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Fibonacci mean and golden section mean

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Abstract

In this paper we show that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean.

Keywords: (Inverse derived) Fibonacci mean; (Inverse) derived mean; Golden section mean

1. Introduction

In this paper we shall discuss a variety of "mean" $M(x, y)$ where $x \geq 0$, $y \geq 0$ and relations among them. Of particular interest is the family of "Fibonacci means"

$$M(x, y) = \frac{a(x + y) + 2bxy}{2a + b(x + y)}$$

where $M(x, x) = \frac{2ax+2bx^2}{2a+2bx} = x$ provided $2a + 2bx \neq 0$.

Particular cases are $a > 0$, $b = 0$, $M(x, y) = \frac{x+y}{2}$, the average (arithmetic mean), $a = 0$, $b > 0$, $M(x, y) = \frac{2xy}{x+y}$, the harmonic mean, and if $q = \frac{1+\sqrt{5}}{2}$, $M_q(x, y) = \frac{q(x+y)+2xy}{2q+(x+y)}$, the golden section mean. Hence the harmonic mean, the arithmetic mean and golden section mean are special cases of the Fibonacci mean.

We shall show below that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean. We shall refer to DM as the derived mean of M, e.g., the derived harmonic mean is the arithmetic mean. We refer to [1,2] for general information.

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2. Derived mean

Definition 2.1. If M is a mean, then DM is defined by the functional equation:

$$\frac{1}{1+DM(x,y)} = M\left(\frac{1}{1+x}, \frac{1}{1+y}\right). \quad (1)$$

We say that DM is the *derived mean* of M.

For example, if $M(x,y) = \sqrt{xy}$ is the geometric mean, then $DM(x,y) = \sqrt{(1+x)(1+y)} - 1$ is the derived geometric mean. Indeed, from the functional equation $\frac{1}{1+DM(x,y)} = \frac{1}{\sqrt{(1+x)(1+y)}}$, the conclusion follows immediately. Note that in this case $DM(x,y) \geq M(x,y)$. Indeed, if $x = m^2-1$, $y = n^2-1$, then $(\sqrt{xy})^2 = (m^2-1)(n^2-1) = m^2n^2+1-(m^2+n^2) \leq (\sqrt{m^2n^2}-1)^2 = m^2n^2+1-2mn$, since $m^2+n^2 \geq 2mn$, i.e., $(m-n)^2 \geq 0$.

Definition 2.2. Let M_i be means where $i = 1, 2$. We define $M_1 \leq M_2$ provided $M_1(x,y) \leq M_2(x,y)$ for any $x, y \geq 0$.

Proposition 2.3. *If $M_1 \geq M_2$, then $DM_1 \leq DM_2$.*

Proof. If $M_1(x,y) \geq M_2(x,y)$, then $\frac{1}{1+DM_1(x,y)} = M_1(\frac{1}{1+x}, \frac{1}{1+y}) \geq M_2(\frac{1}{1+x}, \frac{1}{1+y}) = \frac{1}{1+DM_2(x,y)}$ so that $1+DM_1(x,y) \leq 1+DM_2(x,y)$ and $DM_1(x,y) \leq DM_2(x,y)$. □

Hence if $D^2M(x,y) = D(DM(x,y))$, $M(x,y) = \sqrt{xy}$, then $D^2M(x,y) \leq DM(x,y)$ for example, where we find that $D^2M(x,y) = \frac{2\sqrt{(1+x)(1+y)}-\sqrt{(2+x)(2+y)}}{\sqrt{(2+x)(2+y)}-\sqrt{(1+x)(1+y)}}$, in this case.

Theorem 2.4. *The derived harmonic mean is the arithmetic mean.*

Proof. If $M(x,y) = \frac{2xy}{x+y}$, then $\frac{1}{1+DM(x,y)} = \frac{2}{2+x+y}$ and hence $1+DM(x,y) = (2+x+y)/2$, so that $DM(x,y) = \frac{x+y}{2}$. □

We note that if $M(x,y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$, then $DM(x,y) = \frac{(a+b)(x+y)+2axy}{2(a+b)+a(x+y)}$. Hence if $a = 0$ then $DM(x,y) = (x+y)/2$ as we have already seen. Starting from $M_0(x,y) = \frac{a_0(x+y)+2b_0xy}{2a_0+b_0(x+y)}$, we let $D^{k+1}M_0 = D(D^kM_0)$ and $D^kM_0(x,y) = \frac{a_k(x+y)+2b_kxy}{2a_k+b_k(x+y)}$ so that we obtain an iteration system $a_{k+1} = a_k + b_k$ and $b_{k+1} = a_k$. Combining these results, we have $a_{k+1} = a_k + a_{k-1}$, $b_{k+1} = a_{k-1} + a_{k-2} = b_k + b_{k-1}$, so that the iterations are of the Fibonacci type, thus our terminology "Fibonacci means".

3. Inverse derived mean

Definition 3.1. Given a mean $M(x,y)$, the *inverse derived mean* $D^{-1}M(x,y)$ is given by the formula:

$$\frac{1}{1+M(x,y)} = D^{-1}M\left(\frac{1}{1+x}, \frac{1}{1+y}\right). \quad (2)$$

If $M(x,y) = \sqrt{xy}$, then a computation yields $D^{-1}M(x,y) = \frac{\sqrt{xy}}{\sqrt{xy}+\sqrt{(1-x)(1-y)}}$.

Theorem 3.2. *Given a mean $M(x,y)$, $M(x,y) = DD^{-1}M(x,y) = D^{-1}DM(x,y)$ for any $x, y \geq 0$.*

Proof. By applying (1) and (2) we obtain

$$\begin{aligned}\frac{1}{1+DD^{-1}M(x,y)} &= D^{-1}M\left(\frac{1}{1+x}, \frac{1}{1+y}\right)\\ &= \frac{1}{1+M(x,y)}\end{aligned}$$

and $DD^{-1}M(x,y) = M(x,y)$. Also $\frac{1}{1+DM(u,v)} = D^{-1}DM(\frac{1}{1+u}, \frac{1}{1+v}) = M(\frac{1}{1+u}, \frac{1}{1+v})$ and thus $D^{-1}DM(x,y) = M(x,y)$, $x = \frac{1-u}{u}$, $y = \frac{1-v}{v}$, proving the theorem. □

By Theorem 3.2 we may continue with negative "powers" $D^{-k}M$, $k = 1, 2, \ldots$ so that for $k, l \in Z$, we have a general rule $D^k D^l M = D^{k+l} M$.

Theorem 3.3. *The inverse derived Fibonacci mean is also a Fibonacci mean.*

Proof. If $M(x, y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$, then $D^{-1}M(\frac{1}{1+x}, \frac{1}{1+y}) = \frac{2a+b(x+y)}{2a+(a+b)(x+y)+2bxy}$. If we let $u := \frac{1}{1+x}$, $v := \frac{1}{1+y}$, then

$$D^{-1}M(u, v) = \frac{b(u+v) + 2(a-b)uv}{2b + (a-b)(u+v)} \tag{3}$$

and the conclusion follows. □

If we start a table with $a_0 = 0$, $b_0 = 1$, $M_0(x, y) = \frac{2xy}{x+y}$, then we generate a table:

k	a_k	b_k
−4	−3	5
−3	2	−3
−2	−1	2
−1	1	−1
0	0	1
1	1	0
2	1	1
3	2	1
4	3	2
5	5	3
6	8	5
7	13	8
8	21	13

For example, since $a_{-3} = 2$, $b_{-3} = -3$, we have $D^{-3}M_0(x, y) = \frac{a_{-3}(x+y)+2b_{-3}xy}{2a_{-3}+b_{-3}(x+y)} = \frac{2(x+y)-6xy}{4-3(x+y)}$. Similarly, $D^6 M_0(x, y) = \frac{8(x+y)+10xy}{16+5(x+y)}$. In fact, if we let $M(x, y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$, then by (3) we get

$$D^{-1}M(x, y) = \frac{b(x+y) + 2(a-b)xy}{2b + (a-b)(x+y)}. \tag{4}$$

Subsequently

$$D^{-2}M(x, y) = \frac{(a-b)(x+y) + 2(2b-a)xy}{2(a-b) + (2b-a)(x+y)} \tag{5}$$

and

$$D^{-3}M(x, y) = \frac{(2b-a)(x+y) + 2(2a-3b)xy}{2(2b-a) + (2a-3b)(x+y)}. \tag{6}$$

If we let $a := 0$, $b := 1$ in (4)–(6), then we obtain $D^{-1}M(x, y) = \frac{x+y-2xy}{2-(x+y)}$, $D^{-2}M(x, y) = \frac{1(x+y)+4xy}{-2+2(x+y)}$, and $D^{-3}M(x, y) = \frac{2(x+y)+(-6)xy}{4+(-3)(x+y)} = D^{-3}M(x, y)$, as already noted above.

4. Fibonacci mean and golden section mean

The golden section mean $M_q(x, y)$ is defined by $M_q(x, y) = \frac{q(x+y)+2xy}{2q+(x+y)}$ where $q = \frac{1+\sqrt{5}}{2}$, and we define $M_{q^*}(x, y)$ by $\frac{q^*(x+y)+2xy}{2q^*+(x+y)}$ where $q^* = \frac{1-\sqrt{5}}{2}$, which is called a *conjugate golden section mean*.

Theorem 4.1. *Let* $M(x, y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$ *be a Fibonacci mean. If* $M(x, y) = DM(x, y)$, *then either* $M(x, y) = M_q(x, y)$ *or* $M(x, y) = M_{q^*}(x, y)$.

Proof. If $M(x, y) = DM(x, y)$, then

$$\frac{a(x+y)+2bxy}{2a+b(x+y)} = \frac{(a+b)(x+y)+2axy}{2(a+b)+a(x+y)}.$$

Hence $(a^2 - ab - b^2)(x-y)^2 = 0$ for any $x, y \geq 0$, and thus $a^2 - ab - b^2 = 0$ and $a = (b \pm \sqrt{5} \mid b|)/2$. If $b > 0$, then either $a = bq$ or $a = bq^*$. Hence either $M(x, y) = M_q(x, y)$ or $M(x, y) = M_{q^*}(x, y)$. If $b < 0$, then $a = (b \mp \sqrt{5} \mid b|)/2$, and hence this is the same as the case for $b > 0$. □

Proposition 4.2. *Let* $M(x, y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$ *be a Fibonacci mean. If* $M(x, y) = D^2M(x, y)$*, then either* $M(x, y) = M_q(x, y)$ *or* $M(x, y) = M_{q^*}(x, y)$.

Proof. Similar to Theorem 4.1. □

Theorem 4.3. *If* $M(x, y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$ *is a Fibonacci mean, then* $\lim_{n\to\infty} D^n M$ *is the golden section mean.*

Proof. Consider $D^k M_0(x, y) = \frac{a_k(x+y)+2b_kxy}{2a_k+b_k(x+y)}$. Then $b_k = a_{k-1}$ and $\{a_k\}$ and $\{b_k\}$ are Fibonacci sequences, and hence $a_{k-1}^2 - a_{k-2}a_k = (-1)^k$ and $b_{k-1}^2 - b_{k-2}b_k = (-1)^k$. From this we obtain

$$a_{k+2}b_k - b_{k+2}a_k = (-1)^k. \tag{7}$$

Furthermore,

$$\begin{aligned} D^{k+2}M_0(x, y) - D^k M_0(x, y) &= \frac{(b_k a_{k+2} - a_k b_{k+2})(x-y)^2}{(2a_{k+2} + b_{k+2}(x+y))(2a_k + b_k(x+y))} \\ &= \frac{(-1)^k (x-y)^2}{(2a_{k+2} + b_{k+2}(x+y))(2a_k + b_k(x+y))} \end{aligned}$$

so that for k even we have $D^{k+2}M_0(x, y) \geq D^k M_0(x, y)$, i.e., $M_0 \leq D^2M_0 \leq D^4M_0 \leq D^6M_0$, and for k odd we have $D^{k+2}M_0(x, y) \leq D^k M_0(x, y)$, so that we have a chain $DM_0 \geq DM_0^3 \geq D^5M_0 \geq \cdots$. Next, we consider the term $D^{2n+1}M_0(x, y) - D^{2m}M_0(x, y) = \triangle$ with the simplified expression written as

$$\triangle = \frac{(b_{2m}a_{2n+1} - a_{2m}b_{2n+1})(x-y)^2}{(2a_{2n+1} + b_{2n+1}(x+y))(2a_{2m} + b_{2m}(x+y))}.$$

Now,

$$\begin{aligned} b_{2m}a_{2n+1} - a_{2m}b_{2n+1} &= a_{2m-1}a_{2n+1} - a_{2m}a_{2n} \\ &= a_{2m-1}(a_{2n} + a_{2n-1}) - (a_{2m-1} + a_{2m-2})a_{2n} \\ &= a_{2m-1}a_{2n-1} - a_{2m-2}a_{2n} \\ &= (a_{2m-2} + a_{2m-3})a_{2n-1} - a_{2m-2}(a_{2n-1} + a_{2n-2}) \\ &= a_{2m-3}a_{2n-1} - a_{2m-2}a_{2n-2}, \end{aligned}$$

which turns out to be

$$\begin{cases} a_{2(m-n)-1}a_1 - a_{2(m-n)}a_0 = a_{2(m-n)-1} & \text{if } m > n \\ a_{-1}a_1 - a_0^2 = 1 & \text{if } m = n \\ a_{-1}a_{2(n-m)+1} - a_0a_{2(n-m)} = a_{2(n-m)+1} & \text{if } m < n. \end{cases}$$

Hence, $\triangle \geq 0$, i.e., $D^{2n+1}M_0(x, y) \geq D^{2m}M_0(x, y)$, for any natural numbers m, n.

For any $x, y \geq 0$ (fixed), we compute

$$\lim_{m,n\to\infty} |\triangle| \leq \lim_{m,n\to\infty} \left| \frac{(b_{2m}a_{2n+1} - a_2mb_{2n+1})}{b_{2n+1}b_{2m}} \right| \left(\frac{x-y}{x+y} \right)^2.$$

Now, since

$$\lim_{m,n\to\infty}\left|\frac{b_{2m}a_{2n+1}-a_{2m}b_{2n+1}}{b_{2n+1}b_{2m}}\right| = \begin{cases} \lim_{m,n\to\infty}\left|\frac{a_{2(m-n)-1}}{a_{2n}a_{2m-1}}\right| \leq \lim_{n\to\infty}\left|\frac{1}{a_{2n}}\right| = 0 & \text{if } m > n \\ \lim_{n\to\infty}\left|\frac{1}{a_{2n}a_{2n-1}}\right| & \text{if } m = n \\ \lim_{m,n\to\infty}\left|\frac{a_{2(n-m)+1}}{a_{2n}a_{2n-1}}\right| \leq \lim_{n\to\infty}\left|\frac{1}{a_{2n}}\right| = 0 & \text{if } m < n \end{cases}$$

we obtain $\lim_{m,n\to\infty} | \triangle | = 0$, i.e., $\lim_{k\to\infty} D^k M_0 = L$ exists. Notice that L is a Fibonacci mean, for which $DL = L$, whence it follows that $L(x, y) = \frac{q(x+y)+2xy}{2q+(x+y)}$ as asserted, provided we can show that $D^{2m}M_0(x, y) \leq L(x, y) \leq D^{2n+1}M_0(x, y)$ for all m, n. If $L(x, y) = \frac{r(x+y)+2xy}{2r+(x+y)}$, then $L(x, y) = DL(x, y) = \frac{(r+1)(x+y)+2rxy}{2(r+1)+r(x+y)}$ so that we must have $r^2 - r - 1 = 0$, $r = (1+\sqrt{5})/2 = q$ or $r = (1-\sqrt{5})/2 = q^*$. Now

$$\begin{aligned} D^k M_0(x, y) &= \frac{a_k(x+y)+2b_k xy}{2a_k + b_k(x+y)} \\ &= \frac{(x+y)+2(\frac{a_{k-1}}{a_k})xy}{2+(\frac{a_{k-1}}{a_k})(x+y)}. \end{aligned}$$

Next, consider $\lim_{k\to\infty}\frac{a_k}{a_{k-1}} = 1 + \lim_{k\to\infty}\frac{a_{k-2}}{a_{k-1}}$, i.e., $\lim_{k\to\infty}\frac{a_k}{a_{k-1}} = Q > 1$, means $Q = 1 + \frac{1}{Q}$, $Q^2 - Q - 1 = 0$, $Q = \frac{1+\sqrt{5}}{2} = q$, and thus $\lim_{k\to\infty} D^k M_0(x, y) = \frac{q(x+y)+2xy}{2q+(x+y)} = L(x, y)$ as asserted. Consequently, the inequalities $D^{2m}M_0(x, y) \leq L(x, y) \leq D^{2n+1}M_0(x, y)$ hold and the conclusion follows. □

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References

[1] K. Atanassove, et al., New Visual Perspectives on Fibonacci Numbers, World Sci. Pub. Co., NJ, 2002.
[2] R.A. Dunlap, The Golden Ratio and Fibonacci Numbers, World Sci. Pub. Co., NJ, 1997.