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Optimal Contracts

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Abstract. The model is proposed by incorporating incomplete capital markets into the conventional implicit labor contract model. The optimal contracts and the effects of firm's financial decisions on it's employment level are derived by maximum principle.

1. Introduction

The implicit contract allows the firm and workers to share the risk of fluctuations in the revenues of the firm. This implicit contract approach considers the employment agreement as a contractual arrangement between the firm and its workers, where there is uncertainty about the state of nature. See [6] for a recent survey of the literature in this area.

A number of people have contributed to the implicit contract theory [**1–2,4–5**]. None of them consider the interdependence between labor and capital and the effects of capital utilization on the production process. Thus these models do not allow the firm's financial decisions to affect its ability to share risks with its workers.

The model considers objective functions for three sets of economic agents: the firm, the workers, and the creditors. This set-up differs from the conventional approach in which only a single contractual relationship is analyzed, that is, between the worker-firm or the creditor-firm. The analysis examines optimal labor and loan contracts when the wage and the cost of capital are both state contingent.

2. The Basic Model

This model is characterized by two periods, date 0 (ex ante) and date 1 (ex post). There are three types of agents in the economy: one firm, N workers, and M creditors. It is assumed for simplicity that all workers have identical preferences. Creditors also have identical preferences, though these differ from the worker's preferences. We can therefore analyze the behavior of a representative agent in each group. The firm acquires labor and capital inputs, ex ante, by offering employment and loan contracts.

Each worker is assumed to have a von Neumann–Morgenstern utility function which depends on net income, Y. It can be written as:

$$U^w(Y) = U^w(W(\theta) - V(n(\theta)))$$

where θ is the state of nature, $W(\theta)$ is the wage payment from the firm, $n(\theta)$ is the nonnegative number of working hours supplied by the worker, and $V(n)$ represents the worker's opportunity cost of supplying n labor hours in terms of foregone leisure time. It is assumed to be an increasing convex function of n. Assume that the worker's utility function is twice differentiable and that the marginal utility of net income is diminishing.

Each creditor is assumed to have a von Neumann–Morgenstern utility function which depends on final wealth, R, i.e.

$$U^c(R) = U^c((1 + i(\theta))b(\theta) + (1 + r)(A^c - b(\theta)))$$

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where $i(\theta)$ is the interest earnings per unit of debt, $b(\theta)$ is the amount of debt incurred by the firm, r is the fixed opportunity cost of funds for the creditor, and A^c is the creditor's initial wealth. $U^c(R)$ is twice differentiable with $(U^c)' > 0$ and $(U^c)'' < 0$.

Let the state of nature θ be a nonnegative random variable which reflects the productivity shocks of output. In state θ, the firm hires N identical workers each for $n(\theta)$ number of working hours. Therefore, the total hours employed under the contract can be defined as $L(\theta) = Nn(\theta)$. Note that whereas this model allows for variations in hours of work, it does not permit the possibility of layoffs. The firm borrows $b(\theta)$ from each of M identical creditors. The total amount of external financing in state θ is the loan supplied by the M lenders under contract, i.e., $B(\theta) = Mb(\theta)$. The portion of investment A, which is internally financed is not state contingent, since it is treated as ex ante sunk investment. Note that this cannot exceed E, the value of the firm's total assets. If $A = E$, then this defines the firm's maximum asset participation. The total amount of capital $K(\theta)$ that the firm can acquire in any state is the sum of the internal and external funds. Hence: $K(\theta) = A + B(\theta) = A + Mb(\theta)$.

Let the production function of the firm be $f(L(\theta), K(\theta))$, where f is a concave and twice-differentiable function for all $L \geq 0$ and $K \geq 0$, with $f(0,K) \geq 0$, $f(L,0) \geq 0$, $f_i > 0$, $f_{ii} < 0$. The total output of the firm is given by $y(\theta) = \theta f(L(\theta), K(\theta))$.

The probability density function of θ, denoted by $h(\theta)$, is assumed to be continuous and positive on the interval $(\underline{\theta}, \overline{\theta})$, $\underline{\theta} > 0$. Once the state of nature is observed. The firm cannot hire any additional labor and/or capital. The total cost of production for the firm in state θ is the sum of the total wage payment $NW(\theta)$ and the capital expenditure $M(1 + i(\theta))b(\theta)$. Total profit is: $\pi(\theta) = \theta f(L(\theta), K(\theta)) - NW(\theta) - M(1 + i(\theta))b(\theta)$.

The single-owner firm has a von Neumann–Morgenstern utility function which depends on profits,

$$U^f(\pi(\theta)) = U^f(\theta f(L(\theta), K(\theta)) - NW(\theta) - M(1 + i(\theta))b(\theta)).$$

This is assumed to be twice differentiable with $(U^f)' > 0$ and $(U^f)'' < 0$.

The objective of the firm is to maximize the expected utility of profits. Each worker in this firm seeks to maximize the expected utility of his/her net income. Worker will not accept a contract unless he is guaranteed at least the level of utility $\overline{U}$. On the other hand, the contingent interest and loan levels of the contract must provide a minimum utility $\overline{R}$ to each creditor.

3. Optimal Contracts

Since the state of nature θ becomes public information at date 1, the employment level, wages, interest rate and the amount of loan can be made conditional on θ. An employment contract $\{W(\theta), n(\theta)\}$ for the worker consists of an agreed wage payment W and number of working hours n, in each state of nature. A loan contract $\{i(\theta), b(\theta)\}$ stipulates the interest rate i and amount of loan b in state of nature for the creditor.

The optimal employment and loan contracts maximize the firm's expected utility of profit subject to the constraints that each worker's and creditor's expected utility should be no lower than the alternative utility levels $\overline{U}$ and $\overline{R}$, respectively. The maximization problem can be formulated as follows (P1S):

Max

$$\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)U^f(\pi(\theta))d\theta, \tag{1}$$

Subject to

$$\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)U^w(Y(\theta))d\theta \geq \overline{U}, \tag{2}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)U^c(R(\theta))d\theta \geq \overline{R}. \tag{3}$$

There are labor contracts $\{n(\theta), W(\theta)\}$ and loan contracts $\{i(\theta), b(\theta)\}$ such that $U^w(Y(\theta)) \geq \overline{U}$, $U^c(R(\theta)) \geq \overline{R}$ for all states. Based on the maximization problem (P1S), we are able to set up the Lagrangian function. The first order conditions of this maximization problem result in Proposition 1.

PROPOSITION 1. *In the symmetric information case, the optimal employment and loan contracts will satisfy the following conditions:*

$$\theta f_L - V' = 0, \qquad \text{if} \qquad n > 0; \tag{4}$$

$$\frac{(U^w)'(\theta_s)}{(U^w)'(\theta_t)} = \frac{(U^f)'(\theta_s)}{(U^f)'(\theta_t)} \qquad \forall \ \theta_s, \theta_t, \qquad \text{if} \qquad W > 0; \tag{5}$$

$$\theta f_K - (1+r) = 0, \qquad \text{if} \qquad b > 0; \tag{6}$$

$$\frac{(U^c)'(\theta_s)}{(U^c)'(\theta_t)} = \frac{(U^f)'(\theta_s)}{(U^f)'(\theta_t)} \qquad \forall \ \theta_s, \theta_t, \qquad \text{if} \qquad i > 0. \tag{7}$$

At date 0, the firm chooses the optimal employment and loan levels that satisfy equations (4) and (6). For the risk-averse and risk-neutral firm, the wage and interest payments in each state can be simultaneously determined from equations (5) and (7). Differentiating equations (4) and (6) with respect to the state of nature, θ, and using Cramer's rule, we get:

$$\frac{dn}{d\theta} = \frac{-\theta M f_L f_{KK} + \theta M f_K f_{LK}}{\theta^2 MN f_{LL} f_{KK} - \theta MNV'' f_{KK} - \theta^2 MN f_{KL} f_{LK}} \tag{8}$$

$$\frac{db}{d\theta} = \frac{-\theta N f_K f_{LL} + N f_K V'' + \theta N f_L f_{LK}}{\theta^2 MN f_{LL} f_{KK} - \theta MNV'' f_{KK} - \theta^2 MN f_{KL} f_{LK}} \tag{9}$$

Consider a linear homogeneous production function, then we have:

PROPOSITION 2. *In a productively efficient contract, employment and loan levels are increasing with θ , i.e., $n'(\theta) > 0$, $b'(\theta) > 0$.*

4. REMARK

This paper attempts to integrate state-contingent labor and loan contracts to the analysis of fluctuations of employment over the business cycle. The approach adopted is to incorporate incomplete capital markets into the conventional implicit labor contract model. The model developed shows that the labor and capital are employed at their productively efficient levels. The proposed model also generates several testable hypotheses. The analysis suggests:

(a) Employment and wages move procyclically with output.
(b) Fluctuations in employment will be affected by the firm's investment levels.

We will examine empirical evidence to support these hypotheses and analyze the asymmetric information case to see how the results are affected [3].

REFERENCES

1. Costas Azariadis, *Implicit contracts and underemployment equilibria*, Journal of Political Economy (December 1975), 1183–92.
2. M.N. Baily, *Wages and employment under uncertain demand*, Review of Economic Studies (January 1974), 37–50.
3. M.Y. Chou and E.B. Lin, in preparation.
4. Douglas Gale and Martin Hellwig, *Incentive-compatible debt contracts: The one-period problem*, Review of Economic Studies (October 1985), 647–63.
5. S.J. Grossman and Oliver D. Hart, *Implicit contracts, moral hazard and unemployment*, it The American Economic Review (May 1981), 301–8.
6. S. Rosen, *Implicit contracts: A survey*, Journal of Economic Literature (September 1985), 1144–75.

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