A Furuta-like inequality for spin orbifolds and the minimal genus problem

Daniel J. Acosta

Department of Mathematics, Southeastern Louisiana University, Hammond, LA 70402, USA

Received 28 August 1999

Abstract

We generalize Furuta’s 10/8ths inequality involving the index of the Dirac operator on a smooth spin 4-manifold to the setting of spin orbifolds. These spin orbifolds are obtained by coning off a tubular neighborhood of an embedded sphere representing a characteristic homology class in a smooth 4-manifold. This inequality can be used to obtain minimal genus bounds for characteristic classes in the original manifold. For example, for $X$ with positive definite intersection form of rank $n \geq 2$, we show a smoothly embedded characteristic sphere $\xi$ satisfies $\xi \cdot \xi \leq 9n - 16$. © 2001 Elsevier Science B.V. All rights reserved.

AMS classification: 57R15

Keywords: Furuta inequality; Dirac operator; Minimal Genus; Spin orbifold

1. Introduction

For a closed, connected, oriented, smooth spin 4-manifold $X$ with indefinite intersection form $Q$, one has $Q = \pm 2kE_8 \oplus mH$, with $E_8$ the unique negative definite, even, unimodular form of rank 8. The famous 11/8ths conjecture then states $m \geq 3k$. Although the conjecture has not been proven, Furuta provides a partial result, namely $m \geq 2k + 1$, or equivalently,

$$m \geq \frac{\left|\sigma(X)\right|}{8} + 1,$$

where $\sigma(X)$ is the signature of the manifold $X$ [5]. In this article, we follow a suggestion of Fintushel and adapt the proof of Furuta’s result to the case of spin orbifolds. The smooth 4-orbifolds considered here have the structure of a smooth 4-manifold except at finitely
many orbifold points which are cones on \( L(p, 1) \). These objects possess many of the features of smooth 4-manifolds and indirectly tell us information about the minimal genus problem for smooth 4-manifolds. Orbifolds of this type can be obtained by contracting a smoothly embedded 2-sphere. It is known that any class in \( H_2(X; \mathbb{Z}) \), where \( X \) is a compact oriented smooth 4-manifold, can be represented by a smoothly embedded orientable surface. Given a particular class \( \xi \), the minimal genus problem asks what is minimal genus of a surface representing \( \xi \). In particular, when is \( \xi \) represented by a surface of genus 0, that is, by a 2-sphere? For an account of the minimal genus problem, see Lawson’s excellent article [12].

We first state the results of this article, namely Theorems 1–3 and Corollaries 4–6. The remainder of the paper consists of a section on spin orbifolds and sections devoted to proofs.

**Theorem 1.** Let \( Y \) be a 4-dimensional spin orbifold with a finite number of non-manifold singularities which are cones on lens spaces. Then,

\[
\begin{align*}
\beta^+ (Y) &\geq \text{ind}(D) + 1 \quad \text{if } \text{ind}(D) > 0 \text{ and } b^+_2(Y) > 0; \\
\beta^- (Y) &\geq |\text{ind}(D)| + 1 \quad \text{if } \text{ind}(D) < 0 \text{ and } b^-_2(Y) > 0,
\end{align*}
\]

where \( \text{ind}(D) \) is the (complex) index of the Dirac operator of the orbifold.

Note, the proof of the theorem will require \( \text{ind}(D) \neq 0 \), although the result holds trivially in the case of equality.

For the following two theorems it is assumed that the orientation on \( X \) is chosen so that the signature is nonpositive. The reader can recover the analogous results for the positive signature case by reversing orientations.

**Theorem 2.** Suppose \( \xi \) is a characteristic homology class in an indefinite, smooth, oriented, 4-manifold \( X \) with nonpositive signature which is represented by an embedded 2-sphere. Then

\[
\begin{align*}
|\xi \cdot \xi - \sigma(X)| &\leq 8(b^+_2(X) - 2) \quad \text{if } \xi \cdot \xi \leq \sigma(X) \text{ and } b^+_2(X) > 1, \\
|\xi \cdot \xi - \sigma(X)| &\leq 8(b^-_2(X) - 2) \quad \text{if } 0 \leq \xi \cdot \xi \text{ and } b^-_2(X) > 1, \\
|\xi \cdot \xi - \sigma(X)| &\leq 8(b^+_2(X) - 1) \quad \text{if } \sigma(X) \leq \xi \cdot \xi \leq 0.
\end{align*}
\]

The above theorem is extended to characteristic surfaces of genus \( g \) by using a lemma of due to Yasuhara.

**Theorem 3.** Suppose \( \xi \) is a characteristic homology class in a smooth, oriented, simply connected 4-manifold \( X \) with nonpositive signature which is represented by an embedded surface of genus \( g \). If \( \xi \cdot \xi \equiv \sigma(X) \mod 16 \), then

\[
\begin{align*}
|\xi \cdot \xi - \sigma(X)| &\leq 8(g + b^+_2(X) - 2) \quad \text{if } \xi \cdot \xi \leq \sigma(X) \text{ and } g + b^-_2(X) > 1, \\
|\xi \cdot \xi - \sigma(X)| &\leq 8(g + b^+_2(X) - 2) \quad \text{if } 0 \leq \xi \cdot \xi \text{ and } g + b^-_2(X) > 1, \\
|\xi \cdot \xi - \sigma(X)| &\leq 8(g + b^+_2(X) - 1) \quad \text{if } \sigma(X) \leq \xi \cdot \xi \leq 0 \text{ and } g + b^+_2(X) \geq 1.
\end{align*}
\]
The following theorem has also been proven by Morgan and Szabo [14], and a slightly weaker version by Ruberman [16].

**Corollary 4.** Let $\xi$ be a 2-dimensional homology class in a smooth spin 4-manifold $X$ with $b_2^+ = 3$ and $b_2 \geq 8$. If $\xi \cdot \xi \geq 0$, then $\xi$ cannot be represented by an embedded sphere.

In particular, Corollary 4 shows that any smooth 4-manifold $X$ homeomorphic to $K3$ admits no spheres with nonnegative square.

Other applications of the above theorems include characteristic classes in definite and almost definite manifolds.

**Corollary 5.** Let $X$ be a smooth 4-manifold with positive definite intersection form of rank $n$. Then for a characteristic class $\xi \neq 0$ to be smoothly represented by a sphere we must have

$$\xi \cdot \xi \leq 9n - 16.$$ 

Furthermore, we construct examples of $(\xi, n)$ where equality is achieved for $n = 1, 2, 3, 4$.

Another example involves characteristic spheres in an almost definite manifold. Donaldson has shown there are no such spheres of positive square [2]. The following complements this result and gives a partial answer to a conjecture of Lawson [11].

**Corollary 6.** Suppose there is a characteristic sphere in $X$ with $b_2^+(X) = 1$ such that $\sigma(X) \leq \xi \cdot \xi$. Then, $\xi \cdot \xi = \sigma(X)$.

2. Spin orbifolds

2.1. Ruberman's construction

We now consider $Y$, a smooth 4-orbifold and ask what features such as intersection forms, spin structures, and Dirac operators carry over to $Y$. Following Ruberman [16], objects on $Y$ will be defined as $\mathbb{Z}/p$ equivariant objects on $B^4$ near a singular point $y$, and by their usual definitions on $Y - \{\text{singular points}\}$. For example, an $SO(4)$ bundle on $Y$ with only one singular point $y$ is a $\mathbb{Z}/p$ equivariant $SO(4)$ bundle on $B^4$ (which is described completely by the action of $\mathbb{Z}/p$ on the fiber over 0) and an $SO(4)$ bundle over $Y - \{y\}$ (together with an identification of the two bundles where they overlap). We can thus discuss smooth sections of bundles, and therefore connections and curvature of bundles over such orbifolds $Y$.

Recall that a spin structure on a manifold $M$ is by definition a double covering $\tilde{P} \rightarrow P$ of the oriented frame bundle $P$ of $M$ whose restriction to each fiber is the standard double covering $Spin(4) \rightarrow SO(4)$. If the group $\mathbb{Z}/p$ acts on $M$ by isometries, then it also acts on $P$. By definition, a $\mathbb{Z}/p$-equivariant spin structure on $M$ is a lift of this action to a $\mathbb{Z}/p$ action on $\tilde{P}$. Not all actions lift to actions on $\tilde{P}$ compatible with the $Spin(4)$ action,
and those that do are called spin actions. If \( \mathbb{Z}/p \) acts freely on \( M \), then equivariant spin structures correspond one-to-one with spin structures on the quotient manifold. (We will apply this when \( M \) is a Lens space \( L \).) Ruberman then defines a spin structure on the cone \( C(L) \) as a \( \mathbb{Z}/p \)-equivariant spin structure on \( B^4 \) and furthermore shows existence of this equivariant structure by giving an explicit lifting of a \( \mathbb{Z}/p \) subgroup of \( SO(4) \) to a \( \mathbb{Z}/p \) subgroup of \( Spin(4) \).

**Ruberman Lemma A.5.** Let the group \( \mathbb{Z}/p \) act linearly on \( B^4 \) so that its restriction to the boundary is free. Then there is a \( \mathbb{Z}/p \)-equivariant spin structure on \( B^4 \).

A spin structure on an orbifold with isolated singularities consists of a spin structure on the manifold with boundary \( Y \setminus \{ \text{singular set} \} \), together with an extension of the induced spin structure on the boundary over each of the cone points. Again Ruberman demonstrates the existence of such a structure.

**Ruberman Theorem A.2.** Any spin structure on a \((4n-1)\)-dimensional lens space \( L \) extends over the cone on \( L \).

Still following Ruberman, we now create examples of smooth spin 4-orbifolds. One way to do this is by “blowing down” a collection of 2-spheres in a spin manifold \( X \). It suffices to consider only one 2-sphere to describe this process since it occurs locally. A smoothly embedded 2-sphere representing a class \( \xi \in H_2(X; \mathbb{Z})/\text{torsion} \) with \( \xi \cdot \xi = p \) will have a tubular neighborhood \( N(\xi) \) with boundary \( L = L(p; 1) \). If we remove \( N(\xi) \) and cone off the resulting boundary lens space, we have created an orbifold. Since \( X \) was spin, \( X \setminus N(\xi) \) possesses a spin structure by restriction. In particular, \( \partial(X \setminus N(\xi)) \simeq L \) has an induced spin structure which now extends over the cone by Ruberman’s Theorem A.2.

A second way to arrive at a smooth spin 4-orbifold is to blow down a characteristic sphere in a nonspin smooth 4-manifold \( X \). Such a sphere represents a class \( \xi \in H_2(X) \), and because it is characteristic we also have \( \xi \cdot X \equiv X \cdot \xi \pmod{2} \) for all \( X \in H_2(X; \mathbb{Z})/\text{torsion} \). Because \( \xi \cdot \xi = p \) we can again remove \( N(\xi) \) leaving \( X \setminus N(\xi) \) with boundary \( L \). \( X \setminus N(\xi) \) has a spin structure because we have in effect “killed” the class \( \xi \) which is characteristic and thus the obstruction to obtaining a spin structure.

This second spin structure restricts to a spin structure on the boundary \( L \) which in turn extends over the cone \( C(L) \). A similar technique was used by Kervaire and Milnor in 1961 to show that any characteristic sphere in a spin manifold must have square congruent modulo 16 to the signature. Here, however, the boundary of \( X \setminus N(\xi) \) was a 3-sphere rather than a lens space, yielding another manifold rather than an orbifold when this boundary was capped off by a 4-disk [9]. Both methods result in smooth spin 4-orbifolds which have Dirac operators.

**Ruberman Corollary A.3.** Any 4-dimensional orbifold gotten by blowing down a collection of 2-spheres in a spin manifold has a Dirac operator. If the manifold is not spin, but some collection of spheres is characteristic then the same conclusion applies.
2.2. Index formulas

We now use the work of Kawasaki who in a series of papers in the late 1970s and early 1980s generalized the Atiyah–Singer formulas to the case of orbifolds [7,8]. In those papers, Kawasaki generalizes the notions of bundles, sections, and operators to the category of orbifolds and he proves an index theorem. His formula gives the index of an operator in terms of characteristic numbers (as in the usual index formula [17]) plus a term determined by local data at the orbifold points. For 4-dimensional orbifolds with one orbifold point modeled on a cone on $L(p, \pm 1)$, the index of the signature operator and the index of the Dirac operator are given by:

$$
\sigma(Y) = \int_Y p_1(Y)^3 - \sum_{k=1}^{p-1} \frac{\cot^2 \left( \frac{\pi k}{p} \right)}{p},
$$

$$
\text{ind}(D) = \int_Y -\frac{p_1(Y)}{24} + \sum_{k=1}^{p-1} \frac{\varepsilon_x \csc^2 \left( \frac{\pi k}{p} \right)}{4p}.
$$

The term $p_1(Y)$ refers to the first Pontryagin class of $Y$. We think of the cone on $L(p, \pm 1)$ as gluing in $D_p \simeq B^4/(\mathbb{Z}/p)$ to the boundary lens space. The $\mp$ for the formulas indicate whether this gluing is done in an orientation preserving or orientation reversing manner. The $\varepsilon_x$ is either $\pm 1$ and in general is determined by the action of $\mathbb{Z}/p$ on the spin structure of $X$. It is only through these signs that the choice of a lifting of the action of $\mathbb{Z}/p$ on $P$ to an action on $\tilde{P}$ (if indeed there is a choice) enters into the formula for the local data. A given lifting of the action determines one sign; the opposite lifting determines the opposite set of signs.

2.3. Spin structures over lens spaces

Because any 3-dimensional orientable manifold is parallelizable, $w_2(L) = 0$ for any lens space. In particular, $L = L(p, 1)$ admits a spin structure. In fact, $L$ admits either one or two spin structures, parameterized by:

$$
H^1(L(p, 1); \mathbb{Z}/2) = \begin{cases} 
0, & \text{if } p \text{ is odd;} \\
\mathbb{Z}/2, & \text{if } p \text{ is even.}
\end{cases}
$$

Ruberman explicitly gives the two spin structures on $L$ when $p$ is even [16].

- **Standard spin structure:**
  - $\tilde{P}_1 \simeq \{S^3 \times \text{Spin}(4)\}/\mathbb{Z}/p$
  - $(x, y, z) \simeq \left( x \xi^{-1}, \xi y, z \right)$

- **Nonstandard spin structure:**
  - $\tilde{P}_2 \simeq \{S^3 \times \text{Spin}(4)\}/\mathbb{Z}/p$
  - $(x, y, z) \simeq \left( x \xi^{-1}, -\xi y, -z \right)$

Here, $\xi = e^{2\pi i/p} \in S^3$ generates $\mathbb{Z}/p \subset S^3$. This second spin structure arises by twisting $\tilde{P}_1$ by a line bundle and thus changing the $\mathbb{Z}/2$ cohomology class of the spin structure.
**Lemma 7.** Let $p$ be even. Only the standard spin structure extends over the neighborhood $N_p$. In other words, only $\tilde{P}_1$, which is a spin structure over the sphere bundle of $S^2$, extends over the disk bundle.

**Proof.** Because $N_p$ deformation retracts onto $S^2$ it suffices to look at bundles over $S^2$. $TX|_{S^2} \cong TS^2 \oplus v_p(S^2)$, that is, the tangent bundle of the manifold $X$ (or in fact $TN_p$) restricted to our embedded sphere splits as the Whitney sum of the tangent bundle to $S^2$ and the normal bundle with Euler number $p = 2k$. Now $w_2$ of this bundle over $S^2$ is just $w_2(S^2) + w_2(v_p) = 0 + 0 = 0$ and thus this bundle admits a unique spin structure. (Note if $p$ was odd, $w_2(v_p) = 1$ and no spin structure would exist for this bundle.) Our frame bundle is classified by the Euler numbers $(2, 2k)$, and is therefore just the trivial bundle since $(2, 2k)$ maps to $2 + 2k = 0 \bmod (2)$ in $\mathbb{Z}_2 \cong \pi_1(SO(4))$. Thus, the frame bundle is simply $S^2 \times SO(4)$ and is covered by $S^2 \times Spin(4)$. The spin structure over $L$ which extends is then the product structure (recall $L$ is parallelizable and the normal line bundle of $\partial N \subset N$ is always trivial.) We get the following diagram.

$$
\begin{array}{c}
L \times Spin(4) \rightarrow S^2 \times Spin(4) \\
L \times SO(4) \rightarrow S^2 \times SO(4)
\end{array}
$$

We now show $\tilde{P}_1 \cong L \times Spin(4)$ as spin structures. Define a map from $\tilde{P}_1$ to $L \times Spin(4)$ by

$$(x, y, z) \mapsto ([x], xy, z).$$

Since $(x, y, z) \sim (x\xi^{-1}, \xi y, z)$ which maps to $([x\xi^{-1}], x\xi^{-1}\xi y, z) = ([x], y, z)$, this map is well defined. It is also one-to-one: if $(x, y, z)$ and $(a, b, c)$ map to the same image, i.e., $([x], xy, z) = ([a], ab, c)$ then $z = c$. $[x] = [a] \Rightarrow a = x\xi^t$, and with $xy = ab$ we get $xy = x\xi^t b \Rightarrow y\xi^{-t} = b$. In other words $(x, y, z) \sim (a, b, c)$ and our map is injective. This suffices because our map is fiber preserving and linear. $\square$

Notice we also understand the specific correspondence between index 2 subgroups of $\pi_1(P)$ and spin structures. (Here $P$ is the frame bundle $L \times SO(4)$.) The above commutative diagram induces the following diagram on the fundamental group level:

$$
\begin{array}{c}
\mathbb{Z}/p \rightarrow 0 \\
\mathbb{Z}/p \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2
\end{array}
$$

Thus, $\tilde{P}_1 \leftrightarrow \mathbb{Z}/p \subset \mathbb{Z}/p \oplus \mathbb{Z}/2$ generated by $(1, 0)$, and $\tilde{P}_2 \leftrightarrow \mathbb{Z}/p \subset \mathbb{Z}/p \oplus \mathbb{Z}/2$ generated by $(1, 1)$. These subgroups are distinguished since every element in the latter subgroup has coordinates that sum to even integers.
3. Proofs

We are now ready to prove Theorems 2 and 3, and Corollaries 4–6, assuming Theorem 1 whose proof is found in the next section.

Proof of Theorem 2. Assume $X$ is not spin. Without loss of generality we assume $\xi$ has negative square, $-\rho$. By possibly changing the orientation on $X$ we can always achieve this. The neighborhood of this characteristic sphere now has boundary $L(p; -1)$ which we cone off to achieve a spin orbifold $Y$. The nonstandard spin structure $\tilde{\mathcal{P}}_2$ is used in this case since $X$ is not spin (we have the structure that cannot extend over the disk bundle—see Section 2.3. We apply the index theorem for orbifolds and get the following.

$$\sigma(Y) = \sigma(X) + 1 = \int_Y p_1(Y) - \frac{1}{p} \sum_{k=1}^{p-1} \cot^2 \left( \frac{\pi k}{p} \right),$$

$$\text{ind}(D) = \int_Y -\frac{p_1(Y)}{24} - \frac{1}{4p} \sum_{k=1}^{p-1} \varepsilon_k \csc^2 \left( \frac{\pi k}{p} \right).$$

Here notice we have $-\sum$ in the first formula and $-\sum$ in the second formula as $D_p$ is glued onto the boundary lens space $L(p; -1)$ in an orientation preserving manner. The angles of rotation are $2\pi/p$ and $2\pi/p$. To decide on the spin numbers $\varepsilon_k$, which are intimately linked to the lifting of the action on $P$ to a spin action on $\tilde{P}$, it suffices to know $\varepsilon_1$ as this number will generate the rest. Now $\varepsilon_1$ is $\pm 1$ and corresponds to the lifting of $g$ where $g$ generates $\mathbb{Z}/p$. Carrying the following calculation through with both possibilities yields an integer for the index only in the case of 1. The spin numbers are therefore $\varepsilon_k = (1)^k = 1$. When we use the first formula to solve for $\int_Y p_1$ and substitute into the second formula, we get

$$\text{ind}(D) = -\frac{1}{8} \left\{ \sigma(Y) + \frac{1}{p} \left[ \sum_{k=1}^{p-1} \cot^2 \left( \frac{\pi k}{p} \right) + 2 \sum_{k=1}^{p-1} (-1)^k \csc^2 \left( \frac{\pi k}{p} \right) \right] \right\}$$

$$= -\frac{1}{8} \left\{ \sigma(Y) + \frac{1}{p} \left[ \sum_{k=1}^{p-1} \left( 3 \csc^2 \left( \frac{\pi k}{p} \right) - 1 \right) \right] \right\}$$

$$= -\frac{1}{8} \left\{ \sigma(Y) + \frac{1}{p} \left[ (p-1)(p+1) - (p-1) \right] \right\}$$

$$= -\frac{1}{8} \left\{ \sigma(X) + 1 + (p-1) \right\}$$

$$= \frac{1}{8} (-p - \sigma(X)).$$

Here we used the substitution:

$$\sum_{k=1}^{p-1} \csc^2 \left( \frac{\pi k}{p} \right) = \frac{(p-1)(p+1)}{3}.$$

This identity can be found in Hirzebruch and Zagier and is related to Dedekind sums [6].
Proof for (1). The signature $\sigma(X)$ is negative, the square $\xi \cdot \xi$ is negative, and the index is negative since the term $\xi \cdot \xi - \sigma(X)$ is negative. Applying Theorem 1 then gives,

$$b_2^-(Y) \geq \frac{|\xi \cdot \xi - \sigma(X)|}{8} + 1.$$  

Note that we use the assumption $b_2^-(X) > 1$ to guarantee the orbifold $Y$ is still indefinite.

$$\frac{|\xi \cdot \xi - \sigma(X)|}{8} \leq 8(b_2^-(Y) - 1),$$

$$\frac{|\xi \cdot \xi - \sigma(X)|}{8} \leq 8(b_2^-(X) - 2).$$

The proofs for (2) and (3) repeat the argument above, eventually utilizing the appropriate case from Theorem 1.

For the case when $X$ is spin we first stabilize by internal connected sum with $\mathbb{C}P^1$ in $\mathbb{C}P^2$ or $\mathbb{C}P^2$ to create a new class with nonzero square in a nonspin manifold. Now we just apply the appropriate case above and the same conclusions hold. $\blacksquare$

Proof of Theorem 3. The proof of this theorem follows from the above result (Theorem 2) and a Connecting Lemma of Yasuhara [18].

Yasuhara Connecting Lemma III. Let $X$ be a closed simply connected 4-manifold and $F$ an embedded, closed, orientable surface in $X$ that represents a characteristic homology class. If $\text{Arf}(F) = 0$, i.e., $[F] \cdot [F] \equiv \sigma(X) \pmod{16}$, then there exists an embedded, closed, orientable surface $F_1$ in $M\#S^2 \times S^2$ such that $[F_1]$ is a characteristic homology class, $\text{Arf}([F_1]) = 0$, $[F_1] \cdot [F_1] = [F] \cdot [F]$, and $\text{genus}(F_1) = \text{genus}(F) - 1$.

In his article, Yasuhara uses the hypothesis on the Arf invariant to find an embedded essential loop $C$ in $F$ which bounds a 2-disk in $X$ which is transverse to $F$. A neighborhood $D^4 \cong D^2 \times D^2$ of this disk will then intersect $F$ in one annulus and some 2-disks, the boundary of this intersection consisting of a link $L$ whose diagram is easily drawn. This link acts as a model in the following way. In the punctured manifold $\text{punc}(S^2 \times S^2)$ one can find a collection of mutually disjoint 2-disks that can be connected by strips, the union representing the class $0\alpha + 2m\beta$ ($m \in \mathbb{Z}$) where $\alpha$ and $\beta$ generate the relative homology group

$$H_2(\text{punc}(S^2 \times S^2), \delta \left[ \text{punc}(S^2 \times S^2) \right]; \mathbb{Z}) \cong H_2(S^2 \times S^2; \mathbb{Z}).$$

Here $\text{Arf}(C) = 0$ is used to guarantee this surface with boundary represents an even multiple of a generator and thus a characteristic class in the punctured product. The boundary of this surface is precisely $L$. We cap off the manifold $X$ minus the neighborhood of the 2-disk with the punctured $S^2 \times S^2$, producing a new closed orientable surface $F_1$ in $X\#S^2 \times S^2$ such that the new class represented by $F_1$ is still characteristic and still has the same square as the old class represented by $F$. Likewise the signature is unchanged. However, the essential loop $C$ in $F$ now bounds in $F_1$, thus reducing the genus by 1.

We simply apply this theorem $g$ times, where $g$ is the genus of our embedded surface representing $\xi$, resulting in a class represented by a sphere. Note that although $b_2^-$ and
\( b_2 \) both increase by one after each stabilization, the square of the homology class and the signature of the manifold remain unchanged. Hence applying the inequality obtained in Theorem 2 to the stabilized class represented by the sphere gives us our desired inequality concerning the original class. 

\[ \square \]

**Proof of Corollary 4.** By way of contradiction suppose \( \xi \) is represented by an embedded sphere.

**Case 1.** \( \xi \cdot \xi = p > 0 \). In this case the neighborhood \( N(\xi) \) has boundary \( L(p; 1) \). We remove \( N(\xi) \) and cone off the boundary to achieve an orbifold. This orbifold is spin, and moreover the spin structure on the lens space is given by \( \tilde{P}_1 \), the structure that extends over the disk bundle since \( X \) is spin. The index is computed to be \( -\sigma(X)/8 \) (see below) which is therefore positive since the signature is negative under our hypotheses. We now apply Theorem 1.

\[
 b_2^+(Y) \geq \text{ind}(D) + 1, \\
 2 \geq -\frac{\sigma(X)}{8} + 1 = \frac{\sigma(X)}{8} + 1.
\]

This follows because \( b_2^+(Y) = b_2^+(X) - 1 = 2 \), as we are killing a class of positive square in \( X \). But \( \sigma(X)/8 \) is even by Rokhlin’s Theorem [15] and we thus have a contradiction.

**Case 2.** \( \xi \cdot \xi = 0 \). Ruberman proves this result but again requires no 2-torsion in \( H_1(X; \mathbb{Z}) \). Furthermore he requires the class \( \xi \) to be of odd divisibility. His argument is as follows. Any 2-sphere representing the class \( \xi \) with square zero has trivial normal bundle and thus may be removed by surgery. This creates a new spin manifold \( X' \) with \( b_2^+ = 2 \) and \( b_2 \geq 6 \). This follows because we are losing a class of positive square and a class of negative square when we kill \( \xi \) which has square zero. Now if this class is of odd divisibility, then \( H_1 \) of the new manifold will have no 2-torsion. Such a manifold violates Donaldson’s Theorem C [3]. To prove this case without the above restrictions we simply replace Donaldson’s Theorem C with Furuta’s Theorem. The manifold \( X' \) created by the same surgery technique is spin, indefinite, and thus has intersection form \( Q_{X'} \cong 2kE_8 \oplus mH \) with \( m = 2 \) and \( k \geq 1 \). Such a manifold now violates Furuta’s Theorem which says \( m \geq 2k + 1 \). Of course Furuta’s Theorem requires no hypotheses concerning torsion or divisibility.

Now we perform the index calculation.

**Claim.** \( \text{ind}(D) = -\sigma(X)/8 \).

Using spin structure \( \tilde{P}_1 \) and the index formulas for signature(\( Y \)) and Dirac(\( Y \)) we have the following:

\[
 \sigma(Y) = \sigma(X) - 1 = \int_Y \frac{p_1(Y)}{3} + \frac{1}{p} \sum_{k=1}^{p-1} \cot^2 \left( \frac{\pi k}{p} \right), \\
 \text{ind}(D) = \int_Y \frac{p_1(Y)}{24} + \frac{1}{4p} \sum_{k=1}^{p-1} \varepsilon_k \csc^2 \left( \frac{\pi k}{p} \right).
\]
The +\(\sum\) in the first formula and the +\(\sum\) in the second formula reflect that \(D_p\) is glued onto the boundary lens space \(L(p; 1)\) in an orientation reversing manner. The angles of rotation are \(2\pi/p\) and \(2\pi/p\). Reviewing the spin action of \(\tilde{P}_2\) we see that we’ve twisted the standard spin structure, the actions agreeing on even powers of the generator only. This has the effect of changing the sign of each odd \(\varepsilon_k\) term in the index calculation for the standard spin structure. Thus we have the spin numbers \(\varepsilon_k = (-1)^{k}\) for the standard spin structure. When we use the first formula to solve for \(\int_Y p_1\) and substitute into the second formula, we get

\[
\text{ind}(D) = -\frac{1}{8} \left\{ \sigma(Y) - \frac{1}{p} \left[ \sum_{k=1}^{p-1} \cot^2 \left( \frac{\pi k}{p} \right) + 2 \sum_{k=1}^{p-1} (-1)^k \csc^2 \left( \frac{\pi k}{p} \right) \right] \right\} \\
= -\frac{1}{8} \left\{ \frac{\sigma(Y)}{p} - \frac{1}{3} \left[ \frac{p^2 - 3p + 2}{3} \right] \right\} \\
= -\frac{1}{8} \left\{ \frac{\sigma(X)}{1} - 1 + 1 \right\} \\
= -\frac{\sigma(X)}{8}. \quad \square
\]

In particular, Corollary 4 shows any smooth 4-manifold \(X\) homeomorphic to \(K^3\) admits no spheres with nonnegative square.

**Proof of Corollary 5.** In the case \(0 \leq \sigma(X) \leq \xi \cdot \xi\) the analogue of Theorem 1(1) for the positive signature scenario gives the following restrictions on characteristic spheres when \(n \geq 2\):

\[
|\xi \cdot \xi - \sigma(X)| \leq 8(b^+_2(X) - 2), \\
|\xi \cdot \xi - n| \leq 8(n - 2), \\
|\xi \cdot \xi| \leq 8n - 16.
\]

For a concrete example, let \(X\) be a connect sum of \(n\) copies of \(\mathbb{C}P^2\).

\[
n = 2 \iff \xi \cdot \xi \leq 2, \\
n = 3 \iff \xi \cdot \xi \leq 11, \\
n = 4 \iff \xi \cdot \xi \leq 20, \\
n = 5 \iff \xi \cdot \xi \leq 29.
\]

In each of these cases the characteristic class given by the sum of generators is represented by a sphere. Also, certain characteristic classes are seen to be nonrepresentable due to Rokhlin’s Theorem [15]. For example, the classes \((3, 1)\) and \((3, 1, 1)\) are not represented by spheres as their squares are congruent modulo 8 to the respective signatures, and not modulo 16. Other characteristic classes such as \(\xi = (3, 3, 1)\) are ruled out by our inequality: \(\xi \cdot \xi = 19 > 11\) for this example. The class \(\xi = (3, 1, 3, 1)\) and its permutations, however, are representable. Notice \(\xi \cdot \xi = 20\) which achieves equality in the formula \(\xi \cdot \xi \leq 20\).
for the $n = 4$ case. We describe the construction of such a sphere. We first claim the class $(3, 1)$ is represented by an immersed sphere with one intersection point, which can be chosen to be positive or negative. To see this, note the class $(3, 0)$ is represented by three complex projective lines in general position which meet in three positive intersection points. We form two connected sums in a neighborhood of the intersection points to end up with a single 2-sphere with one positive intersection point remaining. The class $(0, 1)$ is represented by the sum of two copies of the positively oriented complex projective line with one negatively oriented line. They meet in two negative points and one positive point. These spheres are disjoint from the immersed sphere representing the class $(3, 0)$ so that we can form a connected sum by removing neighborhoods of the positive point of the immersed sphere and a negative point from the configuration for the class $(0, 1)$. The result is two spheres intersecting twice—one positive point and one negative point. Using either point to form a connected sum gives an immersed sphere with one intersection point representing the class $(3, 1)$. Given two such configurations, such as $(3, 1, 0, 0)$ and $(0, 0, 3, 1)$, one with the immersed sphere having a positive intersection point, the other with a negative intersection point, we can form a connected sum by identifying the boundary 3-spheres of the neighborhoods of these points by an orientation reversing diffeomorphism. This results in an embedded sphere representing the class $(3, 1, 3, 1)$. Note, this construction also shows the following classes are also representable: $(3, 1, 3, 1, 1, \ldots)$, $(3, 1, 3, 1, 3, 1, 1, \ldots)$, etc.

\[\Box\]

**Proof of Corollary 6.** Suppose there is a characteristic sphere of nonpositive square in $X$ (almost definite) such that $\sigma(X) \leq \xi \cdot \xi \leq 0$. Applying (3) of Theorem 2

\[
|\xi \cdot \xi - \sigma(X)| \leq 8(1 - 1),
\]

\[
|\xi \cdot \xi - \sigma(X)| \leq 0,
\]

and we see $\xi \cdot \xi$ must actually equal $\sigma(X)$. In the definite case, $(b^{+}_2(X) = 1, b^{-}_2(X) = 0)$, we first stabilize to $\xi # \mathbb{C}P^1 \subset X # \mathbb{C}P^2$ and apply Donaldson’s result on the nonexistence of characteristic spheres of positive square in an almost definite manifold [2].

\[\Box\]

4. **Proof of Theorem 1**

4.1. A brief description of Furuta’s argument

The idea of Furuta’s argument can be summarized as follows. Furuta defines a monopole equation involving a linear piece and a quadratic piece. This equation is defined on sections of the spinor bundle $\times$ the cotangent bundle and may be thought of as the standard Seiberg–Witten equations simplified by the choice of a trivial line bundle with trivial flat connection and restricted to a particular slice. This equation possesses a $Pin(2) \subset Spin(4)$ symmetry by construction. Furuta shows the zeros of this map are bounded and proceeds to find a finite-dimensional approximation for this equation in terms of an equivariant map between finite dimensional spheres with group actions. This leads to a $Pin(2)$-equivariant map between
ball-sphere pairs that preserves boundaries. Furuta’s argument utilizes equivariant $K$-theory to produce the inequality. The last step of the argument has been simplified by Bryan who bypasses much of the deep homotopy theory used by Furuta by instead analyzing the complex representation ring of $\text{Pin}(2)$ in greater detail to produce the inequality [1].

4.2. The monopole equation and its finite-dimensional approximation

To begin, we assume without loss of generality, that $\text{ind}(D) > 0$, where $D$ is the Dirac operator. We can always achieve this by changing the orientation on $Y$. Also, we assume without loss of generality that $b_1 = 0$. If indeed this were not the case, we could perform surgery along loops representing elements of infinite order in $\pi_1(Y)$ to produce a new orbifold $Y'$ with the same $\text{ind}(D)$ and with the same signature and $b_2^\pm$. To see this, we choose loops that miss our isolated singular point. The surgery provides a cobordism $Y \times I \cup D^2 \times D^3$ with boundary $Y \coprod Y'$, where we replace an $S^1 \times D^3$ in $Y$ by a $D^2 \times S^2$ to achieve $Y'$. This is a smooth manifold construction away from the isolated singular point and thus the Pontrjagin numbers for $Y$ and $Y'$ must be preserved [13]. Alternatively, one could show this claim by examining the Mayer–Vietoris sequence for the manifold $M = S^1 \times D^3 \cup_{S^1 \times S^2} D^2 \times S^2$. This gives $H_2(M; \mathbb{Z}) = 0$, in fact $M = S^4$, which in turn implies the Pontrjagin numbers on the pieces are the same. Examining the index formula for our orbifold (see Section 3) then shows $\text{ind}(D)$ remains unchanged. Likewise, $b_2^\pm(Y') = b_2^\pm(Y)$, $\Rightarrow$ $\sigma(Y') = \sigma(Y)$. This condition, $b_1 = 0$, is used later in the proof and indeed Furuta makes the same assumption and notes the index of the signature operator remains unchanged by performing surgery along loops to achieve a new spin manifold.

We are now ready to give the monopole equation on the orbifold $Y$. Consider the vector spaces

$$\Gamma(i \Lambda^1 \oplus S^+) \quad \text{and} \quad \Gamma(S^- \oplus i \text{su}(S^+) \oplus i \Lambda^0),$$

where we write $S^\pm = S^\pm_C(\tilde{P})$, the spinor bundles. Let $V$ denote the $L^2_2$ completion of the former, and $W'$ the $L^2_2$ completion of the latter. Furuta’s equations can be written as follows (we are quoting Bryan’s description of these equations [1]):

$$L + Q : V \rightarrow W',$$

$$L(a, \phi) = (D(\phi), \rho(d^+a), d^*a),$$

$$Q(a, \phi) = (\rho(a)\phi, \phi \otimes \phi^* - \frac{1}{2}|\phi|^2 \text{id}, 0).$$

$L$ is thus the linear piece, $Q$ the quadratic piece. Also we are using the isomorphism

$$\rho : \Lambda^+ \rightarrow \text{End}_C(S).$$

The image of $i \Lambda^+_C$ under $\rho$ is the tracefree, Hermitian endomorphisms of $S^+$, denoted by $i \text{su}(S^+)$. The system $L + Q$ is elliptic. These maps also possess certain symmetries. In fact, Furuta shows $L + Q$ is $\text{Pin}(2)$-invariant. $\text{Pin}(2) \subset SU(2)$ is defined as the centralizer of $S^1 \subset SU(2)$, regarded as the group of unit quaternions in $\mathbb{H}$. 
Furuta then develops a finite dimensional approximation to these equations by first giving a compactness argument. To demonstrate the necessary compactness in the orbifold context, we now follow the techniques of Fintushel and Stern [4]. We decompose our orbifold with one isolated singularity as \( Y = Y_0 \cup L \times (\varepsilon, \varepsilon) \cup cL \). Covering this last term is \( B^4 \) where recall \( B^4/\mathbb{Z}/p \simeq cL \). We think of \( B^4 \) embedded in \( S^4 \) as a hemisphere, thus possessing by restriction a metric of positive scalar curvature which is also \( \mathbb{Z}/p \)-invariant. This corresponds to a metric of positive scalar curvature on the quotient \( cL \).

We can now obtain a family of generic metrics on \( Y \): \( g_r = g_0 \cup g_{L,r} \cup g_{cL} \) where \( g_{L,r} \) and \( g_{cL} \) have positive scalar curvature, and \( g_{cL} \) makes the neck isometric to \( L \times (-r, r) \). Let \( Y_0^+ = Y_0 \cup L \times [0, \infty) \) and \( cL^+ = cL \cup L \times [0, \infty) \).

Since the necks have positive scalar curvature, there is a gluing theory for obtaining solutions to the Seiberg–Witten equations on \( Y \) from solutions on \( Y_0^+ \) and \( cL^+ \), and this gluing theory parallels the gluing theory for solutions of the anti-self-duality equations on a connected sum. The moduli space of the orbifold comes from gluing the moduli space on the cylindrical end manifold \( Y_0^+ \) to the equivariant moduli space on the disk along the trivial connection over the lens space. Since the neighborhood \( B^4 \) has positive scalar curvature, the only solution of the Seiberg–Witten equations is the trivial reducible one, i.e., the trivial connection on the trivial bundle with 0 spinor. This is an equivariant solution. For the metric \( g_r \) with \( r \) large, it follows from the gluing theory that

\[
M_{Y, g_r} \simeq M_{Y_0^+} \times \{(0, 0)\},
\]

so in fact, \( cL \) contributes nothing to the moduli space. The compactness of \( M_{Y, g_r} \) comes from the compactness of \( M_{Y_0^+} \) which follows from the work of Kronheimer and Mrowka [10]. Combining this compactness with Furuta’s analysis of \( L + Q \) restricted to the subspace \( V_{\lambda} \) of \( V \) spanned by the eigenspaces of \( L^*L \) with eigenvalues less than or equal to \( \lambda \), we obtain

**Furuta, Kronheimer and Mrowka Lemma.** There exists a finite-dimensional approximation to the monopole equation

\[
f : (BV, SV) \mapsto (BW, SW)
\]

This map is a Pin(2) invariant map on disks preserving boundaries.

Here, \( BV \) is homotopic to a ball in \( V_{\lambda} \otimes \mathbb{C} \) and \( SV \) is the boundary of \( BV \). Similar definitions apply for \( BW \) and \( SW \).

4.3. Representation theory

The remainder of Furuta’s proof has been simplified by Bryan, and in fact, holds in the orbifold setting. To summarize, \( \text{ind}(L) \) is regarded as an element in the complex representation ring \( R(\text{Pin}(2)) \), and by analyzing the finite dimensional approximation to the monopole equations, Furuta is able to show

\[
\text{ind}(L) = (\text{ind}(D))h - (b_2^+)\bar{1}.
\]
The group $\text{Pin}(2)$ has one nontrivial one-dimensional (complex) representation, $\tilde{1}$, defined by multiplication of $\text{Pin}(2)/S^1 = \{\pm 1\}$. Let $1$ denote the trivial one-dimensional representation. $\text{Pin}(2)$ also has a countable series of two-dimensional irreducible representations $h_1, h_2, \ldots$. The representation $h = h_1$ is the restriction of the standard representation of $SU(2)$ to $\text{Pin}(2)$. The representations $h_i$ can be obtained using the relation $h_i \otimes h_j = h_{i+j} \oplus h_{|i-j|}$, where by convention $h_0$ denotes $1 + \tilde{1}$.

Furuta’s finite dimensional approximation $f$ induces a map $f^*$ in $K$-theory:

$$K_{\text{Pin}(2)}(BV, SV) \leftarrow K_{\text{Pin}(2)}(BW, SW).$$

These groups are free modules over the representation ring $R(\text{Pin}(2))$ by the equivariant Thom isomorphism theorem. Each is generated by the Bott class $\lambda(V), \lambda(W)$, respectively. Thus, $f^*$ determines a unique element $\alpha_f \in R(\text{Pin}(2))$ by the equation:

$$f^*(\lambda(W)) = \alpha_f \cdot \lambda(V).$$

This element is called the $K$-theoretic degree of $f$.

Any element $\alpha \in R(\text{Pin}(2))$ can be written

$$\alpha = \alpha_0 1 + \alpha_1 \tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i.$$ 

In particular, $\alpha_f$ can be written in this way. To examine $\alpha_f$ Bryan uses $K$-theoretic techniques to deduce restrictions on the map $f$ and thus avoids Furuta’s use of the equivariant Adam’s operations. The key ingredient needed by Bryan is the following character formula for the degree $\alpha_f$ proved by tom Dieck.

**Tom Dieck Theorem.** Let $f : BV \rightarrow BW$ be a $\Gamma$-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the $K$-theoretic degree. Then

$$\text{tr}_g(\alpha_f) = d(f^g)\text{tr}_g(\lambda_{-1}(W^+_g - V^+_g)),$$

where $\text{tr}_g$ is the trace of the action of an element $g \in \Gamma$.

Here, $\Gamma$ is a compact group, $V_g$ and $W_g$ are the subspaces of $V$ and $W$ fixed by $g \in \Gamma$, and $f^g : V_g \rightarrow W_g$ denotes the restriction of $f$. Note, this restriction is well defined because of equivariance. The expression $d(f^g)$ denotes the ordinary topological degree of $f^g$, and so by definition $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$. Also, for any $\beta \in R(\Gamma), \lambda_{-1}\beta$ denotes the alternating sum $\sum (-1)^i \lambda^i \beta$ of exterior powers.

The two elements in $\text{Pin}(2)$ to which Bryan applies tom Dieck’s formula are $\phi \in S^1 \subset \text{Pin}(2)$ with $\phi$ irrational and thus generating a dense subgroup of $S^1$, and $j \in \text{Pin}(2)$ coming from the quaternions. We have

$$\text{tr}_\phi(\alpha) = 0 = \alpha_0 + \alpha_0^\phi + \sum_{i=1}^{\infty} \alpha_i (\phi^i + \phi^{-i})$$

since $\phi$ acts nontrivially on $h$ and trivially on $\tilde{1}$, which therefore implies $\dim V_\phi \neq \dim W_\phi$ under our assumption $b_2^+ > 0$. Thus, $\alpha_0 = -\alpha_0$ and $\alpha_i = 0$ for $i \geq 1$. Now $j$ acts nontrivially on $h$ and $1$, which implies $\dim V_j = \dim W_j$ and thus $d(f^j) = 1$. 


\[
\text{tr}_j(\alpha) = \text{tr}_j(\lambda - (b_2^+ \tilde{1} - \text{ind}(D)h)) \\
= \text{tr}_j((1 - \tilde{1})b_2^+ (2 - h)^{\text{ind}(D)}) \\
= 2b_2^+ - \text{ind}(D).
\]

Here Bryan used \(\text{tr}_j h = 0\) and \(\text{tr}_j \tilde{1} = -1\). Note also that \(\text{tr}_j(\alpha) = \text{tr}_j(\alpha_0(1 - \tilde{1})) = 2\alpha_0\).

These results yield an expression for the degree:

\[
\text{degree} = \alpha = 2b_2^+ - \text{ind}(D) - 1 (1 - \tilde{1}),
\]

\[
\Rightarrow b_2^+ - \text{ind}(D) - 1 \geq 0,
\]

\[
\Rightarrow b_2^+ \geq \text{ind}(D) + 1.
\]

The last statement of the theorem (Theorem 1) follows by changing orientation of the orbifold which brings us back to the case just proven. We use the formula for \(\text{ind}(D)\) (see Section 3) to see that reversing orientation of the orbifold leads to a sign change of \(\text{ind}(D)\).

References