. Because of the skip-free nature of the birth-death process, one has

$$\begin{aligned} P_{ij}(t) = &\Pr[X(t) = j, \quad 0 \le X(u) \le N-1;\ 0 \le u \le t \mid X(0) = i] \\ &+ \Pr[X(t) = j,\ X(u) = N \text{ for some } u : 0 \le u \le t \mid X(0) = i] \end{aligned} \tag{20}$$

for all $0 \le i, j \le N-1$. The first term in the right hand side of (20) is $g_{ij}(t)$ by definition. The second term there is equal to $f_i(t) * P_{Nj}(t)$ due to the strong Markov property of the birth-death process. It follows that

$$P_{N-1,N-1}(t) = g_{N-1,N-1}(t) + f_{N-1}(t) * b(t) * P_{N-1,N-1}(t). \tag{21}$$

Taking the Laplace transform in (21) and using (18) then yield the lemma, since $f_{N-1}(t) = \lambda g_{N-1,N-1}(t)$. ∎

REMARK 1. When $N = 1$, i.e., an ergodic $M/M/1$ queue, (19) becomes

$$\hat{P}_{00}(s) = \frac{1}{s + \lambda - \lambda \hat{b}(s)},$$

since $f_0(t) = \lambda e^{-\lambda t}$. This together with (13) and (14) leads to (10) and (12).

LEMMA 2. *In the birth-death process under consideration, one has*

$$P_{jj}(t) = \frac{1}{\mu}\, b(t) + \rho\, b^{(2)}(t) * P_{j-1,j-1}(t), \qquad j \ge N. \tag{22}$$

In particular,

$$P_{NN}(t) = \frac{1}{\mu}\,[b(t) + b^{(2)}(t) * m(t)]. \tag{23}$$

PROOF. By the standard renewal argument, one easily sees that, for $j \ge N$,

$$P_{jj}(t) = e^{-(\lambda+\mu)t} + \lambda e^{-(\lambda+\mu)t} * b(t) * P_{jj}(t) + \mu e^{-(\lambda+\mu)t} * f^{+}_{j-1}(t) * P_{jj}(t), \tag{24}$$

where $f_{j-1}^{+}(t)$ denotes the first-passage-time density of $\{X(t)\}$ from state $(j-1)$ to state j. In the Laplace transform, (24) is written as

$$\hat{P}_{jj}(s) = \frac{1}{s + \lambda + \mu - \lambda\hat{b}(s) - \mu\hat{f}_{j-1}^{+}(s)}. \tag{25}$$

This together with (13) yields

$$\hat{P}_{jj}(s) = \frac{1}{\mu}\frac{\hat{b}(s)}{1 - \hat{b}(s)\,\hat{f}_{j-1}^{+}(s)}. \tag{26}$$

On the other hand, the time reversibility implies $\pi_i P_{ij}(t) = \pi_j P_{ji}(t)$ for all $i, j \geq 0$, from which one has

$$\mu\hat{f}_{j-1}^{+}(s)\,\hat{P}_{jj}(s) = \lambda\hat{b}(s)\,\hat{P}_{j-1,j-1}(s). \tag{27}$$

Using (26) and (27) appropriately, one arrives at (22). (23) is immediate from Lemma 1. ∎

REMARK 2. When $j = N$, (26) can be written as

$$\hat{P}_{NN}(s) = \frac{1}{\mu}\frac{\hat{b}(s)}{1 - \hat{b}(s)\,\hat{f}_{N-1}(s)}, \tag{28}$$

since $\hat{f}_{N-1}^{+}(s) = \hat{f}_{N-1}(s)$ by definition. The difference between (19) and (28) should be made clear. Namely, the tasks of the busy period B_n and the first-passage-time T_n in the alternating renewal process are interchanged. Also, note that a similar representation to (18) can be always derived based on the decomposition (20). For example, it is readily shown that

$$\hat{P}_{NN}(s) = \frac{1}{\lambda}\frac{\hat{f}_{N}^{+}(s)}{1 - \hat{b}(s)\,\hat{f}_{N}^{+}(s)},$$

which should be compared with (28). On the other hand, however, (26) holds true only for $j \geq N$ since (13) is the key relation to derive (26) from (25).

REMARK 3. The limiting probability of $P_{N-1,N-1}(t)$ is obtained by the elementary renewal theorem applied to (19) as

$$\lim_{t\to\infty} P_{N-1,N-1}(t) = \frac{1}{\lambda(E[T_1] + E[B_1])} = \frac{\pi_{N-1}}{\sum_{j=0}^{\infty}\pi_j},$$

since

$$E[B_1] = \frac{1}{\mu(1-\rho)}; \qquad E[T_1] = \frac{1}{\lambda\pi_{N-1}}\sum_{j=0}^{N-1}\pi_j$$

(see [10]) and $\pi_{N+n} = \pi_{N-1}\,\rho^{n+1}$ for $n \geq 0$. The limiting probabilities of $P_{jj}(t)$ for $j \geq N$ are obtained from (22) and (23), while for $j < N$, Theorem 1 below should be applied.

We now state our main result.

THEOREM 1. *In the birth-death process described above, every $P_{ij}(t)$ can be represented as a combination of convolutions of $b^{(n)}(t)$, $n \geq 1$, $m(t)$ and $g_{ij}(t)$.*

PROOF. From the time reversibility, it suffices to consider $P_{ij}(t)$ for $i \geq j$. Hence, to prove the theorem, we need to consider the following three cases:

Case 1. $i \geq j \geq N$;
Case 2. $i \geq N > j$;
Case 3. $N > i \geq j$.

For Case 1, it is easy to see that

$$P_{ij}(t) = b^{(i-j)}(t) * P_{jj}(t), \qquad i \geq j \geq N. \tag{29}$$

Since $P_{jj}(t)$, $j \geq N$, are represented in terms of $b^{(n)}(t)$ and $m(t)$ (see Lemmas 1 and 2), one has the desired conclusion. For Case 2, the skip-free property and the strong Markov property at the last exit from state N bear

$$P_{ij}(t) = b^{(i-N)}(t) * P_{Nj}(t); \qquad P_{Nj}(t) = \mu P_{NN}(t) * {}_NP_{N-1,j}(t),$$

where ${}_NP_{N-1,j}(t)$ are the taboo probabilities with taboo state N. But, ${}_NP_{N-1,j}(t)$ is equal to $g_{N-1,j}(t)$ by definition. Hence,

$$P_{Nj}(t) = \mu\, g_{N-1,j}(t) * P_{NN}(t), \qquad N > j. \tag{30}$$

It follows from (23) and (30) that

$$P_{ij}(t) = g_{N-1,j}(t) * \left\{ b^{(i-N+1)}(t) + b^{(i-N+2)}(t) * m(t) \right\}, \qquad i \geq N > j, \tag{31}$$

arriving at the desired conclusion. Finally, to prove the assertion for Case 3, we recall the proof of Lemma 1. Combining (20) and (30), it is easy to see that

$$P_{ij}(t) = g_{ij}(t) + \lambda g_{i,N-1}(t) * g_{N-1,j}(t) * \{b(t) + b^{(2)}(t) * m(t)\}, \qquad N > i \geq j. \tag{32}$$

This completes the proof of the theorem. ∎

4. THE BULK-ARRIVAL $M/M/1$ QUEUE

In this section, we consider an $M/M/1$ queue with bulk arrivals of random size. The arrival and service rates are λ and μ, respectively. Let C be the generic random variable of the bulk size and define the generating function as $A(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \Pr[C = n]$, $n = 0, 1, \ldots$. Note that we allow the possibility of arriving empty bulks. The mean of the bulk size is given by $a = E[C] = A'(1)$. It is assumed that $\rho = a\lambda/\mu < 1$.

Let $b(t)$ be the busy period density of the bulk-arrival $M/M/1$ queue. It is well known (see, e.g., [4] and references therein) that

$$\hat{b}(s) = \frac{\mu}{s + \lambda + \mu - \lambda A(\hat{b}(s))} \tag{33}$$

so that

$$s\hat{b}(s) = \mu(1 - \hat{b}(s)) - \lambda \hat{b}(s)\,(1 - A(\hat{b}(s))). \tag{34}$$

Now, consider the transition probabilities $P_{ij}(t) = \Pr[X(t) = j \mid X(0) = i]$, where $X(t)$ denotes in turn the queue length of the bulk-arrival queue at time t. By the standard renewal argument, it is readily seen that

$$\hat{P}_{00}(s) = \frac{1}{s + \lambda(1 - a_0)} \left[1 + \lambda \sum_{i=1}^{\infty} a_i \hat{b}^i(s)\, \hat{P}_{00}(s) \right]. \tag{35}$$

Hence,

$$\hat{P}_{00}(s) = \frac{1}{s + \lambda - \lambda A\left(\hat{b}(s)\right)}. \tag{36}$$

It follows from (33) and (36) that

$$\hat{P}_{00}(s) = \frac{1}{\mu} \frac{\hat{b}(s)}{1 - \hat{b}(s)}. \tag{37}$$

Here, again, we encounter the renewal structure as in (12). Note that $\hat{P}_{i0}(s) = \hat{b}^i(s)\,\hat{P}_{00}(s)$. Since, from (37), $\hat{P}_{00}(s) = \hat{P}_{10}(s) + \frac{\hat{b}(s)}{\mu}$, one has

$$\hat{P}_{i0}(s) - \hat{P}_{i+1,0}(s) = \frac{\hat{b}^{i+1}(s)}{\mu},$$

so that (5) holds even for the bulk-arrival case.

From (34), one can rewrite (37) as

$$s\hat{P}_{00}(s) = 1 - \rho\,\frac{\hat{b}(s)\,(1 - A(\hat{b}(s)))}{a(1 - \hat{b}(s))}. \tag{38}$$

Note that the Markov chain $\{X(t),\ t \geq 0\}$ is spatially homogeneous with retaining boundary at 0. Hence, its governing generator, after uniformization, is stochastically monotone (see [10]). It follows from [13, Theorem 1] that $P_{00}(t)$ is non-increasing in t. Therefore, since $\lim_{s\to 0} s\hat{P}_{00}(s) = 1 - \rho$, there is a probability density function $\beta(t)$ whose Laplace transform is given by

$$\frac{\hat{b}(s)\,\left(1 - A\left(\hat{b}(s)\right)\right)}{a\left(1 - \hat{b}(s)\right)}$$

and

$$P_{00}(t) = 1 - \rho \int_0^t \beta(u)\,du. \tag{39}$$

Note the resemblance between (4) with $j = 0$ and (39).

EXAMPLE 1. Suppose that all the bulk sizes are the same, say n. Then, $A(z) = z^n$ so that $1 - A(\hat{b}(s)) = (1 - \hat{b}(s)) \sum_{k=0}^{n-1} \hat{b}^k(s)$. It follows from (38) that

$$\beta(t) = \frac{1}{n} \sum_{k=1}^{n} b^{(k)}(t). \tag{40}$$

When $n = 1$, (40) agrees with the $M/M/1$ case, as it should be.

EXAMPLE 2. In this example, we consider the case that the bulk size is geometrically distributed. Suppose that $a_n = (1-\gamma)\,\gamma^n$. Then, as in Example 1, one has the expression

$$s\hat{P}_{00}(s) = 1 - \rho\,\frac{(1-\gamma)\,\hat{b}(s)}{1 - \gamma\hat{b}(s)}.$$

Hence, $\beta(t)$ for this case is a compound geometric distribution associated with the busy period density having the mixing rate γ.

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