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Cohomology of regular differential forms for affine curves

Philippe Bonnet

Mathematisches Institut, Universität Basel Rheinsprung 21, 4051 Basel, Switzerland

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Abstract

Let C be a complex affine reduced curve, and denote by $H^1(C)$ its first truncated cohomology group, i.e. the quotient of all regular differential 1-forms by exact 1-forms. First we introduce a nonnegative invariant $\mu'(C,x)$ that measures the complexity of the singularity of C at the point x, and we establish the following formula:

$$\dim H^1(C) = \dim H_1(C) + \sum_{x\in C} \mu'(C,x)$$

where $H_1(C)$ denotes the first singular homology group of C with complex coefficients. Second we consider a family of curves given by the fibres of a dominant morphism $f: X \to \mathbb{C}$, where X is an irreducible complex affine surface. We analyze the behaviour of the function $y \mapsto \dim H^1(f^{-1}(y))$. More precisely we show that it is constant on a Zariski open set, and that it is lower semi-continuous in general.

Résumé

Soit C une courbe affine complexe réduite. Son premier groupe $H^1(C)$ de cohomologie tronqué est le quotient des 1-formes différentielles régulières sur C par les 1-formes régulières exactes. A tout point x de C, nous attachons un invariant positif $\mu'(C,x)$ qui mesure la complexité de la singularité (C,x). Puis nous montrons la formule suivante :

$$\dim H^1(C) = \dim H_1(C) + \sum_{x\in C} \mu'(C,x)$$

où $H_1(C)$ désigne le premier groupe d'homologie singulière de C à coefficients complexes. Ensuite nous considérons une famille de courbes données par les fibres d'un morphisme dominant $f: X \to \mathbb{C}$, où X est une surface affine complexe irréductible. Nous analysons le comportement de la fonction

E-mail address: philippe.bonnet@unibas.ch (P. Bonnet).

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$y \to \dim H^1(f^{-1}(y))$. Plus précisément, nous montrons qu'elle est constante sur un ouvert de Zariski, et qu'elle est semi-continue inférieurement en général.

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1. Introduction

Let C be a reduced complex affine curve that may be reducible or singular. For any integer k, denote by $\Omega^k(C)$ the space of regular differential k-forms (or Kähler forms) on C. The exterior derivative d is well-defined on $\Omega^k(C)$, and yields a complex:

$$0 \to \mathbb{C} \to \Omega^0(C) \to \Omega^1(C) \to 0.$$

The first truncated De Rham cohomology group $H^1(C)$ is the quotient $\Omega^1(C)/d\Omega^0(C)$. If C is smooth, then C is a non-compact Riemann surface, for which the De Rham cohomology groups $H^k_{DR}(C)$ with complex coefficients are well-defined. Moreover $H^1(C)$ coincides with the algebraic De Rham cohomology group of C (see [5]) and, by a theorem of Grothendieck (see [7]), we have the isomorphism:

$$H^1(C) \simeq H^1_{DR}(C).$$

So truncated De Rham cohomology is always defined and coincides with standard De Rham cohomology if C is smooth. We would like to know to what extend this cohomology reflects the topological properties of C, especially when C has singularities.

Definition 1.1. Let $\widehat{\Omega^k_{C,x}}$ be the space of formal differential k-forms on the germ (C,x). The local De Rham cohomology group of C at x is the quotient:

$$H^1(C,x) = \widehat{\Omega^1_{C,x}}/d\widehat{\Omega^0_{C,x}}.$$

Its dimension $\mu'(C,x)$ is the local Betti number of C at x.

This number characterizes the presence of singularities, in the sense that $\mu'(C,x) = 0$ if and only if x is a smooth point of C. Moreover it coincides with the Milnor number (see [10]) if C is locally a complete intersection (see [2]).

Let $H_1(C)$ be the first singular homology group of C with complex coefficients. We identify this group with the first simplicial homology group of C associated to a triangulation of C whose singular points are vertices. Every regular 1-form on C is closed on every face F of this triangulation, so that its integral along ∂F is always zero. Therefore, integration of differential forms along cycles is well-defined and provides us with a bilinear pairing $\langle , \rangle$ on $H^1(C) \times H_1(C)$ given by:

$$\langle \omega, \gamma \rangle = \int_\gamma \omega.$$

This induces the so-called De Rham morphism $\beta : H^1(C) \to H_1(C)^*$, $\omega \mapsto \langle \omega, . \rangle$. By Poincaré Duality and a theorem of Grothendieck (see [7]), this map is an isomorphism when C is smooth. In the general case, we establish the following formula.

Theorem 1.2. *For any complex affine curve C, we have*:

$$\dim H^1(C) = \dim H_1(C) + \sum_{x\in C} \mu'(C,x).$$

The idea of the proof is the following. For any affine curve C, the morphism β is onto (see [3]) and this yields the exact sequence:

$$0 \to \ker\beta \to H^1(C) \to H_1(C)^* \to 0.$$

For any point x in C, every regular 1-form ω can be seen as a formal 1-form on the germ (C,x). Moreover every exact 1-form on C is exact as a formal 1-form on (C,x). We then have a natural morphism:

$$i_x : H^1(C) \to H^1(C,x).$$

We prove that the morphism α:

$$\alpha : \ker\beta \to \bigoplus_{x\in C} H^1(C,x), \quad \omega \to \big(i_x(\omega)\big)_{x\in C}$$

is an isomorphism, which gives the result by passing to the dimensions.

So local Betti numbers measure the default to Poincaré Duality in the case of singular curves. Theorem 1.2 implies in particular that *a complex affine curve is isomorphic to a disjoint union of copies of $\mathbb{C}$ if and only if $H^1(C)=0$.*

Now we are going to study the behaviour of the function $h_1(y) = \dim H^1(f^{-1}(y))$, where X is a complex affine irreducible surface and $f : X \to \mathbb{C}$ is a dominant morphism. *The following results still hold for any reduced surface X (that is, any equidimensional reduced affine variety of dimension* 2) *as soon as the morphism f is dominant on every irreducible component of X.* Recall that $\mathcal{P}$ holds for every generic point of $\mathbb{C}$ if the set of points y of $\mathbb{C}$ where $\mathcal{P}(y)$ does not hold is finite. We have the following first result.

Proposition 1.3. *Let X be a complex affine irreducible surface and $f : X \to \mathbb{C}$ a dominant morphism. Then there exists an integer $h_f \geqslant 0$ such that, for every generic point y of $\mathbb{C}$*:

$$\dim H^1\big(f^{-1}(y)\big) = h_f.$$

The proof splits in two steps. By a theorem due to Varčenko (see [11]), there exists a Zariski open set U in $\mathbb{C}$ such that $f : f^{-1}(U) \to U$ is a locally trivial topological fibration. In particular, the function $y \mapsto \dim H_1(f^{-1}(y))$ is constant on U. Then there remains to show that the sum of the local Betti numbers is constant on a Zariski open set. For simplicity, denote by $\mathrm{Sing}(f)$ the set of points x of X where X is not smooth or where $df(x) = 0$.

Theorem 1.4. *Let X be a complex affine surface that is locally a complete intersection and $f : X \to \mathbb{C}$ be a dominant morphism. If $f^{-1}(y) \cap \mathrm{Sing}(f)$ is finite, then*:

$$\dim H^1\big(f^{-1}(y)\big) \leqslant h_f.$$

In particular the function h^1 is lower semi-continuous at every point y_0 of $\mathbb{C}$ such that $f^{-1}(y_0) \cap \mathrm{Sing}(f)$ is finite, i.e.:

$$h^1(y_0) \leqslant \underline{\lim}_{y\to y_0} h^1(y).$$

Note the analogy with singular homology. If X is the complex space $\mathbb{C}^2$, then the Euler characteristic $\chi(f^{-1}(y))$ is an upper semi-continuous function when y runs through the non-critical values of f. This is a direct consequence of the expression of $\chi(f^{-1}(y))$ in terms of the Milnor numbers of $f^{-1}(y)$ at infinity (see for instance [4]).

We end up this paper with an example illustrating the necessity for X to a locally complete intersection.

2. Properties of the normalisation

Let C be a complex affine curve and $\mathcal{O}_C$ its ring of regular functions. Let $\widetilde{C}$ be its affine normalisation and $\Pi : \widetilde{C} \to C$ the normalisation morphism. Choose a triangulation $\widetilde{T}$ on $\widetilde{C}$ and let $\widetilde{V}$ be its set of vertices. Since Π is finite, we may refine this triangulation so that the set $V = \Pi(\widetilde{V})$ contains all the singular points of C, and so that $\widetilde{V} = \Pi^{-1}(V)$. By construction, the image $T = \Pi(\widetilde{T})$ defines a triangulation of C. We denote by $\{\tilde{\gamma}_i\}$ the set of edges of $\widetilde{T}$, and set $\gamma_i = \Pi(\tilde{\gamma}_i)$. *We consider this triangulation fixed from now on.*

For any point x in C, $\mathcal{O}_{C,x}$ stands for the ring of germs of regular functions at x. Denote by $\mathcal{O}_{C,V}$ the ring of germs of regular functions at V, i.e. the direct sum:

$$\mathcal{O}_{C,V} = \bigoplus_{x \in V} \mathcal{O}_{C,x}.$$

Let I be the vanishing ideal of the set V in C, and denote by $\widehat{\mathcal{O}_{C,V}}$ the I-adic completion of $\mathcal{O}_{C,V}$. Note that we have the isomorphism:

$$\widehat{\mathcal{O}_{C,V}} = \bigoplus_{x \in V} \widehat{\mathcal{O}_{C,x}}.$$

A *formal function on* (C, V) is an element of $\widehat{\mathcal{O}_{C,V}}$. In a similar way, denote by $\Omega^1_{C,x}$ the space of germs of regular 1-forms on C at x, and by $\Omega^1_{C,V}$ the finite sum:

$$\Omega^1_{C,V} = \bigoplus_{x \in V} \Omega^1_{C,x}.$$

The I-adic completion $\widehat{\Omega^1_{C,V}}$ of $\Omega^1_{C,V}$ is the set of *formal 1-forms on* (C, V). Note that we have the isomorphism:

$$\widehat{\Omega^1_{C,V}} = \bigoplus_{x \in V} \widehat{\Omega^1_{C,x}}.$$

We can define the sets of *formal functions and formal 1-forms on* $(\widetilde{C}, \widetilde{V})$ in exactly the same way. In this section, we are going to describe the relationships between the functions and 1-forms on $\widetilde{C}$ and C.

2.1. Formal functions

Let $\Pi^* : \mathcal{O}_C \to \mathcal{O}_{\widetilde{C}}$ be the morphism induced by the normalisation map. After localization at V and completion, we obtain the following injective map:

$$\widehat{\Pi^*_V} : \widehat{\mathcal{O}_{C,V}} \to \widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}}.$$

Since the germ $(\widetilde{C}, x)$ is smooth for any point x in $\widetilde{C}$, every element R of $\mathcal{O}_{\widetilde{C},\widetilde{V}}$ has a well-defined order $\operatorname{ord}_x(R)$ at x, and thus it defines a divisor:

$$\operatorname{div}(R) = \sum_{x \in \widetilde{V}} \operatorname{ord}_x(R)x.$$

Proposition 2.1. *Let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$ that vanishes at every point of $\widetilde{V}$. Then there exists a regular function S on $\widetilde{C}$, vanishing at every point of $\widetilde{V}$, and a formal function R on (C, V) such that $\widetilde{R} = S + \widehat{\Pi_V^*}(R)$.*

In order to prove this proposition, we need the following lemma.

Lemma 2.2. *With the previous notations, there exists a divisor D on $(\widetilde{C}, \widetilde{V})$ such that, for any formal function $\widetilde{R}$ on $(\widetilde{C}, \widetilde{V})$, we have*: $\operatorname{div}(\widetilde{R}) \geqslant D \Rightarrow \widetilde{R} \in \widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}})$.

Proof. Let A be a conductor of the normalisation, i.e. an element of $\mathcal{O}_C$ that is not a zero-divisor and such that $\Pi^*(A)\mathcal{O}_{\widetilde{C}} \subseteq \Pi^*(\mathcal{O}_C)$. After localisation at V and completion, we obtain that:

$$\widehat{\Pi_V^*}(A)\widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}} \subseteq \widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}}).$$

Set $D = \operatorname{div} \widehat{\Pi_V^*}(A)$ and let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$ such that $\operatorname{div}(\widetilde{R}) \geqslant D$. Then $\widetilde{R}$ is locally divisible by $\widehat{\Pi_V^*}(A)$, and the quotient $S = \widetilde{R}/\widehat{\Pi_V^*}(A)$ is a formal function on $(\widetilde{C}, \widetilde{V})$. Therefore $\widetilde{R} = \widehat{\Pi_V^*}(A)S$ belongs to $\widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}})$. $\square$

Proof of Proposition 2.1. Let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$. For any point x in $\widetilde{V}$, let z_x be a uniformising parameter of $\widetilde{C}$ at x defined on all of $\widetilde{C}$. Then $\widetilde{R}$ has a Taylor expansion $\sum_{k \geqslant 0} R_{k,x} z_x^k$ at x. For any such x, we set:

$$R_x = \sum_{k \leqslant n} R_{k,x} z_x^k.$$

Let $\mathcal{O}_{\widetilde{C}}$ be the ring of regular functions on $\widetilde{C}$, and denote by $\widetilde{I}$ the ideal generated by I in $\mathcal{O}_{\widetilde{C}}$. Since the radical of $\widetilde{I}$ is the vanishing ideal of $\widetilde{V}$, $\widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}}$ is the $\widetilde{I}$-adic completion of $\mathcal{O}_{\widetilde{C}}$. So there exists a regular function S on $\widetilde{C}$, whose Taylor expansion of order n at any point x is equal to R_x. For n large enough, we have the inequality:

$$\operatorname{div}(\widetilde{R} - S) \geqslant D.$$

By Lemma 2.2, there exists a formal function R on (C, V) such that $\widehat{\Pi_V^*}(R) = \widetilde{R} - S$. $\square$

2.2. Formal 1-forms

Let $\Pi^* : \Omega^1(C) \to \Omega^1(\widetilde{C})$ be the morphism induced by normalisation. After localisation at V and completion, we obtain the following morphism:

$$\widehat{\Pi_V^*} : \widehat{\Omega^1_{C,V}} \to \widehat{\Omega^1_{\widetilde{C},\widetilde{V}}}.$$

In this subsection, we consider $\Omega^1(\widetilde{C})$ as a $\mathcal{O}_C$-module via the multiplication rule $(P, \omega) \mapsto \Pi^*(P)\omega$. If M is an $\mathcal{O}_C$-module and $\mathcal{M}$ is an ideal, denote by $M_{\mathcal{M}}$ its localisation with respect to $\mathcal{M}$, and by $\widehat{M_{\mathcal{M}}}$ its $\mathcal{M}$-adic completion. We are going to prove the following proposition.

Proposition 2.3. *Let ω be a formal 1-form on the germ (C,V). Then there exist a formal function R on (C,V), a regular 1-form ω_0 on C and a regular function S in $\mathcal{O}_{\widetilde{C}}$, vanishing at all points of $\widetilde{V}$, such that $\omega = dR + \omega_0$ and $\Pi^*(\omega_0) = dS$.*

Lemma 2.4. *Let R be a noetherian ring, and $L : M \to N$ a morphism of finite R-modules. Let ω be an element of N that belongs to $\operatorname{Im} \widehat{L_{\mathcal{M}}}$ for any maximal ideal $\mathcal{M}$. Then ω belongs to $\operatorname{Im} L$.*

Proof. First we show that ω belongs to $\operatorname{Im} L_{\mathcal{M}}$ for any maximal ideal $\mathcal{M}$. Let $\{e_1,\dots,e_k\}$ be a set of generators of M, i.e. $M = R\langle e_1,\dots,e_k\rangle$. After localisation and completion, we get the equalities:

$$\widehat{M_{\mathcal{M}}} = \widehat{R_{\mathcal{M}}}\langle e_1,\dots,e_k\rangle \quad \text{and} \quad \operatorname{Im} \widehat{L_{\mathcal{M}}} = \widehat{R_{\mathcal{M}}}\bigl\langle L(e_1),\dots,L(e_k)\bigr\rangle = \widehat{\operatorname{Im} L_{\mathcal{M}}}.$$

Since N has finite type, the $\mathcal{M}$-adic topology on N is Hausdorff and we find:

$$\operatorname{Im} L_{\mathcal{M}} = \operatorname{Im} \widehat{L_{\mathcal{M}}} \cap N.$$

So ω belongs to $\operatorname{Im} L_{\mathcal{M}}$, and for any maximal ideal $\mathcal{M}$, there exists an element $P_{\mathcal{M}}$ of $R - \mathcal{M}$ such that $P_{\mathcal{M}}\omega$ belongs to $\operatorname{Im} L$. Let I be the ideal in R generated by all the $P_{\mathcal{M}}$. We claim that $I = (1)$, so that ω belongs to $\operatorname{Im} L$. Indeed if I were not equal to (1), it would be contained in a maximal ideal $\mathcal{M}_0$ by Zorn's Lemma. Since I contains $P_{\mathcal{M}_0}$, $P_{\mathcal{M}_0}$ would be contained in $\mathcal{M}_0$, hence a contradiction. □

Lemma 2.5. *Let $\tilde{\omega}$ be an element of $\Omega^1(\widetilde{C}) \cap \operatorname{Im} \widehat{\Pi^*_V}$. Then $\tilde{\omega}$ belongs to $\operatorname{Im} \Pi^*$.*

Proof. We set $M = \Omega^1(C)$, $N = \Omega^1(\widetilde{C})$ and $L = \Pi^*$. Let $\mathcal{M}$ be a maximal ideal and x the corresponding point in C. If x belongs to V, then $\tilde{\omega}$ belongs to $\operatorname{Im} \widehat{L_{\mathcal{M}}}$ by assumption. If not, then $\tilde{\omega}$ still belongs to $\operatorname{Im} \widehat{L_{\mathcal{M}}}$ because x is a smooth point of C, and then $\widehat{L_{\mathcal{M}}}$ is an isomorphism. By Lemma 2.4, $\tilde{\omega}$ belongs to $\operatorname{Im} \Pi^*$. □

Lemma 2.6. *Under the previous assumptions,* $\dim \ker \Pi^*$ *is finite and the natural map* $\ker \Pi^* \to \ker \widehat{\Pi^*_V}$ *is an isomorphism.*

Proof. For any x in C, denote by $\mathcal{M}$ the vanishing ideal of x and set $L = \Pi^*$. For any x outside V, Π is an isomorphism over an open neighborhood of x. So the map $\widehat{L_{\mathcal{M}}}$ is an isomorphism for all x outside V, and the support of $\ker \Pi^*$ is contained in V. Since V is a finite set and $\ker \Pi^*$ is a finite module, $\ker \Pi^*$ is an artinian module and $\dim \ker \Pi^* < \infty$. So there exists an order n such that $I^n \ker \Pi^* = 0$, and $\ker \Pi^*$ is complete for the I-adic topology. Since completion is an exact functor, we have:

$$\ker \Pi^* \simeq \widehat{\ker \Pi^*} \simeq \ker \widehat{\Pi^*_V}. \qquad \square$$

Proof of Proposition 2.3. Let ω be a formal 1-form on the germ (C,V). Since the germ $(\widetilde{C},\widetilde{V})$ is a disjoint union of smooth curves, the 1-form $\widehat{\Pi^*_V}(\omega)$ is exact on each of these curves. There exists a formal function $\widetilde{R}$ on $(\widetilde{C},\widetilde{V})$ such that:

$$\widehat{\Pi^*_V}(\omega) = d\widetilde{R}.$$

By Proposition 2.1, there exist a regular function S on $\widetilde{C}$, vanishing at all points of $\widetilde{V}$, and a formal function R on (C, V) such that $\widetilde{R} = S + \widehat{\Pi_V^*}(R)$. After derivation, this implies:

$$\widehat{\Pi_V^*}(\omega - d\widetilde{R}) = dS.$$

By Lemma 2.5 applied to $\tilde{\omega} = dS$, there exists a regular 1-form ω_1 on C such that $\Pi^*(\omega_1) = dS$. This yields:

$$\widehat{\Pi_V^*}(\omega - d\widetilde{R} - \omega_1) = 0.$$

By Lemma 2.6, there exists a regular 1-form ω_2 in $\ker \Pi^*$ such that $\omega - d\widetilde{R} - \omega_1 = \omega_2$. Then the 1-form $\omega_0 = \omega_1 + \omega_2$ is regular on C and satisfies the following relations:

$$\omega = d\widetilde{R} + \omega_0 \quad \text{and} \quad \Pi^*(\omega_0) = dS. \qquad \square$$

3. Proof of Theorem 1.2

Let C be a complex reduced affine curve in $\mathbb{C}^n$, and let $\beta : H^1(C) \to H_1(C)^*$ be the map defined in the introduction. Since β is onto, it induces the following complex:

$$0 \to \ker \beta \to H^1(C) \to H_1(C)^* \to 0.$$

Moreover the inclusion of regular 1-forms into formal 1-forms at x induces a morphism:

$$\alpha : \ker \beta \to \bigoplus_{x \in C} H^1(C, x).$$

Since C carries a structure a CW-complex, the vector space $H_1(C)$ is finite dimensional, and the same holds for every $H^1(C, x)$ (see [2]). So for the proof of Theorem 1.2, we only need to show that α is an isomorphism, and the result will follow by passing to the dimensions.

3.1. Injectivity of α

Without loss of generality, we may assume that the curve C is connected. Let ω be an element of $\ker \beta$. Fix a point x_0 in C, and consider the map R defined as follows. For any point x in C, choose a path γ going from x_0 to x, and set:

$$R(x) = \int_\gamma \omega.$$

Since ω has null integral along any closed path in C, this number is well-defined and independent of the path γ chosen. Furthermore the function $S = R \circ \Pi$ is holomorphic on $\widetilde{C}$ because it defines an integral of $\Pi^*(\omega)$ on $\widetilde{C}$. By Grothendieck's Theorem, S is a regular function on $\widetilde{C}$, and S takes the value $R(x)$ on $\Pi^{-1}(x)$.

Assume now that $\alpha(\omega) = 0$. Then for any point x of C, the class of ω in $H^1(C, x)$ is zero, and there exists a formal function R^x on the germ (C, x) such that $\omega = dR^x$. Let $\mathcal{M}$ be the vanishing ideal of x and denote by $\widehat{L_{\mathcal{M}}}$ the morphism induced by Π^* after localisation at $\mathcal{M}$ and completion. The formal function $S - \widehat{L_{\mathcal{M}}}(R^x)$ on $(\widetilde{C}, \Pi^{-1}(x))$ is constant around every point of $\Pi^{-1}(x)$, because S and $\widehat{L_{\mathcal{M}}}(R^x)$ are both integrals of $\Pi^*(\omega)$. Since S and $\widehat{L_{\mathcal{M}}}(R^x)$ are constant on $\Pi^{-1}(x)$, there exists a constant λ such that:

$$S - \widehat{L_{\mathcal{M}}}(R^x) = \lambda$$

on $(\widetilde{C}, \Pi^{-1}(x))$. Up to replacing R^x by $R^x - \lambda$, we may assume that $\lambda = 0$, and so S belongs to $\operatorname{Im} \widehat{L_{\mathcal{M}}}$ for any point x in C. By applying Lemma 2.4 to the morphism $\Pi^* : \mathcal{O}_C \to \mathcal{O}_{\widetilde{C}}$ of finite $\mathcal{O}_C$-modules, we get that S belongs to $\mathcal{O}_C$. Since $S = R^x$ for any x in C, we get by derivation:

$$\omega = dS = dR^x \quad \text{in} \quad \widehat{\Omega^1_{C,x}}.$$

Since $\Omega^1_{C,x}$ is a finite $\mathcal{O}_{C,x}$-module, the $\mathcal{M}$-adic topology is separated and $\omega = dS$ in $\Omega^1_{C,x}$. By Bourbaki result (Commutative Algebra, Chap. 1-7, Corollary 1, p. 88), $\omega = dS$ in $\Omega^1(C)$ and the class of ω in $H^1(C)$ is zero.

3.2. Surjectivity of α

By construction, the set V contains all the singular points of C. Since $H^1(C, x) = 0$ if C is smooth at x, we have the isomorphism:

$$\bigoplus_{x \in C} H^1(C, x) \simeq \bigoplus_{x \in V} H^1(C, x).$$

So every element ω of this sum can be represented by a formal 1-form on (C, V), which we also denote by ω. By Lemma 2.3, there exist a formal function R on (C, V), a regular 1-form ω_0 on C and a regular function S on $\widetilde{C}$, vanishing at all points of $\widetilde{V}$, such that:

$$\omega = dR + \omega_0 \quad \text{and} \quad \Pi^*(\omega_0) = dS.$$

Let γ be a 1-cycle in C. This cycle can be represented as a formal linear combination of the edges γ_i of the triangulation T on C. Since S vanishes at all vertices of $\widetilde{T}$, and these vertices are endpoints of the $\tilde{\gamma}_i$, we have:

$$\int_{\gamma_i} \omega_0 = \int_{\tilde{\gamma}_i} \Pi^*(\omega_0) = \int_{\tilde{\gamma}_i} dS = S\big(\tilde{\gamma}_i(1)\big) - S\big(\tilde{\gamma}_i(0)\big) = 0.$$

By linearity, we get that $\langle \omega_0, \gamma \rangle = 0$ for any cycle γ in C. So ω_0 belongs to $\ker \beta$ and represents the same class as ω in $\bigoplus_{x \in V} H^1(C, x)$. Therefore $\alpha(\omega_0) = \omega$ and α is surjective.

4. Proof of Proposition 1.3

In this section, we are going to prove Proposition 1.3. First by a theorem of Varčenko (see [11]), there exists a non-empty Zariski open set V such that $f : f^{-1}(V) \to V$ is a locally trivial topological fibration. In particular, there exists an integer $q \geqslant 0$ such that, for any y in U:

$$\dim H_1\big(f^{-1}(y)\big) = q.$$

By Theorem 1.2, we only need to prove that the sum of the local Betti numbers along a fibre $f^{-1}(y)$ is constant for generic y. More precisely:

Proposition 4.1. *Let $f : X \to \mathbb{C}$ be a dominant morphism, where X is a complex reduced surface. Then there exists a non-empty Zariski open set U in $\mathbb{C}$ and an integer p such that, for any y in U:*

$$\sum_{x \in f^{-1}(y)} \mu'\big(f^{-1}(y), x\big) = p.$$

The rest of the section will be devoted to the proof of Proposition 4.1, which we will split in several lemmas. Let $f : X \to \mathbb{C}$ be a dominant map, where X is a complex affine reduced surface. Let $N : \widetilde{X} \to X$ be the normalisation map and set $\tilde{f} = f \circ N$. Let S be the singular part of X, I its corresponding ideal in $\mathcal{O}_X$ and set $\widetilde{S} = N^{-1}(S)$. The normalisation morphism N^* induces an exact sequence:

$$0 \to M \to \Omega^1_X / \Omega^0_X df \to \Omega^1_{\widetilde{X}} / \Omega^0_{\widetilde{X}} d\tilde{f}.$$

Lemma 4.2. *Under the previous assumptions, there exists a non-zero polynomial P in $\mathbb{C}[t]$ such that $M_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module.*

Proof. Since N is an isomorphism between $\widetilde{X} - \widetilde{S}$ and $X - S$, M is a finite $\mathcal{O}_X$-module with support in S. So there exists an integer n such that $I^n.M = 0$, and M is a finite $\mathcal{O}_X/I^n$-module. There remains to check that $(\mathcal{O}_X/I^n)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module for a suitable choice of $P \neq 0$.

By assumption, S has dimension $\leqslant 1$. So there exists a non-empty Zariski open set U of $\mathbb{C}$ such that either $S \cap f^{-1}(U)$ is empty or the restriction $f : S \cap f^{-1}(U) \to U$ is a finite morphism. Let P be a non-zero polynomial of $\mathbb{C}[t]$ vanishing on $\mathbb{C} - U$. If $S \cap f^{-1}(U)$ is empty, then $(\mathcal{O}_X/I^n)_{(P(f))} = 0$. If $f : S \cap f^{-1}(U) \to U$ is a finite morphism, then $(\mathcal{O}_X/I)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module. Then it is easy to check via an induction on n that $(\mathcal{O}_X/I^n)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module, by considering the following exact sequence:

$$0 \to I^n/I^{n+1} \to \mathcal{O}_X/I^{n+1} \to \mathcal{O}_X/I^n \to 0$$

and using the fact that I^n/I^{n+1} is a finite $\mathcal{O}_X/I$-module for any $n \geqslant 0$. □

Lemma 4.3. *Let $\Pi : \widetilde{f^{-1}(y)} \to f^{-1}(y)$ be the normalisation morphism. Then for generic y in $\mathbb{C}$, $M/(f - y) \simeq M_y$ where M_y is the kernel of the morphism Π^* in the exact sequence*:

$$0 \to M_y \to \Omega^1\big(f^{-1}(y)\big) \to \Omega^1\big(\widetilde{f^{-1}(y)}\big).$$

Proof. Since N is a finite morphism, its restriction $N : \tilde{f}^{-1}(y) \to f^{-1}(y)$ is finite for any y. By generic smoothness, the fibre $\tilde{f}^{-1}(y)$ is smooth for generic y because $\widetilde{X}$ has at most finitely many singularities. Moreover, for generic y, the fibre $f^{-1}(y)$ intersects S in finitely many points. So for generic y, the curve $\tilde{f}^{-1}(y)$ is smooth and the morphism $N : \tilde{f}^{-1}(y) \to f^{-1}(y)$ is finite and birational. As a consequence, this latter is the normalisation map:

$$\Pi : \widetilde{f^{-1}(y)} \to f^{-1}(y).$$

By generic flatness, there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that localisation yields an exact sequence of flat $\mathbb{C}[f]_{(P(f))}$-modules:

$$0 \to M_{(P(f))} \to (\Omega^1_X/\Omega^0_X\, df)_{(P(f))} \to (\Omega^1_{\widetilde{X}}/\Omega^0_{\widetilde{X}}\, d\tilde{f})_{(P(\tilde{f}))}.$$

In particular, after tensoring by $\mathbb{C}[f]/(f - y)$, we get for any y with $P(y) \neq 0$:

$$0 \to M/(f - y) \to \Omega^1_X/\Omega^0_X\, df + (f - y)\Omega^1_X \to \Omega^1_{\widetilde{X}}/\Omega^0_{\widetilde{X}}\, d\tilde{f} + (\tilde{f} - y)\Omega^1_{\widetilde{X}}.$$

Since the ideal $(f - y)\mathcal{O}_X$ is radical for generic y (see [8]), this yields:

$$0 \to M/(f - y) \to \Omega^1\big(f^{-1}(y)\big) \to \Omega^1\big(\widetilde{f^{-1}(y)}\big).$$

So for generic y, we have the isomorphism $M/(f - y) \simeq M_y$. □

Lemma 4.4. *There exists an integer p such that* $\dim M_y = p$ *for generic* y.

Proof. By Lemma 4.2, $M_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module for a suitable P. If p denotes its rank, then $M/(f-y)$ has dimension p for generic y. Also for generic y, we have the isomorphism $M/(f-y) \simeq M_y$ by Lemma 4.3. Therefore M_y has dimension p for generic y. □

For the sake of simplicity, we introduce the following integer:

$$\mu_1'(f^{-1}(y), x) = \dim \frac{\Omega^1_{f^{-1}(y),x}}{d\Omega^0_{f^{-1}(y),x} + M_y \Omega^1_{f^{-1}(y),x}}.$$

Lemma 4.5. *For any y in $\mathbb{C}$, we have*:

$$\sum_{x \in f^{-1}(y)} \mu'\big(f^{-1}(y), x\big) = \dim M_y + \sum_{x \in f^{-1}(y)} \mu_1'\big(f^{-1}(y), x\big).$$

Proof. By construction and definition of $\mu'(f^{-1}(y), x)$ and $\mu_1'(f^{-1}(y), x)$, we clearly have:

$$\sum_{x \in f^{-1}(y)} \mu'\big(f^{-1}(y), x\big) = \sum_{x \in f^{-1}(y)} \dim \frac{d\Omega^0_{f^{-1}(y),x} + M_y \Omega^1_{f^{-1}(y),x}}{d\Omega^0_{f^{-1}(y),x}} + \sum_{x \in f^{-1}(y)} \mu_1'\big(f^{-1}(y), x\big).$$

Since M_y has a finite support, we have the isomorphism for any y:

$$M_y \simeq \bigoplus_{x \in f^{-1}(y)} M_y \Omega^0_{f^{-1}(y),x}.$$

In particular, we have the equality for any y:

$$\dim M_y = \sum_{x \in f^{-1}(y)} \dim M_y \Omega^0_{f^{-1}(y),x}.$$

So we only have to show that $\dim d\Omega^0_{f^{-1}(y),x} + M_y \Omega^0_{f^{-1}(y),x} / d\Omega^0_{f^{-1}(y),x} = \dim M_y \Omega^0_{f^{-1}(y),x}$, and this is equivalent to saying that $d\Omega^0_{f^{-1}(y),x}$ and $M_y \Omega^0_{f^{-1}(y),x}$ are in direct sum. Let ω be an element of $d\Omega^0_{f^{-1}(y),x} \cap M_y \Omega^0_{f^{-1}(y),x}$, and let R be a formal 1-form such that $\omega = dR$. Then we get:

$$\Pi^*(\omega) = 0 = \Pi^*(dR) = d\Pi^*(R).$$

So $\Pi^*(R)$ is locally constant in a neighborhood of $\Pi^{-1}(x)$, and R is constant in a neighborhood of x. Therefore $dR = 0$ and $\omega = 0$. □

Lemma 4.6. *Under the previous assumptions, there exists an integer N such that for any generic y and any point x in $f^{-1}(y)$, we have*:

$$I^N.\Omega^1_{f^{-1}(y),x} \subseteq d\Omega^0_{f^{-1}(y),x} + M_y.\Omega^0_{f^{-1}(y),x}.$$

Proof. Consider the set $\widetilde{S}$ in $\widetilde{X}$. Since $\widetilde{X}$ is normal, there exists a regular function g on $\widetilde{X}$ that vanishes along $\widetilde{S}$ and whose multiplicity equals 1. Up to cancelling a few fibres of $\tilde{f}$, we may assume that the map $(g, \tilde{f})$ is a submersion at any point x of $\widetilde{S}$, and that $(f - y)$ is a radical ideal in $\mathcal{O}_X$ for any y. According to Lemma 4.3, we may further assume that $\tilde{f}^{-1}(y)$ is the normalisation of $f^{-1}(y)$ for any y. Since $\widetilde{X} - \widetilde{S}$ and $X - S$ are isomorphic, we have the inclusion $\mathcal{O}_{\widetilde{X}} \subseteq (\mathcal{O}_X)_{(g)}$. Since $\mathcal{O}_{\widetilde{X}}$ is a finite $\mathcal{O}_X$-module, there exists an integer N such that $g^N \mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_X$. In particular g^N belongs to $\mathcal{O}_X$.

If x is a smooth point of $f^{-1}(y)$, then the inclusion to prove is obvious because every formal 1-form is exact. Assume that x is not a smooth point, and set $\Pi^{-1}(x) = \{x_1, \ldots, x_r\}$. Let ω be a formal 1-form in $\Omega^1_{f^{-1}(y),x}$ and set $\tilde{\omega} = \Pi^*(\omega)$. Since $(g, \tilde{f})$ is a submersion at any point x_i, we have:

$$\tilde{\omega} = a_i(\tilde{f}, g)\, d\tilde{f} + b_i(\tilde{f}, g)\, dg$$

where a_i, b_i are formal functions in two variables defined at $(\tilde{f}(x_i), g(x_i))$. Since by assumption, ω belongs to $I^N.\Omega^1_{f^{-1}(y),x}$ and $f^{-1}(y)$ is reduced for any y, we find:

$$\tilde{\omega} = g^N.\eta + \alpha\, d\tilde{f} + (\tilde{f} - y)\theta = a_i(\tilde{f}, g)\, d\tilde{f} + b_i(\tilde{f}, g)\, dg$$

where η, α, θ are formal expressions at x. By wedge product with $d\tilde{f}$, we get:

$$b_i(\tilde{f}, g)\, d\tilde{f} \wedge dg = g^N.d\tilde{f} \wedge \eta + (\tilde{f} - y)\, d\tilde{f} \wedge \theta.$$

Since $(g, \tilde{f})$ is a submersion at any point x_i, $b_i(\tilde{f}, g)$ must belong to the ideal $(g^N, \tilde{f} - y)$. We write $b_i(\tilde{f}, g)$ as:

$$b_i(\tilde{f}, g) = g^N d_i(\tilde{f}, g) + (\tilde{f} - y)c_i(\tilde{f}, g)$$

Let R_i be a formal series such that $R_i(\tilde{f}, 0) = 0$ and $\partial R_i/\partial g(\tilde{f}, g) = g^N d_i(\tilde{f}, g)$. Then we find:

$$\tilde{\omega} = d\big(R_i(\tilde{f}, g)\big) - \left(a_i + \frac{\partial R_i}{\partial \tilde{f}}\right)(\tilde{f}, g)\, d\tilde{f} - (\tilde{f} - y)c_i(\tilde{f}, g)\, dg.$$

Therefore $\tilde{\omega} = d(R_i(\tilde{f}, g))$ in $\Omega^1_{\tilde{f}^{-1}(y),x_i}$ for any x_i. Since $R_i(\tilde{f}, g)$ is divisible by g^N for any i, and that $g^N \mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_X$, there exists a formal function R at the point x in X such that:

$$N^*(R) = (R_1, \ldots, R_r)(\tilde{f}, g)$$

where N is the normalisation morphism of X. Since by assumption, it coincides with the normalisation morphism Π on $f^{-1}(y)$, we have $\Pi^*(R) = (R_1, \ldots, R_r)(\tilde{f}, g)$. So $\tilde{\omega} = \Pi^*(\omega) = \Pi^*(dR)$, $\omega - dR$ belongs to $M_y \Omega^0_{f^{-1}(y),x}$ and the result follows. □

Lemma 4.7. *Let y be a complex number such that $f^{-1}(y)$ is reduced. Let J be an ideal of $\mathcal{O}_X$ such that $J \subseteq I^N + (f - y)$ and $V(J)$ is finite. Then we have*:

$$\sum_{x \in f^{-1}(y)} \mu'_1\big(f^{-1}(y), x\big) = \dim \frac{\Omega^1(f^{-1}(y))}{d\Omega^0(f^{-1}(y)) + M_y + J.\Omega^1(f^{-1}(y))}.$$

Proof. Since $J \subseteq I^N + (f - y)$, Lemma 4.6 asserts that for any point x of $f^{-1}(y)$:

$$\frac{\Omega^1_{f^{-1}(y),x}}{d\Omega^0_{f^{-1}(y),x} + M_y.\Omega^0_{f^{-1}(y),x} + J.\Omega^1_{f^{-1}(y),x}} = \frac{\Omega^1_{f^{-1}(y),x}}{d\Omega^0_{f^{-1}(y),x} + M_y.\Omega^0_{f^{-1}(y),x}}.$$

By definition of the $\mu_1'(f^{-1}(y),x)$, it suffices to show that the natural map:

$$L: \frac{\Omega^1(f^{-1}(y))}{d\Omega^0(f^{-1}(y)) + M_y + J.\Omega^1(f^{-1}(y))} \to \bigoplus_{x\in f^{-1}(y)} \frac{\Omega^1_{f^{-1}(y),x}}{d\Omega^0_{f^{-1}(y),x} + M_y.\Omega^0_{f^{-1}(y),x} + J.\Omega^1_{f^{-1}(y),x}}$$

induced by the inclusion is an isomorphism.

First let us show that L is onto. Let ω be an element of the sum on the right and set $\{x_1,\dots,x_r\} = V(J)\cap f^{-1}(y)$. Since $V(J)$ is finite, there exist some regular 1-forms $\omega_1,\dots,\omega_r$ such that $\omega-\omega_i$ belongs to $J.\Omega^1_{X,x_i}$ for any i. Since $V(J)$ is finite, there exist some regular functions $f_1,\dots,f_r$ on X such that f_i-1 belongs to $J.\Omega^0_{X,x_i}$ and f_i belongs to $J.\Omega^0_{X,x_j}$ for $j\neq i$. Consider the following regular 1-form:

$$\Omega = \sum_{i=1}^r f_i\omega_i.$$

By construction, $\Omega-\omega_i$ belongs to $J.\Omega^1_{X,x_i}$ for any i, and $L(\Omega)=\omega$, which proves surjectivity.

Let us show now that L is injective. Let Ω be an element on the left such that $L(\Omega)=0$. Then there exists a formal function $R=(R_1,\dots,R_r)$ on the germ $(f^{-1}(y),\{x_1,\dots,x_r\})$ such that $\Omega = dR$. Since $V(J)$ is finite, there exist some regular functions $S_1,\dots,S_r$ such that R_i-S_i belongs to $J^2.\Omega^0_{X,x_i}$ for any i. Since $V(J)$ is finite, there exist also some regular functions g_i such that g_i-1 belongs to $J^2.\Omega^0_{X,x_i}$ and g_i belongs to $J^2.\Omega^0_{X,x_j}$ for $j\neq i$. Consider the following regular function:

$$S = \sum_{i=1}^r S_i g_i.$$

By construction, we easily check that $dS \equiv dS_i \equiv dR_i [J.\Omega^0_{X,x_i}]$ for any i. Therefore $\Omega = dR = dS$ and its class is zero, which proves injectivity. $\square$

Lemma 4.8. *Assume that the surface X is embedded in $\mathbb{C}^n$, with coordinates $(x_1,\dots,x_n)$. Then there exist some non-zero polynomials $P_1(y,x_1),\dots,P_n(y,x_n)$ such that for generic y, the ideal $J=(P_1,\dots,P_n)$ enjoys the conditions of Lemma* 4.7.

Proof. Since S has dimension $\leqslant 1$, there exist some non-zero polynomials $Q_i(y,x_i)$ such that $Q_i(f,x_i)=0$ on S for any $i=1,\dots,n$. Let N be the integer given by Lemma 4.6 and set $P_i(y,x_i) = Q_i(y,x_i)^N$. By construction the ideal $J=(P_1,\dots,P_n)$ has a finite support for generic y (in fact for any y such that $Q_i(y,.)$ is not identically zero). Moreover $J\subseteq I^N + (f-y)$ for generic y and the conditions of Lemma 4.7 are satisfied. $\square$

Proof of Proposition 1.3. Let $P_1,\dots,P_n$ be the polynomials given by Lemma 4.8. For generic y, the ideal $J=(P_1,\dots,P_n)$ has finite codimension in $\mathbb{C}[x_1,\dots,x_n]$, and $\mathbb{C}[x_1,\dots,x_n]/J$ admits a basis $\{e_1,\dots,e_s\}$ consisting of the monomials of degree $< n_i$ in x_i for any i, where n_i is the degree of P_i in the variable x_i. This is easy to see by a Euclidean division with respect to every P_i. We denote by E_1 the vector subspace of $\mathbb{C}[x_1,\dots,x_n]$ spanned by $e_1,\dots,e_s$, by E_2 the vector subspace spanned by the e_ie_j and by E_3 the vector space spanned by the $e_i\,dx_j$.

Let J' be the ideal defining the surface X in $\mathbb{C}^n$, and let $f_1, \dots, f_r$ be a system of generators of J'. Let $m_1, \dots, m_p$ be a system of generators of the module $M_{(P(f))}$. Every regular 1-form ω on X can be represented by a polynomial 1-form Ω on $\mathbb{C}^n$. Then ω belongs to $d\Omega^0(f^{-1}(y)) + M_y + J.\Omega^1(f^{-1}(y))$ if and only if Ω can be written as:

$$\Omega = dR + \sum_{i=1}^{r} f_i \eta_i + \sum_{i=1}^{r} a_i \, df_i + a \, df + (f - y)\eta + \sum_{k=1}^{p} \alpha_k M^k + \theta$$

where R belongs to E_2, $\eta, \eta_1, \dots, \eta_r$ to E_3, $a, a_1, \dots, a_r$ to E_1, $\alpha_1, \dots, \alpha_p$ to $\mathbb{C}$ and θ to $J.\Omega^1(\mathbb{C}^n)$. This can be done by performing a Euclidean division by the P_i on all the terms of this sum, except for R where we perform a division with respect to the $P_i P_j$. Consider the linear map:

$$F : E_2 \times E_3^{r+1} \times E_1^{r+1} \times \mathbb{C}^p \to E', \quad (R, \eta, \eta_1, \dots, \eta_r, a, a_1, \dots, a_r, \alpha_1, \dots, \alpha_p) \to \omega$$

where ω is the reduction modulo J of the 1-form:

$$\Omega = dR + \sum_{i=1}^{r} f_i \eta_i + \sum_{i=1}^{r} a_i \, df_i + a \, df + (f - y)\eta + \sum_{k=1}^{p} \alpha_k M^k.$$

Note that for generic y, L is a linear map between spaces of finite dimension. In the monomials bases given below, it is represented by a matrix $A(y)$ whose entries belong to $\mathbb{C}(y)$, because reduction modulo J is performed via a Euclidean division with respect to the P_i, which are polynomials with coefficients in $\mathbb{C}[y]$. Then the rank of this matrix is given by the size of its biggest non-zero minors, which belong to $\mathbb{C}(y)$. So there exists an integer l such that for generic y, $A(y)$ has rank l. By Lemma 4.7, we find for generic y:

$$\sum_{x \in f^{-1}(y)} \mu_1'\big(f^{-1}(y), x\big) = \dim \Omega^1(\mathbb{C}^n)/J - l.$$

Therefore by Lemmas 4.4 and 4.5, we get that, for generic y:

$$\sum_{x \in f^{-1}(y)} \mu'(f^{-1}(y), x) = p + \dim \Omega^1(\mathbb{C}^n)/J - l. \qquad \square$$

5. Relative cohomology

Let X be a complex irreducible affine surface, and $f : X \to \mathbb{C}$ a dominant morphism. Denote by $\Omega^k(X)$ the space of regular k-forms on X. The first group of truncated relative cohomology of f is the quotient:

$$H^1(f) = \frac{\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)\, df}.$$

Note that $H^1(f)$ is a $\mathbb{C}[f]$-module via the multiplication $(P(f), \omega) \mapsto P(f)\omega$. In the case of analytic germs f, relative cohomology groups have been extensively used to describe the topological and cohomological properties of f; for more details, see for instance [9]. In the algebraic setting, the relative cohomology of polynomial mappings has been intensively studied, especially via the use of the Gauss–Manin connexion (see for instance [1]).We are going to study some properties of truncated relative cohomology and use them to prove Theorem 1.4.

5.1. Rank of $H^1(f)$

Our purpose in this subsection is to compute the rank of $H^1(f)$ as a $\mathbb{C}[f]$-module. We begin with the following lemma.

Lemma 5.1. *Let X be a complex affine surface and $f : X \to \mathbb{C}$ a dominant morphism. If $(f - y)$ is a radical ideal in $\mathcal{O}_X$, then $H^1(f)/(f-y) \simeq H^1(f^{-1}(y))$. In particular, this holds for generic y.*

Proof. By definition, we have a first isomorphism:

$$H^1(f)/(f-y) \simeq \frac{\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)df + (f-y)\Omega^1(X)}$$
$$\simeq \frac{\Omega^1(X)/\Omega^0(X)\,df + (f-y)\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)\,df + (f-y)\Omega^1(X)/\Omega^0(X)\,df + (f-y)\Omega^1(X)}.$$

Since $(f-y)$ is a radical ideal in $\mathcal{O}_X$, the restriction morphism induces an isomorphism:

$$\Omega^1(X)/\Omega^0(X)\,df + (f-y)\Omega^1(X) \simeq \Omega^1\big(f^{-1}(y)\big).$$

From that we deduce $H^1(f)/(f-y) \simeq \Omega^1(f^{-1}(y))/d\Omega^0(f^{-1}(y)) = H^1(f^{-1}(y))$. $\square$

Proposition 5.2. *The rank of $H^1(f)$ is equal to h_f.*

Proof. First we prove by contradiction that $rk(H^1(f)) \leqslant h_f$. Let $\omega_1, \dots, \omega_{r+1}$ be some $\mathbb{C}[f]$-linearly independent elements of $H^1(f)$, where $r = h_f$. By definition of r and Lemma 5.1, $H^1(f)/(f-y)$ has dimension r for generic y. So for generic y, there exist some constants λ_i^y, not all zero, such that:

$$\lambda_1^y \omega_1 + \dots + \lambda_{r+1}^y \omega_{r+1} \equiv 0\big[(f-y)\big].$$

Then there exists an element η^y of $\Omega^1(X)$ such that:

$$\lambda_1^y \omega_1 + \dots + \lambda_{r+1}^y \omega_{r+1} = (f-y)\eta^y.$$

The collection $\{\eta^y\}$ is uncountable because it is indexed on a non-empty Zariski open set. Since $\Omega^1(X)$ has countable dimension, $\{\eta^y\}$ is not linearly free, and there exists some distinct points $\{y_1, \dots, y_m\}$ together with non-zero constants $\{\delta^1, \dots, \delta^m\}$ such that:

$$\delta^1 \eta^{y_1} + \dots + \delta^m \eta^{y_m} = 0.$$

By replacing each η^y with its expression, we find:

$$\sum_{j=1}^{r+1} \left(\sum_{i=1}^{m} \frac{\delta^i \lambda_j^{y_i}}{f - y_i} \right) \omega_j = 0.$$

Since $\delta^i \neq 0$ for any i, and that the $\lambda_j^{y_i}$ are not all zero for any j, the coefficients of this sum are not all zero. Therefore $\omega_1, \dots, \omega_{r+1}$ are $\mathbb{C}[f]$-linearly dependent, hence a contradiction. In particular the rank of $H^1(f)$ is finite.

Second we prove that $\mathrm{rk}(H^1(f)) \geqslant h_f$. Let $\omega_1, \dots, \omega_s$ be a maximal collection of $\mathbb{C}[f]$-linearly independent elements of $H^1(f)$. Let $\{g_i\}_{i \in \mathbb{N}}$ be a countable set of generators of $\Omega^1(X)$

as a $\mathbb{C}$-vector space. For any i, there exist some polynomials $P_i, a_{i,1}, \dots, a_{i,r}$ with $P_i \neq 0$ such that:

$$P_i(f)g_i = a_{i,1}(f)\omega_1 + \dots + a_{i,s}(f)\omega_s$$

in $H^1(f)$. We fix the P_i and consider the set Z of all roots of all P_i. This set is at most countable by construction. Since any non-empty Zariski open set of $\mathbb{C}$ is uncountable, there exists a complex number y that does not belong to Z and such that $(f - y)$ is radical. By reduction modulo $(f - y)$, we get that:

$$P_i(y)g_i \equiv a_{i,1}(y)\omega_1 + \dots + a_{i,r}(y)\omega_r\big[(f - y)\big].$$

Since every $P_i(y)$ is non-zero, every g_i is spanned by the classes of $\omega_1, \dots, \omega_s$ in $H^1(f)/(f - y)$. Since the g_i form a system of generators of $H^1(f)$, $H^1(f)/(f - y)$ is spanned by the classes of $\omega_1, \dots, \omega_s$. In particular $h_f \leqslant s = \operatorname{rk}(H^1(f))$. □

5.2. The property $\mathcal{P}$

In this subsection, we are going to prove the inequality given in Theorem 1.4 by using a special property of the relative cohomology group $H^1(f)$. This property will enable us to control the dimension of $H^1(f^{-1}(t))$ by means of the rank of $H^1(f)$.

Definition 5.3. A $\mathbb{C}[f]$-module M satisfies the property $\mathcal{P}(y)$ if for any integer r and any element ω of M, we have: $(f - y)^r\omega = 0 \Rightarrow \omega \in (f - y)M$.

Lemma 5.4. *Let M be a $\mathbb{C}[f]$-module satisfying $\mathcal{P}(y)$. Then* $\dim M/(f - y) \leqslant \operatorname{rk} M$.

Proof. Let $e_1, \dots, e_s$ be some elements of M whose classes in $M/(f - y)$ are free. In order to establish the lemma, we prove by contradiction that $e_1, \dots, e_s$ are free in M. Assume there exist some polynomials $P_1(f), \dots, P_s(f)$ not all zero such that $P_1(f)e_1 + \dots + P_s(f)e_s = 0$ in M. Let m be the minimum of the orders of the P_i at y. Every $P_i(f)$ can be written as $P_i(f) = (f - y)^m T_i(f)$ where at least one of the $T_i(y)$ is nonzero. So we get:

$$(f - y)^m\big\{T_1(f)e_1 + \dots + T_s(f)e_s\big\} = 0.$$

By the property $\mathcal{P}(y)$, this implies:

$$T_1(f)e_1 + \dots + T_s(f)e_s \equiv T_1(y)e_1 + \dots + T_s(y)e_s \equiv 0\big[(f - y)\big].$$

Since the e_i are free modulo $(f - y)$, every $T_i(y)$ is zero, hence a contradiction. □

Our purpose in this subsection is to prove:

Proposition 5.5. *Let X be a complex irreducible affine surface, and $f : X \to \mathbb{C}$ a dominant morphism. Assume that X is locally a complete intersection. If $f^{-1}(y) \cap \operatorname{Sing}(f)$ is finite, then $H^1(f)$ satisfies the property $\mathcal{P}(y)$.*

Since X is locally a complete intersection, the finiteness of $f^{-1}(y) \cap \operatorname{Sing}(f)$ implies that $(f - y)$ is a radical ideal in $\mathcal{O}_X$. By Lemma 5.1, we have $H^1(f)/(f - y) \simeq H^1(f^{-1}(y))$. So Theorem 1.4 will follow from Lemma 5.4 and Proposition 5.5. We begin with a few lemmas.

Lemma 5.6. *Let X be a complex affine surface that is locally a complete intersection. Let ω be a regular* 1*-form on X and A a regular function on X such that $(f-y)\omega = A\,df$. If $f^{-1}(y) \cap \operatorname{Sing}(f)$ is finite, there exists a regular function B on X such that $\omega = B\,df$.*

Proof. Let ω be a regular 1-form on X and A a regular function on X such that $(f-y)\omega = A\,df$. Then A vanishes on the set $f^{-1}(y) - \operatorname{Sing}(f)$. Since $f^{-1}(y)$ is equidimensional of dimension 1 and $f^{-1}(y) \cap \operatorname{Sing}(f)$ is finite, A vanishes on $f^{-1}(y)$. Since $f^{-1}(y) \cap \operatorname{Sing}(f)$ is finite and X is locally a complete intersection, $f^{-1}(y)$ defines locally a complete intersection. Hence it is a complete intersection on X, and $(f-y)$ divides A. If $A = (f-y)B$, then $(f-y)(\omega - B\,df) = 0$. Since X is locally a complete intersection, the module $\Omega^1(X)$ is torsion-free (see [6]) and $\omega = B\,df$. □

Lemma 5.7. *Let X be a complex irreducible affine surface and $f : X \to \mathbb{C}$ a dominant morphism. Let $C_1, \dots, C_r$ be the connected components of $f^{-1}(t)$ and n an integer $\geqslant 0$. Then there exist some regular functions $S_{i,n}$ on X such that $S_{i,n} = 1$ on C_i, $S_{i,n} = 0$ on C_j for $j \neq i$ and $dS_{i,n}$ belongs to $(f-t)^{n+1}\Omega^1(X)$.*

Proof. For simplicity, assume that $t = 0$. There exists a regular function T_i on X such that $T_i = 1$ on C_i and $T_i = 0$ on C_j for $j \neq i$. Then $T_i(1-T_i)$ vanishes on $f^{-1}(0)$ and by Hilbert's Nullstellensatz, there exists an integer m such that $T_i^m(1-T_i)^m$ belongs to $f^{n+1}\mathcal{O}_X$. We set:

$$P_i(x) = \int_0^x t^m(1-t)^m\,dt \quad \text{and} \quad R_{i,n} = P_i(T_i).$$

By construction the 1-form $dR_{i,n} = T_i^m(1-T_i)^m\,dT_i$ is divisible by f^{n+1}. Since $P_i(0) = 0$ and T_i vanishes on C_j for $j \neq i$, $R_{i,n}$ vanishes on C_j if $j \neq 0$. Since $P_i(1) \neq 0$, $R_{i,n} = P_i(1) \neq 0$ on C_i. Then choose $S_{i,n} = R_{i,n}/P_i(1)$. □

Lemma 5.8. *Let X be a complex irreducible affine surface and $f : X \to \mathbb{C}$ a dominant morphism. Let R be a regular function on X such that $dR = A\,df + (f-t)\eta$, where A, η are regular on X. Then R is locally constant on $f^{-1}(t)$.*

Proof. Since $dR = A\,df + (f-t)\eta$, the restriction of dR to $f^{-1}(t)$ is zero. So R is singular at any smooth point of $f^{-1}(t)$, and R is constant on every connected component of the smooth part of $f^{-1}(t)$. By continuity and density, R is constant on every connected component of $f^{-1}(t)$, hence it is locally constant on $f^{-1}(t)$. □

Proof of Proposition 5.5. Let X be a complex irreducible affine surface that is locally a complete intersection. Let $f : X \to \mathbb{C}$ be a dominant morphism and assume that $f^{-1}(t) \cap \operatorname{Sing}(f)$ is finite. We may assume that $t = 0$. Let us prove by induction on $n \geqslant 0$ that, if $f^n\omega = 0$ in $H^1(f)$, then ω belongs to $(f)H^1(f)$. This is trivial for $n = 0$. Assume that the assertion holds to the order n. Let ω be a regular 1-form on X such that $f^{n+1}\omega = 0$ in $H^1(f)$. Then there exist some regular functions R, A such that $f^{n+1}\omega = dR + A\,df$ on $\Omega^1(X)$. By Lemma 5.8, R is locally constant on $f^{-1}(0)$. Let $C_1, \dots, C_r$ be the connected components of $f^{-1}(0)$. If R takes the value λ_i on C_i, then the function:

$$R' = R - \sum_i \lambda_i S_{i,n+1}$$

vanishes on $f^{-1}(0)$. By construction, there exists a regular 1-form η such that:

$$f^{n+1}\omega = dR' + A\,df + f^{n+2}\eta.$$

Since $f^{-1}(0) \cap \mathrm{Sing}(f)$ is finite and X is locally a complete intersection, (f) is a radical ideal and R' is divisible by f. If $R' = fS$ with S regular, we obtain:

$$f(f^n\omega - dS - f^{n+1}\eta) = (A+S)\,df.$$

By Lemma 5.6, there exists a regular function B such that:

$$f^n(\omega - f\eta) = dS + B\,df.$$

By induction $(\omega - f\eta)$ belongs to $(f)H^1(f)$, as well as ω, and we are done. □

6. An example

We end this paper with an example of a surface that is not locally a complete intersection. For that surface there exists a map for which the conclusion of Theorem 1.4 fails. Let (u, v, w_1, w_2) be a system of coordinates in $\mathbb{C}^4$, and consider the affine set X of $\mathbb{C}^4$ defined by the equations:

$$u^2w_1 - v^2 = 0, \qquad u^3w_2 - v^3 = 0, \qquad w_1^3 - w_2^2 = 0.$$

Note that X can be reinterpreted as:

$$X = \mathrm{Spec}\big(\mathbb{C}[x, xy, y^2, y^3]\big).$$

So X is an irreducible surface. Moreover 0 is the only singular point of X, but X is not locally a complete intersection. Indeed if it were so, then X would be a normal surface because it is non-singular in codimension 1. Consider the function $h = w_2/w_1 = v/u$ on X. It is well-defined and regular outside the origin, hence h is regular because X is normal. Moreover we have the following relations:

$$v = hu, \qquad w_1 = h^2, \qquad w_2 = h^3.$$

So every regular function on X can be expressed as a polynomial in (u, h), and X is isomorphic to $\mathbb{C}^2$. But this is impossible because X is singular at the origin. Consider now the map $f : X \to \mathbb{C}$ defined by:

$$f(u, v, w_1, w_2) = u.$$

For $y \neq 0$, the fibre $f^{-1}(y)$ is isomorphic to a line, hence $H^1(f^{-1}(y)) = 0$. The fibre $f^{-1}(0)$ is isomorphic to a cusp, hence contractible, and $f^{-1}(0) \cap \mathrm{Sing}(f)$ is reduced to the origin. Moreover its Milnor number coincides with its local Betti number and is equal to 2. With the notations of the previous sections, $h_f = 0$ and $\dim H^1(f^{-1}(0)) = 2$, so that $\dim H^1(f^{-1}(0)) \nleq h_f$.

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