The $q$-numerical radius of weighted shift operators with periodic weights

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Abstract

We deal with the $q$-numerical radius of weighted unilateral and bilateral shift operators. In particular, the $q$-numerical radius of weighted shift operators with periodic weights is discussed and computed.

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1. Introduction

Let $T$ be a bounded linear operator on a complex Hilbert space $H$. For $0 \leq q \leq 1$, the $q$-numerical range $W_q(T)$ of $T$ is defined by

$$W_q(T) = \{ \langle T\xi, \eta \rangle : \|\xi\| = \|\eta\| = 1, \langle \xi, \eta \rangle = q \}.$$

It is known that $W_q(T)$ is a bounded convex subset of $C$. The $q$-numerical range $W_q(T)$ is unitary similarity invariant and it is also transpose invariant. If $q = 1$, $W_q(T)$ reduces to the classical numerical range $W(T)$ of $T$. The $q$-numerical radius $w_q(T)$ of $T$ is defined by

$$w_q(T) = \sup\{|z| : z \in W_q(T)\}.$$
We consider a weighted unilateral shift operator with weights \((s_1, s_2, s_3, \ldots)\) represented by an infinite matrix of the form
\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
s_1 & 0 & 0 & 0 & \ldots \\
0 & s_2 & 0 & 0 & \ldots \\
0 & 0 & s_3 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
(1)
where \(\{s_n : n = 1, 2, 3, \ldots\}\) a bounded complex sequence. The operator (1) is the bounded linear operator on the Hilbert space \(\ell^2(\mathbb{N})\) by the mapping
\[
A(\xi_1, \xi_2, \xi_3, \ldots) = (0, s_1\xi_1, s_2\xi_2, s_3\xi_3, \ldots),
\]
\((\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2(\mathbb{N})\). The operator norm (spectral norm) of the weighted shift operator \(A\) is given by
\[
\|A\| = \sup\{|s_n| : n = 1, 2, 3, \ldots\}.
\]

We also consider a weighted bilateral shift operator with weights \((\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, \ldots)\) represented by an infinite matrix of the form
\[
B = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \ddots \\
\ldots & 0 & 0 & 0 & 0 & \ldots \\
\ldots & s_{-1} & 0 & 0 & 0 & \ldots \\
\ldots & 0 & s_0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & s_1 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & s_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]
(2)
where \(\{s_n : n \in \mathbb{Z}\}\) a bounded complex sequence. Similarly, the operator (2) is the bounded linear operator on the Hilbert space \(\ell^2(\mathbb{Z})\) by the mapping
\[
B(\ldots, \xi_{-3}, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots) = (\ldots, \eta_{-3}, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \eta_2, \ldots),
\]
where \(\eta_{n+1} = s_n\xi_n\), and \((\ldots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots) \in \ell^2(\mathbb{Z})\).

Define a unitary operator
\[
U = \text{diag} \left( c_1, c_1c_2, c_1c_2c_3, \ldots, \prod_{j=1}^{n} c_j \right),
\]
where \(\{c_j : j = 1, 2, 3, \ldots\}\) is a sequence of complex numbers with \(|c_j| = 1, j = 1, 2, 3, \ldots\). Then the operator \(UAU^*\) is a weighted unilateral shift operator with weights \((c_2s_1, c_3s_2, c_4s_3, \ldots, c_{n+1}s_n, \ldots)\). By choosing \(c_1 = 1, c_{n+1} = \frac{s_n}{|s_n|} \) if \(s_n \neq 0\), and \(c_{n+1} = 1\) if \(s_n = 0\), we have \(UAU^* = |A|\). Since the \(q\)-numerical range is unitarily invariant, we may assume, in the remainder of this paper, that \(s_n \geq 0, n = 1, 2, 3, \ldots\). Similarly, the weighted bilateral shift associated with weights \(\{s_n : n \in \mathbb{Z}\}\) is unitarily similar to the weighted unilateral shift associated with weights \(\{|s_n| : n \in \mathbb{Z}\}\).

There have been a number of interesting papers on the \(q\)-numerical range and \(q\)-numerical radius, e.g., [2,5–7,10], but the formulation and computation of the \(q\)-numerical radius are more complicated. A numerical algorithm to compute the numerical radius of a shift matrix is found in
A lower bound for the \( q \)-numerical radius of a weighted shift operator is discussed in [11]. In this paper, we deal with the \( q \)-numerical radius of weighted shift operators. In particular, the \( q \)-numerical radius of weighted shift operators with periodic weights is discussed and computed.

2. Weighted shift operators

Let \( A \) be a weighted unilateral or bilateral shift operator. For a complex number \( c \) with modulus 1, we consider the following four sequences in \( \ell^2(\mathbb{N}) \)

\[
\begin{align*}
x &= (\xi_1, \xi_2, \xi_3, \ldots), \\
y &= (\eta_1, \eta_2, \eta_3, \ldots), \\
\tilde{x} &= (c\xi_1, c^2\xi_2, c^3\xi_3, \ldots), \\
\tilde{y} &= (c\eta_1, c^2\eta_2, c^3\eta_3, \ldots).
\end{align*}
\]

with \( \|x\| = \|y\| = 1 \) and \( \langle x, y \rangle = q \). Then \( \|\tilde{x}\| = \|\tilde{y}\| = 1 \), \( \langle \tilde{x}, \tilde{y} \rangle = q \), and

\[ \langle A\tilde{x}, \tilde{y} \rangle = c\langle Ax, y \rangle. \]

This following result is immediate.

**Theorem 1** (cf. [11, Theorem 3]). Let \( A \) be a weighted unilateral shift operator (1) or bilateral shift operator (2). Then \( W_q(A) \) is a circular disc.

As a consequence of Theorem 1, \( w(A) = \sup(\sigma(\Re(A))) \), where the real Hermitian part \( \Re(A) = (A + A^*)/2 \) of \( A \) is

\[
\Re(A) = \frac{1}{2} \begin{pmatrix}
0 & \overline{s_1} & 0 & 0 & \ldots \\
s_1 & 0 & \overline{s_2} & 0 & \ldots \\
0 & s_2 & 0 & \overline{s_3} & \ldots \\
0 & 0 & s_3 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

By Tsing’s circular union formula [12], we have

\[
w_q(A) = \sup \left\{ qz + \sqrt{1 - q^2w} \sqrt{h(z) - |z|^2} : z \in W(A), w \in \mathbb{C}, |w| \leq 1 \right\},
\]

where

\[ h(z) = \sup\{t : (z, t) \in W(A, A^*A)\} \quad (3) \]

for \( z \in W(A) \), and

\[ W(A, A^*A) = \{(v^*Av, v^*A^*Av) \in \mathbb{C} \times \mathbb{R} : v \in \mathbb{C}^n, v^*v = 1\} \]

is the Davis-Wielandt shell of \( A \). The numerical range \( W(A) \) is a circular disc centered at the origin. The function \( h(z) \) satisfies the invariance \( h(z) = h(|z|) \). By the concaveness of the function \( h(z) \) (cf. [1]),

\[ h(x) = \sup\{h(x + iy) : y \in \mathbb{R}, x + iy \in W(A)\}, \]

for every real point \( x \in W(A) \). We consider the orthogonal projection

\[ \pi : (x, y, z) \in \mathbb{R}^3 \mapsto (x, z) \in \mathbb{R}^2. \]
Then the set
\[ \{ \pi(x, y, z) : (x, y, z) \in \mathbb{R}^3, x + iy \in W(A), z = h(x + iy) \} \]
is convex and its upper boundary is given by
\[ \{ (x, h(x)) : x \in \mathbb{R}, x \in W(A) \} = \{ (x, h(x)) : x \in W(\Re(A)) \}. \]
The function \( h \) on the interval \( W(\Re(A)) \) is an even concave function \( h(-x) = h(x) \). The graph of \( h \) is the upper part of the numerical range \( W(\Re(A) + iA^*A) \).

**Theorem 2.** Let \( A \) be a weighted unilateral shift operator (1) or bilateral shift operator (2). The curve \( \{ (x, h(x)) : 0 \leq x \leq \sup W(\Re(A)) \} \) is the envelope of one-parameter family of straight lines
\[ \{ (x, y) \in \mathbb{R}^2 : y + kx = \lambda(k), \quad k > 0, \]where \( \lambda(k) \) is the supremum of the spectrum of the Hermitian operator \( A^*A + k\Re(A) \).

**Proof.** The numerical range \( W(\Re(A) + iA^*A) \) of the operator \( \Re(A) + iA^*A \) is symmetric with respect to the imaginary axis. The upper part of its boundary is expressed by the concave function \( h \) on the interval \( \Cl(W(\Re(A)) \)
\[ h(x) = \sup \{ y : x + iy \in W(\Re(A) + iA^*A) \}. \]
We consider a support line \( \ell \) of \( W(\Re(A) + iA^*A) \) for which \( \ell \) is a tangent of the graph of \( h \), and \( \ell \) is expressed as
\[ \cos \theta x + \sin \theta y = L(\theta) \tag{4} \]
for \( 0 < \theta < \pi \), where \( L(\theta) \) is supremum of the numerical range of the Hermitian part
\[ \Re(e^{-i\theta}(\Re(A) + iA^*A)) = \cos \theta \Re(A) + \sin \theta A^*A. \]
We rewrite the support line (4) in the form
\[ \cot \theta x + y = M(\theta), \]
where
\[ M(\theta) = \sup(\sigma(A^*A + \cot \theta \Re(A))). \]
Substituting the variable, the graph of \( h \) is the envelope of the one-parameter family of straight lines
\[ y + kx = \sup(\sigma(A^*A + k\Re(A))). \]

3. Periodic weights

In this section, we study unilateral and bilateral shift operators with periodic weights. Suppose that \( A \) is a unilateral (resp. bilateral) shift operator with periodic weights \( \{ \ldots, s_1, s_2, \ldots, s_m, s_1, s_2, \ldots, s_m, \ldots \} \). Then the operator \( A \) is represented by a block matrix form
\[
A = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & s_m S_0 \\
S_1 I_K & 0 & 0 & \ldots & 0 & 0 \\
0 & S_2 I_K & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & S_{m-1} I_K & 0 \\
\end{pmatrix},
\]
where \( K \) is an infinite dimensional separable Hilbert space and \( S_0 \) is the unilateral (resp. bilateral) shift operator on \( \ell^2(\mathbb{N}) \) (resp. \( \ell^2(\mathbb{Z}) \)). We assume that \( s_1 \geq 0, \ldots, s_m \geq 0 \). Observe that the two operators \( A^*A \) and \( 2\Re(A) \) are given by

\[
A^*A = \begin{pmatrix}
 s_1^2 I_K & 0 & \cdots & 0 \\
 0 & s_2^2 I_K & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & s_m^2 I_K
\end{pmatrix}
\]

and

\[
2\Re(A) = \begin{pmatrix}
 0 & s_1 I_K & 0 & \cdots & 0 & s_m S_0 \\
 s_1 I_K & 0 & s_2 I_K & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 s_m S_0^T & 0 & 0 & \cdots & s_{m-1} I_K & 0
\end{pmatrix}.
\]

Consider the \( m \times m \) matrix

\[
T(k) = \begin{pmatrix}
 s_1^2 & k s_1 /2 & 0 & \cdots & 0 & k s_m /2 \\
 k s_1 /2 & s_2^2 & k s_2 /2 & \cdots & 0 & 0 \\
 0 & k s_2 /2 & s_3^2 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 k s_m /2 & 0 & 0 & \cdots & k s_{m-1} / 2 & s_m^2
\end{pmatrix}
\]

(5)

We have the following result.

**Theorem 3.** Let \( A \) be a weighted unilateral or bilateral shift operator with periodic weights \( \{s_1, s_2, \ldots, s_m\} \). Then

\[
\sup \sigma(A^*A + k\Re(A)) = \max \sigma(T(k)).
\]

**Proof.** For \( k \geq 0 \), we see that

\[
\sup \sigma(A^*A + k\Re(A)) = \sup \{ \langle (A^*A + k\Re(A))\xi, \xi \rangle : \|\xi\| = 1 \}
\]

\[
= \sup \{ s_1^2 \xi_1 \bar{\xi}_1 + \cdots + s_m^2 \xi_m \bar{\xi}_m + k s_1 \xi_1 \bar{\xi}_2 + k s_2 \xi_2 \bar{\xi}_3 + \cdots + k s_{m-1} \xi_{m-1} \bar{\xi}_m + k s_m \xi_m \bar{\xi}_1 : \xi_1, \xi_2, \ldots, \xi_m \in \mathbb{R},
\]

\[
\|\xi_1\|^2 + \cdots + \|\xi_m\|^2 = 1, \xi_1 \geq 0, \ldots, \xi_m \geq 0 \}
\]

\[
\leq \sup \{ s_1^2 \|\xi_1\|^2 + \cdots + s_m^2 \|\xi_m\|^2 + k s_1 \|\xi_1\| \|\xi_2\| + k s_2 \|\xi_2\| \|\xi_3\| + \cdots + k s_{m-1} \|\xi_{m-1}\| \|\xi_m\| + k s_m \|\xi_m\| \|\xi_1\| : \xi_1, \xi_2, \ldots, \xi_m \in \mathbb{R},
\]

\[
\xi_1 \geq 0, \ldots, \xi_m \geq 0, \|\xi_1\|^2 + \cdots + \|\xi_m\|^2 = 1 \}.
\]

This inequality implies

\[
\sup \sigma(A^*A + k\Re(A)) \leq \max \sigma(T(k)).
\]

Next, we take nonnegative real numbers \( x_1, x_2, \ldots, x_m \) with \( x_1^2 + x_2^2 + \cdots + x_m^2 = 1 \) so that

\[
\max \sigma(T(k)) = s_1^2 x_1^2 + \cdots + s_m^2 x_m^2 + k s_1 x_1 x_2 + k s_2 x_2 x_3 + \cdots + k s_{m-1} x_{m-1} x_m + k s_m x_m x_1.
\]
For every \( L \in \mathbb{N} \), denote the unit vector \( \xi(L) \in \ell^2 \) by
\[
\xi(L) = \frac{1}{\sqrt{L}} (\ldots, 0, 1, 1, \ldots, 1, 0, \ldots),
\]
where the number of entries 1 is \( L \).
We set
\[
\xi_j = x_j \xi(L) \in \ell^2, \quad j = 1, 2, \ldots, m.
\]
Then
\[
\lim_{L \to \infty} \langle (A^* A + k \Re(A)) \xi_1, \xi_2, \ldots, \xi_m \rangle
= \lim_{L \to \infty} \sum
\begin{align*}
& s_1^2 \langle \xi_1, \xi_1 \rangle + \cdots + s_m^2 \langle \xi_m, \xi_m \rangle + k s_1 \langle \xi_1, \xi_2 \rangle \\
& + k s_2 \langle \xi_2, \xi_3 \rangle + \cdots + k s_{m-1} \langle \xi_{m-1}, \xi_m \rangle + k s_m \langle S_0 \xi_m, \xi_1 \rangle
\end{align*}
= \max \sigma(T(k)). \quad \Box
\]
Consider the \( m \times m \) bilateral shift matrix \( S \) with weights \( \{s_1, s_2, \ldots, s_m\} \)
\[
S = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & s_m \\
s_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & s_2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & s_{m-1} & 0
\end{pmatrix}, \tag{6}
\]
and define the height function associated with \( S \)
\[
H(z) = \max \{ t : (z, t) \in W(S, S^* S) \} \tag{7}
\]
for \( z \in W(S) \).

**Remark 1.** It is easy to see that the results of Theorem 2 and Theorem 3 are valid for the matrix \( S \), namely, the curve \( \{ (x, H(x)) : 0 \leq x \leq \max W(\Re(S)) \} \) is the envelope of one-parameter family, \( k > 0 \), of straight lines
\[
\{(x, y) \in \mathbb{R}^2 : y + kx = \lambda(k)\},
\]
where \( \lambda(k) \) is the maximum of \( \sigma(S^* S + k \Re(S)) \). Indeed we have \( S^* S + k \Re(S) = T(k) \) in Theorem 3.

**Theorem 4.** Let \( A \) be a weighted unilateral or bilateral shift operator with periodic weights \( \{s_1, s_2, \ldots, s_m\} \) and \( S \) be the \( m \times m \) bilateral shift matrix (6) with weights \( \{s_1, s_2, \ldots, s_m\} \). The height functions \( h(z) \) and \( H(z) \) are respectively defined by (3) and (7). Then

(i) \( W(S) \subset \{ z \in \mathbb{C} : |z| \leq \max W(\Re(S)) \} \).
(ii) \( H(z) \leq H(|z|) \) for every \( z \in W(S) \).
(iii) \( \max W(\Re(S)) = \sup W(\Re(A)) \).
(iv) \( H(x) = h(x) \) for every \( 0 \leq x \leq \max W(\Re(S)) \).
Proof. For any unit vector $(\xi_1, \ldots, \xi_m)^T \in \mathbb{C}^n$, we have
\[
|s_1\xi_1\xi_2 + s_2\xi_2\xi_3 + \cdots + s_m\xi_m\xi_1| \leq s_1|\xi_1||\xi_2| + s_2|\xi_2||\xi_3| + \cdots + s_m|\xi_m||\xi_1| \leq \max W(\Re(S)),
\]
and (i) is proved.

Since $S$ is a real matrix, it follows that $H$ has the symmetry property $H(\bar{z}) = H(z)$. We take an arbitrary point $z_1 = |z_1|e^{i\theta}$ with $z_1 \neq 0$, and define $\tilde{H}(x : \theta)$ by
\[
\tilde{H}(x : \theta) = \max \{H(e^{i\theta}(x + iy)) : y \in \mathbb{R}, \ e^{i\theta}(x + iy) \in W(A)\}.
\]
Then the inequality $H(z_1) \leq \tilde{H}(|z_1| : \theta)$ holds. By the symmetry of $H$ with respect to the real axis and its concaveness, we have
\[
\tilde{H}(x : 0) = H(x)
\]
for $x \geq 0$. The function $\tilde{H}(x : \theta)$ is characterized as
\[
\tilde{H}(x : \theta) = \max \{t : (x, t) \in W(\Re(e^{-i\theta}S), S^*S)\}.
\]
Direct computations imply that the maximum of the spectrum of the Hermitian matrix $S^*S + k\Re(e^{-i\theta}S)$ is
\[
\max \{|s_1|^2|\xi_1|^2 + \cdots + |s_m|^2|\xi_m|^2 + k\Re(e^{-i\theta}s_1\xi_1\xi_2) + \cdots + k\Re(e^{-i\theta}s_m\xi_m\xi_1) : (\xi_1, \ldots, \xi_m) \in \mathbb{C}^m, |\xi_1|^2 + \cdots + |\xi_m|^2 = 1\}.
\]
This maximum is less than or equal to the value
\[
\max \{|s_1|^2|\xi_1|^2 + \cdots + |s_m|^2|\xi_m|^2 + ks_1\xi_1\xi_2 + \cdots + ks_m\xi_m\xi_1 : (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, \xi_1 \geq 0, \ldots, \xi_m \geq 0, \xi_1^2 + \cdots + \xi_m^2 = 1\},
\]
which is the maximum of the spectrum of the Hermitian matrix $S^*S + k\Re(S)$. Thus, by Remark 1, we have that
\[
H(z) \leq \tilde{H}(|z| : \theta) \leq \tilde{H}(|z| : 0) = H(|z|),
\]
and (ii) follows. The assertion (iii) is obvious.

By Theorem 2, the graphs of the functions $h$ and $H$ on the common interval $[0, \max W(\Re(S))]$ are respectively the envelopes of the family of straight lines
\[
y + kx = \max \sigma(A^*A + k\Re(A))
\]
and
\[
y + kx = \max \sigma(S^*S + k\Re(S)).
\]
By Theorem 3, the right-hand sides of (8) and (9) coincide for every $k$, we conclude (iv). 

As a consequence of Theorem 4, we have the following result which extends Ridge’s result [8] for the case $q = 1$.

Theorem 5. Let $A$ be a weighted unilateral or bilateral shift operator with periodic weights $\{s_1, s_2, \ldots, s_m\}$ and $S$ be the $m \times m$ bilateral shift matrix (6) with weights $\{s_1, s_2, \ldots, s_m\}$. Then $w_q(A) = w_q(S)$ for every $0 \leq q \leq 1$. 
**Proof.** By using Tsing’s circular union formula [12], we have that

\[
w_q(S) = \max \left\{ \left| qz + \sqrt{1-q^2w\sqrt{H(z)} - |z|^2} \right| : z \in W(S), w \in \mathbb{C}, |w| \leq 1 \right\}
\]

\[
= \max \left\{ qx + \sqrt{1-q^2\sqrt{H(x)} - x^2} : x \in W(S), x \geq 0 \right\}
\]

\[
= \max \left\{ qx + \sqrt{1-q^2\sqrt{h(x)} - x^2} : x \in W(A), x \geq 0 \right\}
\]

\[
= \max \left\{ x \in \mathbb{R} : x \geq 0, x \in W_q(A) \right\}. \quad \Box
\]

Let \( A \) be the weighted bilateral shift operator with weights \( \{ s_n : n \in \mathbb{Z} \} \), and \( \tilde{A} \) the weighted bilateral shift operator with weights \( \{ s_{n-1} : n \in \mathbb{Z} \} \). Define the unitary operator \( V \) on \( \ell^2(\mathbb{Z}) \) by

\[
V(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, x_2, x_1, x_0, x_{-1}, x_{-2}, \ldots).
\]

Then \( \tilde{A} = V^*AV \), and thus \( w_q(A) = w_q(\tilde{A}) \). In general, the \( q \)-numerical radius of a weighted shift operator depends on the ordering of the weights [10]. In the following, we obtain that the \( q \)-numerical radius of the periodic weighted unilateral shift operators is invariant under reversing the order of the weights.

**Theorem 6.** Let \( A \) and \( \tilde{A} \) be respectively the weighted unilateral shift operators with periodic weights \( \{ s_1, s_2, \ldots, s_{m-1}, s_m \} \) and \( \{ s_m, s_{m-1}, \ldots, s_2, s_1 \} \). Then \( w_q(A) = w_q(\tilde{A}) \).

**Proof.** This follows from Theorem 5 and the fact that

\[
W_q \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & s_m \\ s_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & s_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & s_{m-1} & 0 \end{pmatrix} = W_q \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & s_1 \\ s_m & 0 & 0 & \ldots & 0 & 0 \\ 0 & s_{m-1} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & s_2 & 0 \end{pmatrix}. \quad \Box
\]

4. Radius computations

A general formula for the \( q \)-numerical radius of a weighted shift operator is unknown. In this section, we treat of this question for special periodic weighted shift operators. At first, we obtain a result of 2-periodic.

**Theorem 7.** Let \( A \) be a weighted unilateral or bilateral shift operator with periodic weights \( \{ s_1, s_2 \} \). Then

\[
w_q(A) = \frac{s_1 + s_2}{2} + \sqrt{1-q^2}\frac{|s_1 - s_2|}{2}.
\]
Proof. By Theorem 5,

\[ w_q(A) = w_q \left( \begin{pmatrix} 0 & s_2 \\ s_1 & 0 \end{pmatrix} \right). \]

Furthermore, Nakazato [6] shows that

\[ W_q \left( \begin{pmatrix} 0 & s_2 \\ s_1 & 0 \end{pmatrix} \right) \]

is the elliptic disc

\[
\frac{x^2}{((s_1 + s_2)/2 + \sqrt{1 - q^2|s_1 - s_2|/2})^2} + \frac{y^2}{(|s_1 - s_2|/2 + \sqrt{1 - q^2(s_1 + s_2)/2})^2} \leq 1,
\]

and thus the assertion follows. □

For \( m \geq 3 \), we have the following result.

Theorem 8. Let \( A \) be a weighted unilateral or bilateral shift operator with periodic weights \( \{s_1, s_2, \ldots, s_m\}, m \geq 3 \). If \( s_j < s_2 \) for all \( j = 1, 3, \ldots, m \) then

\[ w_q(A) = s_2 - \frac{(s_2^2 - s_1^2)(s_2^2 - s_3^2)}{2s_2(2s_2^2 - s_1^2 - s_3^2)} q^2 + c_3 q^3 + \cdots \]

for sufficiently small \( q \).

Proof. Suppose that \( a_1, a_2, \ldots, a_m \) are distinct real numbers. Decompose the following real symmetric matrix \( A(k) \) into the form \( H_1 + kH_2 \)

\[
A(k) = \begin{pmatrix}
  a_1 & kb_{12} & kb_{13} & \ldots & kb_{1m} \\
  kb_{21} & a_2 & kb_{23} & \ldots & kb_{2m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  kb_{m1} & kb_{m2} & kb_{m3} & \ldots & a_m \\
\end{pmatrix} + k \begin{pmatrix}
  0 & b_{12} & b_{13} & \ldots & b_{1m} \\
  b_{12} & 0 & b_{23} & \ldots & b_{2m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{1m} & b_{2m} & b_{3m} & \ldots & 0 \\
\end{pmatrix}.
\]

Then, the perturbation expansion, as \( k \to 0 \), of the eigenvalues of \( A(k) \) are given by

\[
\lambda_j(A(k)) = a_j - \sum_{1 \leq i \leq m, i \neq j} \frac{b_{ij}^2}{a_i - a_j} k^2 + \mu_j k^3 + \cdots
\]

for small \( k \). Similarly, the expansion of the eigenvalues of \( T(k) \) in (5) are

\[
\lambda_j(T(k)) = s_j^2 - \left( \frac{s_j^2}{4(s_{j+1}^2 - s_j^2)} + \frac{s_j^2}{4(s_{j-1}^2 - s_j^2)} \right) k^2 + \mu_j k^3 + \cdots,
\]

and

\[
\lambda_1(T(k)) = s_1^2 - \left( \frac{s_1^2}{4(s_2^2 - s_1^2)} + \frac{s_1^2}{4(s_m^2 - s_1^2)} \right) k^2 + \mu_1 k^3 + \cdots.
\]
From assumption, \( s_2 > s_j \) for \( j \neq 2 \), we set
\[
\beta = \frac{s_1^2}{4(s_2^2 - s_1^2)} + \frac{s_2^2}{4(s_2^2 - s_3^2)} = \frac{s_2^4 - s_2^3 s_3^2}{4(s_2^2 - s_1^2)(s_2^2 - s_3^2)} > 0. \tag{10}
\]

By Theorem 2, the graph of the height function \( \beta \) where \( x \) for \( \phi(x) = -x \), the expression (15) is rewritten as using Taylor expansion, we have
\[
\text{Moreover, by (17), we have}
\]
Differentiate (13) with respect to \( h \)
\[
\text{Differentiate (13) with respect to } k, \text{ we obtain}
\]
The Taylor expansion of the function \( h \) at \( x = 0 \) is obtained by eliminating \( k \) from the equations (13) and (14). The result is given by
\[
\text{The Taylor expansion of the function } h \text{ at } x = 0 \text{ is obtained by eliminating } k \text{ from the equations (13) and (14). The result is given by}
\]
\[
\text{The expression (15) is rewritten as}
\]
and \( h''(0)/2 = -1/(4\beta) \). Define \( \phi(x) = (h(x) - x^2)^{1/2} \). Substituting (16) into this function, and using Taylor expansion, we have
\[
\text{By (ii) of Theorem 4, the height function } h(x) \text{ and the } q \text{-numerical radius } w_q(A) \text{ of the weight shift } A \text{ is related by the equation}
\]
Moreover, by (17), we have
\[
\text{By (ii) of Theorem 4, the height function } h(x) \text{ and the } q \text{-numerical radius } w_q(A) \text{ of the weight shift } A \text{ is related by the equation}
\]
By (ii) of Theorem 4, the height function \( h(x) \) and the \( q \)-numerical radius \( w_q(A) \) of the weight shift \( A \) is related by the equation
\[
\text{Moreover, by (17), we have}
\]
The maximum of (18) occurs at
\[ x = (-q/\phi''(0))(1 - 1/2q^2)^{-1} + c_3q^3 + \cdots = -q/\phi''(0) + \tilde{c}_3q^3 + \cdots \] (19)
Substituting (19) into (18), we conclude that
\[ w_q(A) = s_2 - \frac{(s_2^2 - s_1^2)(s_2^2 - s_3^2)}{2s_2(2s_2^2 - s_1^2 - s_3^2)}q^2 + c_3q^3 + \cdots = -q/\phi''(0) + \tilde{c}_3q^3 + \cdots \] □

Example 1. Consider \( m = 4, s_2 = 1, s_1 = s_3 = \sqrt{9/10} \) and \( s_4 = 199/200 \). Then, by the formula in Theorem 8, we have
\[ w_q(A) = 3 - \frac{5}{12}q^2 + c_3q^3 + \cdots \] (20)
In [11, Proposition 8], a lower bound of \( w_q(A) \) is estimated by
\[ w_q(A) \geq \frac{1 - q^2}{1 - q^{2m}} \max_{\sigma(1)} s_\sigma(1) + s_\sigma(2)q^2 + \cdots + s_\sigma(m)q^{2(m-1)}. \] (21)
We point out that the lower bound (21) is false. Suppose the inequality (21) holds, it implies
\[ w_q(A) \geq 1 + s_4q^2 + s_1q^4 + s_2q^6 \\
\quad = (1 + s_4q^2 + s_1q^4 + \cdots)(1 - q^2 + r_4q^4 + \cdots) \\
\quad = 1 - (1 - s_4)q^2 + u_4q^4 + \cdots \]
Then
\[ 1 - w_q(A) \leq (1 - s_4)q^2 + v_4q^4 + \cdots \]
However, by (20), we have the estimation
\[ 1 - w_q(A) = (1/40)q^2 - c_3q^3 + \cdots \]
We will modify the estimation (21) in Section 5.

Example 2. Consider \( m = 3, s_2 = 3, s_1 = s_3 = 2 \). Then, again by Theorem 8, we have
\[ w_q(A) = 3 - \frac{5}{12}q^2 + c_3q^3 + \cdots \]
On the other hand, by a computation method used in [1], the \( q \)-numerical radius \( w = w_q(A) \) is a solution of the quartic equation
\[ F(q, w) = 164w^4 + (432q - 924)w^3 + (1081q^2 - 2232q + 864)w^2 \\
\quad + (480q^3 - 4896q^2 + 2808q + 2592)w \\
\quad + (400q^4 - 1040q^3 + 5184q^2 - 3888) = 0, \] (22)
0 \leq q \leq 1. From (22), it implies that
\[ w_q(A) = 3 - \frac{5}{12}q^2 + \frac{5}{54}q^3 - \frac{65}{432}q^4 + \cdots, \]
and in particular
\[ w_1(A) = 3 + \sqrt{41}. \]

Theorem 9. Let \( A \) be a weighted unilateral or bilateral shift operator with periodic weights \( \{s_1, s_2, \ldots, s_m\}, m \geq 3 \). If there exist \( 1 \leq i_1 < i_2 < \cdots < i_p \leq m, p \geq 2 \), such that \( s_{i_1} = s_{i_2} = \cdots = s_{i_p} > s_j \geq 0 \) for all \( j \neq i_1, \ldots, i_p \), then for some \( q_0 \), \( w_q(A) = s_i \) for \( 0 \leq q \leq q_0 \) if and only if
\[ \min\{i_2 - i_1, i_3 - i_2, \ldots, i_p - i_{p-1}, i_1 + m - i_p\} = 1. \] (23)

Proof. At first, we claim that \( w_q(A) \) is constant for \( 0 \leq q \leq q_0 \) if and only if the height function \( h(x) = h(0) \) for \( 0 \leq x \leq x_0 \) for some \( 0 < x_0 \leq w_1(A) \). Since the height function \( h \) on \([-w_1(A), w_1(A)]\) is an even function, the function \( \phi(x) = \sqrt{h(x) - x^2} \) on \([-w_1(A), w_1(A)]\) is also an even function. Setting
\[ G(q) = \frac{w_q(A)}{\sqrt{1 - q^2}}, \quad 0 \leq q < 1, \]
and changing the variable \( \lambda = q/\sqrt{1 - q^2} \), we have, for \( 0 \leq \lambda < +\infty \)
\[ G(\lambda) = \max \left\{ \lambda x + \sqrt{h(x) - x^2} : 0 \leq x \leq w_1(A) \right\}. \]
For \(-\infty < \lambda < 0 \), we define \( G(\lambda) = G(-\lambda) \). Then we have
\[ G(\lambda) = \max \left\{ \lambda x + \sqrt{h(x) - x^2} : -w_1(A) \leq x \leq 0 \right\} \]
\[ = \max \left\{ \lambda x + \sqrt{h(x) - x^2} : -w_1(A) \leq x \leq w_1(A) \right\}. \]
By [1], the function \( \psi(x) = -\sqrt{h(x) - x^2} \) is a continuous convex function on \([-w_1(A), w_1(A)]\). Further, the duality of convex functions [9, p. 34] implies that
\[ \psi(x) = -\sqrt{h(x) - x^2} = \sup\{x\lambda - G(\lambda) : -\infty < \lambda < \infty\}. \]
If \( h(x) = h(0) \), \( 0 \leq x \leq x_0 \) for some \( x_0 \), then for sufficiently small \( q > 0 \), we have
\[ w_q(A) = \max \left\{ qx + \sqrt{1 - q^2}\sqrt{h(x) - x^2} : 0 \leq x \leq x_0 \right\} \]
\[ = \max \left\{ qx + \sqrt{1 - q^2}\sqrt{h(0) - x^2} : 0 \leq x \leq x_0 \right\} \]
\[ = q\sqrt{h(0)}q + \sqrt{1 - q^2}\sqrt{h(0)}\sqrt{1 - q^2} = \sqrt{h(0)}\]
Hence \( w_q(A) \) is constant near \( q = 0 \).
Conversely, we assume that \( w_q(A) = w_0(A) \) for sufficiently small \( q \in [0, q_0] \). Then \( G(\lambda) = G(0)\sqrt{1 + \lambda^2} \) for sufficiently small \( \lambda > 0 \), where \( G(0) > 0 \). For sufficiently small \( x > 0 \), we have
\[ \psi(x) = \max \left\{ \lambda x - G(0)\sqrt{1 + \lambda^2} : 0 \leq \lambda \leq \lambda_0 \right\} \]
\[ = \left( x/\sqrt{G(0)^2 - x^2} \right) x - G(0)\sqrt{1 + \left( x/\sqrt{G(0)^2 - x^2} \right)^2} \]
\[ = -\sqrt{G(0) - x^2}. \]

Hence \( \phi(x) = \sqrt{\phi(0)^2 - x^2} \) for sufficiently small \( x \), and thus \( h(x) = h(0) \) for sufficiently small \( x \). This proves the claim.

The local property of the height function \( h(x), 0 \leq x \leq w_1(A) \) near \( x = 0 \) is determined by the maximum of the spectrum of \( T(k) \) given by (5). We consider the \( p \)-dimensional subspace \( E \) of \( \mathbb{C}^m \) spanned by
\[ e_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0)^T, \quad j = 1, \ldots, p, \]
where the entry 1 is the \( ij \)th coordinate of the vector \( e_{ij} \). We denote by \( P_E \) the orthogonal projection of \( \mathbb{C}^m \) onto \( E \).

Suppose that the condition (23) is invalid. We set
\[ \{f_{p+1}, \ldots, f_m\} = \{e_1, \ldots, e_m\}\backslash\{e_{i_1}, \ldots, e_{i_p}\}. \]
Then the real symmetric matrix
\[
\begin{pmatrix}
0 & s_1/2 & 0 & \ldots & 0 & s_m/2 \\
s_1/2 & 0 & s_2/2 & \ldots & 0 & 0 \\
0 & s_2/2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_m/2 & 0 & 0 & \ldots & s_{m-1}/2 & 0 \\
\end{pmatrix}
\]
is represented by the block matrix
\[
\begin{pmatrix}
0 & M_2 \\
M_2^T & M_1 \\
\end{pmatrix},
\]
where
\[
M_1 = (I - P_E) \begin{pmatrix}
0 & s_1/2 & 0 & \ldots & 0 & s_m/2 \\
s_1/2 & 0 & s_2/2 & \ldots & 0 & 0 \\
0 & s_2/2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_m/2 & 0 & 0 & \ldots & s_{m-1}/2 & 0 \\
\end{pmatrix} (I - P_E)
\]
and
\[
M_2 = P_E \begin{pmatrix}
0 & s_1/2 & 0 & \ldots & 0 & s_m/2 \\
s_1/2 & 0 & s_2/2 & \ldots & 0 & 0 \\
0 & s_2/2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_m/2 & 0 & 0 & \ldots & s_{m-1}/2 & 0 \\
\end{pmatrix} (I - P_E).
\]
The \( p \times (m - p) \) matrix \( M_2 \) is represented by
\[
M_2 f_r = \sum_{j=1}^k b_{jr} e_{ij}, \quad r = k + 1, \ldots, m.
\]
Then we have
\[ M_2^T e_{ij} = \sum_{r=k+1}^{m} b_{jr} f_r, \]
and hence
\[ \sum_{r=k+1}^{m} b_{jr}^2 = \| M_2^T e_{ij} \|^2 = \left\| \begin{pmatrix} 0_k & M_2 \\ M_2^T & M_1 \end{pmatrix} e_{ij} \right\|^2. \]
This quantity does not vanish since
\[ \left( \begin{array}{cc} 0_p & M_2 \\ M_2^T & M_1 \end{array} \right) e_{ij} = \tilde{s}_{ij}^2 e_{ij} + \cdots, \]
We assume that
\[ A^* A f_j = \tilde{s}_{j}^2 f_j, \quad j = p + 1, \ldots, m. \]
The \( p \) eigenvalues of \( T(k) \) near \( s_{i_1}^2 \) are given by
\[ \lambda_{ij}(T(k)) = s_{i_1}^2 - \sum_{r=p+1}^{m} \frac{b_{jr}^2}{s_{ij}^2 - \tilde{s}_{r}^2} k^2 + \mu k^3 + \cdots, \quad j = 1, \ldots, p. \] (24)
Apply the result (24) to an argument used in the proof of Theorem 8, the function (15) becomes
\[ h(x) = s_{i_1}^2 - \frac{1}{4\beta} x^2 + \tilde{\mu} x^3 + \cdots, \]
where
\[ \beta = \sum_{r=p+1}^{m} \frac{b_{jr}^2}{(s_{ij}^2 - \tilde{s}_{r}^2)} > 0. \]
Thus the function \( h \) is strictly decreasing on \([0, x_0]\) for sufficiently small \( x_0 \). Hence \( h(x) \) can not be constant near 0.
Next, if the condition (23) holds then the Hermitian matrix
\[ M_0 = P_E \begin{pmatrix} 0 & s_1/2 & 0 & \cdots & 0 & s_m/2 \\ s_1/2 & 0 & s_2/2 & \cdots & 0 & 0 \\ 0 & s_2/2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_m/2 & 0 & 0 & \cdots & s_{m-1}/2 & 0 \end{pmatrix} P_E \]
has a positive and a negative eigenvalues since \( M_0 \) has a principal submatrix
\[ \left( \begin{array}{cc} 0 & s_{i_1} \\ s_{i_1} & 0 \end{array} \right). \]
We set
\[ \sigma(M_0) = \{\mu_1 \leq \mu_2 \leq \cdots \leq \mu_p\}. \]
Then we have \( \mu_1 < 0 < \mu_p \). The largest eigenvalue of \( \lambda_j(T(k)) \) is given by
\[ s_{i_1}^2 + \mu_p k + c_p k^2 + \cdots, \]
if \( k > 0 \), and
\[
\frac{s_1^2}{\tau_1} + \mu_1 k + c_1 k^2 + \cdots
\]
if \( k < 0 \). Thus the maximal eigenvalue of \( T(k) \) has distinct right-hand and left-hand derivatives at \( k = 0 \). Then, by [3, Theorem 2] \( W(\Re(S) + iS^*S) \) has a horizontal flat portion on the upper part of its boundary. By (iv) of Theorem 4, the graph of \( h \) is the upper part of \( W(\Re(S) + iS^*S) \) on the interval \([0, \max \sigma(\Re(S))]\) and \( h \) is an even function. Hence the function \( h(x) \) is constant near 0. \( \square \)

5. A lower bound

In this section, we provide some lower bounds for the \( q \)-numerical radius of the weighted unilateral shift operator \( A \) with periodic weights \( \{s_1, s_2, \ldots, s_m\} \). For abbreviation, we denote the corresponding radius by \( w_q(\{s_1, s_2, \ldots, s_m\}) \). We have the following invariant properties:

**Theorem 10**

(i) \( w_q(\{s_1, s_2, \ldots, s_m\}) = w_q(\{|s_1|, |s_2|, \ldots, |s_m|\}) \).

(ii) \( w_q(\{c_1 s_1, c_1 s_2, \ldots, c_1 s_m\}) = |c| w_q(\{s_1, s_2, \ldots, s_m\}) \).

(iii) If \( s_1, s_2, \ldots, s_m, s'_1, s'_2, \ldots, s'_m \) are nonnegative real numbers, then \( w_q(\{s_1, s_2, \ldots, s_m\}) \leq w_q(\{s_1 + s'_1, s_2 + s'_2, \ldots, s_m + s'_m\}) \).

(iv) \( w_q(\{1, 1, \ldots, 1\}) = w_q(\{1\}) = 1 \).

(v) \( \min\{|s_1|, \ldots, |s_m|\} \leq w_q(\{s_1, \ldots, s_m\}) \leq \max\{|s_1|, \ldots, |s_m|\} \).

(vi) \( w_q(\{s_m, s_{m-1}, \ldots, s_2, s_1\}) = w_q(\{s_1, \ldots, s_{m-1}, s_m\}) \).

(vii) \( w_q(\{s_2, \ldots, s_m, s_1\}) = w_q(\{s_1, \ldots, s_m\}) \).

**Proof.** The property (i) is verified in Section 1. The assertions (ii) is clear. To prove the assertion (iii), by Theorem 4 and the duality theorem in [5], it suffices to show that

\[
\max \sigma \left( \begin{pmatrix}
\frac{us_1^2}{\tau_1} & v_1 & 0 & \cdots & v_{s_m} \\
v_1 & \frac{us_2^2}{\tau_2} & v_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_s & 0 & 0 & \cdots & \frac{us_m^2}{\tau_m}
\end{pmatrix} \right) \leq \max \sigma \left( \begin{pmatrix}
\frac{u(s_1 + s'_1)^2}{\tau_1} & v(s_1 + s'_1) & 0 & \cdots & v(s_m + s'_m) \\
v(s_1 + s'_1) & \frac{u(s_2 + s'_2)^2}{\tau_2} & v(s_2 + s'_2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v(s_m + s'_m) & 0 & 0 & \cdots & \frac{u(s_m + s'_m)^2}{\tau_m}
\end{pmatrix} \right),
\]

for \( u \geq 0, v \geq 0, u + v > 0 \). If \( v = 0 \) and \( u = 1 \), then the inequality (25) is clear. So we may assume that \( v = 1 \) and \( u \geq 0 \). Since the matrices involved are nonnegative, by [4, Theorem 7.2], there exists a unit eigenvector \( (\xi_1, \xi_2, \ldots, \xi_m)^T \) with nonnegative coordinates such that

\[
\begin{pmatrix}
\frac{us_1^2}{\tau_1} & s_1 & 0 & \cdots & s_m \\
s_1 & \frac{us_2^2}{\tau_2} & s_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_m & 0 & 0 & \cdots & \frac{us_m^2}{\tau_m}
\end{pmatrix}(\xi_1, \ldots, \xi_m)^T, (\xi_1, \ldots, \xi_m)^T
\]

(26)
Then the region for \(0 < q \leq 1\) does depend on the ordering of \(s_1, \ldots, s_m\) if \(m \geq 4\). For \(m = 4\), we have the following result.

**Theorem 11.** Suppose that \(s_4 \geq s_3 \geq s_2 \geq s_1 \geq 0\). Then

\[
w_q([s_2, s_4, s_3, s_1]) \geq w_q([s_1, s_4, s_3, s_2]) \geq w_q([s_1, s_4, s_2, s_3])
\]

for \(0 \leq q \leq 1\).

Before we proceed to prove Theorem 11, we need the following Lemma.

Suppose that \(g(t) = t^4 + at^3 + bt^2 + ct + d\) is a monic polynomial with real coefficients such that \(g''(t) = 12t^2 + 6at + 2b\) has two distinct roots. Then \(b\) is the form \(3a^2/8 - p^2\) for some \(p > 0\). The monic polynomial is then rewritten as

\[
g(t : a, p, c, d) = t^4 + at^3 + (3a^2/8 - p^2)t^2 + ct + d.
\]

**Lemma 12.** For any negative number \(a\) and positive number \(p\), let

\[
g(t : c, d) = t^4 + at^3 + (3a^2/8 - p^2)t^2 + ct + d.
\]

Then the region

\[
\Omega = \{(c, d) \in \mathbb{R}^2 : g(t : c, d) = 0 \text{ has four real roots counting multiplicity}\}
\]
is compact. Moreover, if \((c_1, d_1), (c_2, d_2) \in \Omega\) with \(c_1 < c_2\) and \(d_1 < d_2\), then
\[
\max\{t \in \mathbb{R} : g(t : c_1, d_1) = 0\} \geq \max\{t \in \mathbb{R} : g(t : c_2, d_2) = 0\}.
\]

**Proof.** Fixed \(a\) and \(p\), it is easy to see that the cubic polynomial
\[
g'(t : c, d) = 4t^3 + 3at^2 + (3a^2/4 - 2p^2)t + c
\]
has three real roots if and only if \(c\) belongs to the closed interval
\[
[(9a^2 - 72ap^2 - 32\sqrt{6}p^3)/144, (9a^2 - 72ap^2 + 32\sqrt{6}p^3)/144].
\]
For such \(c\), the quartic polynomial \(g(t : c, d)\) has four real roots if and only if \(d\) belongs to some closed interval. Thus the points \((c, d)\) form a closed region. The boundary of \(\Omega\) lies on the algebraic curve defined by the condition \(g(t : c, d) = 0\) with multiple roots. The algebraic curve has three singular points on \(\partial\Omega\). One singular point is a node at
\[(c, d) = (a(a^2 - 8p^2)/16, (a^2 - 8p^2)^2/256).
\]
The other two singular points are a pair of cusps at
\[(c, d) = ((9a^3 - 72ap^2 - 32\sqrt{6}p^3)/144, (9a^4 - 144a^2p^2 - 128\sqrt{6}ap^3 - 192p^4)/2304),\]
and
\[(c, d) = ((9a^3 - 72ap^2 + 32\sqrt{6}p^3)/144, (9a^4 - 144a^2p^2 + 128\sqrt{6}ap^3 - 192p^4)/2304).
\]
The closed region \(\Omega\) is not convex. Consider the region
\[\Omega_1 = \{(c, d) \in \text{conv}(\Omega) : (c, d + d_0) \in \Omega\text{ for some }d_0 \geq 0\}.
\]
By changing variable \(t = s - a/4\), the polynomial \(g(t : c, d)\) is transformed to a monic polynomial
\[
\hat{g}(s : \gamma, \delta) = s^4 - p^2s^2 + \gamma s + \delta,
\]
where
\[
\gamma = c - a^3/16 + ap^2/2,
\]
\[
\delta = d - ac/4 + 3a^4/256 - a^2 p^2 /16.
\]
The correspondence between \((c, d)\) and \((\gamma, \delta)\) is a bijective affine transformation, and the regions \(\Omega\) and \(\Omega_1\) are transformed into the closed regions \(\hat{\Omega}\) and \(\hat{\Omega}_1\) by this transform. The region \(\hat{\Omega}\) is surrounded by the quartic curve
\[-27\gamma^4 + 256\delta^3 - 144p^2\gamma^2\delta - 128p^4\delta^2 + 4p^6\gamma^2 + 16p^8\delta = 0.
\]
This quartic curve has a node at \((\gamma, \delta) = (0, p^2/4)\), and a pair of cusps at \((\gamma, \delta) = (\pm 2\sqrt{6}p^3/9, -p^3/12)\). The boundary arc of \(\hat{\Omega}_1\) other than the line segment joining the two cusps is parametrized by
\[
\gamma = \pm \gamma(\delta),
\]
where
\[
\gamma(\delta) = \sqrt{2/27}(p^6 - 36p^2\delta + (p^4 + 12\delta)^{3/2})^{1/2}
\]
on the interval \([-p^4/12, p^2/4]\). The positive one of the parametrization is monotone decreasing for \(\gamma'(\delta) < 0\).
By the hypotheses \(a < 0\) and \(c_1 < c_2, d_1 < d_2\), the points \((\gamma_1, \delta_1)\) and \((\gamma_2, \delta_2)\) of \(\tilde{\Omega}\) corresponding to \((c_1, d_1)\) and \((c_2, d_2)\) satisfying the inequalities \(\gamma_1 < \gamma_2\) and \(\delta_1 < \delta_2\). The region \(\tilde{\Omega}_1\) is not convex, but the polygonal lines with vertices \((\gamma_1, \delta_1)\), \((\gamma_2, \delta_1)\), \((\gamma_2, \delta_2)\) are contained in the region \(\tilde{\Omega}_1\). For every point \((\gamma, \delta)\) on the polygonal lines, the largest and smallest real roots of \(\tilde{g}(s : \gamma, \delta)\) are simple.

Consider the case
\[
\tilde{g}(s : \gamma, \delta) = (s - a_1)(s - a_2)(s - a_3)(s - a_4),
\]
with \(a_1 > a_2 \geq a_3 \geq a_4\) and \(a_1 + a_2 + a_3 + a_4 = 0\). Observe that \(a_1 > 0\). Using a perturbation method, the largest roots of \(\tilde{g}(s : \gamma, \delta + k)\) and \(\tilde{g}(s : \gamma + k, \delta)\) are respectively given by
\[
a_1 - \frac{1}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}k + b_2k^2 + \cdots,
\]
and
\[
a_1 - \frac{a_1}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}k + c_2k^2 + \cdots
\]
for sufficiently small \(k\).

Next, we consider the case
\[
\tilde{g}(s : \gamma, \delta) = (s - a_1)(s - a_2)(s^2 + (a_1 + a_2)s + (a_1 + a_2)^2/4 + r^2)
\]
for some \(r > 0\) and \(a_1 > a_2\). Then the largest real roots of \(\tilde{g}(s : \gamma, \delta + k)\) and \(\tilde{g}(s : \gamma + k, \delta)\) are respectively given by
\[
a_1 - \frac{4}{(a_1 - a_2)((3a_1 + a_2)^2 + 4r^2)}k + \tilde{b}_2k^2 + \cdots,
\]
and
\[
a_1 - \frac{4a_1}{(a_1 - a_2)((3a_1 + a_2)^2 + 4r^2)}k + \tilde{b}_2k^2 + \cdots
\]
for sufficiently small \(k\). This completes the proof. \(\square\)

**Proof of Theorem 11.** Let
\[
f(t : s_i, s_j, s_k, s_\ell : u, v) = \det\begin{pmatrix}
(t - us_i^2 & -vs_i & 0 & -vs_\ell \\
-vs_i & (t - us_j^2 & -vs_j & 0 \\
0 & -vs_j & (t - us_k^2 & -vs_k \\
-vs_\ell & 0 & -vs_k & (t - us_\ell^2
\end{pmatrix}
\]
for \(u \geq 0, v \geq 0\) with \(u + v > 0\). By the duality theorem in [5], the assertion inequality of Theorem 11 is equivalent to the inequality
\[
\max\{t \in \mathbb{R} : f(t : s_2, s_4, s_3, s_1 : u, v) = 0\}
\geq \max\{t \in \mathbb{R} : f(t : s_1, s_4, s_3, s_2 : u, v) = 0\}
\geq \max\{t \in \mathbb{R} : f(t : s_1, s_4, s_2, s_3 : u, v) = 0\}
\]
(27)
for every \(u \geq 0, v \geq 0\) with \(u + v > 0\).

In the case \(u = 0\), we may assume that \(v = 1/2\). Then
\[
f(t : s_2, s_4, s_3, s_1 : 0, 1/2)
= t^4 - (s_1^2 + s_2^2 + s_3^3 + s_4^2)t^2/4 + (s_4s_1 - s_3s_2)^2/16,
\]
(28)
\[
f(t : s_1, s_4, s_3, s_2 : 0, 1/2) = t^4 - \frac{1}{4}(s_1^2 + s_2^2 + s_3^2 + s_4^2)t^2 + (s_4s_2 - s_3s_1)^2/16, \tag{29}
\]
\[
f(t : s_1, s_4, s_2, s_3 : 0, 1/2) = t^4 - \frac{1}{4}(s_1^2 + s_2^2 + s_3^2 + s_4^2)t^2 + (s_4s_3 - s_2s_1)^2/16. \tag{30}
\]

The square of the maximum positive roots of (28)–(30) are respectively
\[
(s_1^2 + s_2^2 + s_3^2 + s_4^2 + ((s_1^2 + s_2^2 + s_3^2 + s_4^2)^2 - 4(s_4s_1 - s_3s_2)^2)^{1/2}/8,
\]
\[
(s_1^2 + s_2^2 + s_3^2 + s_4^2 + ((s_1^2 + s_2^2 + s_3^2 + s_4^2)^2 - 4(s_4s_2 - s_3s_1)^2)^{1/2}/8,
\]
\[
(s_1^2 + s_2^2 + s_3^2 + s_4^2 + ((s_1^2 + s_2^2 + s_3^2 + s_4^2)^2 - 4(s_4s_3 - s_2s_1)^2)^{1/2}/8.
\]

Then the inequality (27) follows from the relation
\[
(s_4s_2 - s_3s_1)^2 - (s_4s_1 - s_3s_2)^2 = (s_4 - s_3)(s_2 - s_1)(s_4 + s_3)(s_2 + s_1) \geq 0,
\]
\[
(s_4s_3 - s_2s_1)^2 - (s_4s_1 - s_3s_2)^2 = (s_4 - s_2)(s_3 - s_1)(s_4 + s_2)(s_3 + s_1) \geq 0,
\]
\[
(s_4s_3 - s_2s_1)^2 - (s_4s_2 - s_3s_1)^2 = (s_4 - s_1)(s_3 - s_2)(s_4 + s_1)(s_3 - s_2) \geq 0.
\]

In the case \( v = 0 \), we may assume that \( u = 1 \). In this case we have
\[
f(t : s_i, s_j, s_k, s_\ell : 1, 0) = (t - s_i^2)(t - s_j^2)(t - s_k^2)(t - s_\ell^2)
\]
for every permutation \((i, j, k, \ell)\) of \( \{1, 2, 3, 4\} \). Then the equality holds in (27).

Next we assume that \( u > 0 \) and \( v > 0 \). By direct computations, we have
\[
f(t : s_i, s_j, s_k, s_\ell : u, v) = t^4 - uc_1t^3 + (u^2c_2 - v^2c_1)t^2
\]
\[
+ (-u^3c_3 + uv^2c_2 + uv^2(s_i^2s_k^2 + s_j^2s_\ell^2))t + u^4c_4 - u^2v^2c_3 + v^4(s_is_ks_\ell - s_js_\ell)^2,
\]
where
\[
c_1 = s_1^2 + s_2^2 + s_3^2 + s_4^2,
\]
\[
c_2 = s_1^2s_2^2 + s_1^2s_3^2 + s_1^2s_4^2 + s_2^2s_3^2 + s_2^2s_4^2 + s_3^2s_4^2,
\]
\[
c_3 = s_2s_3s_4^2 + s_1s_2s_4^2 + s_1s_3s_4^2 + s_1^2s_2s_4^2 + s_1^2s_3s_4^2,
\]
\[
c_4 = s_1^2s_2^2s_3^2s_4^2.
\]

Notice that the two coefficients of the polynomial
\[
f(t : s_1, s_4, s_3, s_2 : u, v) - f(t : s_2, s_4, s_3, s_1 : u, v)
\]
\[
= uv^2(s_1 + s_2)(s_3 + s_4)(s_4 - s_3)(s_2 - s_1)t
\]
\[
+ v^4(s_1 + s_2)(s_3 + s_4)(s_4 - s_3)(s_2 - s_1)
\]
are positive, so are the polynomial
\[
f(t : s_1, s_4, s_2, s_3 : u, v) - f(t : s_1, s_4, s_3, s_2 : u, v)
\]
\[
= uv^2(s_1 + s_4)(s_2 + s_3)(s_4 - s_1)(s_3 - s_2)t
\]
\[
+ v^4(s_1 + s_4)(s_2 + s_3)(s_4 - s_1)(s_3 - s_2).
\]
Hence the three quartic monic polynomials $f(t : s_2, s_4, s_3, s_1 : u, v)$, $f(t : s_1, s_4, s_3, s_2 : u, v)$ and $f(t : s_1, s_4, s_2, s_3 : u, v)$ in $t$ have common coefficients of $t^3$ and $t^2$. Each of the three polynomials has four real roots counting multiplicity, and its second derivative has two distinct roots.

The common coefficient $-u\sigma_1(s_2^2, s_3^2, s_1^2)$ of $t^3$ is negative. The assertion then follows Lemma 12. □

By the fact that

$$\langle A (\ldots, 0, 1, 0, 0, \ldots), \left(0, 0, 0, 1, \sqrt{1-q^2}, 0, \ldots\right) \rangle = \langle 0, 0, 0, 1, \sqrt{1-q^2}, 0, \ldots \rangle = s_i \sqrt{1-q^2},$$

$i = 1, 2, \ldots, m$, we are easy to obtain a simple lower bound of $w_q(A)$ by

$$w_q(A) \geq \sqrt{1-q^2} \max\{s_1, s_2, \ldots, s_m\}.$$ 

In [11], it is claimed that for $0 \leq q < 1$ the inequality (21) holds for every permutation $\sigma$ of $\{1, 2, \ldots, m\}$. Unfortunately, the application of the symmetric group in the proof of this formula is not valid, see Example 1. We modify and provide a lower bound of the $q$-numerical radius.

**Theorem 13.** Let $A$ be a weighted unilateral shift operator with periodic weights $\{s_1, s_2, \ldots, s_m\}$. Then

$$w_q(A) \geq \frac{1}{1-q^{2m}} \max\{s_1 + q^2s_2 + \cdots + q^{2(m-1)}s_m, \quad s_m + q^2s_{m-1} + \cdots + q^{2(m-1)}s_1 : j = 0, 1, 2, \ldots, m-1\}$$

for $0 \leq q < 1$.

**Proof.** By Theorem 4 and properties (v), (vi) of Theorem 10, it suffices to show that

$$w_q(A) \geq \frac{1}{1-q^{2m}} (s_1 + q^2s_2 + \cdots + q^{2(m-1)}s_m).$$

By setting

$$\xi = \sqrt{1-q^2(1, q, q^2, q^3, \ldots)},$$

$$\eta = \sqrt{1-q^2(0, 1, q, q^2, \ldots)},$$

we have $\|\xi\| = \|\eta\| = 1$, $(\xi, \eta) = (1-q^2)q(1+q^2+q^4+\cdots) = q$, and
\[ \langle A \xi, \eta \rangle = (1 - q^2)(s_1 + q^2 s_2 + q^4 s_3 + \cdots + q^{2(m-1)} s_m + q^{2m} s_1 + \cdots = (s_1 + q^2 s_2 + \cdots + q^{2(m-1)} s_m)(1 + q^{2m} + q^{4m} + \cdots)(1 - q^2) = \frac{1 - q^2}{1 - q^{2m}} (s_1 + q^2 s_2 + \cdots + q^{2(m-1)} s_m). \]

\[ \square \]

References