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OBSTRUCTION THEORY IN ALGEBRAIC CATEGORIES, II *

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§ 0. Introduction

This paper is a continuation of the work described in [13]. In that paper we considered categories $\mathcal{C}$ satisfying the following eight axioms:

(1). There is a triple $\mathbf{T} = (T, \eta, \mu)$ on S (the category of sets) such that $T(\emptyset) = \{p\}$ (a one point set) and $\mathcal{C}$ is equivalent to $S^{\mathbf{T}}$.

(2). U: $\mathcal{C} \to S_*$ (the category of pointed sets) factors through the category of groups.

(3). All operations in $\mathcal{C}$ are finitary.

(4). There is a generating set Ω for the operations in $\mathcal{C}$, and

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$$

(where Ω_i is the set of i-ary operations in Ω).

(5). If $* \in \Omega_2' = \Omega_2 \setminus \{+\}$ (where $+$ is the group operation arising from (2)), then

$$a * (b + c) = a * b + a * c.$$

(6). If $\omega \in \Omega_1' = \Omega_1 \setminus \{-\}$ (where $-$ is the inverse associated with the group structure), then

$$\omega(a * b) = \omega(a) * b.$$

(7). If $x_1, x_2, x_3 \in X$, an object in $\mathcal{C}$, and $* \in \Omega_2'$, then

$$x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1.$$

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(8). For each ordered pair $(\cdot, *) \in \Omega_2' \times \Omega_2'$, there is a word w such that

$$(x_1 \cdot x_2) * x_3 = w(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,$$
$$x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2)$$

where each juxtaposition represents an operation in Ω_2'.

The categories satisfying (1)–(8) were called *categories of interest*. We associated to each object A in C a class $\mathbf{E}A$ consisting of equivalence classes of sequences of the form

$$0 \to ZA \to A \to E \to M \to 0.$$

(Here ZA is the center of A in the sense defined in [2].)

We considered the problem: Given a diagram

$$\begin{array}{c} R \\ \downarrow \rho \\ 0 \to ZA \to A \to E \to M \to 0, \end{array} \tag{0.1}$$

is there an extension $0 \to A \to T \to R \to 0$ which induces ρ?

We showed that this can be settled by associating with (0.1) a cohomology class $[\rho] \in H^2(R, ZA)$. $[\rho]$ is called the *obstruction* of ρ. The cohomology used is that obtained from the triple on S_*.

In §1 of this paper we will place further restrictions on C and then treat the question of when $H^2(R, Z)$ is precisely the set of obstructions. In §2, we show that to a certain extent relative cohomology groups can also be used to measure obstructions.

The author wishes once again to express her gratitude to Michael Barr for many helpful suggestions.

§1. Elements of $H^2(R, Z)$ as obstructions

In this section, we restrict our attention to those categories of interest in which a more restrictive form of axiom (8) is imposed. We require:

(8)′. For each ordered pair $(\cdot, *) \in \Omega_2' \times \Omega_2'$, there is a word w involving no binary operation except + such that

$$(x_1 \cdot x_2) * x_3 = w(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,$$
$$x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents one of the operations in Ω_2'.

It is clear that categories satisfying (1)–(7) and (8)′ are categories of interest in the sense of [13, Definition 1.14]. Moreover, all the categories of the interest which were provided as examples in [13, §1] satisfy (8)′. In the case of associative algebras with multiplication represented by $*$, it sufficies to let $\Omega_2' = \{*, *^0\}$. We can then take

$$w(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = a_1.$$

For Lie algebras, take $\Omega_2' = \{[\ ,\ ]\}$. Then

$$w(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = a_1 + a_6,$$

as can be seen from the fact that $[a, b] = -[b, a]$ and the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

In the example of groups, we can choose Ω so that $\Omega_2' = \emptyset$, and thus axioms (8) and (8)′ are both vacuous.

The main result of this section gives a criterion for determining when $H^2(R, Z)$ coincides precisely with the set of obstructions. That this is not always the case can be seen in the following example.

Example 1.1. Let K be a ring of global dimension greater than or equal to 2 and C the category of K-modules. There are K-modules R and A such that $\mathrm{Ext}^2(R, A) \neq 0$. Such is the nature of K.

Since + is commutative and we can take $\Omega_2' = \emptyset$, if A is a subobject of T, then $A < T$. Also, $ZA = A$ and $Z(T, A) = T$. Therefore all sequences in $\mathbf{E}A$ have the form

$$0 \to A \xrightarrow{1_A} A \xrightarrow{\lambda} 0 \xrightarrow{\pi} 0 \to 0.$$

Any morphism $\rho: R \to M = 0$ is just the zero map and is unobstructed since there is always the exact sequence

$$0 \to A \to A \oplus R \to R \to 0$$

(which induces ρ). But $\mathrm{Ext}^2(R, A) = H^2(R, A)$ since $\mathrm{Der}(R, ZA) = \mathrm{Hom}_K(R, A)$ in this case.

We will prove:

Theorem 1.2. *Let R be an algebra in C for which there exists a set X and a surjection $\alpha: F(X) \to R$ with $Z(F(X), \ker \alpha) = 0$. Given an R-module Z and a class ξ in $H^2(R, Z)$, there is an object A, a sequence*

$$0 \to ZA \to A \to E \to M \to 0$$

in $\mathbf{E}A$, *and an isomorphism* $\rho : R \to M$ *such that* $Z \simeq ZA$ *as an* R*-module (via* ρ*) and* $[\rho] = \xi$.

The hypotheses of this theorem appear to involve a restriction on R. The condition can however be viewed more accurately as a restriction on $\mathcal{C}$. In Example 1.1, we have seen a category of interest in which Theorem 1.2 is not always valid. More generally, in any category of modules, + is commutative and Ω_2' can be taken to be empty. Hence $Z(F(X), K) = F(X)$ for any $K < F(X)$. Thus the theorem does not apply. However, the obstruction theory is uninteresting since $\mathbf{E}A$ degenerates in the manner described in Example 1.1.

There are several restrictions that we can place on $\mathcal{C}$ in order to insure that Theorem 1 holds for any choice of R and Z in $\mathcal{C}$. These are:

(9a). For each R in $\mathcal{C}$, there is an X in S_* and a surjection $\alpha : F(X) \to R$ with $Z(F(X), \ker \alpha) = 0$, or

(9b). There exist sets Y and X with $Y \subseteq X$ and such that $Z(F(X), F(Y)) = 0$.

Remark 1.3. (9b) implies (9a).

Proof. Let $X = UR \mathbin{\dot\cup} Y$, where Y is as in (9b). Consider $\alpha : F(X) \to R$ determined by

$$\alpha(x) = \begin{cases} \epsilon(x) \text{ if } x \in UR, \\ 0 \quad \text{if } x \in Y. \end{cases}$$

Clearly $F(Y) \subseteq \ker \alpha$ and α is surjective. Therefore, $Z(F(X), \ker \alpha) = 0$.

(9b) is a useful condition because it is easily checked in many special cases.

We will call a category which satisfies (1)–(7), (8)′, and (9a) or (9b) a *special category of interest*. The various categories in which obstruction theory has been studied in the past – groups [6], associative algebras over a field [8], associative algebras [14], and commutative algebras [1; 7] – are special categories of interest.

Example 1.4. (a). Let $\mathcal{C}$ be the category of groups. We have immediately that

$$Z(F(\{a, b, c\}), F(\{a, b, c\})) = Z(F(\{a, b, c\})) = 0.$$

In [6], the case $R = \mathbf{Z}/2\mathbf{Z}$ was handled as a special case in the proof of the theorem corresponding to Theorem 1.2. This is not necessary with our treatment.

(b). Let $\mathcal{C}$ be the category of associative algebras. This case is essentially the same as the case of commutative algebras, which is treated in [1]. If R is an associative algebra, then GR is the polynomial ring whose noncommuting variables are determined by the underlying set of R:

$$Z(GR, \ker \epsilon) = \{x \in GR \mid xa = ax = 0 \text{ and } a + x = x + a \text{ for all } a \in \ker \epsilon\}.$$

The second condition $(a + x = x + a)$ is satisfied trivially since addition is commutative for all objects in the category. Hence,

$$Z(GR, \ker \epsilon) = \{x \in GR \mid xa = ax = 0 \text{ for all } a \in \ker \epsilon\}.$$

As in [1] we note that w, the variable corresponding to 0, is in $\ker \epsilon$ but is not a zero divisor. Thus, since $x \in Z(GR, \ker \epsilon)$ must satisfy $wx = xw = 0$, we conclude that $x = 0$. Hence $\mathcal{C}$ satisfies (9a).

(c). Now let $\mathcal{C}$ be the category of K-Lie algebras where K is a commutative ring. We will show that

$$Z(F(\{a, b\}), F(\{a, b\})) = 0.$$

Let $F_m(\{a, b\})$ be the free K-module on $\{a, b\}$. Since + is commutative,

$$Z(F(\{a, b\}), F(\{a, b\})) = \{x \in F(\{a, b\}) \mid [x, y] = [y, x] = 0 \text{ for all } y \in F(\{a, b\})\}.$$

We note that in a free Lie algebra, if $[x, y] = 0$, then there is an element z in the algebra and $k_1, k_2 \in K$ such that $x = k_1 z$ and $y = k_2 z$. So, if $x \in Z(F(\{a, b\}), F(\{a, b\}))$, then in particular, $[x, a] = [x, b] = 0$. Therefore there exist $k_1, k_2, k_3, k_4 \in K$ and $z_1, z_2 \in F(\{a, b\})$ such that $x = k_1 z_1$, $a = k_2 z_1$, $x = k_3 z_2$ and $b = k_4 z_2$.

$F(\{a, b\})$ has $F_m(\{a, b\})$ as a module direct summand [5]. Since $a, b \in F_m(\{a, b\})$, we may as well assume $k_2 = k_4 = 1$, $z_1 = a$ and $z_2 = b$. Therefore x is a scalar multiple of each generator. By freeness, we conclude $x = 0$.

Thus, our proof of Theorem 1.2 provides a uniform treatment of the body of work mentioned above.

Proof of Theorem 1.2. We will proceed very much as in [1], but instead of relying on the standard resolution, we will use

$$0 \leftarrow R \xleftarrow{\alpha} F(X) \underset{t^0}{\overset{e^0, e^1}{\leftrightarrows}} F(X_2) \underset{t^0, t^1}{\overset{e^0, e^1, e^2}{\leftrightarrows}} F(X_3) \ldots,$$

where α is the surjection described in the statement of the theorem.

Represent ξ by a derivation $p: F(X_3) \to Z$. Since p is a cycle

$$p(e^3 - e^2 + e^1 - e^0) = 0$$

and by the simplicial normalization theorem we may also suppose $pt^0 = pt^1 = 0$. Let

$$V = \{(x, z) \in F(X_2) \times Z \mid e^1 x = 0\}.$$

Let

$$I = \{(e^0 y, -py) \mid y \in F(X_3) \text{ and } e^1 y = e^2 y = 0\}.$$

Then $I \subseteq V$ since $e^1 e^0 y = e^0 e^2 y = 0$. We claim that $I < V$. Let $(x, z) \in V$ and $(e^0 y, -py) \in I$, and take

$$y' = t^0 x + y - t^0 x$$

in $F(X_3)$. Then

$$\begin{aligned}
e^0 y' &= x + e^0 y - x, \\
-py' &= -(pt^0 x + t^0 x + py + y + p(-t^0 x) - y - t^0 x) \\
&= -(t^0 x + py + y - t^0 x - pt^0 x + t^0 x - y - t^0 x) \\
&= t^0 x - py - t^0 x \\
&= \alpha e^1 e^0 t^0 x - py - \alpha e^1 e^0 t^0 x \\
&= \alpha e^1 x - py - \alpha e^1 x \\
&= -py.
\end{aligned}$$

Therefore

$$(x, z) + (e^0 y, -py) - (x, z) = (x + e^0 y - x, -py) = (ey', -py')$$

which is in I. We can also check that for any $* \in \Omega_2'$,

$$(x, z) * (e^0 y, -py) = (e^0(t^0 x * y), -p(t^0 x * y))$$

and this is in I also.

Let $A = V/I$. Next, consider the composite $Z \to V \to A$ which we call ι.

$$\iota(z) = (0, z) + I.$$

It is not hard to see that this is an injection.

Finally, we must show that the image of Z in A is actually ZA. Clearly it is contained in ZA. To see the reverse inclusion, choose any $(x, z) + I \in ZA$. This part of the proof requires the hypothesis that $Z(F(x), \ker \alpha) = 0$. For any $(x', z') \in V$ we are assuming that

$$(x', z') * (x, z) = (x' * x, 0)$$

and

$$(x, z) + (x', z') - (x, z) - (x', z') = (x + x' - x - x', 0)$$

belong to I (where $* \in \Omega'_2$). In particular, there exist $y, y' \in F(X_3)$ such that

$$e^1 y = e^2 y = e^1 y' = e^2 y' = 0,$$
$$e^0 y = x' * x,$$
$$e^0 y' = x + x' - x - x'.$$

Hence,

$$e(x' * x) = e e^0 y = e^0 e^1 y - e^0 e^2 y = 0$$

and similarly

$$e(x + x' - x - x') = 0.$$

Thus, for any $x' \in \ker e^1$, we have

$$e^0 x' * e^0 x = 0,$$
$$e^0 x + e^0 x' = e^0 x' + e^0 x.$$

By the simplicial normalization theorem this means $e^0 x$ is an element of $Z(F(X), \ker \alpha) = 0$. Hence, $ex = 0$ and so $x = ey$. Using the simplicial normalization theorem again, we can assume $e^1 y = e^2 y = 0$. Therefore $ZA \subseteq Z + I$.

Starting with R and an R-module Z we have found A such that $ZA \simeq Z$. Next we construct a diagram (see (0.1) above) for which $[\rho] = \xi$.

A is seen to be an $F(X)$-structure by specifying a set of derived actions. The actions of $F(X)$ on A are induced by the following actions of $F(X)$ on V. For $y \in F(X)$ and $(x, z) \in V$,

$$y * (x, z) = (ty * x, \alpha y * z),$$
$$y + (x, z) - y = (ty + x - ty, \alpha y + z - \alpha y).$$

One easily checks that these actions restrict to actions on I. Thus $F(X)$ acts on $A = V/I$. A computation shows that $T = \overline{F(X) \times A}$ is an object in $\mathcal{C}$. We note that this computation depends upon the substitution of axiom (8)′ for axiom (8). It is easy to see that A can be regarded as an ideal in T by identifying it with $0 \times A$.

Next, let $E = T/Z(T, A)$. We observe that $(x, a) \in Z(T, A)$ if and only if

$$(1.5)\qquad \begin{aligned} &-(a' * x) = a' * a, \\ &-x + a' + x = a + a' - a \end{aligned}$$

for all $a' \in A$ and $* \in \Omega_2'$. Let $\lambda : A \to E$ be given by $\lambda(a) = (0, a) + Z(T, A)$; then it is easy to check that

$$0 \to ZA \to A \xrightarrow{\lambda} E \xrightarrow{\pi} M \to 0$$

is exact. Here $M = T/(Z(T, A) + A)$.

It remains for us to define $\rho : R \to M$. To do this, we first define $\rho_0 : F(X) \to E$. Let

$$\rho_0(x) = (x, 0) + Z(T, A).$$

Given $r \in R$, there exists $y \in F(X)$ such that $r = \alpha(y)$. So let

$$\rho(r) = \pi\rho_0(y).$$

It remains to show that ρ is well-defined. Suppose $\alpha(y) = \alpha(y')$. We will show that for any $x \in F(X_2)$, $\rho_0(e^0 - e^1)x \in \lambda(A)$. Thus $\alpha(y - y') = 0$ implies $y - y' = e(x)$ for some $x \in F(X_2)$, and so

$$\pi\rho_0 y - \pi\rho_0 y' = \pi\rho_0 ex \in \pi\lambda A = 0.$$

Moreover, ρ is onto because $\rho\alpha = \pi\rho_0$, and $\pi\rho_0$ is onto since

$$\pi(x, a) + Z(T, A)) = \pi(x, 0) + Z(T, A))$$

for any $(x, a) \in T$.

We must now fill the gap in the above argument. Take any $x \in F(X_2)$. We will show that

$$\rho_0(e^0 - e^1)x = \lambda a,$$

where

$$a = (x - te^1 x, 0) + I.$$

To begin, we note that

$$\rho_0(e^0 - e^1)x - \lambda a = (e^0x - e^1x, -a) + Z(T, A)$$

and then verify that $(e^0x - e^1x, -a)$ satisfies (1.5). Let $a' = (x', z) + I$ be any element of A. We must show

$$(1.6) \qquad -(a' * (e^0x - e^1x)) = a' * (-a) \quad \text{for any } * \in \Omega'_2,$$

$$(1.7) \qquad -(e^0x - e^1x) + a' + (e^0x - e^1x) = -a + a' + a.$$

For (1.6), we use the element

$$y = (1 - t^0e^1)(t^0x' * t^1x)$$

which is in $F(X_3)$. Observe that

$$e^1y = 0,$$
$$e^2y = (e^2 - e^2t^0e^1)(t^0x' * t^1x) = 0,$$

since $e^2t^0x' = te^1x'$ and $e^1x' = 0$. Also,

$$e^0y = x' * (e^0t^1x - x).$$

Further,

$$py = 0.$$

since $pt^i = 0$. So

$$(e^0y, -py) = (x' * e^0t^1x - x' * x, 0)$$

is in I. Hence

$$\begin{aligned}
-(a' * (e^0x - e^1x)) &= (x' * (te^1x - te^0x), z * (\alpha e^1x - \alpha e^0x)) + I \\
&= [(x' * (te^1x - te^0x), 0) + I] + [(e^0y, -py) + I] \\
&= (x' * (te^1x - x), 0) + I \\
&= (x', z) * (te^1x - x, 0) + I \\
&= a' * (-a).
\end{aligned}$$

To verify (1.7), let

$$y = (1 - t^0e^1)(-t^1x + t^0x' + t^1x).$$

One easily checks that

$$e^1y = e^2y = py = 0,$$
$$e^0y = -te^0x + x' + te^0x - x - x' + x.$$

Therefore

$$(e^0y, -py) = (-te^0x + x' + te^0x - x - x' + x, 0)$$

is in I and since the actions of $F(X)$ on V restrict to actions on I,

$$\begin{aligned} 0 &= (e^1x + (e^0y, -py) - e^1x) + I \\ &= (te^1x - te^0x + x' + te^0x - x - x' + x - te^1x, 0) + I. \end{aligned}$$

That is,

$$(te^1x - te^0x + x' + te^0x - te^1x, z) + I = (te^1x - x + x' + x - te^1x, z) + I.$$

Hence

$$\begin{aligned} &-(e^0x - e^1x) + a' + (e^0x - e^1x) \\ &\quad = (-(te^0x - te^1x) + x' + (te^0x - te^1x), z) + I \\ &\quad = -(x - te^1x, 0) + (x', z) + (x - te^1x, 0) + I \\ &\quad = -a + a' + a. \end{aligned}$$

A more surprising fact about ρ is that it is an isomorphism. To show this we use the hypothesis that $Z(F(X), \ker \alpha) = 0$. Suppose $\rho r = 0$. Take $x \in F(X)$ such that $\alpha x = r$. Then

$$\pi\rho_0 x = \rho\alpha x = \rho r = 0.$$

Therefore there is an $a \in A$ such that $\lambda a = \rho_0 x$. Say

$$a = (y, z) + I.$$

We see that

$$\begin{aligned} (x, 0) + Z(T, A) &= \rho_0 x \\ &= \lambda a \\ &= (0, a) + Z(T, A), \end{aligned}$$

and so $(x, -a) \in Z(T, A)$. Thus,

$$a' * x = a' * a$$

for all a' in A and $*$ in Ω_2', and

$$-x + a' + x = -a + a' + a.$$

More precisely, for any $(y', z') + I \in A$,

$$(y' * tx, z' * x) + I = (y' * y, 0) + I$$

for all $* \in \Omega_2'$, and

$$(-tx + y' + tx, -x + z' + x) + I = (-y + y' + y, z') + I.$$

Therefore, for any $y' \in \ker e^1$, there exist $w, w' \in \ker e^1 \cap \ker e^2$ such that

$$y' * (tx - y) = e^0 w,$$
$$-tx + y' + tx - y - y' + y = e^0 w'.$$

Applying e^0, we obtain

$$e^0 y' * (x - e^0 y) = 0,$$
$$-x + e^0 y' + x - e^0 y - e^0 y' + e^0 y = 0$$

for all $y' \in \ker e^1$, and, indeed,

$$(x - e^0 y) + e^0 y' - (x - e^0 y) - e^0 y' = x - e^0 y + e^0 y' + e^0 y - x - e^0 y'$$
$$= x + (-x + e^0 y' + x) - x - e^0 y' = 0$$

for all $y' \in \ker e^1$. This means that $x - e^0 y$ belongs to $Z(F(X), \ker \alpha) = 0$. Thus $r = \alpha x = \alpha e^0 y = \alpha e^1 y = 0$.

Finally we must check that $[\rho] = \xi$. Define $\rho_1 : F(X_2) \to P = \overline{E \times A}$ by

$$\rho_1 x = (\rho_0 e^1 x, (-te^1 x + x, 0) + I).$$

Note that

$$d^0 \rho_1 x = \rho_0 e^1 x + \lambda((-te^1 x + x, 0) + I)$$
$$= \rho_0 e^1 x - (\rho_0 e^0 - \rho_0 e^1)(-x) = \rho_0 e^0 x$$

and

$$d^1 \rho_1 x = \rho_0 e^1 x.$$

Choose $\rho_2 : F(X_3) \to B$ so that the appropriate diagrams commute. It is easy to check that if $x \in F(X_3)$, then

$$\partial \rho_2 x = (0, (-te^1 ex + ex, 0) + I).$$

Let $y = (1 - t^0 e^1)(-t^1 e^2 + 1)x$. Then

$$\begin{aligned}
e^1 y &= 0, \\
e^2 y &= (e^2 - e^2 t^0 e^1)(-t^1 e^2 + 1)x \\
&= -e^2 x + e^2 x - t^0 e^1 e^1 x + t^0 e^1 e^2 x = 0, \\
e^0 y &= (e^0 - e^1)(-t^1 e^2 + 1)x \\
&= -e^0 t^1 e^2 x + e^0 x - e^1 x + e^1 t^1 e^2 x \\
&= -t^0 e^1 e^0 x + ex, \\
py &= px.
\end{aligned}$$

So $(e^0 y, -py) + I = 0$, and therefore

$$\begin{aligned}
(-te^1 e^0 x + ex, 0) + I &= (e^0 y, 0) + I \\
&= (0, py) + I \\
&= (0, px) + I.
\end{aligned}$$

That is,

$$\partial \rho_2 x = px.$$

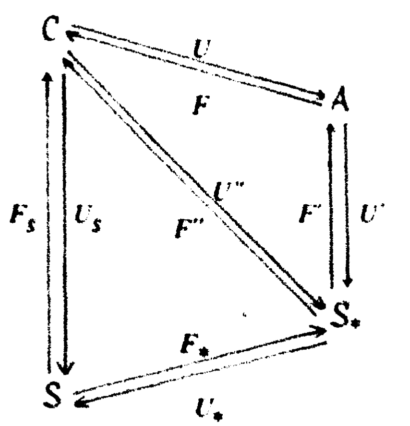
Therefore $[\rho] = [p] = \xi$.

§ 2. Relative theory

In his thesis [4], Beck showed that if $U : \mathcal{C} \to \mathcal{A}$ is tripleable, then the cohomology group $H^1(X, Y)$ relative to U classifies extensions which are split in $\mathcal{A}$ [1, p. 67]. In [13] and § 1 we showed that if $\mathcal{C}$ is a category of interest with $\mathcal{A} = S_*$, then $H^2(X, ZY)$ relative to U classifies obstructions to extensions of X by Y. In this section we turn to the case in which $\mathcal{A}$ is a category with more structure than S_*. We will show that if A is suitably restricted, then $H^2(X, ZY)$ relative to U classifies obstructions to A-split extensions.

The only case in which we have been able to apply the approach used by Barr in [1] and generalized here is when C and A are categories of interest in which + is commutative. Since there are now two categories of interest under consideration, we will use appropriate labels as needed. For example, ${}_{C}\Omega$ refers to the generating set of operations for C and ${}_{A}\Omega$ to the corresponding set for A. In addition to the assumption that + is commutative in both C and A, we assume that ${}_{A}\Omega'_2$ can be chosen empty.

For future reference we establish the following notation:



The cotriple **G** referred to below is the one associated with $G = FU$.

The category A is easily seen to be an abelian category. Its objects are abelian groups whose structure may be enriched by some unary operations. Since the object $Y = F_* F'(\{x\})$, where $\{x\}$ is just some one-point set, is a small projective generator for A, and A is cocomplete by [13, Remark 1.1] we know that A is equivalent to the category of modules over the ring $K = \mathrm{End}(Y)$ [12, p. 104].

Let A be an abject in C for which the inclusion $0 \to ZA \xrightarrow{\iota} A$ splits in A. By this we mean that there exists $\zeta\colon UA \to U(ZA)$ such that $\zeta(U\iota) = \mathrm{id}_{U(ZA)}$. For convenience, we write $\zeta\colon A \dashrightarrow ZA$.

Let $\overline{EA}$ consist of equivalence classes of exact sequences in C of the form

$$0 \to ZA \underset{\zeta}{\overset{\iota}{\leftrightarrows}} A \underset{\nu}{\overset{\lambda}{\leftrightarrows}} E \xrightarrow{\pi} M \to 0,$$

where there is an object T such that $A < T$, $E \simeq T/Z(T, A)$, $M \simeq T/(Z(T, A) + A)$, λ and π are the natural morphisms, and the following identities are satisfied:

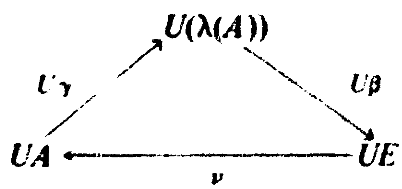
$$\zeta(U\iota) = \mathrm{id}_{U(ZA)},$$

$$(U\iota)\zeta + \nu(U\lambda) = \mathrm{id}_{UA}.$$

Using results of Manes, we see that there is an object in C which can reasonably be called $\lambda(A)$. Since all triples on S are regular [10, 2.1.2] and $\mathbf{T}_s$ is a triple on sets, we have that $T_s = U_s F_s$ preserves regular coimage factorizations [11, 2.7]. But $U^{\mathbf{T}_s}$ (with which we can replace U_s) creates regular coimage factorizations [11, 2.6] and regular coimage factorizations in S are just ordinary image factorizations [10, p. 74]. So, $U_s(\lambda(A))$ is just the set

$$\{x \in U_s(E) \mid x = U_s(\lambda)(a) \text{ for some } a \text{ in } U_s(A)\}$$

and associated with $\lambda(A)$ are surjection $\gamma: A \to \lambda(A)$ and an injection $\beta: \lambda(A) \to E$ such that $\beta\gamma = \lambda$. Thus, in A we have:



Since $U_* U' U\beta = U_s\beta$ is one-one and $U_* U' U\gamma = U_s\gamma$ is onto, we conclude that $U\beta$ is monic and $U\gamma$ is epic. Moreover,

$$\begin{aligned} U\lambda &= (U\lambda)\,\mathrm{id}_{UA} = (U\lambda)((U\iota)\zeta + \nu(U\lambda)) \\ &= U(\lambda\iota)\zeta + (U\lambda)\nu(U\lambda) = (U\lambda)\nu(U\lambda). \end{aligned}$$

That is,

$$(U\beta)(U\gamma) = (U\beta)(U\gamma)\nu(U\beta)(U\gamma);$$

and since $U\beta$ is monic and $U\gamma$ is epic,

$$\mathrm{id}_{U(\lambda(A))} = ((U\gamma)\nu)(U\beta).$$

Thus, in A, we have a split exact sequence:

$$0 \to U(\lambda(A)) \underset{(U\gamma)\nu}{\overset{U\beta}{\rightleftarrows}} UE \xrightarrow{U\pi} UM \to 0.$$

The objects of A, recall, may be viewed as modules over a ring K. In the usual way (see for example [9, p. 15]), we obtain a morphism $\mu: UM \to UE$ such that

$$\begin{aligned} (U\pi)\mu &= \mathrm{id}_{UM}, \\ (U\gamma)\nu\mu &= 0, \qquad (2.1) \\ UE &\simeq (U(\lambda(A)) \oplus UM. \end{aligned}$$

Since C itself is a category of interest, the object $P = \overline{E \times A}$ can be formed precisely as in [13, §4]. We can also form the sequence

$$B \overset{d^0,\,d^1,\,d^2}{\overrightarrow{\overrightarrow{\longrightarrow}}} P \overset{d^0,\,d^1}{\overrightarrow{\longrightarrow}} E \overset{\pi}{\to} M \to 0$$

in C and the derivation $\partial : B \to ZA$ as before.

In the light of the above discussion, if $e \in E$ (i.e., $e \in U_s(E)$), then, viewed as an element of UE, it can be written uniquely in the form $e = \gamma a + m$ for some a in A and m in M. Hence, in A, the morphisms $d^0, d^1 : P \to E$ are given by

$$(2.2) \qquad \begin{aligned} d^0(\gamma a + m, a') &= \gamma(a + a') + m, \\ d^1(\gamma a + m, a') &= \gamma a + m. \end{aligned}$$

As mentioned at the beginning of this section, we will show that $H^2(X, ZY)$ relative to U classifies obstructions to extensions of X by Y which are split in A. We find however than an arbitrary sequence

$$0 \to A \rightleftarrows T \rightleftarrows R \to 0$$

gives rise to a sequence

$$0 \to ZA \underset{\zeta}{\overset{\iota}{\rightleftarrows}} A \to T/Z(T, A) \to T/(Z(T, A) + A) \to 0,$$

which is the representative of an element of EA, but not necessarily of $\overline{\mathrm{E}A}$. It is easy to construct examples by noting that any class of sequences in EA, represented, say, by

$$0 \to ZA \to A \to E \to M \to 0$$

arises from a short exact sequence which is split in A, namely,

$$0 \to A \to P \to P/A \to 0$$

where $P = \overline{E \times A}$.

Although it is not the case that short exact sequences which are split in A give rise to sequences in $\overline{\mathrm{E}A}$, we can nevertheless ask whether a surjection $\rho : R \to M$ where

$$0 \to ZA \underset{\zeta}{\overset{\iota}{\rightleftarrows}} A \underset{\nu}{\overset{\lambda}{\rightleftarrows}} E \overset{\pi}{\to} M \to 0$$

represents a sequence in $\overline{\mathrm{E}A}$ is induced by a short exact sequence in C; whether we can classify such extensions if there are any; and whether indeed they are necessarily split in A.

Let

$$0 \leftarrow R \underset{w}{\overset{e}{\leftrightarrows}} X_1 \underset{t}{\overset{e^0,\, e^1}{\leftleftarrows}} X_2 \underset{t^0,\, t^1}{\overset{e^0,\, e^1,\, e^2}{\leftleftarrows}} X_3 \ \dots$$

be a G-projective resolution of R [3] with the special property that e is split in A. We will use the following fact:

Remark 2.3. If X is G-projective and $A \underset{t}{\overset{f}{\rightleftarrows}} B$ is a surjection in C which is split by t in A, then for any $g: X \to B$ there exists $g': X \to A$ such that $fg' = g$.

Proof. Since X is G-projective, there exists $s: X \to GX$ such that $\epsilon_X s = \mathrm{id}_X$. The morphism

$$g' = \epsilon_A\, F(t)G(g)s$$

has the desired property.

Our first task in developing an obstruction theory is to construct ρ_0, ρ_1, ρ_2 making the following diagram commutative:

$$\begin{array}{ccccccccc}
X_3 & \overset{e^0,\, e^1,\, e^2}{\Longrightarrow} & X_2 & \underset{e^1}{\overset{e^0}{\rightrightarrows}} & X_1 & \xrightarrow{e} & R & \to & 0 \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho} & & \\
B & \overset{d^0,\, d^1 d^2}{\Longrightarrow} & P & \underset{d^1}{\overset{d^0}{\rightrightarrows}} & E & \xrightarrow{\pi} & M & \to & 0
\end{array}$$

This can be done as in [1]. Recall the morphism $\mu: UM \to UE$ satisfying (2.1). By Remark 1.3, there exists ρ_0 making the following square commutative:

$$\begin{array}{ccccc}
X_1 & \xrightarrow{e} & R & & \\
\downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho} & & \\
E & \underset{\mu}{\overset{\pi}{\rightleftarrows}} & M & \to & 0
\end{array}$$

If $\widetilde{d}^0, \widetilde{d}^1: \widetilde{P} \to E$ is the kernel pair of π, then by the universal mapping property of $\widetilde{P}$, there exist $u: P \to \widetilde{P}$ such that $\widetilde{d}^0 u = \widetilde{d}^0$ and $\widetilde{d}^1 u = d^1$ and $\widetilde{\rho}_1: G^2R \to \widetilde{P}$ such that $d^i \widetilde{\rho}_1 = \rho_0 e^i$ for $i = 0$ and 1. In the present context we must show that u is split in A. With this we can use Remark 2.3 to conclude the existence of $\rho_1: X_1 \to P$ satisfying $u\rho_1 = \widetilde{\rho}_1$. But recall that in A, E (that is $U(E)$) can be represented as $\lambda(A) \oplus M$.

Thus,

$$\begin{aligned}\tilde{P} &= \{(e_1, e_2) \mid e_i \in E \text{ and } \pi e_1 = \pi e_2\} \\ &= \{(\gamma a_1 + m, \gamma a_2 + m) \mid a_1, a_2 \in A \text{ and } m \in M\}.\end{aligned}$$

Suppose $x = (\gamma a + m', a')$ belongs to $P = \overline{E \times A}$ and $ux = (\gamma a_1 + m, \gamma a_2 + m) \in \tilde{P}$. Then

$$\gamma a_1 + m = \tilde{d}^0 ux = d^0 x = \gamma(a + a') + m'.$$

By the uniqueness of representation in $E = \lambda(A) \oplus M$, we have

$$\begin{aligned}\gamma a_1 &= \gamma(a + a'), \\ m &= m'.\end{aligned}$$

Similarly,

$$\gamma a_2 + m = \tilde{d}^1 ux = d^1 x = \gamma a + m'$$

so that

$$\gamma a_2 = \gamma a$$

as well. Thus,

$$\begin{aligned}ux &= u(\gamma a + m', a') \\ &= (\gamma(a + a') + m', \gamma a + m').\end{aligned}$$

We seek an additive map $\nu: \tilde{P} \to P$ which also preserves the unary operations in ${}_A\Omega_1$ and such that

$$(Uu)\nu = \mathrm{id}_{U\tilde{P}}.$$

Notice that

$$0 \to ZA \underset{\zeta}{\overset{\iota}{\rightleftarrows}} A \xrightarrow{\gamma} \lambda(A) \to 0$$

is a short exact sequence that splits in $\mathbb{A}$ and that there is a morphism $\xi: U(\lambda(A)) \to UA$ in $\mathbb{A}$ such that $(U\gamma)\xi = \mathrm{id}_{U(\lambda(A))}$. Let

$$\nu(\gamma a_1 + m, \gamma a_2 + m) = (\gamma a_2 + m, \xi\gamma(a_1 - a_2)).$$

It is easy to check that υ preserves + and ω in ${}_A\Omega_1$. Moreover,

$$\begin{aligned} u\upsilon(\gamma a_1 + m, \gamma a_2 + m) &= u(\gamma a_2 + m, \xi\gamma(a_1 - a_2)) \\ &= (\gamma(a_2 + \xi\gamma(a_1 - a_2)) + m, \gamma a_2 + m) \\ &= ((\gamma a_2 + \gamma a_1 - \gamma a_2) + m, \gamma a_2 + m) \\ &= (\gamma a_1 + m, \gamma a_2 + m). \end{aligned}$$

ρ_2 exists as in [1]. This depends only on the universal property of B and not on the projectivity of X_3. The proof that the cohomology class of $\partial\rho_2$ in $\mathrm{Der}(X_3, ZA)$ does not depend on the choices of ρ_0, ρ_1 and ρ_2 is essentially the same as the proof of [1, Proposition 2.1]. It uses only universal mapping properties of certain objects in $\mathcal{C}$ and the existence of υ.

As usual we say that ρ is *obstructed* if the cohomology class of $\partial\rho_2$ is not 0 and *unobstructed* if it is 0.

Corresponding to [1, Theorem 2.2] and [13, Theorem 5.4] we have:

Theorem 2.4. *A surjection $\rho : R \to M$ arises from an extension which is split in $\mathbb{A}$ iff ρ is unobstructed.*

Proof. Suppose ρ arises from

$$0 \to A \underset{\chi}{\overset{\xi}{\rightleftarrows}} T \underset{\psi}{\overset{\sigma}{\rightleftarrows}} R \to 0.$$

As before, we have

$$\begin{array}{ccccccc}
0 \to K & \underset{u_0}{\overset{f^0, f^1}{\rightrightarrows}} & T & \underset{\psi}{\overset{\sigma}{\rightleftarrows}} & R & \to & 0 \\
\downarrow \nu_1 & & \downarrow \nu_0 & & \downarrow & & \\
B \rightrightarrows P & \underset{s^0}{\overset{d^0, d^1}{\rightrightarrows}} & E & \underset{\mu}{\overset{\pi}{\rightleftarrows}} & M & \to & 0
\end{array}$$

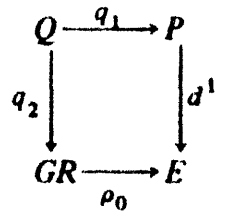
where $f^0, f^1 : K \rightrightarrows T$ is the kernel pair of $\sigma : T \to R$, u_0 is the diagonal map, and ν_0, ν_1 are the projections $\nu_0 : T \to E = T/Z(T, A)$ and $\nu_1 : K \to P = K/\Delta_Z$. Of course, ν_0 and ν_1 are surjections.

Since σ is split in $\mathbb{A}$, there exists $\sigma_0 : X_1 \to T$ such that $\sigma\sigma_0 = e$. Thus,

$$\pi\nu_0\sigma_0 = \rho\sigma\sigma_0 = \rho e.$$

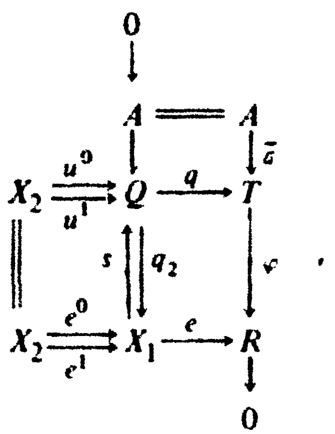
Let $\rho_0 = \nu_0\sigma_0$. The conclusion that $\partial\rho_2 = 0$ can be reached precisely as in [1].

Next, let ρ be unobstructed. As before we can assume $\partial\rho_2 = 0$. In the pullback diagram



q_2 is easily seen to be split in $\mathcal{C}$ since $\rho_0 = \rho_0\,\mathrm{id}_{X_1} = d^1(s^0\rho_0)$. Call the splitting map s. $q_2 s = \mathrm{id}_{X_1}$ implies that q_2 is surjective.

Consider the diagram:



We need only check that φ is split in $\mathcal{A}$. But,

$$U\varphi(Uq)(Us)w = (Ue)(Uq_2)(Us)w = (Ue)w = \mathrm{id}_{UR}.$$

The rest is the same as in [13, §5]. The constructions are carried out in $\mathcal{C}$, which is a category of interest and G is only used in a formal way. That is, the identities used are common to all cotriples.

Next we prove an analogue of [13, Theorem 6.1].

Theorem 2.5. *Let $\rho: R \to M$ be unobstructed. Let $\Sigma = \Sigma_\rho$ denote the equivalence classes of extensions*

$$0 \to A \rightleftarrows T \rightleftarrows R \to 0$$

which induce ρ. Then the group $H^1(R, ZA)$ (cohomology with respect to **G**) *acts on Σ_ρ as a principal homogeneous set.*

Proof. From [4], we know that $H^1(R, ZA)$ can be thought of as equivalence classes of extensions of R by ZA which are split in A and which induce the same module structure on ZA as that arising from ρ. Let $\mathbf{\Lambda}$ denote this set of equivalence classes.

The proof proceeds as in [1]. We note that there is already an addition given in $\mathbf{\Lambda} = H^1(R, ZA)$. The operation $\mathbf{\Lambda} \times \mathbf{\Sigma} \to \mathbf{\Sigma}$ denoted by $\Lambda + \Sigma$ is described as before. Let

$$0 \to ZA \rightleftarrows U \underset{\alpha}{\overset{\psi}{\rightleftarrows}} R \to 0$$

and

$$0 \to A \rightleftarrows T \underset{\beta}{\overset{\varphi}{\rightleftarrows}} R \to 0$$

represent classes in $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$, respectively. Assume $ZA < U$ and $A < T$ to simplify notation.

If $V = \{(t, u) | \varphi(t) = \psi(u)\}$ is the pullback of ψ and φ and $I = \{(z, -z) | z \in ZA\} < V$, then we claim that

$$0 \to A \xrightarrow{\zeta} V/I \xrightarrow{\varphi'} R \to 0,$$

with ζ and φ' given by

$$\zeta(a) = (a, 0) + I,$$

$$\varphi'((t, u) + I) = \varphi(t) = \psi(u),$$

is in $\mathbf{\Sigma}$. The splitting map β': $UR \to U(V/I)$ is given by

$$\beta'(r) = (\beta(r), \alpha(r)) + I.$$

It is easy to check that this is correct.

Similarly, we can show that $\mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbf{\Lambda}$ as defined in [1] works in the present setting. If $\Sigma_i \in \mathbf{\Sigma}_\rho$ are presented by

$$0 \to A \underset{\nu_i}{\overset{\iota}{\rightleftarrows}} T_i \underset{\beta_i}{\overset{\varphi_i}{\rightleftarrows}} R \to 0$$

for $i = 1, 2$, and $\tau_i : T_i \to T_i/Z(T_i, A) \simeq E$, then

$$0 \to ZA \xrightarrow{j} W/J \xrightarrow{\psi} R \to 0$$

is an extension where

$$W = \{(t_1, t_2) | \tau_1 t_1 = \tau_2 t_2 \text{ and } \varphi_1 t_1 = \varphi_2 t_2\},$$

$$J = \{(a, a) | a \in A\},$$

$$j(z) = (z, 0) + J,$$

$$\psi((t_1, t_2) + J) = \varphi_1(t_1) = \varphi_2(t_2).$$

We need only check that it is split in $\mathbb{A}$.

The following diagram is commutative for $i = 1$ and 2:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & A & \underset{\longleftarrow}{\overset{\iota_i}{\longrightarrow}} & T_i & \underset{\beta_i}{\overset{\varphi_i}{\rightleftarrows}} & R \\
 & & & & \Vert & & \downarrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \rho} \\
0 & \to & ZA & \rightleftarrows & A & \underset{\nu}{\overset{\lambda}{\rightleftarrows}} & E & \underset{\mu}{\overset{\pi}{\rightleftarrows}} & M \to 0
\end{array}$$

Therefore, $\pi\tau_i\beta_i = \rho\varphi_i\beta_i = \rho$ for $i = 1$ and 2, and so $(\tau_2\beta_2 - \tau_1\beta_1)(r) \in \lambda A$ for all $r \in U_s(R)$. For each $r \in U_s(R)$ choose an element a_r in $U_s(A)$ such that

$$(\tau_2\beta_2 - \tau_1\beta_1)(r) = \lambda(a_r).$$

Now define $\beta\colon R \to T$ by

$$\beta(r) = (\beta_1(r) + \nu\lambda(a_r), (\beta_2(r)) + J).$$

First we check that $\beta(r) \in T$.

$$\begin{aligned}\tau_1(\beta_1(r) + \nu\lambda(a_r)) &= \tau_1\beta_1(r) + \tau_1\nu\lambda(a_r) = \tau_1\beta_1(r) + \tau_1\iota_1\nu\lambda(a_r)\\ &= \tau_1\beta_1(r) + \lambda\nu\lambda(a_r) = \tau_1\beta_1(r) + \lambda(a_r)\\ &= \tau_1\beta_1(r) + \tau_2\beta_2(r) - \tau_1\beta_1(r) = \tau_2\beta_2(r),\end{aligned}$$

and

$$\varphi_1(\beta_1(r) + \nu\lambda(a_r)) = \varphi_1\beta_1(r) + \varphi_1\iota_1\nu\lambda(a_r) = r = \varphi_2\beta_2(r).$$

β is readily seen to be a morphism in $\mathbb{A}$.

Finally,

$$\varphi\beta(r) = \varphi(\beta_1(r) + \nu\lambda(a_r), \beta_2(r)) = \varphi_1(\beta_1(r) + \nu\lambda(a_r)) = \varphi_2\beta_2(r) = r.$$

So $\Sigma_1 - \Sigma_2 \in \mathbf{A}$.

The proof that

(a) $(\Lambda_1 + \Lambda_2) + \Sigma = \Lambda_1 + (\Lambda_2 + \Sigma)$,

(b) $(\Sigma_1 - \Sigma_2) + \Sigma_2 = \Sigma_1$,

(c) $(\Lambda + \Sigma) - \Sigma = \Lambda$

proceeds as in [13, §6]. It depends on nothing but the construction of several morphisms in C, and this is done without reference to $\mathbb{A}$ or $\mathbf{G}$.

It remains only to show that every element of $H^2(R, Z)$ is an obstruction. As in §1 we must restrict our attention to categories of interest that satisfy axiom (8)$'$.

For the last theorem it is convenient to use the standard resolution for computing cohomology. First we establish several useful facts.

Lemma 2.6. *For each $R \in \mathcal{C}$ and each n, there is a morphism $w^n : UG^nR \to UG^{n+1}R$ in $\mathcal{A}$ such that*

$$(U\epsilon^i)w^n = \begin{cases} w^{n-1}(U\epsilon^i) & \text{for } 0 \leq i \leq n-1, \\ \mathrm{id}_{UG^nR} & \text{for } i = n. \end{cases}$$

Proof. As mentioned before, a simplicial set that underlies an acyclic group complex satisfies the full box condition. We can use the standard resolution of R to construct an acyclic group complex in which we will apply this condition.

We proceed by induction on n. If $n = 0$, let $w^0 = \eta_{UR} : UR \to UGR$. Then

$$(U\epsilon^0)w^0 = (U\epsilon^0)\eta_{UR} = \mathrm{id}_{UR}$$

as desired. Now assume the lemma holds for $m \leq n$. We can form a group complex

$$\ldots Y_n \ldots \underset{s^0,\, s^1}{\overset{d^0,\, d^1,\, d^2}{\rightrightarrows}} Y_1 \overset{d^0,\, d^1}{\rightleftarrows} Y_0 \xrightarrow{d^0} Y_{-1}$$

by letting

$$Y_i = \mathcal{A}(UG^{n+1}R,\, UG^{i+1}R)$$

and $d^i : Y_m \to Y_{m-1}$ be given by

$$d^i(f) = (U\epsilon^i)f,$$

where $f: UG^{n+1}R \to UG^{m+1}R$ and $\epsilon^i: G^{m+1}R \to G^mR$ for $0 \leq i \leq m$. Since $\mathcal{A}$ is an abelian category, Y_i is an abelian group. The complex is readily seen to be acyclic. Next, let

$$y^j = \begin{cases} w_n\,(U\epsilon^j) & \text{for } 0 \leq j \leq n, \\ \mathrm{id}_{UG^{n+1}R} & \text{for } j = n+1. \end{cases}$$

These are elements of Y_n. If $i < j \leq n$, then we can show that

$$d^iy^j = d^{j-1}y^i.$$

If $j = n + 1$, then

$$d^j y^j = d^{j-1} y^i.$$

Therefore there is a $y \in Y_{n+1}$ such that $d^j y = y^j$ for $0 \leqslant j \leqslant n + 1$. Let $w_{n+1} = y$. If $0 \leqslant i \leqslant n$ then

$$(U\epsilon^i) w_{n+1} = d^i y = y^i = w_n(U\epsilon^i).$$

For $i = n + 1$,

$$(U\epsilon^i) w_{n+1} = d^{n+1} y = y^{n+1} = \mathrm{id}_{UG^{n+1}R}.$$

The next lemma concerns the standard resolution. Following [13, (3.5)], we note that

$$\epsilon_n x = \begin{cases} \sum_{j=0}^{n} (-1)^j (U\epsilon^{n-j}) x & \text{if } n \text{ is odd,} \\ \sum_{j=0}^{n} (-1)^j (U\epsilon^j) x & \text{if } n \text{ is even.} \end{cases}$$

We also remark that we are using the convention

$$\epsilon^i = G^{n-i} \epsilon_{G^i R}.$$

Lemma 2.7. *If* $\epsilon_n x = 0$, *then* $x \in \operatorname{im} \epsilon_{n+1}$.

Proof. Let $h_n = \eta_{UG^nR}$ where η is the unit associated with $F \dashv U$. Since η is a natural transformation,

$$(TU\epsilon^i)\, \eta_{UG^nR} = \eta_{UG^{n-1}R}\, (U\epsilon^i)$$

for $0 \leqslant i \leqslant n$. That is,

$$(U\epsilon^i) h_n = h_{n-1}(U\epsilon^i)$$

for $0 \leqslant i < n$, and

$$(U\epsilon^n) h_n = \mathrm{id}_{UG^nR}.$$

If n is odd and

$$0 = \epsilon_n x = \sum_{j=0}^{n} (-1)^j (U\epsilon^{n-j})x,$$

then

$$\begin{aligned}
\epsilon_{n+1} h_{n+1} x &= \sum_{j=0}^{n+1} (-1)^j (U\epsilon^j) h_{n+1} x \\
&= \sum_{j=0}^{n} (-1)^j (U\epsilon^j) h_{n+1} x + (U\epsilon^{n+1}) h_{n+1} x \\
&= \sum_{j=0}^{n} (-1)^j h_n (U\epsilon^j) x + x \\
&= h_n \sum_{j=0}^{n} (-1)^j (U\epsilon^j) x \quad + x \\
&= h_n \left(- \sum_{j=0}^{n} (-1)^j (U\epsilon^{n-j}) x \right) + x \\
&= h_n(0) + x \\
&= x.
\end{aligned}$$

Similarly if n is even. Note that we use the fact that $\mathcal{A}$-morphisms preserve + in the above computation.

Theorem 2.8. *Let R be an algebra in $\mathcal{C}$ and suppose $Z(G(R), \ker \epsilon) = 0$. Then given any R-module Z and class $\xi \in H^2(R, Z)$, there is an object A, a sequence*

$$0 \to ZA \rightleftarrows A \rightleftarrows E \to M \to 0$$

in $\overline{\mathcal{E}\mathcal{A}}$, and an isomorphism $\rho : R \to M$ such that $Z \simeq ZA$ as R-module (via ρ) and $[\rho] = \xi$.

Proof. The proof is very much like that of Theorem 1.2. A is defined as in that proof. Because + is commutative, some of the computations are much simplified in the present context.

Recall that, given

$$V = \{(x, z) \in G^2R \times Z \mid \epsilon^1 x = 0\},$$

$$I = \{(\epsilon^0 y, -py) \mid y \in G^3R \text{ and } \epsilon^1 y = \epsilon^2 y = 0\},$$

where $p: G^3R \to Z$ is a cocycle that represents ξ, then $I < V$; and, letting $A = V/I$, it can be shown that Z is embedded in A as ZA. We must also show that this embedding splits in the category $\mathcal{A}$. For this we use the morphism w_2 which is described in Lemma 2.6. Let $\zeta: UA \to UZ$ be given by

$$\zeta((x, z) + I) = pw_2x + z.$$

To see that ζ is well-defined we show that if $y \in \ker \epsilon^1 \cap \ker \epsilon^2 \subseteq G^3R$, then

$$pw_2\epsilon^0y - py = 0.$$

We first note that

$$\begin{aligned}\epsilon(w_2\epsilon^0y - y) &= (\epsilon^0 - \epsilon^1 + \epsilon^2)(w_2\epsilon^0y - y)\\ &= \epsilon^0\delta^2\epsilon^0y - \epsilon^0y + \epsilon^1y - \epsilon^1w_2\epsilon^0y + \epsilon^2w_2\epsilon^0y - \epsilon^2y\\ &= w_1\epsilon^0\epsilon^0y - \epsilon^0y - w_1\epsilon^1\epsilon^0y + \epsilon^0y\\ &= 0.\end{aligned}$$

Therefore there is a $v \in G^4R$ such that

$$\epsilon(v) = w_2\epsilon^0y - y,$$

and so

$$pw_2\epsilon^0y - py = p(w_2\epsilon^0y - y) = p\epsilon(v) = 0$$

since p is a cocycle. ζ is easily seen to be a morphism in $\mathcal{A}$ and it is easy to check that $\zeta(U\iota) = \mathrm{id}_Z$

$$\zeta(U\iota)(z) = \zeta((0, z) + I) = z.$$

The rest of the proof is exactly like the proof of Theorem 1. We must, however, show that the sequence

$$0 \to Z \underset{\zeta}{\overset{\iota}{\rightleftarrows}} A \xrightarrow{\lambda} E \xrightarrow{\pi} M \to 0$$

constructed there is actually a sequence in $\overline{\mathbf{E}\mathcal{A}}$. It suffices to construct $\mu: UM \to UE$ in $\mathcal{A}$ such that $(U\pi)\mu = \mathrm{id}_{UM}$. But, recall that ρ is an isomorphism in $\mathcal{C}$. We can therefore define μ by

$$\mu(m) = (U\rho_0)\eta_{UR}(U\rho^{-1})(m),$$

and check that

$$(U\pi)\mu(m) = (U\pi)(U\rho_0)\eta_{UR}(U\rho^{-1})(m)$$
$$= (U\rho)(U\epsilon^0)\eta_{UR}(U\rho^{-1})(m) = m.$$

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