Journal of Number Theory 129 (2009) 3020–3025

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Badly approximable affine forms and Schmidt games

Jimmy Tseng

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, USA

ARTICLE INFO

Article history:
Received 12 December 2008
Revised 5 May 2009
Available online 8 August 2009
Communicated by David Goss

ABSTRACT

For any real number θ, the set of all real numbers x for which there exists a constant $c(x) > 0$ such that $\inf_{p\in\mathbb{Z}} |\theta q - x - p| \geqslant \frac{c(x)}{|q|}$ for all $q \in \mathbb{Z}\backslash\{0\}$ is a 1/8-winning set.

1. Introduction

Let $M_{m,n}(\mathbb{R})$ denote the set of $m \times n$ real matrices and $\widetilde{M}_{m,n}(\mathbb{R})$ denote $M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$. The element in $\widetilde{M}_{m,n}(\mathbb{R})$ corresponding to $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$ will be expressed as $\langle A, \mathbf{b}\rangle$. Consider the following well-known sets from the theory of Diophantine approximation [8]:

$$\mathbf{Bad}(m,n) := \left\{ \langle A, \mathbf{b}\rangle \in \widetilde{M}_{m,n}(\mathbb{R}) \;\middle|\; \exists c(A,\mathbf{b}) > 0 \text{ s.t. } \|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} \geqslant \frac{c(A,\mathbf{b})}{\|\mathbf{q}\|^{n/m}}\ \forall \mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\} \right\}$$

where $\|\cdot\|$ is the sup norm on $\mathbb{R}^k$ and $\|\cdot\|_{\mathbb{Z}}$ is the norm on $\mathbb{R}^k$ given by $\|\mathbf{x}\|_{\mathbb{Z}} := \inf_{\mathbf{p}\in\mathbb{Z}^k} \|\mathbf{x} - \mathbf{p}\|$. The set $\mathbf{Bad}(m,n)$ is called the *set of badly approximable systems of* **m** *affine forms in* **n** *variables*. For any $\mathbf{b} \in \mathbb{R}^m$, let $\mathbf{Bad}^{\mathbf{b}}(m,n) := \{A \in M_{m,n}(\mathbb{R}) \mid \langle A, \mathbf{b}\rangle \in \mathbf{Bad}(m,n)\}$, and, for any $A \in M_{m,n}(\mathbb{R})$, let $\mathbf{Bad}_A(m,n) := \{\mathbf{b} \in \mathbb{R}^m \mid \langle A, \mathbf{b}\rangle \in \mathbf{Bad}(m,n)\}$.

The set $\mathbf{Bad}^{\mathbf{0}}(m,n)$ is called the *set of badly approximable systems of* **m** *linear forms in* **n** *variables* and is an important and classical object of study in the theory of Diophantine approximation. Although it is a Lebesgue null set (Khintchine, 1926), it has full Hausdorff dimension and, even stronger, is winning (Schmidt, 1969). Winning sets have a few other properties besides having full Hausdorff dimension; see Section 1.2 for more details.

For the larger set $\mathbf{Bad}(m,n)$, however, less is known. Among its known properties are that it has Lebesgue measure zero, but full Hausdorff dimension. The former property follows from the doubly

E-mail address: tseng@math.ohio-state.edu.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.05.006

metric inhomogeneous Khintchine–Groshev Theorem [3, Chapter VII, Theorem II]. The latter property is a result of D. Kleinbock (1999) proved using mixing of flows on the space of lattices [8]. Recently (2008), Y. Bugeaud, S. Harrap, S. Kristensen, and S. Velani have given a simpler proof of Kleinbock's result; their main result is that, for every A, $\mathbf{Bad}_A(m,n)$ (and some related sets) has full Hausdorff dimension [2]. Using the Marstrand slicing theorem [5, Theorem 5.8], Kleinbock's result follows. In view of these results, a natural question that arises is whether, like $\mathbf{Bad}^{\mathbf{0}}(m,n)$, these sets $\mathbf{Bad}_A(m,n)$ and $\mathbf{Bad}(m,n)$ are winning instead of just having full Hausdorff dimension. In this note, we show that $\mathbf{Bad}_\theta(1,1)$ is winning for every real number θ.[1] For results and open questions concerning general n and m, see Remark 2.3 below.

1.1. Statement of results

Our main result, which generalizes the $m = n = 1$ case of the aforementioned main result in [2] (their main result is Theorem 1 of [2]), is the following (see Section 1.2 for the definition of 1/8-winning):

Theorem 1.1. *For any real number θ, $\mathbf{Bad}_\theta(1,1)$ is a $1/8$-winning set.*

This theorem is proved in Section 2 below. A number of corollaries will follow immediately because of the properties of winning sets (see Section 1.2). A model one is:

Corollary 1.2. *For any countable set $\{\theta_n\} \subset \mathbb{R}$ and any countable family $\{f_m\}$ of invertible affine maps $\mathbb{R} \to \mathbb{R}$, the set $\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} f_m(\mathbf{Bad}_{\theta_n}(1,1))$ is $1/8$-winning and thus has full Hausdorff dimension.*

1.2. Background on winning sets and continued fractions

The proof of our result requires two tools: Schmidt games (see [9] for a reference) and continued fractions (see [6] for a reference). We will discuss both.

W. Schmidt introduced the games which now bear his name in [9]. Let $0 < \alpha < 1$ and $0 < \beta < 1$. Let S be a subset of a complete metric space M. Two players, Black and White, alternate choosing nested closed balls $B_1 \supset W_1 \supset B_2 \supset W_2 \cdots$ on M. The radius of W_n must be α times the radius of B_n, and the radius of B_n must be β times the radius of W_{n-1}. The second player, White, *wins* if the intersection of these balls lies in S. A set S is called (α,β)*-winning* if White can always win for the given α and β. A set S is called α*-winning* if White can always win for the given α and any β. A set S is called *winning* if it is α-winning for some α. Schmidt games have four important properties for us [9]:

- The sets in $\mathbb{R}^n$ which are α-winning have full Hausdorff dimension.
- Countable intersections of α-winning sets are again α-winning.
- The bilipschitz image of an α-winning set is α-winning.
- Let $0 < \alpha \leqslant 1/2$. If a set in a Banach space of positive dimension is α-winning, then the set with a countable number of points removed is also α-winning.

Let us now discuss continued fractions. Let p_i/q_i be the i-th order convergent of an irrational number θ. Define

$$\Delta_i := \|\theta q_i\|_{\mathbb{Z}}.$$

[1] For $\mathbf{Bad}_\theta(1,1)$, we have a slight strengthening of the aforementioned consequence of the Khintchine–Groshev Theorem: $\mathbf{Bad}_\theta(1,1)$ has Lebesgue measure zero for every irrational number θ [7]. This result is essentially a corollary of two elementary facts from the theory of continued fractions (see [10] for this short, second proof and for a connection with shrinking targets). There is yet a third proof of this result; see [1].

We will use the following well-known facts:

- For all $i \in \mathbb{N}$, $\frac{1}{2}\Delta_{i-1}^{-1} < q_i < \Delta_{i-1}^{-1}$.
- Let $0 \leqslant j < k < q_i$. Then, $\|\theta k - \theta j\|_{\mathbb{Z}} > \Delta_{i-1}$.

1.3. The setup

Let $\theta \in \mathbb{R}$. Define

$$\mathbf{Bad}_\theta^+ := \left\{ x \in \mathbb{R} \;\middle|\; \exists c(x) > 0 \text{ s.t. } \|\theta q - x\|_{\mathbb{Z}} \geqslant \frac{c(x)}{q} \; \forall q \in \mathbb{N} \right\}.$$

Note that $\mathbf{Bad}_\theta(1,1) = \mathbf{Bad}_\theta^+ \cap -\mathbf{Bad}_\theta^+$; thus showing $\mathbf{Bad}_\theta^+$ is 1/8-winning will prove Theorem 1.1. Also, we may assume that these sets are restricted to the circle $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$, as they are invariant under integral translations.

Henceforth, let us consider $\mathbf{Bad}_\theta^+$. If θ is rational, then the set is just $\mathbb{T}^1$ with a finite number of points removed and hence is winning. Therefore, we assume that θ is irrational henceforth.

For convenience, let us call the elements in

$$\left\{ \theta q \in \mathbb{T}^1 \;\middle|\; q_i \leqslant q < q_{i+1} \right\}$$

the *elements of generation i*.

Finally, we note a simple property of continued fractions.

Lemma 1.3. *Let $q_{i+1} \leqslant q < q_{i+2}$. Given a $0 < r < 1/2$ such that, for all elements θp of generations $\leqslant i$, $\|\theta q - \theta p\|_{\mathbb{Z}} \geqslant r\Delta_i$, then $q \geqslant \frac{r}{2} q_{i+2}$.*

Proof. There are unique numbers $0 \leqslant s < q_{i+1}$ and $1 \leqslant n \leqslant \lfloor \frac{q_{i+2}}{q_{i+1}} \rfloor$ such that $q = nq_{i+1} + s$. Thus, $n\Delta_{i+1} = \|\theta q - \theta s\|_{\mathbb{Z}} \geqslant r\Delta_i$. Hence, $q \geqslant r\frac{\Delta_i}{\Delta_{i+1}} q_{i+1} \geqslant \frac{r}{2} q_{i+2}$. $\square$

2. A proof of Theorem 1.1

Let $\alpha = 1/8$ and $c = (\frac{(\alpha\beta)}{4})^3$. We will play an (α, β)-game on $\mathbb{T}^1$. Let us start with the following lemma, which tells us how to choose W_m given B_m (note that the radius of a ball B is denoted $\rho(B)$):

Lemma 2.1. *Let U be any union of balls on $\mathbb{T}^1$ with radius $\leqslant (\alpha\beta)\Delta_N/4$ around the elements of generations $\leqslant N$. If*

$$(\alpha\beta)\Delta_N < 2\rho(B_m) \leqslant \Delta_N,$$

then one can choose W_m disjoint from U.

Proof. *Case*: *B_m does not intersect any ball of U.*

Pick any allowed W_m.

Case: *B_m intersects exactly one ball of U.*

Even if B_m contains the whole ball of U, there is, at least, a subinterval in B_m of length 1/4 of the length of B_m that misses U. Pick W_m to be in this subinterval.

Case: *B_m intersects more than one ball of U.*

Note that B_m cannot intersect more than one element of generations $\leqslant N$ (unless one has exactly two elements of generations $\leqslant N$, one at each end). Thus, at least a subinterval in B_m of length

$(1-(\alpha\beta)/2)\Delta_N \geqslant 1/2\Delta_N$ does not meet U. Now $\alpha 2\rho(B_m) \leqslant 1/8\Delta_N$. Therefore, we can choose W_m to be in this subinterval. $\square$

Since the Schmidt game can be played until, for some $J \in \mathbb{N}$, $2\rho(B_J) \leqslant \Delta_1$, we may assume without loss of generality that $J = 1$. Note that there exists an $N_0 \geqslant 2$ such that $2\rho(B_1) \leqslant \Delta_{N_0-1}$, but that $2\rho(B_1) > \Delta_{N_0}$ (follows since $\Delta_{N_0} < \Delta_{N_0-1}$).

Also, there exists an $n_0 \in \mathbb{N}$ such that $2(\alpha\beta)^{n_0-1}\rho(B_1) > \Delta_{N_0}$ and $2(\alpha\beta)^{n_0}\rho(B_1) \leqslant \Delta_{N_0}$. Thus,

$$(\alpha\beta)\Delta_N < 2(\alpha\beta)^{n_0}\rho(B_1) \leqslant \Delta_N \tag{2.1}$$

where $N \geqslant N_0$ is the largest natural number for which (2.1) holds.

We intend to use induction. In the initial induction step, consider the disjoint union of balls around each element of generations $\leqslant N$ of radius $(\alpha\beta)\Delta_N/4$; call this union U. By Lemma 2.1, we may pick W_{n_0+1} to miss U. For any other step of the induction, W_{n_0+1} is already chosen.

As an aside for clarity, note that there are two infinite "processes" that are intertwined in this proof. One is the count of the generations given by the convergents of θ and denoted in the proof by the indices of Δ. The other is the count of the iterations of the Schmidt game and denoted in the proof by the indices of W. The goal of the proof is to fit these two processes together by making astute choices of White's balls. To accomplish this fitting, one must consider the size of $\alpha\beta\Delta_N$ from (2.1) in relation to Δ_{N+1}. There are two possible cases.

2.1. *Case:* $\alpha\beta\Delta_N > \Delta_{N+1}$

The condition implies that there exists an $n_1 \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+1} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leqslant \Delta_{N+1}.$$

Also, there exists a maximal $M \geqslant 1$ such that

$$(\alpha\beta)\Delta_{N+M} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leqslant \Delta_{N+M}.$$

Moreover, $(\alpha\beta)\Delta_{N+1} < \Delta_{N+M}$.

For any element θq of generation $N+1$ in W_{n_0+1}, $q \geqslant \frac{(\alpha\beta)}{8}q_{N+2}$ by Lemma 1.3. For any element θq of generations $> N+1$ in W_{n_0+1}, it is obvious that $q \geqslant \frac{(\alpha\beta)}{8}q_{N+2}$. Thus, for all such θq,

$$\frac{c}{q} \leqslant \frac{(\alpha\beta)^2\Delta_{N+1}}{4} \leqslant \frac{(\alpha\beta)\Delta_{N+M}}{4}.$$

Now play freely until $B_{n_0+n_1+1}$ is chosen. Again by Lemma 2.1, we can choose $W_{n_0+n_1+1}$ to miss the balls of radius $(\alpha\beta)\Delta_{N+M}/4$ around the elements of generations $N+1$ to $N+M$.

2.2. *Case:* $\alpha\beta\Delta_N \leqslant \Delta_{N+1}$

It is easy to see from the theory of continued fractions that there exist a $K \in \mathbb{N}$ such that $(\alpha\beta)\Delta_n > \Delta_{n+K}$ for all $n \in \mathbb{N}$. Therefore, the condition implies that there exists a $1 \leqslant m \leqslant K-1$ such that

$$\Delta_{N+m+1} < \alpha\beta\Delta_N \leqslant \Delta_{N+m}.$$

Thus, we have

$$(\alpha\beta)^2\Delta_{N+m} < (\alpha\beta)^2\Delta_N < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leqslant \alpha\beta\Delta_N \leqslant \Delta_{N+m}.$$

If $(\alpha\beta)^2\Delta_{N+m} < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leqslant (\alpha\beta)\Delta_{N+m}$, then

$$(\alpha\beta)\Delta_{N+m} < 2(\alpha\beta)^{n_0}\rho(B_1) \leqslant \Delta_{N+m}.$$

Since N is the largest natural number for which (2.1) holds, we obtain that $m = 0$, a contradiction.

Thus, we must conclude that

$$(\alpha\beta)\Delta_{N+m} < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leqslant \Delta_{N+m}.$$

Now, there exists an $n_1 \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+m+1} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leqslant \Delta_{N+m+1}.$$

Also, there exists a maximal $M \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+m+M} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leqslant \Delta_{N+m+M}.$$

Moreover, $(\alpha\beta)\Delta_{N+m+1} < \Delta_{N+m+M}$.

If $n_1 = 1$, then even more is true: $(\alpha\beta)\Delta_{N+m} < \Delta_{N+m+M}$. Now note that, for the elements θq of generations $N+1$ to $N+m+M$, we have

$$\frac{c}{q} \leqslant \frac{c}{q_{N+1}} \leqslant \frac{(\alpha\beta)\Delta_{N+m+M}}{4}.$$

Consider the disjoint union of balls around each element of generations $\leqslant N+m+M$ of radius $(\alpha\beta)\Delta_{N+m+M}/4$; call this union U. Again by Lemma 2.1, we can pick W_{n_0+2} to miss U.

Otherwise, $n_1 \geqslant 2$. Now note that, for the elements θq of generations $N+1$ to $N+m$, we have

$$\frac{c}{q} \leqslant \frac{c}{q_{N+1}} \leqslant \frac{(\alpha\beta)\Delta_{N+m}}{4}.$$

Consider the disjoint union of balls around each element of generations $\leqslant N+m$ of radius $(\alpha\beta)\Delta_{N+m}/4$; call this union U. Again by Lemma 2.1, we can pick W_{n_0+2} to miss U.

For any element θq of generation $N+m+1$ in W_{n_0+2}, $q \geqslant \frac{(\alpha\beta)}{8} q_{N+m+2}$ by Lemma 1.3. For any element θq of generations $> N+m+1$ in W_{n_0+2}, it is obvious that $q \geqslant \frac{(\alpha\beta)}{8} q_{N+m+2}$. Thus, for all such θq,

$$\frac{c}{q} \leqslant \frac{(\alpha\beta)^2\Delta_{N+m+1}}{4} \leqslant \frac{(\alpha\beta)\Delta_{N+m+M}}{4}.$$

Now play freely until $B_{n_0+n_1+1}$ is chosen. Again by Lemma 2.1, we can choose $W_{n_0+n_1+1}$ to miss the balls of radius $(\alpha\beta)\Delta_{N+m+M}/4$ around the elements of generations $N+m+1$ to $N+m+M$.

Using these two cases inductively, one can show that the set

$$\left\{ x \in \mathbb{R} \;\middle|\; \exists c(x) > 0 \text{ s.t. } \|\theta q - x\|_{\mathbb{Z}} \geqslant \frac{c(x)}{q} \ \forall q \geqslant q_{N+1} \right\}$$

is 1/8-winning. By shrinking $c(x)$ for each x, we note that this set is $\mathbf{Bad}_\theta^+$. The proof is complete.

Remark 2.2. If θ is a badly approximable number,[2] one can easily see from the continued fraction expansion of θ that there exists an upper bound for Δ_n/Δ_{n+1} independent of n. This uniform bound allows us to simplify the above proof for θ badly approximable (however, we conclude that the set is α-winning for an α depending on this uniform bound).

Remark 2.3. In very recent joint work [4], M. Einsiedler and the author have, using a method different from the one presented in this note, generalized Theorem 1 of [2] to conclude winning instead of just having full Hausdorff dimension. Thus, as a special case, we can show that $\mathbf{Bad}_A(m,n)$ is winning for every $A \in M_{m,n}(\mathbb{R})$. Related results are also presented in [4]. Whether $\mathbf{Bad}(m,n)$ is winning, however, is still an open question. The techniques developed in [4] may be useful in answering this question (see [4] for more details).

Acknowledgments

The author would like to thank Manfred Einsiedler, Dmitry Kleinbock, and the referee for their helpful comments. The author is particularly grateful to Dmitry Kleinbock for pointing out an improvement to the statement of Corollary 1.2.

References

[1] V. Beresnevich, V. Bernik, M. Dodson, S. Velani, Classical metric Diophantine approximation revisited, in: Analytic Number Theory, Cambridge Univ. Press, Cambridge, UK, 2009, pp. 38–61.
[2] Y. Bugeaud, S. Harrap, S. Kristensen, S. Velani, On shrinking targets for $\mathbb{Z}^m$ actions on tori, preprint, arXiv:0807.3863v1, 2008.
[3] J. Cassels, An Introduction to Diophantine Approximation, Cambridge Tracts in Math., vol. 45, Cambridge University Press, Cambridge, UK, 1957.
[4] M. Einsiedler, J. Tseng, Badly approximable systems of affine forms, fractals, and Schmidt games, preprint, 2009.
[5] K. Falconer, The Geometry of Fractal Sets, Cambridge Tracts in Math. and Math. Phys., vol. 85, Cambridge University Press, Cambridge, UK, 1986.
[6] A. Khinchin, Continued Fractions, The University of Chicago Press, Chicago, 1964.
[7] D. Kim, The shrinking target property of irrational rotations, Nonlinearity 20 (2007) 1637–1643.
[8] D. Kleinbock, Badly approximable systems of affine forms, J. Number Theory 79 (1999) 83–102.
[9] W. Schmidt, Badly approximable numbers and certain games, Trans. Amer. Math. Soc. 123 (1966) 178–199.
[10] J. Tseng, On circle rotations and the shrinking target properties, Discrete Contin. Dyn. Syst. 20 (2008) 1111–1122.

[2] In our notation, $\theta \in \mathbf{Bad}^0(1,1)$.