Algebras with involution with linear codimension growth

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Received 5 September 2005
Available online 6 September 2006
Communicated by Efim Zelmanov

Abstract

We study the *-varieties of associative algebras with involution over a field of characteristic zero which are generated by a finite-dimensional algebra. In this setting we give a list of algebras classifying all such *-varieties whose sequence of *-codimensions is linearly bounded. Moreover, we exhibit a finite list of algebras to be excluded from the *-varieties with such property. As a consequence, we find all possible linearly bounded *-codimension sequences.

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Keywords: *-Polynomial identity; T*-*ideal; *-Codimensions

1. Introduction

Let A be an associative algebra with involution over a field F of characteristic zero and let \text{Id}^*(A) be the T*-*ideal of all *-polynomial identities of A. One associates to A, in a natural way, a numerical sequence \( c_n^*(A), \ n = 1, 2, \ldots \), called the sequence of *-codimensions of A which is the main tool for the quantitative investigation of the polynomial identities of the algebra A (see [12]). Recall that \( c_n^*(A), \ n = 1, 2, \ldots \), is the dimension of the space of multilinear polynomials in \( n \) *-variables in the corresponding relatively free algebra with involution of countable rank. In case A satisfies a non-trivial identity, it was proved in [8] that, as in the ordinary case, \( c_n^*(A) \) is exponentially bounded. Moreover, an explicit bound related to the ordinary identities of the...
algebra $A$ was found in [1]. In case of polynomial growth, the following characterization was given in [7]: the $*$-codimension sequence of an algebra $A$ is polynomially bounded if and only if its $*$-identities are not a consequence of the $*$-identities of $F \oplus F$, endowed with the exchange involution, and of $M$, a suitable 4-dimensional subalgebra of the algebra of $4 \times 4$ upper triangular matrices. As a consequence, no intermediate growth is allowed. The above 2 algebras play the role of the infinite-dimensional Grassmann algebra and the algebra of $2 \times 2$ upper triangular matrices in the ordinary case [13,14].

The aim of this paper is to refine the above result in the case of $*$-codimensions with at most linear growth. Concerning the ordinary case, already in [5] the authors gave a classification of the algebras, up to PI-equivalence, whose sequence of codimensions is linearly bounded. They also exhibited a finite list of algebras to be excluded from a variety with such property.

These results depend on the structure theory of T-ideals developed by Kemer. In the proofs one reduces the study to the case of finite-dimensional algebras and superalgebras. Since in the moment there is no complete analogue of the results of Kemer for algebras with involution, we restrict our consideration to finite-dimensional algebras only.

Here we obtain the following characterization: let $A$ be a finite-dimensional algebra and let $\text{var}^*(A)$ be the variety of algebras with involution generated by $A$. Then $\text{var}^*(A)$ has linear growth, i.e., the correspondence sequence of $*$-codimensions is linearly bounded, if and only if it does not contain an explicit list of 11 algebras with suitable involutions. Moreover, we classify, up to $*$-PI-equivalence, the algebras $A$ having such property. We give two algebras, $U_1$ and $U_2$, of dimension 2 and 3, respectively, with the property that the multilinear $*$-polynomial identities of sufficiently high degree of every finite-dimensional algebra $A$ with linear codimension growth coincide with the identities of one of the algebras $0$, $C$, $U_1$, $U_2$, $U_1 \oplus U_2$, where $C$ is a commutative algebra with trivial involution. As a consequence of our classification, we find that for $n$ large enough the only sequences of $*$-codimensions allowed are $c_n^*(A) = 0$, $1$, $n+1$, $3n-1$, $4n-1$. As a bi-product of our considerations we give complete information for the $*$-polynomial identities of several of the 11 considered algebras with non-linear codimension growth.

2. Generalities

Throughout this paper $F$ will be a field of characteristic zero.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F(X, \ast) = F(x_1, x_1^*, x_2, x_2^*, \ldots)$ be the free algebra with involution on $X$ over $F$. It is useful to regard $F(X, \ast)$ as generated by symmetric and skew variables: if for $i = 1, 2, \ldots$, we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, then $F(X, \ast) = F(y_1, z_1, y_2, z_2, \ldots)$. We also define $P_n^*$ as the space of multilinear polynomials of degree $n$ in $y_1, z_1, \ldots, y_n, z_n$; hence for every $i = 1, 2, \ldots, n$ either $y_i$ or $z_i$ appears in every monomial of $P_n^*$ at degree 1 (but not both).

Let $\mathbb{Z}_2 \wr S_n$ be the hyperoctahedral group of degree $n$, i.e., the wreath product of the multiplicative group of order two with $S_n$. The space $P_n^*$ has a natural structure of left $\mathbb{Z}_2 \wr S_n$-module induced by defining for $h = (a_1, \ldots, a_n; \sigma) \in \mathbb{Z}_2 \wr S_n$, $h y_i = y_{\sigma(i)}$, $h z_i = z_{\sigma(i)}$.

Let $A$ be an $F$-algebra with involution $\ast$. Let $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew elements of $A$, respectively. Recall that $f = f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F(X, \ast)$ is a $*$-identity of $A$ if $f(s_1, \ldots, s_n, k_1, \ldots, k_m) = 0$ for all $s_1, \ldots, s_n \in A^+, k_1, \ldots, k_m \in A^-$. We denote by $\text{Id}^*(A) = \{f \in F(X, \ast) \mid f \equiv 0 \text{ on } A\}$ the $T^*$-ideal of $*$-identities of $A$. Since $P_n^* \cap \text{Id}^*(A)$ is invariant under the $\mathbb{Z}_2 \wr S_n$ action, the space $P_n^*/(P_n^* \cap \text{Id}^*(A)) = P_n^*(A)$ has a structure of left $\mathbb{Z}_2 \wr S_n$-module and its dimension, $c_n^*(A)$, is...
called the $n$th $\ast$-codimension of $A$. By complete reducibility the character $\chi_n^\ast(A)$, called the $n$th $\ast$-cocharacter of $A$, decomposes as

$$
\chi_n^\ast(A) = \sum_{r=0}^{n} \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda,\mu} \chi_{\lambda,\mu},
$$

where $\lambda$ and $\mu$ are partitions of $r$ and $n-r$, respectively, $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to the pair $(\lambda,\mu)$, and $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity.

For $0 \leq r \leq n$ let $P_{r,n-r}^\ast$ denote the space of multilinear polynomials in the variables $y_1, \ldots, y_r, z_{r+1}, \ldots, z_n$. It is clear that in order to study $P_n^\ast \cap \text{Id}^\ast(A)$ it is enough to study $P_{r,n-r}^\ast \cap \text{Id}^\ast(A)$ for all $r \geq 0$, and this can be done through the representation theory of $S_r \times S_{n-r}$.

We consider the permutation action of the group $S_r$ on the variables $y_1, \ldots, y_r$ and of the group $S_{n-r}$ on the variables $z_{r+1}, \ldots, z_n$. These in turn induce a (left) action of $S_r \times S_{n-r}$ on $P_{r,n-r}^\ast$. Since $T^\ast$-ideals are invariant under permutations of symmetric (respectively skew) variables, we get that $P_{r,n-r}^\ast(A) = P_{r,n-r}^\ast / (P_{r,n-r}^\ast \cap \text{Id}^\ast(A))$ has an induced structure of left $S_r \times S_{n-r}$-module and we write $\chi_{r,n-r}^\ast(A)$ for its character. By complete reducibility we have

$$
\chi_{r,n-r}^\ast(A) = \sum_{\lambda \vdash r, \mu \vdash n-r} \tilde{m}_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu),
$$

where $\chi_\lambda$ (respectively $\chi_\mu$) denotes the ordinary $S_r$-character (respectively $S_{n-r}$-character), $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_r \times S_{n-r}$-character associated to the pair $(\lambda,\mu)$ and $\tilde{m}_{\lambda,\mu} \geq 0$ is the corresponding multiplicity.

The relation between $\mathbb{Z}_2 \wr S_n$-characters and $S_r \times S_{n-r}$-characters is given in the following.

**Theorem 1.** [3, Theorem 1.3] If $A$ is an algebra with involution then, for all $r \leq n$,

$$
\chi_n^\ast(A) = \sum_{r=0}^{n} \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda,\mu} \chi_{\lambda,\mu} \quad \text{and} \quad \chi_{r,n-r}^\ast(A) = \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu).
$$

As a consequence if $c_{r,n-r}^\ast(A) = \dim P_{r,n-r}^\ast(A)$ we have that

$$
c_n^\ast(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}^\ast(A).
$$

By the main result of [4] there is also a precise connection between $\mathbb{Z}_2 \wr S_n$-representations and $GL \times GL$-representations.

Let $F_m \langle X, \ast \rangle = F \langle y_1, z_1, \ldots, y_m, z_m \rangle$ denote the free subalgebra of $F \langle X, \ast \rangle$ of rank $m$ and let $U = \text{Span}_F \{y_1, \ldots, y_m\}$, $V = \text{Span}_F \{z_1, \ldots, z_m\}$. The group $GL(U) \times GL(V) \cong GL_m \times GL_m$ acts naturally on the left on the space $U \oplus V$ and we can extend this action diagonally to get an action on $F_m \langle X, \ast \rangle$. 
For every algebra $A$ with involution, the space $F_m^* \langle X, * \rangle \cap \text{Id}^*(A)$ is invariant under the above action, hence the space

$$F_m^* (A) = \frac{F_m^* \langle X, * \rangle}{F_m^* \langle X, * \rangle \cap \text{Id}^*(A)}$$

has a structure of left $GL_m \times GL_m$-module. Let $F_m^{n*}$ be the space of all homogeneous polynomials of degree $n$ in the variables $y_1, \ldots, y_m, z_1, \ldots, z_m$. Then

$$F_m^{n*} (A) = \frac{F_m^{n*}}{F_m^{n*} \cap \text{Id}^*(A)}$$

is a $GL_m \times GL_m$-submodule of $F_m^* (A)$ and we denote its character by $\psi_n^* (A)$. Write

$$\psi_n^* (A) = \sum_{\lambda \vdash r, \mu \vdash n - r} m^r_{\lambda, \mu} \psi_{\lambda, \mu},$$

where $h(\lambda)$ (respectively $h(\mu)$) denotes the height of the Young diagram corresponding to $\lambda$ (respectively $\mu$), $\psi_{\lambda, \mu}$ is the irreducible $GL_m \times GL_m$-character associated to the pair $(\lambda, \mu)$ and $m^r_{\lambda, \mu}$ is the corresponding multiplicity. The $\mathbb{Z}_2 \wr S_n$-module structure of $P_n^* (A)$ and the $GL_m \times GL_m$-module structure of $F_m^{n*} (A)$ are related by the following.

**Theorem 2.** [4, Theorem 3] If the $n$th $*$-cocharacter of $A$ has the decomposition given in (1) and the $GL_m \times GL_m$-character of $F_m^{n*} (A)$ has the decomposition given in (2) then $m^r_{\lambda, \mu} = m^r_{\lambda, \mu}$, for all $\lambda, \mu$ with $h(\lambda), h(\mu) \leq m$.

It is well known (see, for instance, [2]) that any irreducible submodule of $F_m^{n*} (A)$ corresponding to the pair $(\lambda, \mu)$, is cyclic and is generated by a non-zero polynomial $f_{\lambda, \mu}$, called highest weight vector, of the form

$$f_{\lambda, \mu} (y_1, \ldots, y_p, z_1, \ldots, z_q)$$

$$= \prod_{i=1}^{\lambda_1} S_{h_i(\lambda)} (y_1, \ldots, y_{h_i(\lambda)}) \prod_{i=1}^{\mu_1} S_{h_i(\mu)} (z_1, \ldots, z_{h_i(\mu)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma,$$

where $\alpha_\sigma \in F$, $S_{k} (x_1, \ldots, x_k) = \sum_{\sigma \in S_k} (\text{sgn} \sigma) x_{\sigma(1)} \cdots x_{\sigma(k)}$ is the standard polynomial of degree $k$, $S_n$ acts from right by permuting places in which the variables occur, and $h_i(\lambda)$ (respectively $h_i(\mu)$) is the height of the $i$th column of the Young diagram corresponding to the partition $\lambda$ (respectively $\mu$).

Let $T_\lambda$ and $T_\mu$ be two Young tableaux. We denote by $f_{T_\lambda, T_\mu}$ the highest weight vector obtained from (3) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \ldots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of $T_\lambda$, $\sigma(h_1(\lambda) + 1), \ldots, \sigma(h_1(\lambda) + h_2(\lambda))$ the second column of $T_\lambda$, $\ldots, \sigma(h_1(\lambda) + \cdots + h_{\lambda_1 - 1}(\lambda) + 1), \ldots, \sigma(r)$ the last column of $T_\lambda$; also $\sigma(r + 1), \ldots, \sigma(r + h_1(\mu))$ fill in the first column of $T_\mu$, $\ldots, \sigma(r + h_1(\mu) + \cdots + h_{\mu_1 - 1}(\mu) + 1), \ldots, \sigma(n)$ the last column of $T_\mu$.

The following results hold.
Remark 3. If

\[ \psi^n_m(A) = \sum_{\lambda \vdash r, \mu \vdash n-r} m_{\lambda, \mu} h(\lambda), h(\mu) \leq m \]

is the \( GL_m \times GL_m \)-character of \( F_m^n(A) \), then \( m_{\lambda, \mu} \neq 0 \) if and only if there exists a pair of tableaux \((T_\lambda, T_\mu)\) such that the corresponding highest weight vector \( f_{T_\lambda, T_\mu} \) is not a \(*\)-polynomial identity for \( A \).

Recall the following result given in [11, Lemma 2] on the decomposition of the Jacobson radical of a finite-dimensional algebra.

Lemma 4. Let \( A \) be a finite-dimensional algebra over \( F \) and suppose that \( A = B + J \) where \( B \) is a semisimple subalgebra and \( J = J(A) \) is its Jacobson radical. Then \( J \) can be decomposed into the direct sum of \( B \)-bimodules

\[ J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}, \]

where for \( i \in \{0, 1\} \), \( J_{ik} \) is a left faithful module or a 0-left module according as \( i = 1 \) or \( i = 0 \), respectively. Similarly, \( J_{ik} \) is a right faithful module or a 0-right module according as \( k = 1 \) or \( k = 0 \), respectively. Moreover, for \( i, k, l, m \in \{0, 1\} \), \( J_{ik}J_{lm} \subseteq \delta_{kl}J_{im} \) where \( \delta_{kl} \) is the Kronecker delta and \( J_{11} = BN \) for some nilpotent subalgebra \( N \) of \( A \) commuting with \( B \).

Notice that if the algebra \( A \) has an involution, then in the above lemma \( J_{00} \) and \( J_{11} \) are stable under the involution whereas \( J_{10}^* = J_{01} \).

Definition 5. Let \( A \) and \( B \) be algebras with involution. We say that \( A \) is \(*\)-PI-equivalent to \( B \) in case \( \text{Id}^*(A) = \text{Id}^*(B) \).

Given an algebra \( A \) with involution let us denote by \( \text{var}^*(A) \) the variety of algebras with involution (or \(*\)-variety) generated by \( A \). Hence \( A \) is \(*\)-PI-equivalent to \( B \) if and only if \( \text{var}^*(A) = \text{var}^*(B) \).

Lemma 6. Let \( \bar{F} \) be the algebraic closure of the field \( F \) and let \( A \) be a finite-dimensional algebra with involution over \( \bar{F} \). Then \( A \) is \(*\)-PI-equivalent to \( B \) for some finite-dimensional algebra with involution \( B \) over \( F \) with \( \dim_{\bar{F}} A/J(A) = \dim_F B/J(B) \).

Proof. By the Wedderburn–Malcev theorem we can write \( A = A_1 \oplus \cdots \oplus A_k + J \) where \( A_1, \ldots, A_k \) are \(*\)-simple algebras and \( J \) is the Jacobson radical of \( A \) [9]. Since \( \bar{F} \) is algebraically closed each \( A_i \) has a \(*\)-basis \( \{u_{i1}, \ldots, u_{imi}\} \) over \( \bar{F} \) (i.e., consisting of symmetric and skew elements) with rational structure constants. Hence, if \( B_i \) is the linear span of \( \{u_{ij}, \ldots, u_{imi}\} \) over \( F \), \( B_i \) is still \(*\)-simple over \( F \). We now take a \(*\)-basis \( \{w_1, \ldots, w_p\} \) of \( J \) over \( \bar{F} \) and we let \( B \) be the algebra with involution generated by \( B_1 \oplus \cdots \oplus B_k \) and \( \{w_1, \ldots, w_p\} \) over \( F \).

Since \( J \) is nilpotent, then \( B \) is finite-dimensional over \( F \). Moreover,

\[ \dim_F B/J(B) = \dim_F (B_1 \oplus \cdots \oplus B_k) = \dim_{\bar{F}} (A_1 \oplus \cdots \oplus A_k) = \dim_{\bar{F}} A/J(A). \]
It is clear that $\text{Id}^n(A) \subseteq \text{Id}^n(B)$. On the other hand, if $f$ is a multilinear $*$-identity of $B$, $f$ vanishes on the set of elements $u_1, \ldots, u_{m_1}, \ldots, u_{mk}, w_1, \ldots, w_p$ which is a basis of $A$ over $\overline{F}$. Hence $\text{Id}^n(B) \subseteq \text{Id}^n(A)$ and $A$ is $*$-PI-equivalent to $B$. □

Let $A$ be an algebra with involution. We say that its sequence of $*$-codimensions is polynomially bounded if for all $n$, $c^n_n(A) \leq an^t$ for some constants $a, t$. In this case we say that $\text{var}^*(A)$ has polynomial growth.

**Proposition 7.** Let $A$ be a finite-dimensional algebra with involution and suppose that $c^n_n(A)$ is polynomially bounded. Then $A$ is $*$-PI-equivalent to a finite direct sum of algebras $A_1 \oplus \cdots \oplus A_m$ where $A_1, \ldots, A_m$ are finite-dimensional algebras with involution over $F$ and $\dim A_i/J(A_i) \leq 1$, for all $i = 1, \ldots, m$.

**Proof.** Suppose first that $F$ is algebraically closed. By the Wedderburn–Malcev decomposition of a finite-dimensional algebra and [9, Theorem 6] we may assume that $A = B + J$ where $B$ is a semisimple subalgebra such that $b^* = b$, for all $b \in B$, and $J$ is the Jacobson radical of $A$. This readily implies that $B = B_1 \oplus \cdots \oplus B_m$ where $B_1 \cong \cdots \cong B_m \cong F$. Moreover, since for all $n$, $c^n_n(A) \leq c^n_n(A)$ by the characterization of the exponent given in [10] it follows that $B_i J B_k = 0$ for all $i \neq k, 1 \leq i, k \leq n$. Set $A_1 = B_1 + J, \ldots, A_m = B_m + J$. We claim that

$$\text{Id}^n(A) = \text{Id}^n(A_1) \cap \cdots \cap \text{Id}^n(A_m) \cap \text{Id}^n(J).$$

Now one direction is obvious. Let $f \in \text{Id}^n(A_1) \cap \cdots \cap \text{Id}^n(A_m) \cap \text{Id}^n(J)$ and suppose that $f$ is not a $*$-identity of $A$. We may clearly assume that $f$ is multilinear. Moreover, by choosing a basis of $A$ as the union of a basis of $B$ and a basis of $J$ consisting of symmetric and skew elements, it is enough to evaluate $f$ on this basis. Let $r_1, \ldots, r_t$ be symmetric or skew elements of this basis such that $f(r_1, \ldots, r_t) \neq 0$. Since $f \in \text{Id}^n(J)$ at least one element, say $r_k$, does not belong to $J$. Then $r_k \in B_i$, for some $i$. Recalling that $B_i J B_j = B_j B_i = B_j J B_i = 0$, for all $j \neq i$, we must have that $r_1, \ldots, r_t \in B_i \cup J$. Thus $r_1, \ldots, r_t \in B_i + J = A_i$ and this contradicts the fact that $f$ is a $*$-identity of $A_i$. This proves the claim.

Since $\text{Id}^n(A_1 \oplus \cdots \oplus A_m \oplus J) = \text{Id}^n(A_1) \cap \cdots \cap \text{Id}^n(A_m) \cap \text{Id}^n(J)$, the proof is completed by noticing that $\dim A_i/J = 1$.

In case $F$ is arbitrary, we consider the algebra $\tilde{A} = A \otimes_F \overline{F}$, where $\overline{F}$ is the algebraic closure of $F$ and $\tilde{A} = A \otimes_F \overline{F}$ is endowed with the induced involution $(a \otimes \alpha)^* = a^* \otimes \alpha$, for $a \in A$, $\alpha \in \overline{F}$. Clearly $A$ is $*$-PI-equivalent to $\tilde{A}$. Moreover, the $*$-codimensions of $A$ over $\overline{F}$ coincide with the $*$-codimensions of $A$ over $\overline{F}$. By the hypothesis it follows that the $*$-codimensions of $\tilde{A}$ are polynomially bounded. But then by the first part of the proof, $\tilde{A} = A_1 \oplus \cdots \oplus A_m$ where $A_1, \ldots, A_m$ are finite-dimensional algebras with involution over $\overline{F}$ and $\dim A_i/J(A_i) \leq 1$, for all $i = 1, \ldots, m$. By the previous lemma there exist finite-dimensional algebras $C_1, \ldots, C_m$ over $F$ such that for all $i$, $\dim_F C_i/J(C_i) = \dim_{\overline{F}} A_i/J(A_i) \leq 1$. It follows that $\text{var}^*(A) = \text{var}^*(A_1 \oplus \cdots \oplus A_m) = \text{var}^*(C_1 \oplus \cdots \oplus C_m)$. □

The following remark will be useful in what follows.

**Remark 8.** Let $A$ be an algebra with involution. If $z_1 \cdots z_m \equiv 0$ is a $*$-identity of $A$, for some $m \geq 1$, then

$$z_1 w_1 z_2 w_2 \cdots w_{m-1} z_m \equiv 0$$

is a $*$-identity of $A$ where $w_1, \ldots, w_{m-1}$ are (eventually trivial) monomials of $F(X, *)$. 

Proof. For $s \in A^+, k \in A^-$ we have that $sk + ks \in A^-$; hence $ks = -sk + k'$ for some $k' \in A^-$. If we now evaluate the polynomial $z_1 w_1 z_2 w_2 \cdots w_{m-1} z_m$ in $A$, after a repeated application of the relation $ks = -sk + k'$ we can write the corresponding evaluation as a linear combination of monomials each containing at least $m$ consecutive skew elements. The proof is completed by recalling that the product of $m$ skew elements of $A$ is zero. □

In [7] the authors proved a much stronger result in characteristic zero, namely that if an algebra with involution satisfied the $\ast$-identity $z^m \equiv 0$, then there exists a positive integer $M$ such that $z_1 w_1 z_2 w_2 \cdots w_{M-1} z_M \equiv 0$ is an identity of $A$ where $w_1, \ldots, w_{M-1}$ are (eventually trivial) monomials of $F\langle X, \ast \rangle$.

The algebra $UT_n$ of $n \times n$ upper triangular matrices over $F$ has an involution that we shall denote $\ast$ defined as follows: if $a \in UT_n$, $a^\ast = ba^t b^{-1}$, where $a^t$ denotes the usual transpose and $b$ is the following permutation matrix:

\[
b = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.
\]

Clearly $a^\ast$ is the matrix obtained from $a$ by reflecting $a$ along its secondary diagonal. Hence, if $a = (a_{ij})$ then $a^\ast = (a^\ast_{ij})$ where $a^\ast_{ij} = a_{n+1-j, n+1-i}$.

Definition 9. The involution $\ast$ on $UT_n$ defined above will be called the (canonical) reflection involution.

In what follows, when we consider subalgebras of $UT_n$, unless otherwise stated, we shall assume that they are endowed with reflection involution.

3. Algebras whose $\ast$-codimensions are bounded

In this section we classify the algebras whose sequence of $\ast$-codimensions is bounded by a constant.

Given polynomials $f_1, \ldots, f_n \in F\langle X, \ast \rangle$ let us denote by $\langle f_1, \ldots, f_n \rangle_{T^\ast}$ the $T^\ast$-ideal generated by $f_1, \ldots, f_n$.

We start with the following.

Lemma 10. Let

\[
U_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in F \right\}
\]

be endowed with the involution

\[
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^\ast = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.
\]
Then

- \( \text{Id}^*(U_1) = \langle [y_1, y_2], [y, z], z_1 z_2 \rangle T^* \).
- \( \{ y_1 \ldots y_n, z_i y_1 \ldots y_{i-1} \mid i_1 < \cdots < i_{n-1} \} \) is a basis of \( P_n^* \) (mod \( P_n^* \cap \text{Id}^*(U_1) \)).
- \( c_n^*(U_1) = n + 1 \).

**Proof.** Let \( I = \langle [y_1, y_2], [y, z], z_1 z_2 \rangle T^* \). It is readily checked that \( I \subseteq \text{Id}^*(U_1) \). Let \( f \in P_n^* \) be a multilinear polynomial of degree \( n \). Since \( z_1 z_2 \in I \), by Remark 8, we may write \( f \), modulo \( I \), as a linear combination of the polynomials

\[
y_1 \cdots y_n, z_i y_1 \cdots y_{i-1}, \quad i_1 < \cdots < i_{n-1}.
\]

We next show that these polynomials are linearly independent modulo \( \text{Id}^*(U_1) \). Let \( f \in P_n^* \cap \text{Id}^*(U_1) \) be a linear combination of the polynomials in (4). By the multihomogeneity of \( T^* \)-ideals, we may assume that either \( f = a y_1 \cdots y_n \) or \( f = \beta z_n y_1 \cdots y_{n-1} \). By choosing \( y_1 = \cdots = y_n = e_{11} + e_{22} \) we get that \( a = 0 \). Also the evaluation \( z_n = e_{12}, y_1 = \cdots = y_{n-1} = e_{11} + e_{22} \) gives \( \beta = 0 \). Thus the elements in (4) are linearly independent modulo \( P_n^* \cap \text{Id}^*(U_1) \). Since \( P_n^* \cap I \subseteq P_n^* \cap \text{Id}^*(U_1) \), it follows that \( \text{Id}^*(U_1) = I \) and the elements in (4) are a basis of \( P_n^* \) (mod \( P_n^* \cap \text{Id}^*(U_1) \)). Hence \( c_n^*(U_1) = n + 1 \). \( \square \)

**Lemma 11.** Let

\[
U_2 = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & a \end{pmatrix} \right| a, b, c \in F \right\}
\]

with reflection involution. Then

- \( \text{Id}^*(U_2) = (S_3(y_1, y_2, y_3), y_1 z y_2, z_1 z_2) T^* \).
- \( \{ y_1 \cdots \hat{y}_i \cdots y_{n-1}, y_1 y_i \} \mid 1 \leq i \leq n - 1 \} \) is a basis of \( P_n^* \) (mod \( P_n^* \cap \text{Id}^*(U_2) \)), where the symbol \( \hat{y}_i \) means that the variable \( y_i \) is omitted.
- \( c_n^*(U_2) = 3n - 1 \).

**Proof.** Write \( I = (S_3(y_1, y_2, y_3), y_1 z y_2, z_1 z_2) T^* \). It is clear that \( I \subseteq \text{Id}^*(U_2) \). By the Poincaré–Birkhoff–Witt theorem (see [2]) every multilinear monomial in \( y_1, \ldots, y_n \) can be written as a linear combination of products of the type

\[
y_{i_1} \cdots y_{i_k} w_1 \cdots w_m
\]

where \( w_1, \ldots, w_m \) are left normed Lie commutators in the \( y_i \)s and \( i_1 < \cdots < i_k \). Recall that by Remark 8 \( z_1 w z_2 \in I \) (and, so, also \( [y_1, y_2] w [y_3, y_4] \in I \)) for any monomial \( w \) in the \( y_i \)s. Then modulo \( [y_1, y_2][y_3, y_4] \), only \( w_1 \) can appear in (5). Also, modulo \( y_1[y_2, y_3], y_4 \) we have that

\[
[y_1, \ldots, y_7] = [y_1, y_2] y_3 \cdots y_7 \pm y_1 \cdots y_3 [y_1, y_2].
\]
Thus, modulo \((y_1 z y_2, z_1 z_2)\) every multilinear monomial in \(P^*_{n,0}\) can be written as a linear combination of elements of the type

\[
y_1 \cdots y_{n-1} [y_i, y_j].
\]

(6)

Now, since \([y_1, y_2] w [y_3, y_4] \in I\), where \(w\) is a monomial in \(y_i\)'s and \(y_2 y_1 = y_1 y_2 + [y_2, y_1]\) we obtain that the variables on the left-hand side of the commutator in (6) can be ordered. Moreover, since \(S t_3 (y_1, y_2, y_3) = y_1 [y_2, y_3] + y_2 [y_3, y_1] + y_3 [y_1, y_2]\) belongs to \(I\), then \(y_3 [y_1, y_2] \equiv y_1 [y_3, y_2] + y_2 [y_1, y_3]\) can be applied and we obtain that the polynomials

\[
y_1 \cdots \hat{y}_i \cdots y_{n-1} [y_n, y_i], \quad 1 \leq i \leq n - 1,
\]

(7)

span \(P^*_{n,0} \mod (P^*_{n,0} \cap \text{Id}^*(U_2))\). We claim that these polynomials are a basis of \(P^*_{n,0} \mod (P^*_{n,0} \cap \text{Id}^*(U_2))\). Let \(\sum_{i=1}^{n-1} \alpha_i y_1 \cdots \hat{y}_i \cdots y_{n-1} [y_n, y_i]\) be an identity of \(U_2\). Then for a fixed \(j\), make the substitution \(y_j = e_{12} + e_{34}\) and \(y_i = e_{11} + e_{44}\) for all \(i \neq j\). We get \(\alpha_j = 0\) and the claim is proved. Notice that \(\dim P^*_{n,0} / (P^*_{n,0} \cap \text{Id}^*(U_2)) = n - 1\).

We now consider \(P^*_{n-1,1}\). Since \(y_1 z y_2 \in I\), then \(P^*_{n-1,1}\) can be generated \(\mod (P^*_{n-1,1} \cap I)\) by the monomials \(z_n y_1 \cdots y_{n-1}\) and \(y_1 \cdots y_{n-1} z_n\). On the other hand, using the equality \(y_2 y_1 = y_1 y_2 + [y_2, y_1]\) and Remark 8, it turns out that \(P^*_{n-1,1}\) is generated \(\mod (P^*_{n-1,1} \cap I)\) by the monomials

\[
z_n y_1 \cdots y_{n-1}, \ y_1 \cdots y_{n-1} z_n.
\]

(8)

We claim that the polynomials in (8) are a basis of \(P^*_{n-1,1} \mod \text{Id}^*(U_2)\). In fact, if \(\alpha z_n y_1 \cdots y_{n-1} + \beta y_1 \cdots y_{n-1} z_n \in \text{Id}^*(U_2)\), then the substitution \(y_1 = \cdots = y_{n-1} = e_{11} + e_{44}\), \(z_n = e_{12} - e_{34}\) gives \(-\alpha e_{34} + \beta e_{12} = 0\), and \(\alpha = \beta = 0\) follows. Hence, by the multihomogeneity of \(T^*\)-ideals, \(\text{Id}^*(U_2) = I, \{ y_1 \cdots \hat{y}_i \cdots y_{n-1} [y_n, y_i] \mid 1 \leq i \leq n - 1 \} \cup \{ z_i y_1 \cdots \hat{y}_i \cdots y_n, y_1 \cdots \hat{y}_i \cdots y_n z_i \mid 1 \leq i \leq n \}\) is a basis of \(P^* \mod (P^* \cap \text{Id}^*(U_2))\) and \(c_n^*(U_2) = 3n - 1\).

Let \(D = F \oplus F\) be the two-dimensional algebra isomorphic to the group algebra \(F \mathbb{Z}_2\) endowed with the exchange involution \((a, b) = (b, a)\). Then it is easily seen [6] that \(\text{Id}^*(D) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T^*}\). Also let \(M\) be the algebra

\[
M = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid u, r, s, v \in F \right\}
\]

endowed with reflection involution. The algebra \(M\) was extensively studied in [15]. In particular \(\text{Id}^*(M) = \langle z_1 z_2 \rangle_{T^*}\).

By the main theorem of [7] an algebra with involution has \(*\)-codimensions polynomially bounded if and only if \(D, M \notin \var^*(A)\). Thus we should be able to detect these algebras among the algebras we are considering. In fact we have the following

**Remark 12.** \(U_1 \in \var^*(D) \cap \var^*(M)\).

In the following lemmas we shall assume that \(A\) is a finite-dimensional algebra with involution of the type \(A = F + J\) where \(J\) has the decomposition given in Lemma 4.
Lemma 13. If $U_1 \notin \text{var}^* (A)$ then $A' = F + J_{11}$ is a commutative algebra with trivial involution (i.e., $z \equiv 0$ in $A'$).

Proof. Suppose that there exists $a \in J_{11}^-$, $a \not= 0$, and let $B$ be the subalgebra generated by 1 and $a$ over $F$. If $I$ is the ideal of $B$ generated by $a^2$, then $B/I$ has induced involution and is isomorphic to $U_1$ through the map that sends $1$ to $e_{11} + e_{22}$ and $a$ to $e_{12}$. Thus $U_1 \in \text{var}^* (A)$, a contradiction. Therefore $J_{11}^- = 0$ and $J_{11}$ consists of symmetric elements. This implies that $J_{11}$ is commutative. In fact if $a, b \in J_{11}$, $ab = (ab)^* = b^*a^* = ba$. □

Lemma 14. Suppose that $U_2 \notin \text{var}^* (A)$. Then $J_{10} = J_{01} = 0$.

Proof. Suppose that there exists $a \in J_{10}$, $a \not= 0$. Then $a^2 = 0$ and let $B$ be the subalgebra generated by 1, $a$, $a^*$. Let $I$ be the *-ideal of $B$ generated by $aa^*$ and $a^*a$ and consider the quotient algebra $\bar{B} = B/I$. It is straightforward to check that $\bar{B}$ is the linear span of $1, \bar{a}, \bar{a}^*$. Also this algebra is isomorphic to $U_2$, a contradiction. Hence $J_{10} = 0$ and also $J_{01} = J_{10}^* = 0$. □

We are now able to characterize the *-identities of algebras with *-codimensions bounded by a constant. Throughout we shall tacitely use the following observations: if $A$ and $B$ are algebras with involution then $\text{Id}^* (A \oplus B) = \text{Id}^* (A) \cap \text{Id}^* (B)$ and $c_n^* (A \oplus B) \leq c_n^* (A) + c_n^* (B)$. Also, for $C$ and $D$ algebras if $C \in \text{var}^* (D)$, then $c_n^* (C) \leq c_n^* (D)$.

Theorem 15. For a finite-dimensional algebra with involution $A$ the following conditions are equivalent.

1. There exists a constant $k$ such that $c_n^* (A) \leq k$, for all $n \geq 1$.
2. $U_1, U_2 \notin \text{var}^* (A)$.
3. $A$ is *-PI-equivalent to $N$ or to $C \oplus N$ where $N$ is a nilpotent algebra with involution and $C$ is a commutative algebra with trivial involution.

Proof. Suppose first that the sequence of *-codimensions of $A$ is bounded by a constant. Then by Lemmas 10 and 11, $U_1, U_2 \notin \text{var}^* (A)$ and (2) is proved. Suppose now that (2) holds. Then by Remark 12, $D, M \notin \text{var}^* (A)$ and by [7], $c_n^* (A)$ is polynomially bounded. But then by Proposition 7, we may assume that

$$A = A_1 \oplus \cdots \oplus A_m$$

where $A_1, \ldots, A_m$ are finite-dimensional algebras with involution over $F$ and $\dim A_i / J(A_i) \leq 1$, for all $i = 1, \ldots, m$. This means that for every $i$, either $A_i$ is a nilpotent algebra or $A_i = F + J(A_i)$. If $A_i$ is nilpotent for all $i$, then $A$ is a nilpotent algebra and we are done. Therefore we may assume that there exists $i \in \{1, \ldots, m\}$ such that $A_i = F + J(A_i)$ and let $J(A_i) = J_{11} + J_{10} + J_{01} + J_{00}$.

By the lemmas above we have that $A_i = F + J_{11} \oplus J_{00}$ is the direct sum of a commutative algebra with trivial involution and a nilpotent algebra. Hence, since $A = A_1 \oplus \cdots \oplus A_m$ it turns out that $A$ is *-PI-equivalent to $C \oplus N$, where $C$ is a commutative algebra with trivial involution and $N$ is a nilpotent algebra. Thus (3) holds.

Since for all $n$, $c_n^* (C) = 1$ and for $n$ large enough $c_n^* (N) = 0$ then (3) implies (1). □
4. Some algebras with quadratic growth of the *-codimensions

In this section we give a list of algebras with involution whose sequence of *-codimensions has quadratic growth.

Recall that the infinite-dimensional Grassmann algebra \( G \) together with the algebra of \( 2 \times 2 \) upper triangular matrices characterize the varieties of polynomial growth (see [13,14]). In the case of algebras with involution the Grassmann algebra \( G_2 \) on a 2-dimensional vector space with a suitable involution comes into play.

Recall that \( G_2 \) is the algebra with 1 generated by the elements \( e_1, e_2 \) over \( F \) subject to the condition \( e_1e_2 + e_2e_1 = e_1^2 = e_2^2 = 0 \). Thus \( G_2 = \text{span}\{1, e_1, e_2, e_1e_2\} \) and let \( G_2^* \) be the algebra \( G_2 \) endowed with the involution * such that \( e_1^* = e_1, e_2^* = e_2 \). We have the following

**Lemma 16.** For the algebra \( G_2^* \) we have

\[
- \text{Id}^*(G_2^*) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3 \rangle_{T^*},
\]

\[
- \{y_1, \ldots, y_n\} \cup \{z_i y_1, \ldots, y_n | 1 \leq i \leq n\} \cup \{z_i z_j y_1, \ldots, y_n | 1 \leq i < j \leq n\}
\] is a basis of \( P_n^* / \text{Id}^*(G_2^*) \).

\[
c_n^*(G_2^*) = 1 + n + \frac{n(n-1)}{2}.
\]

**Proof.** Let \( Q = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3 \rangle_{T^*} \). Clearly \( Q \subseteq \text{Id}^*(G_2^*) \). Moreover, as in the proof of lemmas of the previous section, it is easy to see that the polynomials

\[
y_1, \ldots, y_n, \quad z_i y_1, \ldots, y_n, \quad 1 \leq i \leq n,
\]

\[
z_i z_j y_1, \ldots, y_n, \quad 1 \leq i < j \leq n,
\]

span \( P_n^* / \text{Id}^*(G_2^*) \). We claim that they are linearly independent modulo \( \text{Id}^*(G_2^*) \). If \( f \in P_n^* / \text{Id}^*(G_2^*) \) is a linear combination of the above polynomials, by multihomogeneity of \( T^* \)-ideals we may think that either \( f = \alpha y_1, \ldots, y_n \) or \( f = \beta z_n y_1, \ldots, y_{n-1} \) or \( f = \gamma z_{n-1} z_n y_1, \ldots, y_{n-2} \). If we evaluate \( y_1, \ldots, y_n = 1 \) and \( z_n = e_1, y_1, \ldots, y_{n-1} = 1 \) we get \( \alpha = \beta = 0 \). Moreover, by evaluating \( z_{n-1} = e_1, z_n = e_2 \) and \( y_1, \ldots, y_{n-2} = 1 \) we obtain \( \gamma = 0 \). Since \( P_n^* / Q \subseteq P_n^* / \text{Id}^*(G_2^*) \) the equality \( \text{Id}^*(G_2^*) = Q \) follows; the above polynomials form a basis of \( P_n^* / \text{Id}^*(G_2^*) \) and \( c_n^*(G_2^*) = 1 + n + \frac{n(n-1)}{2} \). \( \square \)

Following our agreement for the subalgebras of \( UT_n \), the algebras \( M_2, M_4, M_5, M_8, M_9, M_{10} \) introduced below are endowed with the reflection involution.

In the following three lemmas we introduce three unitary algebras with involution. These algebras together with the algebra \( G_2^* \) will be crucial for understanding the structure of the space \( J_{11} \) in the following section.

We do not present the proof of the next lemma since it is very similar to the proof of the previous lemma.

**Lemma 17.** Let

\[
M_1 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F \right\}
\]
be endowed with the involution
\[
\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b & c \\ 0 & a & -b \\ 0 & 0 & a \end{pmatrix}.
\]

Then

- \( \text{Id}^*(M_1) = ([y_1, y_2], [y, z], [z_1, z_2], z_1 z_2 z_3)_{T^*} \).
- \( \{y_1 \cdots y_n \} \cup \{z_i y_1 \cdots \hat{y}_i \cdots y_n \mid 1 \leq i \leq n \} \cup \{z_i z_j y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n \mid 1 \leq i < j \leq n \} \) is a basis of \( P_n^* \) (mod \( P_n^* \cap \text{Id}^*(M_1) \)).
- \( c_n^*(A) = 1 + n + \frac{n(n-1)}{2} \).

**Lemma 18.** Let

\[
M_2 = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in F \right\}.
\]

Then

- \( \text{Id}^*(M_2) = ([y_1, y_2], z_1 z_2)_{T^*} \).
- \( \{y_1 \cdots y_n \} \cup \{z_i y_1 \cdots \hat{y}_i \cdots y_n \} \cup \{y_i, z_j \} y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n, \ i, j = 1, \ldots, n \} \) is a basis of \( P_n^* \) (mod \( P_n^* \cap \text{Id}^*(M_2) \)).
- \( c_n^*(M_2) = n^2 + 1 \).

**Proof.** Let \( Q = ([y_1, y_2], z_1 z_2)_{T^*} \). It is immediate that \( Q \subseteq \text{Id}^*(M_2) \). By Remark 8, \( P_n^* \) is generated, mod \( P_n^* \cap Q \), by the monomials

\[ y_1 \cdots y_n, \quad w_i z_{j} w_{n-i-1} \quad (j = 1, \ldots, n), \]

where \( w_i \) and \( w_{n-i-1} \) are multilinear monomials in the only variables \( y_i \)'s (except \( y_j \)) of length \( i \) and \( n - i - 1 \), respectively. Notice that \( y_j z = [y_j, z] + z y_j \) and \( [y_j, z] \in F(X, \ast)^+ \). Hence by a repeated substitution of the above in the monomial \( w_i z w_{n-i-1} \), recalling that \( [y_1, y_2] \in Q \), we obtain that \( P_n^* \) can be generated, modulo \( P_n^* \cap Q \), by the monomials

\[ y_1 \cdots y_n, \quad z_{j} y_{1} \cdots \hat{y}_i \cdots y_n, \quad [y_j, z_j] y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n, \quad i, j = 1, \ldots, n. \quad (9) \]

We next show that the above polynomials are linearly independent modulo \( \text{Id}^*(M_2) \). Let \( f \in P_n^* \cap \text{Id}^*(M_2) \) be a linear combination of the element in (9). By the multihomogeneity of \( T^* \)-ideals we may assume that either \( f = \alpha y_1 \cdots y_n \) or

\[ f = \beta z_n y_1 \cdots y_{n-1} + \sum_{i=1}^{n-1} \beta_i [y_i, z_n] y_1 \cdots \hat{y}_i \cdots y_{n-1}. \]

By making the evaluations \( y_1 = \cdots = y_n = 1_{M_2} \) and \( z_n = e_{12} - e_{34}, \ y_1 = \cdots = y_{n-1} = 1_{M_2} \), we get \( \alpha = \beta = 0 \). Also the evaluation \( z_n = e_{12} - e_{34}, \ y_i = e_{13} + e_{24}, \ y_j = 1_{M_2}, \) for \( j \neq i, \)
gives $\beta_i = 0$. Hence the polynomials in (9) are a basis of $P^*_n \pmod{P^*_n \cap \text{Id}^*(M_2)}$, $\text{Id}^*(M_2) = \langle [y_1, y_2], z_1 z_2 \rangle_{T^*}$ and $c_{n}^{*(n)}(M_2) = n^2 + 1$. \hfill \Box

We do not give the proof of the following lemma since it can be deduced by using the strategy of proof given in the previous lemma.

Lemma 19. Let

$$M_3 = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & a \end{pmatrix} \right| a, b, c, d \in F \right\}$$

be endowed with the involution

$$\left( \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & a \end{pmatrix} \right)^* = \left( \begin{pmatrix} a & b & c & -d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & a \end{pmatrix} \right)^*$$

- $\text{Id}^*(M_3) = \langle [y, z], z_1 z_2 \rangle_{T^*}$.
- $\{ y_1 \cdots y_n, \ y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n[y_i, y_j], \ 1 \leq i < j \leq n \} \cup \{ y_1 \cdots \hat{y}_i \cdots y_n z_i, \ i = 1, \ldots, n \}$ is a basis of $P^*_n \pmod{P^*_n \cap \text{Id}^*(M_3)}$.
- $c_{n}^{*(n)}(M_3) = 1 + n + \frac{n(n-1)}{2}$.

The algebras introduced and studied in the following lemmas are all algebras without 1 and will be crucial in the next section for studying the structure of the spaces $J_{10}$ and $J_{01}$. Notice that unlike the ordinary case treated in [5], all the upper triangular matrix algebras appearing here have some obvious symmetries coming from the involution defined by reflecting a matrix along its secondary diagonal.

Lemma 20. Let

$$M_4 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} \right| a, b, c, d \in F \right\}$$

Then $c_{n}^{*(n)}(M_4) \geq n(n-2)$, for all $n \geq 3$.

Proof. By Theorem 2 and Remark 3 it is enough to show that there exists a pair of tableaux $(T_\lambda, T_\mu)$, with $\deg_{\chi_{\lambda, \mu}} \geq n(n-2)$, such that the corresponding highest weight vector $f_{T_\lambda, T_\mu}$ is not a $*$-polynomial identity for $M_4$. We have that $M_4^+ = \text{span}\{e_{11} + e_{33}, e_{12} + e_{23}, e_{13} \}$ and $M_4^- = \text{span}\{e_{12} - e_{23} \}$. Let

$$(T_{(n-2, 1)}, T_{(1)}) = \begin{pmatrix} 2 & 4 & \cdots & n - 1 & n \\ 3 & \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}$$
and let $f_{T(n-2,1),T(1)} = zy_1y_2y_1^{n-3} - zy_2y_1^{n-2}$ be the corresponding highest weight vector. Now we exhibit a non-zero evaluation of $f_{T(n-2,1),T(1)}$. Consider $s_1 = e_{11} + e_{33}$, $s_2 = e_{12} + e_{23}$ and $k_1 = e_{12} - e_{23}$. A direct computation shows that $f_{T(n-2,1),T(1)}(s_1, s_2, k_1) = -e_{13} \neq 0$. Hence $f_{T(n-2,1),T(1)} \not\in \text{Id}^n(M_4)$ and $c^*_n(M_4) \geq \deg \chi_{\lambda,\mu} = \binom{n}{1} \deg \chi_\lambda \deg \chi_\mu = n(n-2)$. \hfill \Box

Lemma 21. Let

$$M_5 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d \in F \right\}.$$  

Then $c^*_n(M_5) \geq n(n-2)$, for all $n \geq 3$.

Proof. We have that $M_5^+ = \text{span}\{e_{22}, e_{12} + e_{23}, e_{13}\}$ and $M_5^- = \text{span}\{e_{12} - e_{23}\}$. Let $f_{T(n-2,1),T(1)} = zy_1^{n-2}y_2 - zy_1^{n-3}y_2y_1$ be the highest weight vector corresponding to the following pair of tableaux:

$$(T(n-2,1), T(1)) = \left( \begin{array}{cccc} n-1 & 2 & 3 & \cdots & n-2 \\ n & \end{array} \right).$$

By making the evaluation $y_1 = e_{22}$, $y_2 = e_{12} + e_{23}$, $z = e_{12} - e_{23}$ we get that $f_{T(n-2,1),T(1)} = e_{13}$. This says that $\chi_{(n-2,1),(1)}$ appears in the decomposition of the $*$-cocharacter into irreducibles with non-zero multiplicity. Hence $c^*_n(M_5) \geq \deg \chi_{(n-2,1),(1)} = \binom{n}{1} \deg \chi_{(n-2,1)} \deg \chi_{(1)} = n(n-2)$. \hfill \Box

Lemma 22. Let

$$M_6 = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e, f \in F \right\}.$$  

be endowed with the involution

$$\begin{pmatrix} a & b & c & d \end{pmatrix}^* = \begin{pmatrix} a & -f & e & -d \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \end{pmatrix}.$$  

Then $c^*_n(M_6) \geq n(n-2)$, for all $n \geq 3$.

Proof. We have $M_6^+ = \text{span}\{e_{11} + e_{44}, e_{12} - e_{34}, e_{13} + e_{24}\}$ and $M_6^- = \text{span}\{e_{12} + e_{34}, e_{13} - e_{24}, e_{14}\}$. If we consider the pair of Young tableaux

$$(T(n-2,1), T(1)) = \left( \begin{array}{cccc} 2 & 4 & \cdots & n-1 \\ 3 & n \end{array} \right).$$
then the corresponding highest weight vector is \( f_{T(n-2,1), T(1)} = zy_1 y_2 y_1^{n-3} - z y_2 y_1^{n-2} \). By choosing \( y_1 = e_1 + e_4, y_2 = e_1 + e_2 + e_4 \) and \( z = e_1 + e_4 \) we get that \( f_{T(n-2,1), T(1)} = -e_4 \not\in \text{Id}^*(M_6) \) and so, \( c_n^*(M_6) \geq n(n-2) \). 

**Lemma 23.** Let

\[
M_7 = \left\{ \begin{pmatrix} 0 & b & c & d \\ 0 & a & 0 & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d, e, f \in F \right\}
\]

be endowed with the involution

\[
\begin{pmatrix} 0 & b & c & d \\ 0 & a & 0 & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -f & e & -d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Then \( c_n^*(M_7) \geq n(n-2) \), for all \( n \geq 3 \).

**Proof.** We have that \( M_7^+ = \{ e_{22} + e_{33}, e_{12} - e_{34}, e_{13} + e_{24} \} \) and \( M_7^+ = \{ e_{12} + e_{34}, e_{13} - e_{24}, e_{14} \} \). Let \( f_{T(n-2,1), T(1)} = zy_1 y_2 y_1^{n-3} - z y_2 y_1^{n-2} \) be the highest weight vector corresponding to the pair of tableaux

\[
(T(n-2,1), T(1)) = \begin{pmatrix} 2 & \cdots & n-2 & n-1 \\ n \end{pmatrix}, [1]
\]

A direct computation shows that \( f_{T(n-2,1), T(1)} (e_{22} + e_{33}, e_{12} + e_{24}, e_{13} + e_{24}) = e_{14} \). Hence \( f_{T(n-2,1), T(1)} \not\in \text{Id}^*(M_7) \) and

\[
c_n^*(M_7) \geq \deg \chi(n-2,1), (1) = \binom{n}{1} \deg \chi(n-2,1) \deg \chi(1) = n(n-2). \]

In order to obtain our main result we still need to construct three more algebras of upper triangular matrices. We do this in the next lemmas.

**Lemma 24.** Let

\[
M_8 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d, e, f \in F \right\}
\]

Then \( c_n^*(M_8) \geq \frac{(n-1)(n-2)}{2} \), for all \( n \geq 3 \).
Proof. Since \( M_8^+ = \text{span}\{e_{11} + e_{66}, e_{12} + e_{56}, e_{23} + e_{45}, e_{13} + e_{46}\} \) and \( M_8^- = \text{span}\{e_{12} - e_{56}, e_{13} - e_{46}\} \) we consider the highest weight vector \( f_{\mathcal{T}(n-2,1)}(\emptyset) = y_1^{n-3} \mathcal{S}_3(y_1, y_2, y_3) \) corresponding to the following pair of tableaux:

\[
(T_{(n-2,1)}, \emptyset) = \begin{pmatrix}
\begin{array}{cccccccc}
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
n-1 & 0 & 0 & 1 \\
n & 2 & \cdots & n-3 \\
\end{array}
\end{pmatrix}, \emptyset
\]

A direct computation shows that \( (e_{11} + e_{66})^{n-3} \mathcal{S}_3(e_{11} + e_{66}, e_{12} + e_{56}, e_{23} + e_{45}) = e_{13} \neq 0 \). Hence \( f_{\mathcal{T}_{(n-2,1)}(\emptyset)} \notin \text{Id}^*(M_8) \). It follows that \( c_n^*(M_8) \geq \deg \chi_{\lambda, \emptyset} = (n-1)(n-2)/2 \).

Lemma 25. Let

\[
M_9 = \left\{ \begin{pmatrix}
a & b & c \\
0 & 0 & d \\
0 & 0 & 0
\end{pmatrix} \right| a, b, c, d, e, f \in F \right\}.
\]

Then \( c_n^*(M_9) \geq n(n - 2) \), for all \( n \geq 3 \).

Proof. We have that \( M_9^+ = \text{span}\{e_{11} + e_{66}, e_{12} + e_{56}, e_{13} + e_{46}\} \) and \( M_9^- = \text{span}\{e_{12} - e_{56}, e_{23} - e_{45}, e_{13} - e_{46}\} \). We consider the highest weight vector \( f_{\mathcal{T}_{(n-2,1)}(\emptyset)} = y_1^{n-3}[y_1, y_2] \mathcal{S}_3(\emptyset) \) corresponding to the following pair of tableaux:

\[
(T_{(n-2,1)}, T_{(1)}) = \begin{pmatrix}
\begin{array}{cccccccc}
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
n-1 & 0 & 0 & 1 \\
n & 2 & \cdots & n-3 \\
\end{array}
\end{pmatrix}
\]

A direct computation shows that \( (e_{11} + e_{66})^{n-3}[e_{11} + e_{66}, e_{12} + e_{56}](e_{23} - e_{45}) = e_{13} \neq 0 \). Hence \( f_{\mathcal{T}_{(n-2,1)}(\emptyset)} \notin \text{Id}^*(M_9) \). It follows that \( c_n^*(M_9) \geq \deg \chi_{\lambda, \emptyset} = \binom{n}{1} \deg \chi_{\lambda} \deg \chi_{\mu} = n(n - 2) \).

Lemma 26. Let

\[
M_{10} = \left\{ \begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & 0
\end{pmatrix} \right| a, b, c, d, e, f \in F \right\}.
\]

Then \( c_n^*(M_{10}) \geq n(n - 2) \), for all \( n \geq 3 \).
Proof. We have $M_{10}^+ = \text{span}\{e_{11} + e_{22} + e_{55} + e_{66}, e_{23} + e_{45}, e_{13} + e_{46}\}$ and $M_{10}^- = \text{span}\{e_{12} - e_{56}, e_{23} - e_{45}, e_{13} - e_{46}\}$. We take the highest weight vector $f_{T(n-2,1),T(1)} = y_1^{n-3}[y_1, y_2]z$ corresponding to the following pair of tableaux:

$$(T_{(n-2,1)}, T_{(1)}) = \left(\begin{array}{c}
\frac{n-2}{n-1} \\
1 \cdots n-3 \\
\end{array}, y\right).$$

A direct computation shows that

$$(e_{11} + e_{22} + e_{55} + e_{66})^{n-3}[e_{11} + e_{22} + e_{55} + e_{66}, e_{23} + e_{45}](e_{12} - e_{56}) = e_{46} \neq 0.$$

Hence $f_{T(n-2,1),T(1)} \notin \text{Id}^*(M_{10})$. It follows that $e_{n}^*(M_{10}) \geq n(n-2)$. □

5. Algebras with linear growth of the $*$-codimensions

Throughout this section, unless otherwise stated, we assume that $A$ is a finite-dimensional algebra with involution and $A = F + J$ where $J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$. We start our study by analyzing algebras of the type $F + J_{11}$.

Lemma 27. If $M_1 \notin \text{var}^*(A)$ then for all $a \in J_{11}$ we have $a^2 = 0$.

Proof. Suppose that there exists $a \in J_{11}$ such that $a^2 \neq 0$ and consider the subalgebra $B$ of $A$ generated by 1 and $a$ over $F$. Then if $I$ is the $*$-ideal generated by $a^2$, we have that the algebra $B = B/I$ has induced involution and $B = \text{span}\{1, \bar{a}, a^2\}$. It is easily seen that $B \cong M_1$ through the isomorphism $\varphi$ such that $\varphi(1) = e_{11} + e_{22} + e_{33}$, $\varphi(\bar{a}) = e_{12} + e_{23}$. Hence $M_1 \in \text{var}^*(A)$ and we have reached a contradiction. □

Lemma 28. Let $A = F + J_{11}$ with $J_{11} \neq 0$ and suppose that $M_1, M_2, M_3, G_2^* \notin \text{var}^*(A)$. Then $A$ is a commutative algebra and either $A$ has trivial involution (i.e., $z \equiv 0$ on $A$) or $A$ is $*$-PI-equivalent to the algebra $U_1$.

Proof. Write $J_{11} = J$. Our aim is to show that $[J, J] = 0$.

Suppose first that $[J^-, J^-] \neq 0$. Let $a, b \in J^-$ be such that $[a, b] \neq 0$. Since $a, b, a + b \in J_{11}^-$ by Lemma 27, $a^2 = b^2 = (a + b)^2 = 0$. It follows that $ab + ba = 0$ and, so, $ab = -ba$. Consider the subalgebra $B$ of $A$ generated by the elements 1, $a, b$. Clearly $B$ has an induced involution and $B = \text{span}\{1, a, b, ab\}$. It is clear that the algebra $B$ is isomorphic to the algebra $G_2^*$ if we map, for instance, $a$ to $e_1$ and $b$ to $e_2$. Since by hypothesis $G_2^* \notin \text{var}^*(A)$ we get a contradiction. Hence $[z_1, z_2] \equiv 0$ on $A$.

Recall that by Lemma 27 since $M_1 \notin \text{var}^*(A)$, $z^2 \equiv 0$ is an identity of $A$. After linearizing we get that $z_1z_2 + z_2z_1 \equiv 0$ on $A$. This together with $[z_1, z_2] \equiv 0$ gives that $z_1z_2 \equiv 0$ is an identity of $A$.

We next claim that $[J^+, J^+] = 0$. Our proof will be by induction on the index of nilpotence $k$ of $J$.

If $k = 2$ the conclusion is clearly true. Suppose $k > 2$ and let $\tilde{A} = A/J^{k-1}$. Then $\tilde{A}$ is an algebra with induced involution and has still a decomposition of the type $\tilde{A} = F + \tilde{J}$ where $\tilde{J} = J_{11}$ is the Jacobson radical of $A$. Also $J^{k-1} = 0$ and by the inductive hypothesis we have $[\tilde{J}^+, \tilde{J}^+] = 0$. Therefore $[\tilde{J}^+, J^+] = 0$. Hence $[J^+, J^+] = 0$. □
Thus \([J^+, J^+] J = J[J^+, J^+] = 0\). Since the algebra \(A\) already satisfies the identity \(z_1z_2 = 0\), from the above we get that \(\{[y_1, y_2], y_3\} = 0\) is an identity of \(A\).

Let \(a, b \in J^+\) be such that \([a, b] \neq 0\) and let \(B\) be the subalgebra generated by 1, \(a, b\) over \(F\). \(B\) has induced involution and from \([a, b]a = a[a, b] = 0\) and \([a, b]b = b[a, b] = 0\) it easily follows that

\[
B^+ = \text{span}\{1, a^i b^j, ab + ba \mid 0 \leq i, j \leq k, \ (i, j) \neq (1, 1)\}
\]

and \(B^- = \text{span}\{ab - ba\}\).

We claim that \(\text{Id}^*(B) = \text{Id}^*(M_3)\). One inclusion is clear since \([y, z] \equiv 0, z_1z_2 \equiv 0\) are \(*\)-identities of \(B\) and, by Lemma 19, \(\text{Id}^*(M_3) = \langle [y, z], z_1z_2 \rangle \rangle\). Let \(f \in \text{Id}^*(B)\) be a multi-linear polynomial of degree \(n\). By Lemma 19 and by multihomogeneity of \(T^*\)-ideals, we may assume that, modulo \(\text{Id}^*(M_3)\), either \(f = \beta y_1 \cdots y_{n-1} z_n\) or

\[
f = \alpha y_1 \cdots y_n + \sum_{i < j} \alpha_{ij} y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n[y_i, y_j].
\]

If we substitute \(z_n = ab - ba, y_1 = \cdots = y_{n-1} = 1\) we obtain \(\beta = 0\). If we substitute \(y_1 = \cdots = y_n = 1\) and \(y_i = a, y_j = b, y_k = 1\) for \(k \neq i, j\) we get \(\alpha = 0\) and \(\alpha_{i,j} = 0\) for all \(i, j = 1, \ldots, n\). This proves the claim.

Thus the algebra \(A\) satisfies the \(*\)-identities \([y_1, y_2] \equiv 0\) and \(z_1z_2 \equiv 0\) and, so, \(\text{Id}^*(M_2) \subseteq \text{Id}^*(A)\).

Suppose now that \([J^+, J^-] \neq 0\) and let \([a, b] \neq 0\) with \(a \in J^+, b \in J^-\). Let \(f \in P_n^*\) be a \(*\)-identity of \(A\) modulo \(\text{Id}^*(M_2)\). By Lemma 18 and by the multihomogeneity of \(*\)-ideals we may assume that either \(f = \beta y_1 \cdots y_n\) or

\[
f = \alpha z_n y_1 \cdots y_{n-1} + \sum_{j=1}^{n-1} \alpha_j [y_j, z_n] y_1 \cdots \hat{y}_j \cdots y_{n-1}.
\]

By choosing \(y_1 = \cdots = y_n = 1\), we obtain that \(\beta = 0\). Hence we may assume that \(f\) has the form given in (10). We shall prove that \(\alpha = \alpha_j = 0\) for all \(j = 1, \ldots, n-1\). By choosing \(z_n = b\) and \(y_1 = \cdots = y_{n-1} = 1\), we get \(\alpha = 0\). Now, from the evaluations \(z_n = b, y_j = a, y_k = 1, k \neq j\) it follows that \(\alpha_j = 0\) for all \(1 \leq j \leq n - 1\). Hence \(f \in \text{Id}^*(M_2)\) and \(M_2\) is \(*\)-PI-equivalent to \(A\).

We have proved that \([J, J] = 0\). Hence \(A\) is a commutative algebra and satisfies the identities \([y_1, y_2] \equiv 0, [y, z] \equiv 0, z_1z_2 \equiv 0\). If \(z \equiv 0\) holds in \(A\), then \(A\) has trivial involution, a desired conclusion. In case \(z \neq 0\), by Lemma 13, \(U_1 \in \text{var}^*(A)\). Moreover, since \(\text{Id}^*(U_1) \subseteq \text{Id}^*(A)\), it follows that \(A\) is \(*\)-PI-equivalent to \(U_1\). \(\square\)

**Lemma 29.** For the algebra \(A = F + J\), the following holds.

1. If \(M_4 \notin \text{var}^*(A)\), then \(aa^* = 0\) for all \(a \in J_{10}\).
2. If \(M_5 \notin \text{var}^*(A)\), then \(aa^* = 0\) for all \(a \in J_{01}\).

**Proof.** Suppose that there exists \(a \in J_{10}\) such that \(aa^* \neq 0\) and consider the \(*\)-subalgebra \(B\) generated by 1 and \(a\) over \(F\). If \(\overline{B} = B/1\) where 1 is the \(*\)-ideal generated by \(a^*a\), then \(\overline{B}\) is linearly spanned by the elements \(1, \bar{a}, \bar{a}^*, \bar{a}a^*\). It follows that \(\overline{B}\) is isomorphic to the algebra \(M_4\).
through the isomorphism of algebras with involution such that \( \bar{1} \to e_{11} + e_{33} \) and \( \bar{a} \to e_{12} \). Thus \( M_4 \in \text{var}^*(A) \) and the proof of the first part is complete. The second part of the lemma is proved similarly. \( \square \)

**Lemma 30.** If \( M_4, M_5, M_6 \notin \text{var}^*(A) \), then \( J_{10}J_{01} = 0 \).

**Proof.** Since \( M_4, M_5 \notin \text{var}^*(A) \), by Lemma 29, for all \( a \in J_{10}, aa^* = a^*a = 0 \).

Suppose by contradiction that there exist \( a \in J_{10}, b \in J_{01} \) such that \( ab \neq 0 \). Then \( aa^* = b^*b = 0 \) and, since \( a + b^* \in J_{10} \), also \( (a + b^*)(a^* + b) = 0 \). This says that \( ab + b^*a^* = 0 \) and \( ab \in (J_{10}J_{01})^- \subseteq J_{11} \) follows. Notice that from \( ab = -b^*a^* \) we get \( aba = -b^*a^*a = 0 \) and \( bab = -bb^*a^* = 0 \). Hence, since also \( a^2 = b^2 = 0 \), if \( B \) is the \(*\)-subalgebra of \( A \) generated by \( 1, a, b \), we have that \( B = \text{span}\{1, a, b, a^*, b^*, ab, ba\} \). If we now take the quotient with the \(*\)-ideal generated by \( ba \), we obtain an algebra \( \bar{B} \) spanned by the non-zero images of \( 1, a, b, a^*, b^* \). It is easily checked that such algebra is isomorphic to the algebra \( M_6 \) through the isomorphism of algebras with involution \( \varphi \) such that \( \bar{a} = e_{12} \) and \( \bar{\varphi}(\bar{b}) = e_{24} \). Hence \( M_6 \in \text{var}^*(A) \) and we have reached a contradiction. \( \square \)

**Lemma 31.** If \( M_4, M_5, M_7 \notin \text{var}^*(A) \), then \( J_{01}J_{10} = 0 \).

**Proof.** Since \( M_4, M_5 \notin \text{var}^*(A) \), by Lemma 29, for all \( a \in J_{10}, aa^* = a^*a = 0 \).

Suppose by contradiction that there exist \( a \in J_{01}, b \in J_{10} \) such that \( ab \neq 0 \). Then the proof proceeds as in the previous lemma by constructing an algebra \( B \) generated by \( 1, a, b \) and then by taking the quotient with the \(*\)-ideal generated by \( ba \). One obtains an algebra \( \bar{B} \) spanned by the non-zero images of \( 1, a, b, a^*, b^* \) which is isomorphic to the algebra \( M_7 \) through the isomorphism of algebras with involution \( \varphi \) such that \( \bar{a} = e_{12} \) and \( \bar{\varphi}(\bar{b}) = e_{24} \). Hence \( M_7 \in \text{var}^*(A) \), a contradiction. \( \square \)

**Lemma 32.** If \( M_i \notin \text{var}^*(A) \), \( 4 \leq i \leq 9 \), then \( J_{10}J_{00} = J_{00}J_{10} = 0 \).

**Proof.** Suppose that there exist \( a \in J_{10}, b \in J_{00} \) with \( ab \neq 0 \). By taking the largest power of \( b \) such that \( ab^k = 0 \) and \( ab^{k-1} \neq 0 \), we may assume that \( ab^2 = 0 \). Also, in case \( a(b + b^*) = ab^2 = 0 \), one would get \( ab = ab^* = -ab^* \) and, so, \( ab = 0 \), a contradiction. Therefore we may assume that \( b \in J_{00} \).

Since \( M_i \notin \text{var}^*(A) \), \( 4 \leq i \leq 9 \), then \( J_{10}J_{01} = J_{01}J_{10} = 0 \). Moreover, being \( a^2 = ab^2 = a(b^*)^2 = 0 \) we get that the \(*\)-subalgebra \( \bar{B} \) generated by \( 1, a, b \) over \( F \) is linearly spanned by the elements \( 1, a, a^*, b, b^*, \ldots, b^i, ab, ba^* \). Also if we let \( \bar{B} \) be the quotient algebra \( B/I \) where \( I \) is the \(*\)-ideal generated by \( b^2 \), then \( \bar{B} \) is spanned by the images of the elements \( 1, a, a^*, b, ab, ba^* \).

It is easily checked that the above elements are linearly independent over \( F \) and the algebra \( \bar{B} \) is isomorphic to the algebra \( M_8 \) or \( M_9 \) according as \( b \in J_{00}^+ \) or \( b \in J_{00}^- \), a contradiction. Hence \( J_{10}J_{00} = J_{00}J_{10} = 0 \). \( \square \)

**Lemma 33.** Suppose that \( (J_{11}^-)^2 = 0 \). If \( G_{2}^+, M_i \notin \text{var}^*(A) \), \( 4 \leq i \leq 10 \), then \( J_{01}J_{11}^- = J_{11}^-J_{10} = 0 \).

**Proof.** Suppose that \( J_{01}J_{11}^- \neq 0 \) and pick \( a \in J_{01}, b \in J_{11}^- \) with \( ab \neq 0 \). Recall that by the previous lemmas we have that \( J_{10}J_{00} = J_{00}J_{11}^- = J_{11}^-J_{10} = 0 \). Then the \(*\)-subalgebra \( B \) generated by \( 1, a, b \) is the linear span of the elements \( 1, a, a^*, b, ab, ba^*, b^i, i \geq 2 \). We shall prove that \( M_{10} \in \text{var}^*(A) \). To this end, by taking the quotient with the \(*\)-ideal generated by \( b^2 \),
we may assume that $B$ is the linear span of the elements $1, a, a^*, b, ab, ba^*$ and they are linearly independent over $F$. This algebra is isomorphic to the algebra $M_{10}$.

In the next lemma we compute the $*$-codimension sequence of the direct sum of the only two algebras with non-trivial involution, so far encountered, whose $*$-codimensions grow linearly.

**Lemma 34.** For the algebra $U_1 \oplus U_2$ we have

$\text{Id}^*(U_1 \oplus U_2) = \langle \text{St}_3(y_1, y_2, y_3), y_1[y_2, y_3]y_4, y_1zy_2 - y_2zy_1, y_1[y_2, z]y_3, z_1z_2 \rangle$.

This algebra is isomorphic to the algebra $\langle y_1 \cdots \hat{y}_i \cdots y_{n-1}[y_n, y_i] | 1 \leq i \leq n-1 \rangle$ is a basis of $P_n^*$ (mod $P_n^* \cap \text{Id}^*(U_1 \oplus U_2)$).

$C_n(U_1 \oplus U_2) = 4n - 1$.

**Proof.** Let $Q = \langle \text{St}_3(y_1, y_2, y_3), y_1[y_2, y_3]y_4, y_1zy_2 - y_2zy_1, y_1[y_2, z]y_3, z_1z_2 \rangle$. It is immediate that $Q \subseteq \text{Id}^*(U_1 \oplus U_2)$. Now, as in the proof of Lemma 11, any multilinear polynomial of degree $n$ in the $y_i$’s can be written, modulo $Q$, as a linear combination of the following polynomials

$z_i y_1 \cdots \hat{y}_i \cdots y_n,  \quad y_1z_i \cdots \hat{y}_i \cdots y_n,  \quad y_1 \cdots \hat{y}_i \cdots y_n z_i,  \quad 1 \leq i \leq n.$

We claim that they are linearly independent modulo $\text{Id}^*(U_1 \oplus U_2)$. Let $f \in P_n^* \cap \text{Id}^*(U_1 \oplus U_2)$ be a linear combination of the above polynomials. Then we may assume that $f = \alpha z_n y_1 \cdots y_{n-1} + \beta y_1z_n \cdots y_{n-1} + \gamma y_1 \cdots y_{n-1} z_n$. The substitutions $y_1 = \cdots = y_{n-1} = (0, e_{11} + e_{44}), z_n = (0, e_{12} - e_{34})$ and $y_1 = \cdots = y_{n-1} = (e_{11} + e_{22}, 0), z_n = (e_{12}, 0)$ give $\alpha = \beta = \gamma = 0$. Hence $\text{Id}^*(U_1 \oplus U_2) = Q, \{y_1 \cdots \hat{y}_i \cdots y_{n-1}[y_n, y_i] | 1 \leq i \leq n-1 \} \cup \{z_i y_1 \cdots \hat{y}_i \cdots y_n, y_1z_i \cdots \hat{y}_i \cdots y_n, y_1 \cdots \hat{y}_i \cdots y_n z_i | 1 \leq i \leq n \}$ is a basis of $P_n^*$ (mod $P_n^* \cap \text{Id}^*(U_1 \oplus U_2)$) and $c_n(U_1 \oplus U_2) = 4n - 1$.

**Lemma 35.** Suppose that $G_5^*, M_i \notin \text{var}^*(A), 1 \leq i \leq 10, \text{and } J_0 \neq 0, J_1 \neq 0$. Then $A$ is $*-\text{PI}$-equivalent to the algebra $U_1 \oplus U_2 \oplus N$, for some nilpotent algebra $N$.

**Proof.** By the previous lemmas we have that $(J_1)^2 = J_{10}J_{00} = J_{00}J_{01} = J_{10}J_{01} = J_{01}J_{10} = 0$. It follows that $A = F + J_{11} + J_{01} + J_{10} \oplus 0$ and $J_{00} = N$ is a nilpotent two-sided ideal of $A$. Also by Lemmas 28 and 33, $J_{11}$ is commutative and $J_{01}J_{11} = J_{11}J_{10} = 0$.

Let $B = F + J_{11} + J_{00} + J_{10}$. A direct calculation shows that $B$ satisfies the identities of the algebra $U_1 \oplus U_2$. Hence $\text{Id}^*(U_1 \oplus U_2) \subseteq \text{Id}^*(B)$. On the other hand, the subalgebras $F + J_{11}$ and $F + J_{01} + J_{10}$ are isomorphic to the algebras $U_1$ and $U_2$, respectively. Hence $\text{Id}^*(B) \subseteq$
\[ \text{Id}^*(U_1) \cap \text{Id}^*(U_2) = \text{Id}^*(U_1 \oplus U_2) \] and equality holds. We have proved that the algebra \( B \) is \(*\)-PI-equivalent to the algebra \( U_1 \oplus U_2 \). \( \square \)

Recall that by [7], an algebra with involution \( A \) has polynomially bounded \(*\)-codimensions if and only if the algebras \( D \) and \( M \) defined before in Remark 12 do not lie in \( \text{var}^*(A) \). We have

**Remark 36.**

1. \( M_1 \in \text{var}^*(D) \).
2. \( M_2, M_3 \in \text{var}^*(M) \).

We can now prove the main result of this section.

**Theorem 37.** For a finite-dimensional algebra with involution \( A \) the following conditions are equivalent.

1. There exists a constant \( k \) such that \( c_n^*(A) \leq kn \), for all \( n \geq 1 \).
2. \( G^*_j, M_i \notin \text{var}^*(A) \), for all \( i \in \{1, \ldots, 10\} \).
3. \( A \) is \(*\)-PI equivalent to \( N \) or \( C \oplus N \) or \( U_1 \oplus N \) or \( U_2 \oplus N \) or \( U_1 \oplus U_2 \oplus N \), where \( N \) is a nilpotent algebra with involution and \( C \) is a commutative algebra with trivial involution.

**Proof.** In case (1) holds, then the algebras \( G^*_j, M_i, 1 \leq i \leq 10 \), do not lie in \( \text{var}^*(A) \) since their \(*\)-codimensions have at least quadratic growth. Hence (2) holds.

Suppose now that the algebras \( G^*_j, M_i, \) \( 1 \leq i \leq 10 \), do not lie in \( \text{var}^*(A) \). Then by Remark 36, \( D, M \notin \text{var}^*(A) \) and by [7], \( c_n^*(A) \) is polynomially bounded. But then by Proposition 7, \( A \) is \(*\)-PI-equivalent to a finite direct sum of algebras \( A_1 \oplus \cdots \oplus A_m \) where \( A_1, \ldots, A_m \) are finite-dimensional algebras with involution over \( F \) and \( \dim A_i / J(A_i) \leq 1 \), for all \( i = 1, \ldots, m \). This means that for every \( i \), either \( A_i \) is a nilpotent algebra or \( A_i \) has a decomposition of the type \( A_i = F + J = F + J_{11} + J_{10} + J_{01} + J_{00} \). By the lemmas above we have that \( A_i = F + J_{11} + J_{10} + J_{01} + J_{00} \) and \( F + J_{11} + J_{10} + J_{01} + J_{00} \) is \(*\)-PI-equivalent to \( U_1 \) in case \( J_{10} = 0 \) and \( J_{11} \neq 0 \), to \( U_2 \) in case \( J_{10} \neq 0 \) and \( J_{11} = 0 \), to \( U_1 \oplus U_2 \) in case \( J_{10} \neq 0 \) and \( J_{11} \neq 0 \) and to a commutative algebra with trivial involution in case \( J_{10} = 0 \) and \( J_{11} = 0 \). Thus (3) holds.

From Lemmas 10 and 11 it follows that the algebras \( U_1 \) and \( U_2 \) have linear growth of the codimensions, hence all the algebras described in (3) have \(*\)-codimensions bounded by a linear function. This proves (1). \( \square \)

As a consequence of the previous theorem we can now classify all possible linearly bounded \(*\)-codimension sequences.

**Corollary 38.** If \( A \) is a finite-dimensional algebra with involution whose sequence of codimensions is linearly bounded, then there exists \( n_0 \) such that for all \( n > n_0 \) we have either \( c_n^*(A) = 0 \) or \( 1 \) or \( n + 1 \) or \( 3n - 1 \) or \( 4n - 1 \).

**References**