Computers Math. Applic. Vol. 26, No. 1, pp. 35–52, 1993
Printed in Great Britain.

0898-1221/93 \$6.00 + 0.00

Existence of Similarity Solutions for Surface-Tension Driven Flows in Floating Rectangular Cavities*

Y.-Y. CHEN
Institute of Applied Mathematics, National Chiao Tung University
Hsinchu, Taiwan 300, R.O.C.

T.-W. HWANG
Department of Mathematics, National Kaohsiung Normal University
Kaohsiung, Taiwan 802, R.O.C.

C.-A. WANG**
Department of Mathematics, National Chung Cheng University
Minghsiung, Chiayi, Taiwan 621, R.O.C.

(Received September 1992; accepted October 1992)

Abstract—We consider the following problem:

$$f''' + Q \cdot [Aff'' - (f')^2] = \beta, \qquad (Q, \beta \in \mathbb{R},\ A \geq 0),$$
$$f(0) = f(1) = f''(1) = f''(0) + 1 = 0,$$

which arises from the steady surface-tension driver flows in floating rectangular cavities. In this paper, we first classify all possible solutions and obtain that the given problem can only possess, at most, three types of solutions for $A \geq 1$. By the classification, we further verify that for every $Q \geq 0$ or $\beta \geq 0$, the problem has at least one solution if $A \geq 1$. Moreover, if $1 \leq A \leq 3/2$, multiple solutions also exist for sufficiently large $Q > 0$.

Keywords—Similarity solutions, Classification, Shooting method.

1. INTRODUCTION

We study the following boundary value problem:

$$f''' + Q[Aff'' - (f')^2] = \beta, \qquad \left(' = \frac{d}{d\eta}\right) \tag{1}$$

for $0 \leq \eta \leq 1$, subject to the boundary conditions

$$f(0) = f(1) = f''(1) = f''(0) + 1 = 0. \tag{2}$$

The given problem arises from a reduction by similarity of the boundary layer formulation of the Navier-Stokes system under the assumption of negligible "end effect" by Gill *et al.* [1,2]. The Navier-Stokes system was applied to describe the steady state for the distributions of velocity in a low Prandtl (Pr) number fluid in floating rectangular cavities. By means of floating, it was assumed that two opposite surfaces of the cavity are free, and due to the physical consideration,

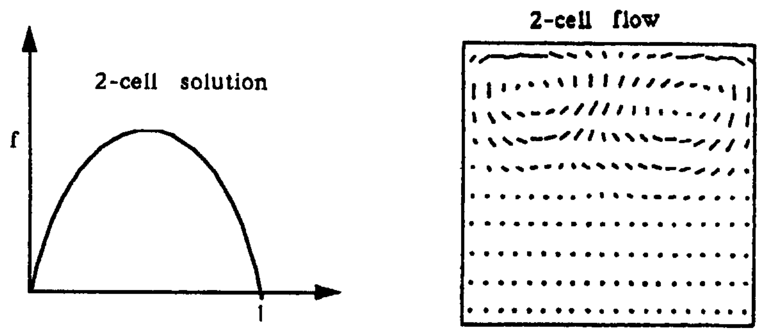
*The work is supported in part by the National Science Councils of R.O.C.
**The author is to receive correspondence.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

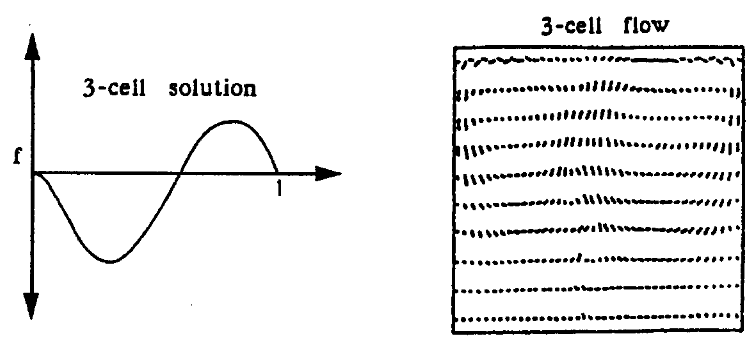
26:1-D

the floating zones are assumed to be in a microgravity environment. Usually, the lateral solid surfaces (wall) of the cavity and the free surfaces confine the flow in a low Prandtl number liquid. On the free surfaces, a temperature radiation, due to the difference of temperatures at the mid-line of the cavity and at the lateral solid wall, is assumed, and the radiation drives the surface-tension driven flows.

In [1], it was obtained that the velocity profiles (u, v) of the plane flow lying on the upper (lower) half of a vertical cross section, in the rectangular coordinate system (x, y) while the x-direction is normal to the free surface, can be described by $(u, v) = (c_1\delta_1(x)f'(\eta), (-c_2\eta f'(\eta) + c_3 f(\eta))\delta_2(x))$ with the similarity variable $\eta = y/\delta_3(x)$ for some positive $\delta_i(x)$'s and constants c_i's. A solution f of (1), (2) is called *two-cell* if f' has exactly one zero in (0,1), as in Figure 1(a), which yields the resultant flow with two cells lying on the half of the cross section. Correspondingly, f is called *three-cell* if f' has exactly two zeros in (0,1). Then, as in Figure 1(b), the surface-tension driven flow yields the third cell lying near the mid-line. In [1,2], 2-cell or 3-cell flows were found numerically when $A = 1, 2$. Also, multiple solutions were obtained when $A = 1$ as shown in Figure 2. However, only a small portion of mathematical results had been provided by Lu *et al.* and Lu [3,4].



(a) Two-cell solution.



(b) Three-cell solution.

Figure 1. The selected graph of solutions of (1), (2) and contour plot of the resultant plane flow.

In this paper, it is our main purpose to extend the study in [3,4] to the case of $A \geq 1$. In Section 2, we classify solutions of (1), (2) by studying an equivalent boundary value problem with the shooting scheme. Then, by the classification, the existence of solutions has been verified in Section 3. Our study implies the following result:

I. *The problem (1), (2) can only possess two- and three-cell solutions for* $1 \leq A < 2$ *and two-cell solutions if* $A \geq 2$.
II. *Let* $A \geq 1$. *For* $Q \geq 0$, *the problem (1), (2) has at least one two-cell solution. Moreover, there is at least one* $Q < 0$ *such that (1), (2) has a two-cell solution for* $\beta > 1$.

III. *Let $1 \leq A \leq 2/3$. The problem (1), (2) has at least one three-cell solution for sufficiently large $Q > 0$.*

2. CLASSIFICATION

Note that for $A \geq 0$, the 2-cell solution $f_0(\eta) = \eta(\eta-1)(\eta-2)/6$ solves (1), (2) uniquely if $Q = 0$, and then $\beta = 1$. Therefore, we shall consider the case of $Q \neq 0$ in the following discussions. By applying the transformation as in [3,5,6], let $y = b(1-\eta)$ and $g(y) = (Q/b)f(\eta)$, for any nonzero Q and positive b. Then, (1), (2) is equivalent to

$$g''' + (g')^2 - Agg'' = -\frac{Q\beta}{b^4}, \qquad \left(' = \frac{d}{dy}\right), \tag{3}$$

$$g(0) = g(b) = g''(0) = g''(b) + \left(\frac{Q}{b^3}\right) = 0. \tag{4}$$

Let $B = g'(0)$ and $E = -Q\beta/b^4$. By assuming values A, B, and E, we can integrate an initial value problem of (3) subject to the conditions

$$g(0) = g'(0) - B = g''(0) = 0. \tag{5}$$

Suppose the solution $g(y; A, B, E)$ of (3), (5) meets the y-axis at some positive y_*. Then, by setting $b = y_*$, (1), (2) has a solution when $Q = -y_*^3 g''(y_*)$ and $\beta = Ey_*/g''(y_*)$. Meanwhile, the number of positive zeros of f' for (1), (2) is the same as the one of g' for (3), (4).

Therefore, we may classify solutions of (1), (2) by assigning values of A, B, and E to (3), (5). It is clear that $g(y; A, B, B^2) = By$ solves (3), (5) uniquely and it has no positive zero for $A \geq 0$. This reduces the study on following regions:

$$D_1 = \{(B,E) \mid B \leq 0,\ E < B^2\},$$
$$D_2 = \{(B,E) \mid B > 0,\ E < B^2\},$$
$$D_3 = \{(B,E) \mid B \geq 0,\ E > B^2\},$$

and

$$D_4 = \{(B,E) \mid B < 0,\ E > B^2\},$$

for given $A \geq 0$. For the simplicity, we write $g(y; B, E) = g(y; A, B, E)$ or $g(y) = g(y; A, B, E)$. Also, let $[0, M)$ be the corresponding maximal interval of $g(y)$ for some $M = M(A, B, E) \leq \infty$. In fact, $g(y)$ can only blow up to either $+\infty$ or $-\infty$ if $M < \infty$.

2.1. $(B, E) \in D_1$

It is clear that g, g', g'', and $g''' < 0$ initially. Then we have the following theorem.

THEOREM 2.1.1. *For $A \geq 0$, $g(y) < 0$ on $(0, M)$.*

PROOF. Suppose we verify that $g'' < 0$ on $(0, M)$. Then g, g' are negative on $(0, M)$ and the desired result follows immediately.

Assume that g'' has the first zero at $y = y_0''$ on $(0, M)$. It is clear that $g'''(y_0'') \geq 0$ and $g'(y_0'') < g'(0) = B$. But, from (3), we get that $g'''(y_0'') < E - B^2 < 0$. This is a contradiction and completes the proof. ∎

2.2. $(B,E) \in D_2$

Now, g, $g' > 0$ and g'', $g''' < 0$ initially.

THEOREM 2.2.1. *If $A \geq 0$ and $E \leq 0$, then $g(y)$ has exactly one positive zero.*

PROOF. Suppose we verify that $g'' < 0$ and $g'(y)$ has exactly one zero on $(0, M)$. Then the desired result is obtained.

Assume that y_0'' is the first positive zero of g''. Then, as in Theorem 2.1.1, $g'''(y_0'') \geq 0$. But, at $y = y_0''$, we get $g''' = E - (g')^2 \leq 0$ since $E \leq 0$. This implies that $g''' < 0$ at $y = y_0''$ if $E < 0$ and it violates the definition of y_0''. Suppose $E = 0$. This implies that $g''' = g' = 0$ at $y = y_0''$. Then $g(y) \equiv g(y_0'')$ is the solution of (3) subject to $g' = g'' = g''' = 0$ at $y = y_0''$, and this violates the condition $g'(0) = B > 0$. Thus, $g'' < 0$ on $(0, M)$.

Now suppose $g' > 0$ on $(0, M)$. It is clear that $M = \infty$ since g' is bounded. Then $g > 0$ and $g''' = E - (g')^2 + Agg'' < 0$ on $(0, \infty)$. This implies that g' is concave on $(0, \infty)$ and it leads to a contradiction. Hence, g' must have a unique zero on $(0, M)$ and g has a unique positive zero, say at $y = y_0$. ∎

By setting $b = y_0$, the corresponding pair (Q, β), satisfying $Q > 0$, $\beta \geq 0$, yields a 2-cell solution f of (1), (2) with $f''' > 0$ on $(0, 1)$.

COROLLARY 2.2.1. *For each $A \geq 0$, there exist constants $Q > 0$, $\beta \geq 0$ such that the problem (1), (2) possesses a two-cell solution f with $f''' > 0$ on $(0, 1)$.*

For the case of $E > 0$, it has been shown [3] that $g(y)$ possesses exactly one positive zero for $1 < A < 2$. We can extend the result to the case of $A \geq 1$.

For $A > 2$, it is required to show that g''' can only possess at most one positive zero on $(0, M)$. Suppose $g''' < 0$ on $(0, M)$. Then $g'' < 0$, g' is concave on $(0, M)$, and then, g has a unique zero, say at $y = y_1 > 0$. Now, we assume that g''' has the first positive zero at $y = y_0'''$ and then, $g''(y_0''') < 0$.

LEMMA 2.2.1. *$g(y)$ has exactly one zero on $(0, y_0''')$.*

PROOF. Assume that $g > 0$ on $(0, y_0''')$. Differentiating (3), we get that

$$\begin{aligned} g^{(\text{iv})} &= Agg''' + (A-2)g'g'' && (6) \\ &= 2gg''' + (A-2)(gg'')'. && (7) \end{aligned}$$

Therefore, $g^{(\text{iv})} < (A-2)(gg'')'$ and $g''' - (A-2)gg''$ is decreasing on $(0, y_0''')$. Then, $g''' - (A-2)gg'' < E - B^2 < 0$, on $(0, y_0''')$. This yields that $(A-2)gg''$, $g'' \geq 0$ at $y = y_0'''$ and contradicts the fact that $g''(y_0''') < 0$. Thus, g has a unique zero at $y = y_1$ on $(0, y_0''')$ since $g'' < 0$. This proves Lemma 2.2.1.

To show that y_1 is unique on $(0, M)$, it is required to verify the next lemma.

LEMMA 2.2.2. *$g''(y) < 0$ on $(0, M)$.*

PROOF. Suppose that g'' has the first positive zero at $y = y_0'' > y_0'''$. Then, $g''' \geq 0$ and $g < 0$ at $y = y_0''$. Also, we have that $g' < 0$ and $(g')^2 = E - g''' + Agg'' > E$ at $y = y_0''$. Therefore, $g'(y_0'') < g'(y_0''') < -\sqrt{E}$ and

$$\begin{aligned} g'''(y_0'') &= E - (g'(y_0''))^2 \\ &< E - (g'(y_0'''))^2 < 0. \end{aligned}$$

This leads to a contradiction again and completes the proof of Lemma 2.2.2.

We turn to show that g''' has no positive zero other than y_0'''. Suppose $y_1''' > y_0'''$ is the second positive zero of g'''. Then, $g' < 0$, $g^{(\text{iv})} > 0$ at $y = y_0'''$ and $g^{(\text{iv})} \leq 0$ at $y = y_1'''$. However, from (6), $g^{(\text{iv})} = (A-2)g'g'' > 0$ at $y = y_1'''$. This is impossible. Thus, g''' possesses exactly one positive zero and y_1 is the unique zero of g on $(0, M)$.

For the case $A = 1$, we first show that $g^{(\mathrm{iv})} < 0$ on $(0, M)$. Differentiating (6), we get that

$$g^{(\mathrm{v})} = Agg^{(\mathrm{iv})} + (2A-2)g'g''' + (A-2)(g'')^2, \tag{8}$$
$$g^{(\mathrm{vi})} = Agg^{(\mathrm{v})} + (3A-2)g'g^{(\mathrm{iv})} + (4A-6)g''g''', \tag{9}$$

and

$$g^{(\mathrm{vii})} = Agg^{(\mathrm{vi})} + (4A-2)g'g^{(\mathrm{v})} + (7A-8)g''g^{(\mathrm{iv})} + (4A-6)(g''')^2. \tag{10}$$

It is clear that $g^{(\mathrm{iv})} = g^{(\mathrm{v})} = g^{(\mathrm{vi})} = 0$ and $g^{(\mathrm{vii})} < 0$ at $y = 0$. This implies $g^{(\mathrm{iv})} < 0$ initially. Suppose $g^{(\mathrm{iv})}$ has the first positive zero at $y = d$, and then, $g^{(\mathrm{v})}(d) \geq 0$. If $g''(d) \neq 0$, then, from (8), $g^{(\mathrm{v})}(d) < 0$ and $g^{(\mathrm{iv})}$ is strictly decreasing in a neighborhood of d. However, this violates the definition of d. Hence, $g''(d) = 0$. If $g'''(d) \neq 0$, then, from (7)–(10), $g^{(\mathrm{iv})} = g^{(\mathrm{v})} = g^{(\mathrm{vi})} = 0$ and $g^{(\mathrm{vii})} < 0$ at $y = d$. This contradicts the definition of d again, and then, $g'''(d) = 0$. But $g(y) = g(d) + g'(d)(y-d)$ solves (3), subject to $g'' = g''' = 0$ at $y = d$, uniquely. However, the condition $g'''(0) = E - B^2 \neq 0$ is violated. Then $g^{(\mathrm{iv})}$, $g''' < 0$ on $(0, M)$ and g has exactly one positive zero, say at $y = y_1$.

For $A = 2$, from (6), $g'''(y) = (E - B^2)\exp(-\int_0^y g) < 0$, then the desired result follows immediately. Hence, we have verified the following theorem.

THEOREM 2.2.2. *If $A \geq 1$ and $E > 0$, then $g(y)$ has exactly one positive zero.*

Note that the property that $g^{(\mathrm{iv})} < 0$ on $(0, M)$ if $A = 1$ is always valid for (B, E) in D_i, $i = 1, \dots, 4$. Now, by setting $b = y_1$, the corresponding pair (Q, β) with $Q > 0$, $\beta < 0$ yields a 2-cell solution f of (1), (2) with $f''' > 0$ on $(0, 1)$.

COROLLARY 2.2.2. *For $A \geq 1$, there exist constants $Q > 0 > \beta$ such that the problem (1), (2) possesses a two-cell solution f with $f''' > 0$ on $(0, 1)$.*

2.3. $(B, E) \in D_3$

Note that g, g', g'', and $g''' > 0$ initially for $B \geq 0$, $E > B^2$. We first consider the case of $A \geq 3/2$.

THEOREM 2.3.1. *For each $A \geq 3/2$, $g(y) > 0$ on $(0, M)$.*

PROOF. Case (i): $B > 0$. From (6), (8), we get that $g^{(\mathrm{iv})} = 0$ and $g^{(\mathrm{v})} > 0$ at $y = 0$. Therefore, $g^{(k)} > 0$ initially for $k = 0, \dots, 5$. Suppose $\tilde{y}$ is the first positive zero of $g^{(\mathrm{v})}$. Then $g^{(k)} > 0$ at $y = \tilde{y}$ for $k = 0, \dots, 4$, and $g^{(\mathrm{vi})}(\tilde{y}) \leq 0$. However, from (9), $g^{(\mathrm{vi})} = (3A-2)g'g^{(\mathrm{iv})} + (4A-6)g''g''' > 0$ at $y = \tilde{y}$. This is a contradiction. Hence, $g^{(\mathrm{v})} > 0$ on $(0, M)$ and g has no positive zero.

Case (ii): $B = 0$. Suppose $A = 3/2$. Then $g(y) = Ey^3/6$ solves (3), (5) uniquely and $g > 0$ on $(0, M)$. Suppose $A > 3/2$. Then, from (8)–(10), we have $g^{(\mathrm{v})} = g^{(\mathrm{vi})} = 0$ and $g^{(\mathrm{vii})} > 0$ at $y = 0$. Suppose that $g^{(\mathrm{vii})}$ has the first zero at $y = \xi$, and then, $g^{(\mathrm{viii})}(\xi) \leq 0$. Differentiating (10), we get that

$$g^{(\mathrm{viii})} = Agg^{(\mathrm{vii})} + (5A-2)g'g^{(\mathrm{vi})} + (11A-10)g''g^{(\mathrm{v})} + (15A-20)g'''g^{(\mathrm{iv})} \tag{11}$$

and $g^{(\mathrm{viii})}(\xi) > 0$. This is a contradiction again, and then, $g^{(\mathrm{vii})} > 0$ on $(0, M)$. Thus, g possesses no positive zero. ∎

For the case of $1 \leq A < 3/2$, $g(y)$ may possess a positive zero.

THEOREM 2.3.2. *For $1 \leq A < 3/2$, $g(y)$ has at most one positive zero. Moreover, if either (i) $A = 1$, or (ii) $B = 0$ and $1 < A < 3/2$, then $g(y)$ possesses exactly one positive zero.*

PROOF. Suppose $g''' > 0$ on $(0, M)$. It is trivial that g, g', and $g'' > 0$ on $(0, M)$. Now, suppose g''' possesses the first positive zero at $y = y_2'''$. Then, from (6), $g^{(\mathrm{iv})} < 0$ at $y = y_2'''$

and g', $g'' > 0$ on $(0, y_2''')$. Suppose g''' has the second zero at $y = y_3''' > y_2'''$. Then, $g^{(\text{iv})}$, $(A-2)g'g'' \geq 0$ at $y = y_3'''$. This implies that $g' \geq 0$, $g'' < 0$ at $y = y_3'''$. Moreover, $g' > 0$ and $g'g''' < 0$ on $[y_2''', y_3''']$. Also, we get that

$$g^{(\text{v})} - Agg^{(\text{iv})} = (2A-2)g'g''' + (A-2)(g'')^2 < 0$$

on (y_2''', y_3'''). This implies that

$$\left(g^{(\text{iv})}(y)\exp\left(-A\int_{y_2'''}^{y} g(s)\,ds\right)\right)' < 0$$

on (y_2''', y_3'''), and then

$$0 \leq g^{(\text{iv})}(y_3''')\exp\left(-A\int_{y_2'''}^{y_3'''} g(s)\,ds\right) < g^{(\text{iv})}(y_2''') \leq 0.$$

This contradicts the definition of y_3''', and then, g''' has a unique positive zero.

Now we turn to show that $g''(y)$ has exactly one zero at $y_1'' > y_2'''$. Suppose $g'' > 0$ on $(0, M)$. From (6), $g^{(\text{iv})} = Agg''' + (A-2)g'g'' < 0$ and g'' is concave on (y_2''', M). This is a contradiction, and then, y_1'' exists uniquely. Hence, g has exactly one zero at $y = y_3 > y_1'$, where y_1' is the unique zero of g'.

For the case $A = 1$, suppose $g(y) > 0$ on $(0, 1)$. Then, from (8), $g^{(\text{v})} < 0$ on $(0, M)$ since $g^{(\text{iv})} < 0$. This implies that g''' is concave and strictly decreasing on $(0, M)$. Consequently, this violates the assumption of $g > 0$. Hence, g has exactly one positive zero.

Finally, let $B = 0$. Suppose we verify that g''' has a unique positive zero. Then g has exactly one positive zero on $(0, M)$. Assume that $g''' > 0$ on $(0, M)$. Then g, g', and $g'' > 0$ on $(0, M)$. But, from (10), $g^{(\text{vii})}(0) < 0$, and then, $^{(\text{iv})}$, $g^{(\text{v})}$, and $g^{(\text{vi})} < 0$ initially. Also, from (9), $g^{(\text{iv})}$, $g^{(\text{v})} < 0$ as long as $g^{(\text{iv})} < 0$. This implies that $g^{(\text{vi})} < 0$ on $(0, M)$. Therefore, g''' is concave, decreasing on $(0, M)$, and this is a contradiction. ∎

In fact, there exists a region in D_3 on which g has exactly one positive zero.

COROLLARY 2.3.1. *For each $A \in (1, 3/2)$, there exists a constant $\delta_1 = \delta_1(A) > 0$ such that $g(y)$ has exactly one positive zero if the pair (B, E) satisfies $B^2 < \delta_1^2 \cdot E$.*

PROOF. From Theorem 2.3.2, there is sufficiently large $\bar{y} > 0$ such that $g(\bar{y}; 0, 1) < 0$. Let $\epsilon = |g(\bar{y}; 0, 1)|/2$. Then, by the continuous dependence on initial data, there is a $\delta_1 = \delta_1(A) > 0$ such that $|g(\bar{y}; \bar{B}, 1) - g(\bar{y}; 0, 1)| < \epsilon$ if $0 < \bar{B} < \delta_1$. This yields that $g(\bar{y}; \bar{B}, 1) < g(\bar{y}; 0, 1)/2 < 0$. However, $g(y; \bar{B}, 1) > 0$ initially and $g(y; A, \bar{B}, 1)$ has exactly one positive zero at $y = y_3 < \bar{y}$. Now, as in [6], the homogeneity property of g is obtained by

$$g(y; B, E) = \lambda g\left(\lambda y; \frac{B}{\lambda^2}, \frac{E}{\lambda^4}\right), \qquad \text{for } \lambda > 0.$$

Then, $g(y; B, E) = \sqrt[4]{E} \cdot g(\sqrt[4]{E}y; B/\sqrt{E}, 1)$ has exactly one positive zero if $B^2 < \delta_1^2 E$. ∎

Hence, if the pair (B, E) is chosen with $B^2 < \delta_1^2 E$, then, by setting $b = y_3$, the corresponding pair (Q, β), $Q > 0$, $\beta < 0$, yields a 2-cell solution f of (1), (2) where f''' changes signs once on $(0, 1)$. Thus, we have the following corollary.

COROLLARY 2.3.2. *For $1 \leq A < 3/2$, there exist $Q > 0$, $\beta < 0$ such that the problem (1), (2) possesses a two-cell solution f where f''' changes signs once on $(0, 1)$.*

2.4. $(B,E) \in D_4$

It is true that g, $g' < 0$ and g'', $g''' > 0$ initially.

THEOREM 2.4.1. *For $A \geq 2$, $g(y)$ has exactly one zero on $(0,M)$.*

PROOF. Case (i): $g''' > 0$ on $(0,M)$. It is clear that $g'' > 0$ on $(0,M)$. Then g' is increasing, convex on $(0,M)$ and g has exactly one positive zero at $y = y_4$.

Case (ii): g''' has a positive zero. The case is only valid when $A > 2$ since $g'''(y) = (E - B^2)\exp(-\int_0^y g) > 0$ on $(0,M)$ if $A = 2$. Let y_4''' be the first zero of g'''. Then, $g'' > 0$ on $(0, y_4''']$ and $g^{(\mathrm{iv})} \leq 0$ at $y = y_4'''$. This implies, from (6), $g'(y_4''') \leq 0$.

Suppose $g'(y_4''') = 0$. Then, from (8), $g^{(\mathrm{v})} > 0$ at $y = y_4'''$ and $g''' > 0$ on $(y_4''', y_4''' + \epsilon)$ for sufficiently small $\epsilon > 0$. Assume that y_5''' is the second zero of g''', and then, $g^{(\mathrm{iv})}(y_5''') \leq 0$. This is impossible since, from (6) again, $g^{(\mathrm{iv})} > 0$ at $y = y_5'''$. Therefore, $g''' > 0$ on (y_4''', M) and $g'' > 0$ on $(0,M)$. Hence, as in case (i), g has exactly one positive zero, say at $y = y_4$.

Suppose $g'(y_4''') < 0$. Then g, g', $g^{(\mathrm{iv})} < 0$ on $[0, y_4''']$ and $(g'(y_4'''))^2 < E$. The desired result will follow immediately if we verify that $g'' > 0$ on $(0,M)$ and g' has a positive zero.

Assume that g'' has the first positive zero at $y = y_1''$. Then, $g'''(y_1'') < 0$ and $y_1'' > y_4'''$ since $g'' > 0$ on $(0, y_4''']$. Now, a contradiction can be obtained by verifying the following lemmas.

LEMMA 2.4.1. $g''' < 0$ *on* (y_4''', y_1'').

PROOF. Suppose g''' has at least one zero in $(y_4''', y_1'']$. Then, at each zero, g''' can not reach any local maximum since $g^{(\mathrm{v})} > 0$ and $g^{(\mathrm{iv})} = 0$. However, $g'''(y) < 0$ as y nears y_4''' and y_1''. This implies that g''' has at least two zeros. Let $y_5''' < y_6'''$ be the first two zeros of g'''. Then $g^{(\mathrm{iv})}(y_5''') \geq 0$, $g^{(\mathrm{iv})}(y_6''' \leq 0$ and $g'(y_6''') \leq 0 \leq g'(y_5''')$. This is a contradiction since $g'' > 0$ on $(0, y_1'')$ and proves Lemma 2.4.1.

LEMMA 2.4.2. $g < 0$ *on* $[y_4''', y_1'']$.

PROOF. Suppose g has the first zero at $y = \kappa$ on $[y_4''', y_1'']$. Then, on $[y_4''', \kappa]$, from (7), $g^{(\mathrm{iv})} > (A-2)(gg'')'$ and $g''' - (A-2)gg''$ is increasing. Therefore, $g''' - (A-2)gg'' > 0$ on $(y_4''', \kappa]$ since $g < 0$, $g'' > 0$ at $y = y_4'''$. Hence, $g'''(\kappa) > 0$ and it violates Lemma 2.4.1. This proves Lemma 2.4.2.

Now, from Lemma 2.4.1, 2.4.2, we obtain that $g''' - (A-2)gg'' > 0$ on $(y_4''', y_1'']$. By applying similar arguments as in Lemma 2.4.2, we obtain $g'''(y_1'') > 0$ and it is again a contradiction. Hence, $g'' > 0$ on $(0,M)$.

Finally, we want to verify that g' has a positive zero. Suppose $g' < 0$ on $(0,M)$. Then $\lim_{y\to\infty} g'(y)$ exists since $g'' > 0$ on $(0,\infty)$. Let $\lim_{y\to\infty} g'(y) = K > B$. Also, from (6), we get that $(g'''(y)\exp(-A\int_{y_4'''}^y g(s)\,ds))' < 0$ on $[y_4''', \infty)$. Then $g''' < 0$ on (y_4''', ∞), and this implies that $\lim_{y\to\infty} g''(y) = 0$ since $g' < 0$ on $(0,\infty)$. Suppose $g^{(\mathrm{iv})} < 0$ on (y_4''', ∞). This implies that g'' is decreasing, concave on (y_4''', ∞), and it is a contradiction. Hence, $g^{(\mathrm{iv})}$ has a zero at $y = d_1 \epsilon (y_4''', \infty)$. Also, from (9), $g^{(\mathrm{iv})} > 0$ on (d_1, ∞) since $(g^{(\mathrm{iv})}(y)\exp(-A\int_{d_1}^y g(s)\,ds))' > 0$. Then $\lim_{y\to\infty} g'''(y) = 0$. Taking limit to (3), we obtain that

$$L = \lim_{y\to\infty} gg'' = -\frac{(E-K^2)}{A} < -\frac{(E-B^2)}{A} < 0. \tag{12}$$

Thus, there exists a $P > 0$, such that

$$gg'' < \frac{L}{2} \tag{13}$$

on (P,∞). Also, from (12), $\lim_{y\to\infty} g(y) = -\infty$. Now multiplying both sides of (14) by g'/g, and integrating from P to y,

$$[g'(y)]^2 - [g'(P)]^2 < L(\ln|g(y)| - \ln|g(P)|). \tag{14}$$

By taking limit to (14), we obtain a contradiction again. Hence, the proof of Theorem 2.4.1 is complete. ∎

Therefore, by choosing $b = y_4$, the corresponding pair (Q, β), satisfying $Q < 0$, $\beta > 0$, leads to a 2-cell solution of (1), (2).

COROLLARY 2.4.1. *For $A \geq 2$, there exists constants $Q < 0$, $\beta > 0$ such that the problem (1), (2) possesses a two-cell solution.*

Now, we consider the case of $3/2 < A < 2$. In fact, we want to show that g''' has at most one positive zero. Suppose $g''' > 0$ on $(0, M)$. Then g has exactly one positive zero by applying similar arguments as in Theorem 2.4.1. Suppose g''' has at least one positive zero. Then the following key lemma is required.

LEMMA 2.4.3. *g''' has exactly one zero on $(0, M)$.*

PROOF. The result is obvious if $g^{(\mathrm{iv})} < 0$ on $(0, M)$. Suppose $g^{(\mathrm{iv})}$ has the first zero at $y = y_0^{\mathrm{iv}}$ in $(0, M)$. Then, at $y = y_0^{\mathrm{iv}}$, gg''', $g'g''$ have the same sign and $g'g''' > 0$ since $g^{(\mathrm{v})}(y_0^{(\mathrm{iv})}) \geq 0$. The determination of sign property for g, g', g'', and g''' at $y = y_0^{(\mathrm{iv})}$ is crucial for the desired result.

We begin with determining the sign of $g'''(y_0^{\mathrm{iv}})$. Suppose $g'''(y_0^{\mathrm{iv}}) > 0$. Then, g'', $g''' > 0$ on $(0, y_0^{\mathrm{iv}})$ and g, g', $g'' > 0$ at $y = y_0^{\mathrm{iv}}$. Therefore, from (9), $g^{(\mathrm{iv})}$, $g^{(\mathrm{v})} > 0$ on (y_0^{iv}, M). This implies that g''' possesses no zero on $(0, M)$ and it is a contradiction. Hence, $g'''(y_0^{\mathrm{iv}}) < 0$, and consequently $g'(y_0^{\mathrm{iv}}) < 0$.

Now we turn to determine the signs of g and g'' at $y = y_0^{\mathrm{iv}}$. In fact, g and g'' have the same sign. Suppose g, $g'' > 0$ at $y = y_0^{\mathrm{iv}}$. Then $g'' > 0$ and g', $g < 0$ on $(0, y_0^{\mathrm{iv}})$. This contradicts the assumption of $g(y_0^{\mathrm{iv}}) > 0$. Hence, g and $g'' < 0$ at $y = y_0^{\mathrm{iv}}$ and g''' has exactly one zero in $(0, y_0^{\mathrm{iv}}]$.

The remaining part is to show that g''' has no zero in (y_0^{iv}, M). Suppose y_7''' is the first zero of g''' in (y_0^{iv}, M). Then, at $y = y_7'''$, we have $g^{(\mathrm{iv})} \geq 0$ and g'', $g' < 0$. But $g^{(\mathrm{iv})} = (A-2)g'g'' < 0$ at $y = y_7'''$. This is a contradiction and the proof of Lemma 2.4.3 is complete.

Hence, from Lemma 2.4.3, we obtain that g'' and g''' have exactly one positive zero, say at $y = y_2'', y_7'''$, respectively. It is clear, $g^{(\mathrm{iv})} < 0$ at $y = y_2''$, y_7''' and $y_7''' < y_2''$. But, $g^{(\mathrm{iv})} = Agg''' < 0$ and $g > 0$ at $y = y_2''$. This implies that g has exactly one zero in $(0, y_2'')$. Furthermore, g has the second zero in (y_2'', M) by the concavity of g. Thus, we have verified the next theorem.

THEOREM 2.4.2. *For $3/2 < A < 2$, $g(y)$ has at least one but at most two zeros on $(0, M)$.*

Note that the existence of unique positive zero of g relies on the fact that $g''' > 0$ on $(0, M)$. Also, from Theorem 2.3.1, $g^{(\mathrm{i})}(y; 0, 1) > 0$ for $y > 0$, $i = 0, \ldots, 5$. Then, existence of a region in D_4 on which g has exactly one zero on $(0, M)$ is also found by applying similar arguments as in Corollary 2.3.1. We omit the proof of the next corollary.

COROLLARY 2.4.2. *For $A \in (3/2, 2)$, there exists a $\delta_2 = \delta_2(A)$ such that $g(y; B, E)$ has exactly one positive zero if $B^2 \leq \delta_2^2 E$.*

Therefore, by choosing such (B, E), the zero of g leads to a 2-cell solution of (1), (2) with $Q < 0$, $\beta > 0$. Hence, we have the following corollary.

COROLLARY 2.4.3. *For $A \in (3/2, 2)$, there exists a pair (Q, β), satisfying $Q < 0$, $\beta > 0$, such that the problem (1), (2) possesses a two-cell solution.*

It should be pointed out that the classification for the case $3/2 < A < 2$ is not completely clear, especially for possible regions in D_4 on which g has exactly two zeros. However, it is complete for $1 \leq A \leq 3/2$ which is given as follows.

As in Theorem 2.4.2, we have to show that both g''' and g'' have exactly one zero on $(0, M)$ and $g^{(\mathrm{iv})} < 0$ at the zero of g''. The desired properties can be obtained by the following lemmas.

LEMMA 2.4.4. *g''' has exactly one positive zero.*

PROOF. Let $g''' > 0$ on $(0, M)$. Then, by the convexity, g' has exactly one positive zero, say, at $y = y_0'$. Consequently, g has exactly one zero at $y = y_3 > y_0'$. Also, $(g^{(\mathrm{iv})} \exp(-A \int_0^y g(s)ds))' < 0$ on $[0, y_0']$ since $g'g''' \leq 0$. Then, $g^{(\mathrm{iv})} < 0$ on $[0, y_0']$. Moreover, $g^{(\mathrm{iv})} < 0$ on $(y_0', y_3]$ since $g' > 0$ and $g \leq 0$.

Now we turn to prove that $g^{(\mathrm{iv})} < 0$ on (y_3, M). Suppose $g^{(\mathrm{iv})}$ has the first zero at $y = y_1^{\mathrm{iv}}$. Then, $2(A-1)g'g^{(\mathrm{iv})} + (4A-6)g''g''' < 0$ on $[y_0', y_1^{\mathrm{iv}}]$ since $g^{(\mathrm{iv})} \leq 0$. This implies that $2(A-1)g'g''' + (A-2)(g'')^2$ is decreasing on $[y_0', y_1^4]$. Then, we get

$$\begin{aligned} g^{(\mathrm{v})}(y_1^{\mathrm{iv}}) &= 2(A-1)g'(y_1^{\mathrm{iv}})g'''(y_1^{\mathrm{iv}}) + (A-2)(g''(y_1^{\mathrm{iv}}))^2 \\ &\leq 2(A-1)g'(y_0')g'''(y_0') + (A-2)(g''(y_0'))^2 < 0. \end{aligned}$$

This contradicts the fact that $g^{(\mathrm{v})} \geq 0$ at $y = y_1^{\mathrm{iv}}$. Then, $g^{(\mathrm{iv})} < 0$ on $(0, \infty)$, and consequently, $g^{(\mathrm{v})} \leq Agg^{(\mathrm{iv})} < 0$ on (y_3, ∞). This implies that g''' is concave on (y_0', ∞), which is a contradiction, and g''' has at least one positive zero.

Suppose $y_8''' < y_9'''$ are the first two positive zeros of g'''. Then $g^{(\mathrm{iv})}(y_8''') \leq 0$, $g^{(\mathrm{iv})}(y_9''') \geq 0$, and $g''' \leq 0$ on $[y_8''', y_9''']$ since g''' cannot reach any local minimum at $y = y_8'''$. Moreover, $g' > 0$ on (y_8''', y_9''') by applying the facts that $g''' \leq 0$ on $[y_8''', y_9''']$, $g'(y_8''')g''(y_8''') \geq 0$ and $g'(y_9''')g''(y_9''') \leq 0$. Therefore, $g'g''' \leq 0$ and $(g^{(\mathrm{iv})} \exp(-A \int_{y_8'''}^y g(s)\,ds))' < 0$ on $[y_8''', y_9''']$. Hence, we obtain that

$$g^{(\mathrm{iv})}(y_9''') \exp\left(-A \int_{y_8'''}^{y_9'''} g(s)\,ds\right) < g^{(\mathrm{iv})}(y_8''') \leq 0$$

and $g^{(\mathrm{iv})}(y_9''') < 0$. But it is impossible, and this proves Lemma 2.4.4.

LEMMA 2.4.5. *g'' has exactly one positive zero.*

PROOF. By Lemma 2.4.4, g''' has exactly one positive zero at $y = y_8'''$. Then $g'(y_8''') \geq 0$ since $g^{(\mathrm{iv})} \leq 0$ and $g'' > 0$ at $y = y_8'''$. Suppose g'' has no zero in (y_8''', ∞). Then g' has no zero on (y_8''', ∞). By applying similar arguments as in Lemma 2.4.4, we obtain that g'' has exactly one zero on (y_8''', ∞) since $g^{(\mathrm{iv})} \leq 0$. This is a contradiction. Hence, Lemma 2.4.5 is obtained.

Now, by similar arguments as in Theorem 2.4.2, g has exactly two positive zeros $y_4 < y_5$. This verifies the following theorem.

THEOREM 2.4.3. *For $1 \leq A \leq 3/2$, $g(y)$ has exactly two positive zeros.*

By choosing $b = y_4$, the corresponding (Q, β) with $Q < 0$, $\beta > 0$ leads to a 2-cell solution of (1), (2). The second zero y_5 leads to a 3-cell solution of (1), (2) with $Q > 0$, $\beta < 0$. Therefore, we have the following corollary.

COROLLARY 2.4.4. *For $1 \leq A \leq 3/2$, there exists two pairs (Q, β) such that the problem (1), (2) possesses a two-cell solution with $Q < 0$, $\beta > 0$ and a three-cell solution with $Q > 0$, $\beta < 0$.*

Therefore, the classification has verified the following first main result.

THEOREM A.

(a) *If the problem (1), (2) has a two-cell solution for $A \geq 1$, then the corresponding parameter (Q, β) must satisfy that either (i) $Q < 0$, $\beta > 0$, (ii) $Q > 0$, $\beta > 0$, or (iii) $Q > 0$, $\beta \leq 0$.*
(b) *The problem (1), (2) has a three-cell solution for $1 \leq A \leq 3/2$, then the corresponding parameter (Q, β) must satisfy $Q > 0$, $\beta < 0$.*
(c) *For $1 \leq A \leq 3/2$, $A \geq 2$, the problem (1), (2) possesses no solution if $Q \leq 0$ and $\beta \leq 0$.*

3. EXISTENCE OF SOLUTIONS

Recall in Section 2 that the desired (Q, β) can be obtained when choosing (B, E) from some D_i, $i = 2, 3, 4$ by applying

$$Q(B, E) = -(y_*(B, E))^3 g''(y_*(B, E))$$

and

$$\beta(B,E) = -Ey_*(B,E)/g''(y_*(B,E)),$$

where $y_*(B,E)$ is a positive zero of g. In fact, $y_*(B,E)$, $Q(B,E)$, and $\beta(B,E)$ are C^1 in $(B,E) \in D_i$'s since g'' never vanishes at $y = y_*(B,E)$. Moreover, by the homogeneity of g, we also obtain the following property.

PROPERTY 3.1.1. *Let $y_*(B,E)$ be a positive zero of g for some (B,E) in D_i, $i = 2,3,4$, and $A \geq 0$. Then, for $\lambda > 0$,*

$$y_*(B,E) = \frac{y_*\left(\frac{B}{\lambda^2}, \frac{E}{\lambda^4}\right)}{\lambda},$$
$$Q(B,E) = Q\left(\frac{B}{\lambda^2}, \frac{E}{\lambda^4}\right)$$

and

$$\beta(B,E) = \beta\left(\frac{B}{\lambda^2}, \frac{E}{\lambda^4}\right).$$

Property 3.3.1 has simplified our study by considering a certain path in D_i's. That is, we may integrate (3), (5) directly to locate y_* and then compute (Q,β) by choosing (B,E) along the perimeter of $|B| + |E| = 1$ counterclockwise in D_i's. The numerical computation with a FORTRAN subroutine SDRIV2 [7] has been performed on CONVEX C220 at NCTU and NCCU for various A's. The results are consistent with the classification in Section 2, and the selected bifurcation diagrams in the Q-β plane are shown in Figures 2–5 for $A = 1$, 1.25, 1.5, 1.8, respectively. It should be pointed out that our bifurcation diagram when $A = 1$ is consistent with the one obtained in [1] by solving (1), (2) with some boundary value problem codes. Also, the bifurcation diagram of $A = 1.8$ is similar to the one when $A = 2$, obtained in [1], except for the difference of the position. From Figures 2–5, further observations should be given as follows:

(O-1) *There exists an A^*, $1 < A^* < 3/2$, such that*
  - *(i) the problem (1), (2) has a two-cell solution for $Q \in R$ when $A > A^*$;*
  - *(ii) there is a negative $Q^*(A)$ such that the problem (1), (2) has a two-cell solution for $Q \geq Q^*(A)$ and no solution for $Q < Q^*(A)$, when $1 \leq A \leq A^*$.*

(O-2) *For $1 \leq A < 3/2$, there are two additional continuous branches of solutions of the problem (1), (2) when Q is sufficiently large, as shown in Figures 2–4, which are corresponding to two- and three-cell solutions, respectively. Moreover, the connection point of these two families in the Q-β plane are moving toward infinity along the upper branch as A increases from 1.*

(O-3) *Let $A = 3/2$, the problem (1), (2) has at least two three-cell solutions for sufficiently large $Q > 0$.*

The observations have clearly indicated existence of multiple solutions of (1), (2) when $1 \leq A \leq 3/2$. However, (1), (2) may only possess a "unique" two-cell solution for every given real Q when $A > 3/2$. In this section, we shall verify a portion of the observations by presenting existence of some connected sets in the Q-β plane on which solutions of (1), (2) exist.

Recall that $g(y)$ possesses at least one positive zero at $y = y_*$ when $(B,E) \in D_2$, D_3, or D_4. Define $\vec{s}(B,E) = (Q(B,E), \beta(B,E)) = (-y_*^3 g''(y_*), Ey_*/g''(y_*))$ and $\vec{s}^+(B,E) = ((-y_*^+)^3 g''(y_*^+), Ey_*^+/g''(y_*^+))$, where $y_* < y_*^+$ are the first and second positive zeros of $g(y)$, respectively. Also, define the sets

$$\Gamma_i = \{\vec{s}(B,E);\ (B,E) \in D_{i+1}\}, \qquad i = 1,2,3,$$

and

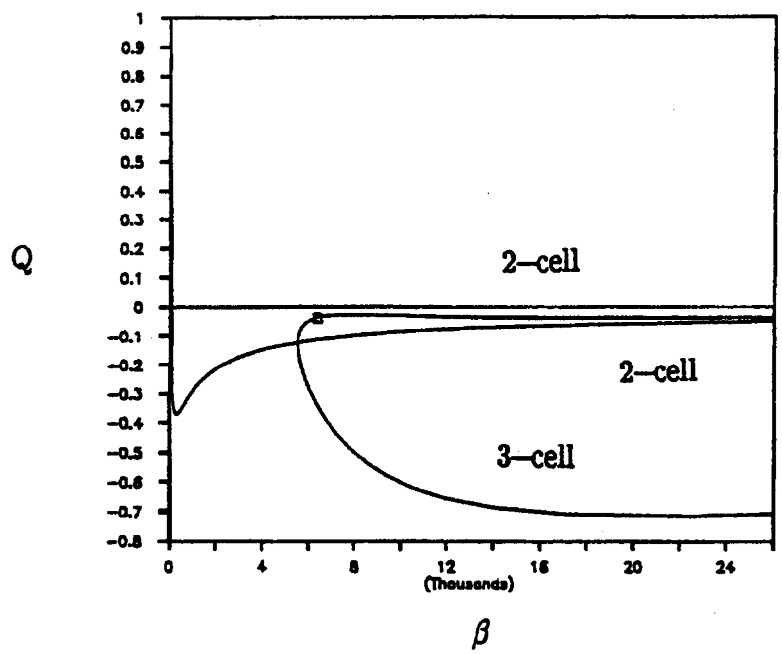
$$\Gamma_4 = \{\vec{s}^+(B,E);\ (B,E) \in D_4\}.$$

It is clear that Γ_4 may be empty if $A > 3/2$, while Γ_1, Γ_3 exist for $A \geq 1$.

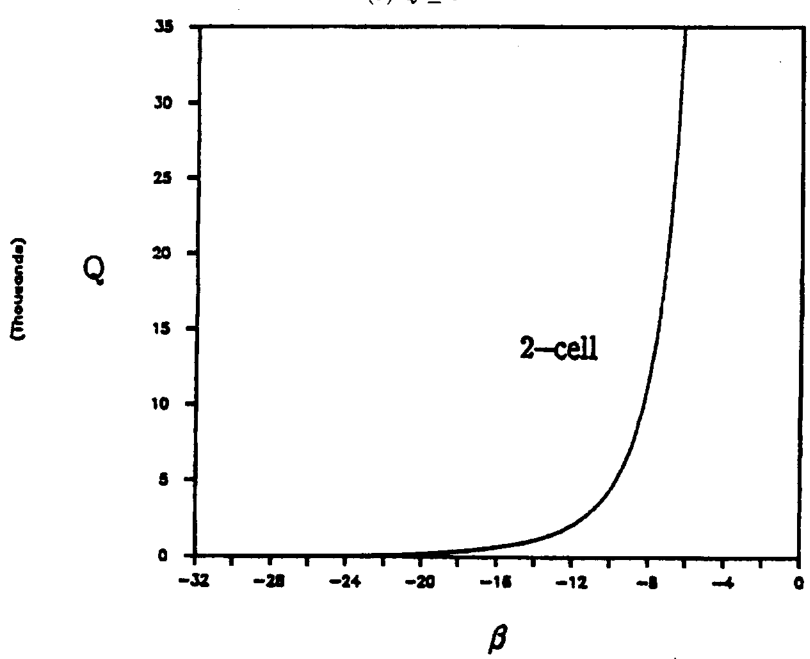
3.1. Existence of Γ_1 and Γ_3

By the homogeneity in Property 3.1.1, we write $\Gamma_1 = \{\vec{s}(1,r);\ -\infty < r < 1\} = \{\vec{s}(1,r);\ -1 \le r < 1\} \cup \{\vec{s}(\omega,-1);\ 0 < \omega < 1\}$ and $\Gamma_3 = \{\vec{s}(\omega,1);\ -1 < \omega < 0\}$. It is clear that Γ_1 and Γ_3 are connected sets since $\vec{s}(B,E)$ is C^1 in (B,E). We may verify the existence of Γ_1 on which 2-cell solutions f exist with $Q > 0$ and $f''' > 0$ on $(0,1)$.

THEOREM 3.1.1. *Suppose $A \in [1,\infty)$. The problem (1), (2) has a two-cell solution f with $Q > 0$ and $f''' > 0$ on $(0,1)$ if and only if $(Q,\beta) \in \Gamma_1$.*

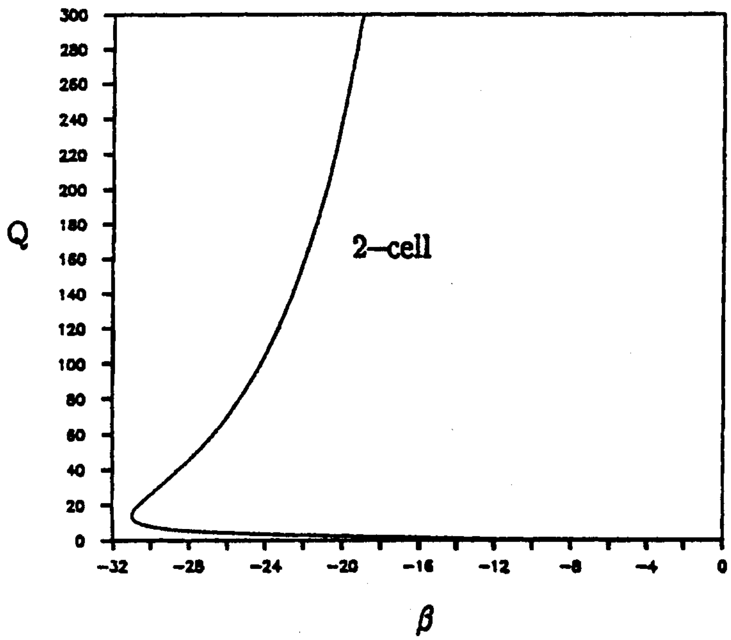


(a) $Q \ge 0$.



(b) $Q \le 0$.

Figure 2. The bifurcation diagram of solution space (Q,β) of (1), (2) when $A = 1$.



(c) Detail portion of $Q \le 0$.

Figure 2. (Continued).

PROOF. The verification of the sufficient condition is easy since for each $r \in (-\infty, 1)$, $f(\eta) = y_*(1,r)g(y_*(1,r)(1-\eta); A, 1, r)/Q$ is a solution of (1), (2) with $Q > 0$.

Conversely, let f be the desired solution of (1), (2) and $g(y) = Qf(1-y)$. Then g is a solution of (3) with $g(1) = 0$, $g'(0) = B = -Qf'(1) > 0$, $g''(1) = -Q$ and $E = -Q\beta$. By the classification, we have $E - B^2 < 0$ since g has a positive zero at $y = 1$ and $g''(1) = -Q < 0$. Hence, $Q(1,r)$ and $\beta(1,r)$ are well-defined. Moreover, $-y_*(B,E)^3 g''(y_*(B,E); B, E) = Q$ and $-E \cdot y_*(B,E)^4/Q = \beta$. ∎

Now the asymptotic behavior of Γ_1 can be obtained by the following corollaries.

COROLLARY 3.1.1.

(a) $\lim_{r \to 1^-} y_*(1,r) = \infty$;
(b) $\lim_{\omega \to 0} y_*(\omega, -1) = 0$.

PROOF. It is clear that $g(y;1,1) = y$ and $g'(y;1,1) \equiv 1$. Then, by the continuous dependence on initial data, given any $P > 0$ there exists a $\delta > 0$ such that $|g'(y;1,r) - g'(y;1,1)| < 1/2$, for all $r \in (1-\delta, 1)$ and $y \in [0, P]$. Thus, $g'(y;1,r) > 1/2$, and then $g(y;1,r) > y/2$ for all $y \in [0, M]$ and $r \in (1-\delta, 1)$. This implies $y_*(1,r) > P$ for $r \in (1-\delta, 1)$.

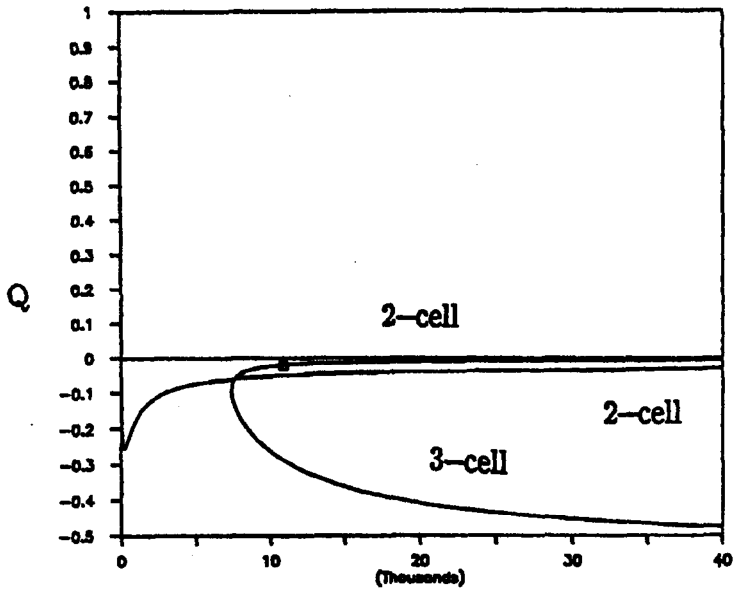
To verify the second assertion, we first consider the function $g(y;0,-1)$, which solves (3),(5). From Theorem 2.1.1, $g(y;0,-1) < 0$ on $(0, M)$. Hence, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|g(\epsilon;\omega,-1) - g(\epsilon;0,-1)| < |g(\epsilon;0,-1)|/2$, for all $\omega \in (0,\delta)$. This shows that $g(\epsilon;\omega,-1) < g(\epsilon;0,-1)/2 < 0$. However, $g(y;\omega,-1) > 0$ for y sufficiently close to 0. This implies that $y_*(\omega,-1) < \epsilon$ whenever $\omega < \delta$ and hence completes the proof. ∎

COROLLARY 3.1.2.

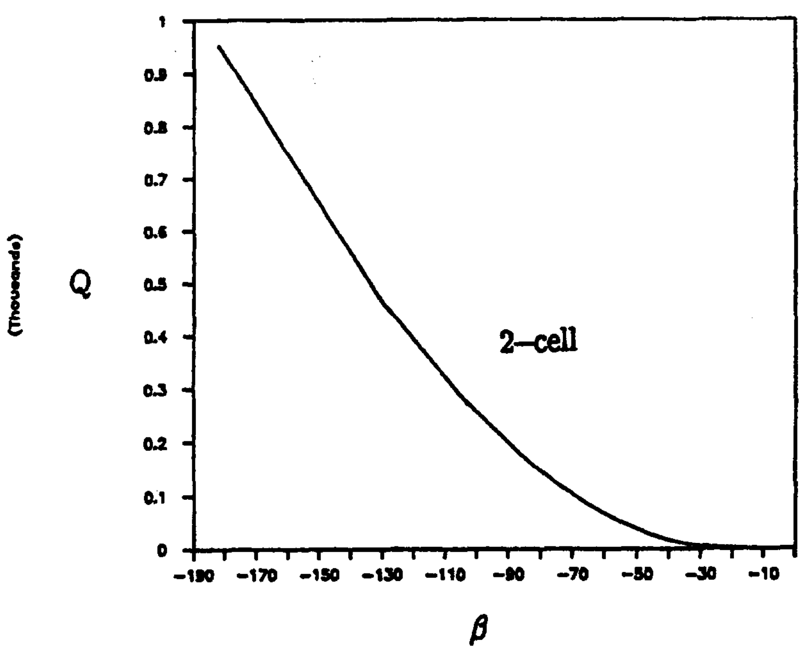
(a) $\lim_{r \to 1^-} Q(1,r) = +\infty$;
(b) $\lim_{\omega \to 0} Q(\omega,-1) = 0$ *and* $\lim_{\omega \to 0} \beta(\omega,-1) = 1$.

PROOF. From Theorem 2.2.2, it is clear that $g'''(y;1,r) < 0$ on $[0, y_*(1,r)]$ and $g'(y_*(1,r);1,r) < 0$ for $0 < r < 1$. From (3), we obtain that

$$g'^2(y_*(1,r);1,r) > g'''(y_*(1,r);1,r) + (g'(y_*(1,r);1,r))^2 = r > 0.$$



(a) $Q \geq 0$.



(b) $Q \leq 0$.

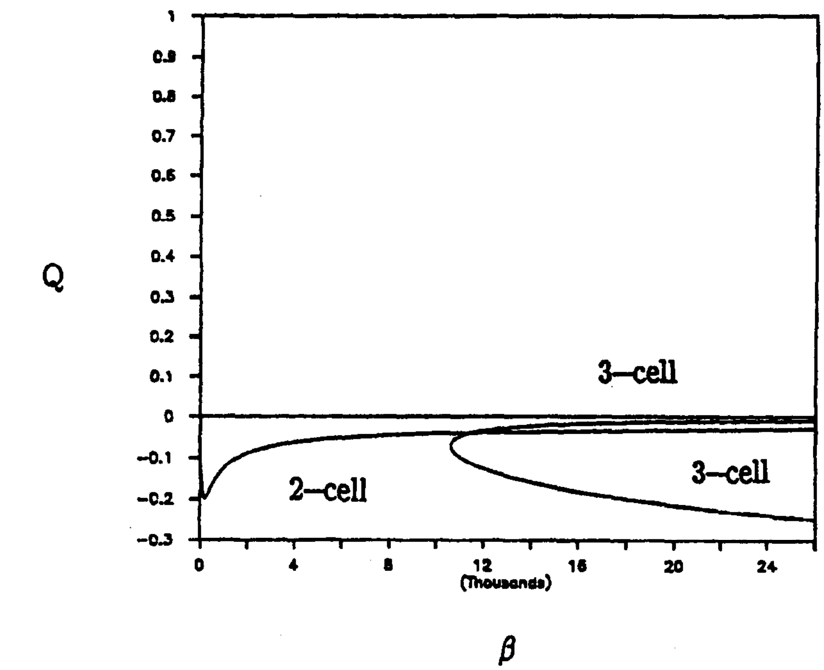
Figure 3. The bifurcation diagram of solution space (Q, β) of (1), (2) when $A = 1.25$.

Then, $g'(y_*(1,r);1,r) < -\sqrt{r}$. Since $g'(0;1,r) = 1$, we have

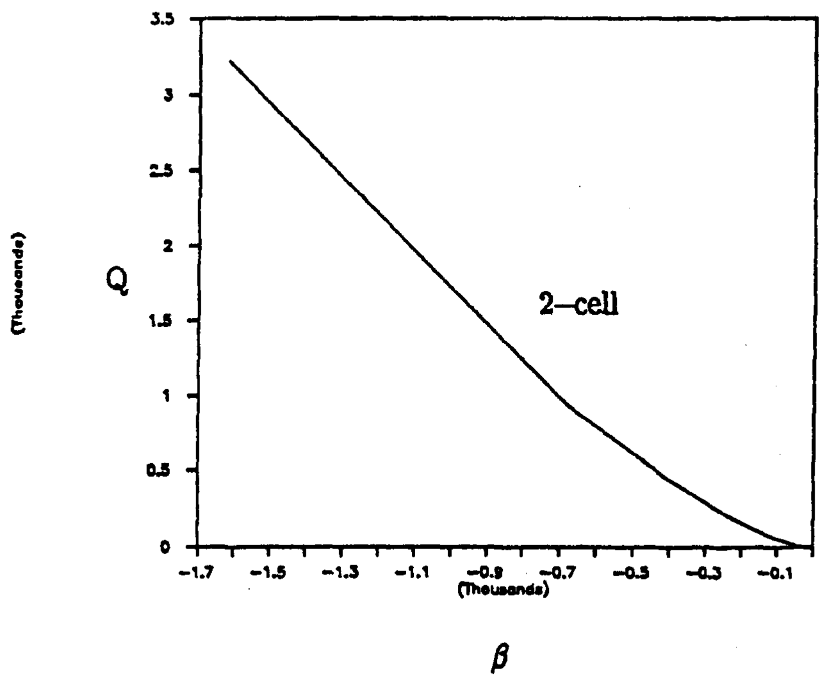
$$-\sqrt{r} - 1 > g'(y_*(1,r);1,r) - g'(0;1,r) = \int_0^{y_*(1,r)} g''(t;1,r)\,dt \geq y_*(1,r)g''(y_*(1,r);1,r).$$

Hence, $Q(1,r) > y_*(1,r)^2(\sqrt{r}+1)$, and, from Corollary 3.1.1, assertion (a) is obtained.

Again, by using Corollary 3.1.1 and the continuity of $g''(y;\omega,-1)$, the second result of Q in assertion (b) is easily obtained. We now turn to prove that $\lim_{\omega\to 0}\beta(\omega,-1) = 1$. Integrating (3),



(a) $Q \geq 0$.

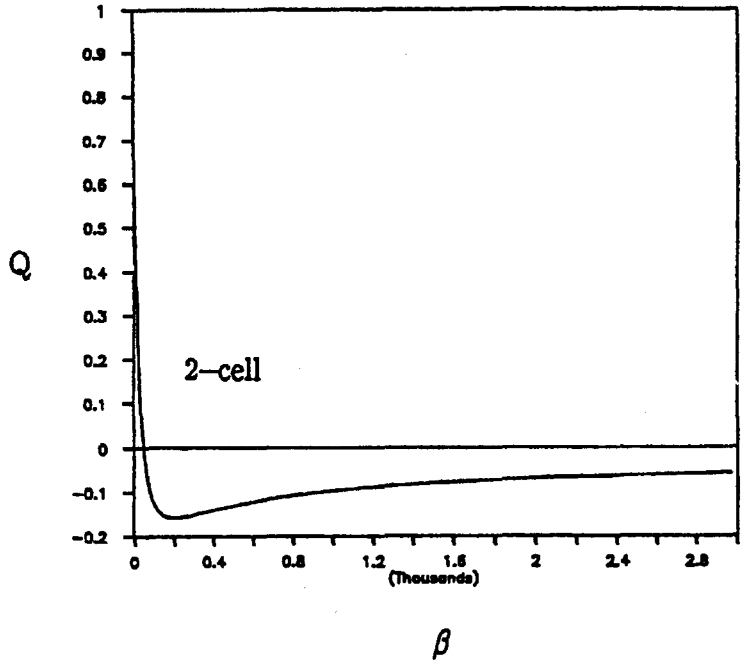


(b) $Q \leq 0$.

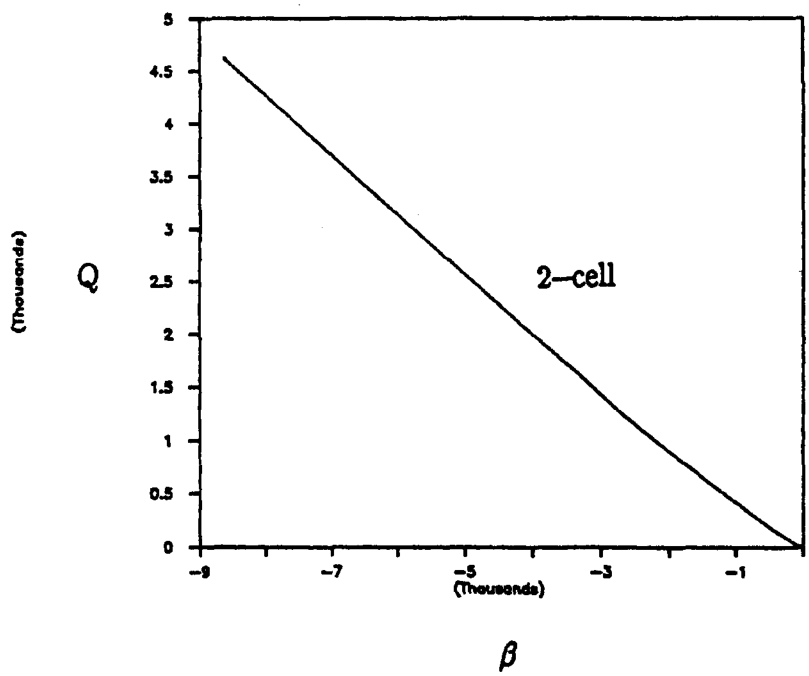
Figure 4. The bifurcation diagram of solution space (Q, β) of (1), (2) when $A = 1.5$.

we get that

$$
\begin{aligned}
0 < 1 - \beta(\omega, -1) &= \int_0^{y_*} \frac{[Agg'' - (g')^2]\, dt}{g''(y_*)} \\
&= \int_0^{y_*} \frac{[(A-1)gg'']\, dt}{g''(y_*)}
\end{aligned}
$$



(a) $Q \geq 0$.



(b) $Q \leq 0$.

Figure 5. The bifurcation diagram of solution space (Q, β) of (1), (2) when $A = 1.8$.

$$< \frac{[(A-1)g''(y_*)\int_0^{y_*} g\,dt]}{g''(y_*)} \longrightarrow 0 \text{ as } \omega \longrightarrow 0.$$

since, from Lemma 2.2.1, g'' is decreasing on $(0, y_*)$. Hence, this completes the proof. ∎

In fact, the pairs $(B, 0)$ for $B > 0$ correspond to the unique $(\tilde{Q}(A), 0)$ for some $\tilde{Q}(A) > 0$ and $\Gamma_1' = \{\vec{s}(1,r);\ -1 < r \leq 0\} \cup \{\vec{s}(\omega, -1);\ 0 < \omega < 1\}$ exists if $A \geq 0$. This implies that, as obtained in [3,4] by a topological method, (1), (2) possesses at least one 2-cell solution for

$\beta \in [0,1]$, provided that $A \geq 0$. Also, from Theorem 2.2.2, Γ_1 is a connected set, lying in the plane $Q > 0$, $\beta \in \mathbb{R}$.

COROLLARY 3.1.3. *For all $Q > 0$, there exists at least one β such that the problem (1), (2) has a two-cell solution provided that $A \geq 1$.*

For $(Q,\beta) \in \Gamma_3$, as in Theorem 3.1.1, we have the following theorem.

THEOREM 3.1.2. *Suppose $A \in [1,\omega)$. The problem (1), (2) has a two-cell solution with $Q < 0$ if and only if $(Q,\beta) \in \Gamma_3$.*

By the property of $g(y;-1,1)$, $g(y;0,1)$, the asymptotic behavior of $y_*(B,E)$ can be obtained from the next corollary.

COROLLARY 3.1.4.

(a) $\lim_{\omega\to 0} y_*(\omega,1) = 0;$
(b) $\lim_{\omega\to -1} y_*(\omega,1) = \infty.$

PROOF. From Theorem 2.3.1, we have that $g(y;0,1) > 0$ on $(0,M)$. But $g(y;\omega,1) < 0$ initially. Then, by the continuous dependence on initial data, assertion (a) follows. Consider $g(y;-1,1) = -y$, by the continuous dependence on initial data, again, assertion (b) is clear. ∎

COROLLARY 3.1.5.

(a) $\lim_{\omega\to 0} Q(\omega,1) = 0$ *and* $\lim_{\omega\to 0} \beta(\omega,1) = 1;$
(b) $\lim_{\omega\to -1} \beta(\omega,1) = \infty.$

PROOF. As in Corollary 3.1.2, assertion (a) follows. If we show that $g''(y_*)$ is bounded above, then, from Corollary 3.1.4, the assertion (b) is obtained since $\beta = y_*/g''(y_*)$. Multiplying both sides of (3) by g'' and the integrating, we get

$$\frac{(g''(y))^2}{2} = g'(y) - \omega + \int_0^y [Ag(g'')^2 - g''(g')^2]\,dt$$

for $y \in (0,y_*)$. From Lemma 2.4.1, $g < 0$, $g'' > 0$ and $g'' < (2g' - 2\omega)^{1/2}$ in $(0,y_*)$. Suppose $g'''(y_*) > 0$. Then, from (3), $(g')^2 = 1 - g''' < 1$ at $y = y_*$. Otherwise, consider the maximum $g''(y''')$ of g'', where y''' is the zero of g'''. Then, $(g')^2 = 1 + Agg'' < 1$ at $y = y'''$. This leads to $g'' < 2$ at $y = y_*$, and the proof is complete. ∎

Recall in Section 2.4 that $Q < 0$ for $(Q,\beta) \in \Gamma_3$. Then, Γ_3 is a connected subset of the quadrant $Q < 0$, $\beta > 0$.

COROLLARY 3.1.6. *For all $\beta > 1$, there exists a $Q < 0$ such that the problem (1), (2) has a two-cell solution provided that $A \geq 1$.*

Note that $\Gamma_1 \cup (0,1) \cup \Gamma_3$ is a connected set, lying in the plane $\mathbb{R}^2/\{Q \leq 0,\ \beta \leq 0\}$, which passes the point $(\tilde{Q}(A),0)$. This implies our second main result.

THEOREM B. *For $Q \geq 0$ or $\beta \geq 0$, the problem (1), (2) possesses at least one 2-cell solution if $A \geq 1$. Moreover, for $0 \leq \beta \leq 1$, (1), (2) has at least one two-cell solution if $A \geq 0$.*

3.2. Existence of Γ_2 and Γ_4

By the classification in Section 2, Γ_2, Γ_4 are empty if $A > 3/2$. Now, from Corollary 2.3.1, there exists a region in D_3 on which g has exactly one positive zero for $1 \leq A < 3/2$. By the homogeneity and continuity, there is an $\alpha \geq \delta_1$ such that $g(y;\alpha,1)$ has no zero on $(0,M)$ and $g(y;B,E)$ has a unique positive zero for $(B,E) \in D_3' = \{(B,E) : (B,E) \in D_3 \text{ and } B^2 < \alpha^2 E\}$. Again, define $\Gamma_2' = \{\vec{s}(B,E); (B,E) \in D_3'\}$, or equivalently, $\Gamma_2' = \{\vec{s}(r,1);\ 0 \leq r < \alpha\}$. Although the numerical result has indicated that $\Gamma_2 = \Gamma'_2$, we only obtain the sufficient condition in the next theorem since the classification is yet complete in D_3.

THEOREM 3.2.1. *Suppose $A \in [1, 3/2]$. The problem (1), (2) has a two-cell solution f with $Q > 0$ and f''' changes signs once on $(0,1)$ if $(Q, \beta) \in \Gamma'_2$.*

To obtain the asymptotic behavior of Γ'_2, it is necessary to get the following corollaries.

COROLLARY 3.2.1.

(a) $\lim_{\omega \to \alpha^1} y_*(\omega, 1) = +\infty$;

(b) $\lim_{\omega \to 0^+} y_*(0,1)$ *for some* $y_*(0,1) = y_*(A, 0, 1) > 0$.

PROOF. Suppose $\liminf_{\omega \to \alpha^-} y_*(\omega, 1) = k < \infty$, then, by the continuity of solution, $g(k; \alpha, 1) = 0$. It contradicts the fact that $g(y; \alpha, 1)$ is positive. Then, $\lim_{\omega \to \alpha^-} y_*(\omega, 1) = \infty$. The remaining part of the corollary can be easily obtained by continuity of y_* in ω. ∎

COROLLARY 3.2.2.

(a) $\lim_{\omega \to \alpha^-} Q(\omega, 1) = +\infty$;

(b) $\lim_{\omega \to 0^+} Q(\omega, 1) = Q(0,1)$ *and* $\lim_{\omega \to 0^+} \beta(\omega, 1) = \beta(0,1)$, *for some* $Q(0,1) > 0$ *and* $\beta(0,1) < 0$, *depending on A.*

PROOF. From Theorem 2.3.2, g''' and g'' have exactly one zero at c and b, respectively. Moreover, $c < b < y_*(\omega, 1)$ and $g'(b; \omega, 1) > 0 > g'(y_*; \omega, 1)$. Hence, from (3), we get $g'(b; \omega, 1) \geq 1$ and $g'(y_*; \omega, 1) < -1$. Thus,

$$-2 \geq g'(y_*; \omega, 1) - g'(b; \omega, 1) = \int_b^{y_*} g''(t; \omega, 1)\, dt \geq (y_* - b) g''(y_*; \omega, 1) \geq y_* \cdot g''(y_*; \omega, 1).$$

This leads to $Q(\omega, 1) \geq 2y_*^2$ and the assertion (a) is obtained. We omit the proof of assertion (b). ∎

Recall in Section 2.3 that the corresponding (Q, β) must satisfy $Q > 0$, $\beta < 0$ for $(B, E) \in D'_3$. This implies that Γ'_2 is a connected set lying in the quadrant $Q > 0$, $\beta < 0$ with an endpoint $(Q(0,1), \beta(0,1))$.

COROLLARY 3.2.3. *Let $Q^*(A) = Q(A, 0, 1) > 0$ if $1 \leq A \leq 3/2$. For all $Q \geq Q^*(A)$, there exists at least one $\beta < 0$ such that the problem (1), (2) has a two-cell solution f where f''' changes signs once on $(0,1)$.*

Now, suppose $(B, E) \in D_4$. Again, we write $\Gamma_4 = \{\vec{s}^{\,+}(\omega, 1);\ -1 < \omega < 0\}$. Then, we have the following theorem.

THEOREM 3.2.2. *Suppose $A \in [1, 3/2)$. The problem (1), (2) has three-cell solutions if only if $(Q, \beta) \in \Gamma_4$.*

COROLLARY 3.2.4.

(a) $\lim_{\omega \to -1^+} Q(\omega, 1) = +\infty$;

(b) $\lim_{\omega \to 0^-} Q(\omega, 1) = Q(0,1)$ *and* $\lim_{\omega \to 0^-} \beta(\omega, 1) = \beta(0,1)$, *where* $Q(0,1)$ *and* $\beta(0,1)$ *are defined in Corollary 3.2.2.*

PROOF. The desired result in (a) is similar to the one in Corollary 3.2.2. The assertion (b) is obtained if we verify that $\lim_{\omega \to 0^-} y_*^+(\omega, 1) = y_*(0,1)$. In fact, from Theorem 2.3.2, $g'(y; 0, 1)$ has exactly one zero, say, at $y = a > 0$. Let $\epsilon_0 = \min\{y_*(0,1) - a, M - y_*(0,1)\}$. Then, for all $0 < \epsilon < \epsilon_0/2$, let $y^\pm = y_*(0,1) \pm \epsilon$ and $m = \min\{|g(y^\pm; 0, 1)|\}$. By the continuous dependence on initial data, there exists a $\delta > 0$, such that $|g(y^\pm; \omega, 1) - g(y^\pm; 0, 1)| < m/2$ provided

$\omega \in (-\delta, 0)$. Hence, we get $g(y^-;\omega,1) > 0 > g(y^+;\omega,1)$ and this implies $y_*^+(\omega,1) \in (y^-,y^+)$. Thus, $\lim_{\omega\to 0^-} y_*^+(\omega,1) = y_*(0,1)$. ∎

Recall in Section 2.4 again that the (Q,β), which yields a 3-cell solution of (1), (2), must satisfy $Q > 0$, $\beta < 0$. Then, Γ_4 is a connected set lying in the quadrant $Q > 0$, $\beta < 0$ with one limit point at $(Q(0,1), \beta(0,1))$.

COROLLARY 3.2.5. *For all $Q > Q^*(A)$ if $1 \le A < 3/2$, there exists at least one $\beta < 0$ such that the problem (1), (2) possesses a three-cell solution.*

Note that $\Gamma_2' \cup \Gamma_4$ is a connected set, lying in the quadrant $Q > 0$, $\beta < 0$ which passes through the point $(Q(0,1), \beta(0,1))$. Also, from Corollary 3.1.3, 3.2.3, and 3.2.5, we have verified the following theorem.

THEOREM 3.2.3. *Let $1 \le A < 3/2$. The problem (1), (2) possesses at least one three-cell and a pair of two-cell solutions for $Q > Q^*(A)$.*

We turn to consider the case of $A = 3/2$. It is easy to verify that $\lim_{\omega\to 0^-} Q(\omega,1) = +\infty$ since $g(y) > 0$ on $(0,M)$ for $(B,E) \in D_3$. Now, suppose $Q(\omega_1,1) = Q(\omega_2,1)$ and $\beta(\omega_1,1) = \beta(\omega_2,1)$ for some $\omega_1 \neq \omega_2$ in $(-1,0)$. Then, the corresponding solutions f_i's must be distinct since $f_1'(1) \neq f_2'(1)$. This proves the next theorem.

THEOREM 3.2.4. *Let $A = 3/2$. The problem (1), (2) possesses at least a two-cell solution and a pair of three-cell solutions for sufficiently large $Q > 0$.*

Moreover, we have verified the final main result.

THEOREM C. *The problem (1), (2) possesses at least three solutions for sufficiently large $Q > 0$ if $1 \le A \le 3/2$.*

4. CONCLUDING REMARKS

Although the results in [3,4] and Section 3 have shown the existence of two-cell and three-cell solutions, the verification of (O-1) is yet completely clear. Also, for $1 \le A \le 3/2$, our numerical result indicated that (1), (2) possesses 'exactly' three solutions for sufficiently large $Q > 0$. Furthermore, it is also interesting to study the case of $A < 1$. We may expect that the bifurcation diagram for $A < 1$ will be much more complicated than the ones in this paper. This requires more delicate numerical and mathematical study.

REFERENCES

1. W.N. Gill, N.D. Kazarinoff, C.C. Hsu, M.A. Noack and J.D. Verhoeven, Thermalcapillary-driven convection in supported and floating-zone driven convection, *Adv. Space Research* **4**, 15–22 (1984).
2. W.N. Gill, N.D. Kazarinoff and J.D. Verhoeven, Convective diffusion in zone refining of low Prandtl number liquid metals and semiconductors, In *Integrate Circuits: Chemical and Physical Processing*, (Edited by P. Stroeve), Amer. Chem. Soc. Symposium Series, No. 290, pp. 47–69, (1985).
3. C. Lu, Existence, bifurcation, and limit of solutions of the similarity equations for floating rectangular cavities and disks, *SIAM J. Math. Anal.* **21** (3), 721–728 (1990).
4. C. Lu, N.D. Kazarinoff, J.B. Mcleod and W.C. Troy, Existence of solutions of the similarity equations for floating rectangular cavities and disks, *SIAM J. Math. Anal.* **19** (5), 1119–1126 (1988).
5. T.W. Hwang, T.H. Kuo and C.A. Wang, Similarity solutions for surface-tension driven flows in a slot with an insulated bottom, *Comput. Math. Applic.* **17** (12), 1573–1586 (1989).
6. T.W. Hwang and C.A. Wang, Existence and classification of similarity solutions for a problem on surface-tension driven flows in a slot with an insulated bottom, *Comput. Math. Applic.* **19** (12), 1–8 (1990).
7. D. Kahaner, C. Moler and S. Nash, *Numerical Methods and Software*, Prentice Hall Inc., New York, (1989).