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On the Mixed Problem for Some Quasilinear Hyperbolic System with Fully Nonlinear Boundary Condition

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1. INTRODUCTION

As is well known, the Neumann or third kind problem for a nonlinear wave equation is written as follows:

$$\text{(N.W)} \begin{cases} \partial_t^2 u(t) - \sum_{i=1}^{n} \partial_i(\partial_i u(t)/\sqrt{1+|\nabla_x u(t)|^2}) \\ \qquad = f_\Omega(t) \qquad \text{in } (0, T)\times\Omega; \\ \sum_{i=1}^{n} \nu_i(\partial_i u(t)/\sqrt{1+|\nabla_x u(t)|^2}) + a(u(t)) \\ \qquad = f_\Gamma(t) \qquad \text{on } (0, T)\times\Gamma; \\ u(0) = u_0 \qquad \text{and} \qquad \partial_t u(0) = u_1 \qquad \text{in } \Omega. \end{cases}$$

Here and hereafter, Ω is a domain in $\mathbb{R}^n$, its boundary Γ being a C^∞ and compact hypersurface; $x = (x_1, ..., x_n)$ and t denote points of $\mathbb{R}^n$ and a time, respectively; $\partial_t = \partial_0 = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$ $(i = 1, ..., n)$; $\nabla_x u$ is the gradient of u in x; $\nu = (\nu_1(x), ..., \nu_n(x))$ denotes the unit outer normal to Γ at $x \in \Gamma$ (for the sake of simplicity, we assume that $\nu_i(x) \in C_0^\infty(\mathbb{R}^n)$ below). The local existence in time of classical solutions to (N.W) was proved by the first

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author [7]. The first equation of (N.W) is quasilinear, but the boundary condition is fully nonlinear. This full nonlinearity causes the derivative loss which breaks down the usual iteration process. And also, the boundary condition of the linearized problem does not satisfy the so-called uniform Lopatinski condition. The difficulties of solving (N.W) come from these facts essentially and we must construct the iteration scheme with the greatest care.

In the present paper, we shall prove the local existence theorem in time of classical solutions to the following mixed problem for second-order systems:

$$\text{(N)} \begin{cases} \partial_t^2 \mathbf{u}(t) - \partial_i(\mathbf{P}^i(t, D^1\mathbf{u}(t))) + \mathbf{Q}_\Omega(t, D^1\mathbf{u}(t)) = \mathbf{f}_\Omega(t) & \text{in } (0, T)\times\Omega; \\ \nu_i \mathbf{P}^i(t, D^1\mathbf{u}(t)) + \mathbf{Q}_\Gamma(t, D^1\mathbf{u}(t)) = \mathbf{f}_\Gamma(t) & \text{on } (0, T)\times\Gamma; \\ \mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \partial_t \mathbf{u}(0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases}$$

Here and hereafter, $\mathbf{u} = {}^t(u_1, ..., u_m)$ denotes an m-vector (${}^tM$ means the transpose M); $\mathbf{P}^i$, $\mathbf{Q}_\Omega$, and $\mathbf{Q}_\Gamma$ are m-vectors of nonlinear functions in t, x, and $D^1\mathbf{u} = (\partial_t\mathbf{u}, \nabla_x\mathbf{u}, \mathbf{u})$ of the forms: $\mathbf{P}^i = {}^t(P_1^i, ..., P_m^i)$ and $\mathbf{Q}_V = {}^t(Q_{V1}, ..., Q_{Vm})$ ($i = 1, ..., n$; $V = \Omega$ and Γ); the summation convection is understood such as the sub- and superscripts i, j take all values 1 to n; the functions are always assumed to be real valued.

Such a problem (N) belongs to a physically reasonable problem, typical if not the most general. In fact, if we put $m = 1$, $\mathbf{P}^i = \partial_i u/\sqrt{1 + |\nabla_x u|^2}$, $\mathbf{Q}_\Omega = 0$, and $\mathbf{Q}_\Gamma = a(u)$, then (N.W) is described by (N). Another important example is a model for a three-dimensional nonlinear elastodynamical equation with some applied surface force which is not dead load, in which the unknown is actually 3-vector valued (cf. [1]). This will be treated in Section 8 below.

(N) was already treated and the local existence theorem was proved by the first auther and G. Nakamura [9]. But, the order of Sobolev spaces in which solutions exist was not best possible. After the works [7, 9], T. Kato [2] treated also mixed problems of the same type as in (N) in his abstract framework. And, when $m = 1$, the nonlinear functions $\mathbf{P}^i$, $\mathbf{Q}_\Omega$ and $\mathbf{Q}_\Gamma$ do not depend on t and $\partial_t\mathbf{u}$, and $\mathbf{f}_\Gamma(t) \equiv 0$; applying his abstract theory to (N), he gave some improvements of the results due to [7] regarding the minimal order of the Sobolev spaces in which the solutions exist. Our purpose in the present paper is to give the same improvements as in Kato [2] when $m \geqslant 1$, where the nonlinear functions depend on t and $\partial_t\mathbf{u}$, and $\mathbf{f}_\Gamma(t) \not\equiv 0$. Our approach below is concrete and elementary and different from Kato's in [2]. Another advantage of the approach in the present paper is that some hyperbolic–parabolic coupled system containing a model for a higher dimensional nonlinear thermoelastodynamical equation as a physical example can be handled in a similar fashion, which will be published elsewhere.

One of the essential points of solving (N) lies in the simple reduction of (N) to some "hyperbolic–elliptic" coupled system for unknown $\mathbf{u}$ and $\partial_t \mathbf{u}$. This reduction was first developed by the first auther [7]. The same reduction was used in [9, 2] and will be also used in the present paper (cf. Sect. 4 below).

Now, let us introduce our assumptions. Throughout the present paper, it is assumed that the spatial dimension $n \geqslant 2$, because the case when $n = 1$ was already treated by the second author [3]. Let u_{0a}, u_{ia}, and u_{n+1a} denote independent variables corresponding to $\partial_t u_a$, $\partial_i u_a$, and u_a, respectively. Here and hereafter, the sub- or superscripts i and j (resp. subscripts a and b) refer to all integers 1 to n (resp. 1 to m). Put $U = (u_{0a}, u_{ia}, u_{n+1a})$. The first assumption is that

(A.1) the $P^i_a = P^i_a(t, x, U)$ and $Q_{Va} = Q_{Va}(t, x, U)$ are in $\mathscr{B}^\infty([-T_0, T_0] \times \bar{\Omega} \times D(U_0))$ and satisfy the condition:

(*) $P^i_a(t, x, 0) = Q_{Va}(t, x, 0) = 0 \quad$ for $\quad (t, x) \in [-T_0, T_0] \times \bar{\Omega}$.

Here and hereafter, U_0 and T_0 are given positive constants; the subscript V always refers to Ω and Γ; $D(U_0) = \{U \in \mathbb{R}^{(n+2)m} \mid |U| < U_0\}$. The condition (*) guarantees that the composed function $P^i_a(t, x, U(x))$ and so on belong to $L^2(\Omega)$ for each t provided that $U(x) \in L^2(\Omega)$. When Ω is bounded, we need not assume (*). But, in the present paper, we consider where Ω is bounded and unbounded.

Put

$$A^{ik}_{ab} = \partial P^i_a / \partial u_{kb}, \qquad B^k_{Vab} = \partial Q_{Va} / \partial u_{kb},$$
$$A^{ik} = (A^{ik}_{ab}), \qquad B^k_V = (B^k_{Vab}) \qquad (k = 0, 1, ..., n, n+1), \tag{1.1}$$

where A^{ik} and B^k_V are $m \times m$ matrices and the subscripts a and b denote the row and column, respectively. The second assumption is that

(A.2) ${}^tA^{ij} = A^{ji}$ and ${}^tA^{i0} = A^{i0}$ on $[-T_0, T_0] \times \bar{\Omega} \times D(U_0)$;

${}^tB^0_\Gamma = B^0_\Gamma$ and ${}^tB^i_\Gamma + B^i_\Gamma = 0$ on $[-T_0, T_0] \times \Gamma \times D(U_0)$.

Roughly speaking, the final assumption in (A.2) means that the boundary condition does not contain oblique derivatives. In fact, when $m = 1$, $\mathbf{Q}_\Gamma$ does not depend on $\partial_i \mathbf{u}$ by the condition: ${}^tB^i_\Gamma + B^i_\Gamma = 0$.

The third assumption is that

(A.3) there exist positive constants δ_0 and δ_1 such that $(A^{ij}(t, \cdot, U(\cdot)) \partial_j \mathbf{v}, \partial_i \mathbf{v}) + \langle B^i_\Gamma(t, \cdot, U(\cdot)) \partial_i \mathbf{v}, \mathbf{v} \rangle \geqslant \delta_1 \|\mathbf{v}\|^2_1 - \delta_0 \|\mathbf{v}\|^2_0$ for $t \in [-T_0, T_0]$, $\mathbf{v} \in H^2(\Omega)$, and $U(x) \in H^{\infty,1}(\bar{\Omega}, D(U_0))$.

Here and hereafter, $H^s(G)$ denotes the usual Sobolev space on G of order s with norm $\|\cdot\|_{s,G}$ and for any function space S equipped with norm $|\cdot|$ we denote a product space $S\times\cdots\times S$ and its norm simply by S and $|\cdot|$, respectively; we write $\|\cdot\|_{s,\Omega}=\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}=\langle\!\langle\cdot\rangle\!\rangle_s$; the $(\ ,\ )$ and $\langle\ ,\ \rangle$ denote the usual inner products of $L^2(\Omega)=H^0(\Omega)$ and $L^2(\Gamma)=H^0(\Gamma)$, respectively; $H^{\infty,1}(\bar{\Omega}, D(U_0)) = \{u\in L^\infty(\Omega) \mid |(\nabla_x u(x), u(x))| < U_0$ for $x\in\bar{\Omega}\}$. Assumption (A.3) is stronger than the assumption that the A^{ij} are strongly elliptic. But, many important physical examples satisfy (A.3).

The fourth assumption is that

(A.4) $\quad \nu_l(x)\, B^l_\Gamma(t, x, U)=0$ for $(t, x, U)\in[-T_0, T_0]\times\Gamma\times D(U_0)$.

Assumption (A.4) means that the operators B^l_Γ do not contain the normal derivative of $\mathbf{u}$ on Γ.

The final assumption is that

(A.5) $\quad \{-\nu_i(x)\, A^{i0}(t, x, U)+2B^0_\Gamma(t, x, U)\}\,\xi\cdot\xi\geqslant 0$ for $(t, x, U)\in[-T_0, T_0]\times\Gamma\times D(U_0)$ and $\xi\in\mathbb{R}^m$.

Here and hereafter, $\xi\cdot\eta=\xi_1\eta_1+\cdots+\xi_m\eta_m$ for any m-vectors ξ and η in $\mathbb{R}^m$. Assumption (A.5) is one of the conditions in order that the energy of the corresponding linear problem to (N) does not increase (cf. Majda [4, p. 145]).

Let J and X be an interval of $\mathbb{R}$ and a Hilbert space, respectively. By $C^k(J, X)$ and $\mathrm{Lip}(J, X)$ we denote the set of all X-valued functions which are in C^k and Lipschitz continuous in J, respectively. Put

$$X^{L,M}(J, G)=\bigcap_{N=0}^{L} C^N(J, H^{L+M-N}(G)).$$

The purpose of the present paper is to prove

THEOREM 1.1. *Assume that* $n\geqslant 2$ *and* (A.1)–(A.5) *are valid. Let* K *be an integer* $\geqslant[n/2]+3$ *and let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *satisfy the following conditions*:

$$\mathbf{u}_0\in H^K(\Omega), \qquad \mathbf{u}_1\in H^{K-1}(\Omega),$$

$$\mathbf{f}_\Omega(t)\in X^{K-2,0}([0, T_0], \Omega), \qquad \mathbf{f}_\Gamma(t)\in X^{K-2,1/2}([0, T_0], \Gamma); \tag{1.2}$$

$$\partial_t^{K-2}\mathbf{f}_\Omega(t)\in \mathrm{Lip}([0, T_0], L^2(\Omega));$$

$$\partial_t^{K-2}\mathbf{f}_\Gamma(t)\in \mathrm{Lip}([0, T_0], H^{1/2}(\Gamma)); \tag{1.3}$$

$\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *satisfy the compatibility condition of order* $K-2$ (*this notion will be defined exactly in* (3.4) *of* Sect. 3 *below*); (1.4)

$$(\mathbf{u}_1(x), D^1_x\mathbf{u}_0(x))\in H^{\infty,1}(\bar{\Omega}, D(U_0)) \qquad (D^1_x\mathbf{u}=(\nabla_x\mathbf{u}, \mathbf{u})). \tag{1.5}$$

Let $\mathbb{B}$ *be a positive constant such that*

$$\|\mathbf{u}_0\|_K + \|\mathbf{u}_1\|_{K-1} + |\mathbf{f}_\Omega|_{K-2,0,[0,T_0]} + \langle \mathbf{f}_\Gamma \rangle_{K-2,1/2,[0,T_0]} \\ + \operatorname*{ess\,sup}_{t\in[0,T_0]} \|\partial_t^{K-1}\mathbf{f}_\Omega(t)\|_0 + \operatorname*{ess\,sup}_{t\in[0,T_0]} \langle\!\langle \partial_t^{K-1}\mathbf{f}_\Gamma(t) \rangle\!\rangle_{1/2} \leqslant \mathbb{B}, \tag{1.6}$$

where the $|\cdot|_{K-2,0,[0,T_0]}$ *and* $\langle\cdot\rangle_{K-2,1/2,[0,T_0]}$ *are norms of* $X^{K-2,0}([0,T_0],\Omega)$ *and* $X^{K-2,1/2}([0,T_0],\Gamma)$, *respectively, which will be defined in* Section 2 *below. Then, there exist a short time* $T>0$ *and a constant* $A>0$ *depending essentially only on* K *and* $\mathbb{B}$ *such that* (N) *admits a unique solution* $\mathbf{u}(t)\in X^{K,0}([0,T],\Omega)$ *satisfying the properties*: $|\mathbf{u}|_{K,0,[0,T]}\leqslant A$ *and* $D^1\mathbf{u}(t)\in H^{\infty,1}(\bar{\Omega}, D(U_0))$ *for* $t\in[0,T)$.

Remark. Since $K\geqslant[n/2]+3$, by Sobolev's imbedding theorem we see that the present solution $\mathbf{u}(t)\in C^2([0,T)\times\bar{\Omega})$. And, as was stated in [2], this order seems to be the best possible to get C^2 solutions. In [7, 9], it was assumed that $K\geqslant[n/2]+8$. To get our improvement, $K\geqslant[n/2]+3$, we treat the problem in a more delicate fashion than in [7, 9] and need some new results on the linear hyperbolic theory obtained by the first auther [8]. The uniqueness of solutions in $X^{K,0}([0,T],\Omega)$ and the existence of C^∞-solutions were already proved in [7, 9]. Hence, we shall prove the existence of solutions in $X^{K,0}([0,T],\Omega)$ only. Below, K will always refer to an integer $\geqslant[n/2]+3$.

The present paper is organized as follows. In Section 2, basic notations are given. In Section 3, the compatibility condition for (N) is defined. In Section 4, the iteration scheme for solving (N) is defined. In Sections 5 and 6, as preparations for proving the convergence of our iteration scheme, we give some results on the linear hyperbolic and elliptic theories. In Section 7, we prove Theorem 1.1. In Section 8, we give two examples containing (N.W) and the three-dimensional nonlinear elastodynamics. In the Appendix, we give some estimations of nonlinear terms and supplementary lemmas used in the text.

Below, (k.h), (Ap.h), Theorem k.h, and Theorem Ap.h will always refer to the formula (k.h) in Section k, the formula (Ap.h) in Appendix, Theorem k.h in Section k, and Theorem Ap.h in the Appendix, respectively.

2. NOTATIONS

In this section, we explain our basic notations. For any k-vector $v=(v_1,...,v_k)$ and multi-index $\alpha=(\alpha_1,...,\alpha_k)$ we put $v^\alpha=v_1^{\alpha_1}\cdots v_k^{\alpha_k}$ and $|\alpha|=\alpha_1+\cdots+\alpha_k$. For differentiations, we use the symbols: $\partial_x=$

$(\partial_1, \ldots, \partial_n)$; $\partial_t^j \partial_x^\alpha v = (\partial_t^j \partial_x^\alpha v_1, \ldots, \partial_t^j \partial_x^\alpha v_k)$; $D^L D_x^M v = (\partial_t^j \partial_x^\alpha v; j + |\alpha| \leqslant L + M, j \leqslant L)$; $D^L v = D^L D_x^0 v$; $D_x^M v = D^0 D_x^M v$. Put

$$\|v\|_{\infty, L} = \sup_{x \in \Omega} |D_x^L v(x)|;$$

$$|v|_{\infty, L, T} = \sup\{|D^L v(t, x)| \mid (t, x) \in [-T, T) \times \bar{\Omega}\}.$$

Let $L^\infty(J, X)$ be the set of all X-valued functions which are measurable in J and bounded *everywhere* in J in the strong topology of X, where J and X are an interval of $\mathbb{R}$ and a Hilbert space, respectively. Put $Y^{0,s}(J, G) = L^\infty(J, H^s(G))$ and for $L \geqslant 1$, $Y^{L,s}(J, G) = \{u(t) \in X^{L-1,s}(J, G) \mid \partial_t^M u(t) \in L^\infty(J, H^{L+s-M}(G)) \cap \mathrm{Lip}(J, H^{L+s-M-1}(G))$ for $0 \leqslant M \leqslant L-1\}$. Note that $Y^{L,s} \subset Y^{L-M,s+M}$ and $X^{L,s} \subset X^{L-M,s+M}$ for $0 \leqslant M \leqslant L$. As the norms of $Y^{L,s}(J, G)$, we use the following:

$$|v|_{0,s,J,G} = \sup_{t \in J} \|v(t)\|_{s,G};$$

$$|v|_{L,s,J,G} = |v|_{0,L+s,J,G} + \sum_{M=0}^{L-1} \sup_{\substack{t,s \in J \\ t \neq s}} \frac{\|(\partial_t^M v)(t) - (\partial_t^M v)(s)\|_{L+s-M-1}}{|t-s|}$$

for $L \geqslant 1$.

If $v(t) \in X^{L,s}(J, G)$, then

$$\sum_{M=0}^{L} \sup_{t \in J} \|\partial_t^M v(t)\|_{L+s-M,G} = |v|_{L,s,J,G}.$$

Hence, we also use $|\cdot|_{L,s,J,G}$ as the norms of $X^{L,s}(J, G)$. Put $|v|_{L,s,J} = |v|_{L,s,J,\Omega}$ and $\langle v \rangle_{L,s,J} = |v|_{L,s,J,\Gamma}$. For the matrix-valued functions, we use the same notations to denote their differentiations, norms, and so on.

We use the same letter C to denote different constants depending on the same set of arguments. $C(\cdots)$ denotes a constant depending essentially on the quantities appearing in the parentheses. In particular, by using the subscripts $l = 1, 2, \ldots$, we distinguish the important constants. For example, C_1, C_2, $C_1(\cdots)$, $C_2(\cdots)$, and so on.

For any nonlinear function $F(t, x, v)$, we write

$$(\partial_t^k \partial_x^\alpha d^h F)(t, x, v)(w_1, \ldots, w_h) = \frac{d^h}{d\theta_1 \cdots d\theta_h} \left[(\partial_t^k \partial_x^\alpha F)\left(t, x, v + \sum_{l=1}^{h} \theta_l w_l\right) \right] \Bigg|_{\theta_1 = \cdots = \theta_h = 0}; \quad (2.1)$$

$$(F)_1(t, x, v) = F(t, x, v) - F(t, x, 0) = \int_0^1 (dF)(t, x, \theta v)\, v d\theta.$$

Note that $(F)_1(t, x, 0) = 0$ and $F(t, x, v) = F(t, x, 0) + (F)_1(t, x, v)$.

Let $R^i(x)$ be $m \times m$ matrices of functions in $\mathscr{B}^1(\bar{\Omega})$ such that

$$\nu_i(x)\, R^i(x) = 0 \qquad \text{for } x \in \Gamma. \tag{2.2}$$

Finally, we find some bilinear forms $S_1(R)[\mathbf{v}, \mathbf{w}]$ on $H^1(\Omega) \times H^1(\Omega)$ and $S_2(R)[\mathbf{v}, \mathbf{w}]$ on $H^1(\Omega) \times L^2(\Omega)$ such that

$$\langle R^i \partial_i \mathbf{v}, \mathbf{w} \rangle = S_1(R)[\mathbf{v}, \mathbf{w}] + S_2(R)[\mathbf{v}, \mathbf{w}]$$
$$\text{for any } \mathbf{v} \in H^2(\Omega) \text{ and } \mathbf{w} \in H^1(\Omega), \tag{2.3}$$

where $R = (R^1, ..., R^n)$. These are used to define the first energy of the linearized problem and to solve the elliptic boundary value problem by using the well-known Lax–Milgram theorem. To get (2.3), first we prepare some notations. Since Γ is a compact and C^∞ hypersurface, we may assume that there exist a finite number of open sets G_l in $\mathbb{R}^n$, positive numbers σ_l, and C^∞ diffeomorphisms Ψ_l from G_l' onto G_l for $l = 1, ..., p$, such that $G_l' = \{ y = (y_1, ..., y_n) \in \mathbb{R}^n \mid |y'| = |(y_1, ..., y_{n-1})| < \sigma_l$ and $|y_n| < \sigma_l \}$, $\Omega \cap G_l = \Psi_l(\{ y \in G_l' \mid y_n > 0 \})$, and $\Gamma \cap G_l = \Psi_l(\{ y \in G_l' \mid y_n = 0 \})$. Let $\Phi_l = (\Phi_{l1}, ..., \Phi_{ln})$ be the inverse maps of Ψ_l. If we put $Y_{li}^j(y) = (\partial \Phi_{lj} / \partial x_i)(\Psi_l(y))$ and $J_l(y') = |(Y_{l1}^n(y', 0), ..., Y_{ln}^n(y', 0))|$, we have that $\nu_i(x) = -Y_{li}^n(y', 0)/J_l(y')$ and $d\Gamma_x = J_l(y')\, dy'$ for $x = \Psi_l(y', 0) \in G_l \cap \Gamma$ where $d\Gamma_x$ is the surface element of Γ. In particular, by (2.2) we see that

$$R^i(\Psi_l(y', 0))\, Y_{li}^n(y', 0) = 0 \qquad \text{for } (y', 0) \in G_l'. \tag{2.4}$$

Let $\phi_l(x) \in C_0^\infty(G_l)$ $(l = 1, ..., p)$ be the partition of unity on Γ and put $\psi_l(y) = \phi_l(\Psi_l(y)) \in C_0^\infty(G_l')$. By the change of variables $x = \Psi_l(y)$ and (2.4), we have

$$\langle R^i \partial_i \mathbf{v}, \mathbf{w} \rangle = \sum_{l=1}^{p} \sum_{q=1}^{n-1} \int_{\mathbb{R}^{n-1}} \psi_l(y', 0)\, S_l^q(R, y')\, \partial_q' \mathbf{v}'(y', 0) \cdot \mathbf{w}'(y', 0)\, dy',$$

where $\partial_j' = \partial / \partial y_j$, $\mathbf{v}'(y) = \mathbf{v}(\Psi_l(y))$ and $S_l^q(R, y') = R^i(\Psi_l(y', 0))\, Y_{li}^q(y', 0)$ $J_l(y')$. If we put

$$S_1(R)[\mathbf{v}, \mathbf{w}] = \sum_{l=1}^{p} \sum_{q=1}^{n-1} \int_{\mathbb{R}_+^n} \Psi_l(y) \{ S_l^q(R, y')\, \partial_n' \mathbf{v}'(y) \cdot \partial_q' \mathbf{w}'(y)$$
$$- S_l^q(R, y')\, \partial_q' \mathbf{v}'(y) \cdot \partial_n' \mathbf{w}'(y) \}\, dy; \tag{2.5a}$$

$$S_2(R)[\mathbf{v}, \mathbf{w}] = \sum_{l=1}^{p} \sum_{q=1}^{n-1} - \int_{\mathbb{R}_+^n} \{ \psi_l(y)(\partial_q' S_l^q(R, y'))\, \partial_n' \mathbf{v}'(y) \cdot \mathbf{w}'(y)$$
$$- (\partial_n' \psi_l(y))\, S_l^q(R, y')\, \partial_q' \mathbf{v}'(y) \cdot \mathbf{w}'(y) \}\, dy, \tag{2.5b}$$

where $\mathbb{R}^n_+ = \{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid y_n > 0\}$, noting the formula

$$\langle R^i \partial_i \mathbf{v}, \mathbf{w} \rangle = \sum_{l=1}^{p} \sum_{q=1}^{n-1} -\int_{\mathbb{R}^n_+} \partial'_n \{\psi_l(y)\, S^q_l(R, y')\, \partial'_q \mathbf{v}'(y) \cdot \mathbf{w}'(y)\}\, dy,$$

by integration by parts we have (2.3). Furthermore, by Schwarz's inequality, we have

$$|S_1(R)[\mathbf{v}, \mathbf{w}]| \leqslant C \left\{ \sum_{i=1}^{n} \|R^i\|_{\infty,0} \right\} \|\mathbf{v}\|_1 \, \|\mathbf{w}\|_1; \tag{2.6a}$$

$$|S_2(R)[\mathbf{v}, \mathbf{w}]| \leqslant C \left\{ \sum_{i=1}^{n} \|R^i\|_{\infty,1} \right\} \|\mathbf{v}\|_1 \, \|\mathbf{w}\|_0. \tag{2.6b}$$

The $S_1(R)$ and $S_2(R)$ are continuous bilinear forms on $H^1(\Omega) \times H^1(\Omega)$ and $H^1(\Omega) \times L^2(\Omega)$, respectively.

3. Compatibility Conditions

In this section, we shall define the compatibility condition which the $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, and $\mathbf{f}_\Gamma(t)$ should satisfy in order that solutions to (N) exist. To do this, first we shall prepare some notations. Let $\mathbf{u}(t) \in X^{K,0}([0,T],\Omega)$. Since $\mathbf{P}^i(t, D^1\mathbf{u}(t))$ and $\mathbf{Q}_V(t, D^1\mathbf{u}(t))$ belong to $X^{K-1,0}([0,T],\Omega)$ as follows from Theorem Ap.3, we can write

$$\partial_t^M \mathbf{P}^i(t, D^1\mathbf{u}) = (\partial_t^M \mathbf{P}^i)(t, x, D^1\mathbf{u}) + \sum_{h=1}^{M} \sum{}' P^{i,M,h}_{\alpha^h,\beta^h}(t, x, D^1\mathbf{u}) \times (D^1_x \partial_t \mathbf{u})^{\alpha^h_1} \cdots (D^1_x \partial_t^h \mathbf{u})^{\alpha^h_h} (\partial_t^2 \mathbf{u})^{\beta^h_1} \cdots (\partial_t^{h+1} \mathbf{u})^{\beta^h_h}; \tag{3.1a}$$

$$\partial_t^M \mathbf{Q}_V(t, x, D^1\mathbf{u}) = (\partial_t^M \mathbf{Q}_V)(t, x, D^1\mathbf{u}) + \sum_{h=1}^{M} \sum{}' Q^{M,h}_{V,\alpha^h,\beta^h}(t, x, D^1\mathbf{u}) \times (D^1_x \partial_t \mathbf{u})^{\alpha^h_1} \cdots (D^1_x \partial_t^h \mathbf{u})^{\alpha^h_h} (\partial_t^2 \mathbf{u})^{\beta^h_1} \cdots (\partial_t^{h+1} \mathbf{u})^{\beta^h_h}. \tag{3.1b}$$

Here, $P^{i,M,h}_{\alpha^h,\beta^h}$ and $Q^{M,h}_{V,\alpha^h,\beta^h}$ are some nonlinear functions in t, x, and $D^1\mathbf{u}$; $\alpha^h = (\alpha^h_1, ..., \alpha^h_h)$ and $\beta^h = (\beta^h_1, ..., \beta^h_h)$; α^h_s and β^h_s are all multi-indices; and the summation $\sum'$ is taken over all (α^h, β^h) such that

$$\sum_{s=1}^{h} (|\alpha^h_s| + |\beta^h_s|) s = h. \tag{3.2}$$

Let us define $\mathbf{u}_{M+2}$, $0 \leqslant M \leqslant K-2$, successively by

$$\begin{aligned}\mathbf{u}_{M+2} = {} & (\partial_t^M \mathbf{f}_\Omega)(0) + \partial_l \{(\partial_t^M \mathbf{P}^l)(0, (\mathbf{u}_1, D_x^1 \mathbf{u}_0))\} - (\partial_t^M \mathbf{Q}_\Omega)(0, (\mathbf{u}_1, D_x^1 \mathbf{u}_0)) \\ & + \sum_{h=1}^{M} \sum{}' \, [\partial_l \{ P^{l,M,h}_{\alpha^h, \beta^h}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0) \\ & \times (D_x^1 \mathbf{u}_1)^{\alpha_1^h} \cdots (D_x^1 \mathbf{u}_h)^{\alpha_h^h} (\mathbf{u}_2)^{\beta_1^h} \cdots (\mathbf{u}_{h+1})^{\beta_h^h} \} \\ & - Q^{M,h}_{\Omega, \alpha^h, \beta^h}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0) \\ & \times (D_x^1 \mathbf{u}_1)^{\alpha_1^h} \cdots (D_x^1 \mathbf{u}_h)^{\alpha_h^h} (\mathbf{u}_2)^{\beta_1^h} \cdots (\mathbf{u}_{h+1})^{\beta_h^h}]. \end{aligned} \tag{3.3}$$

Obviously, if $\mathbf{u}(t) \in X^{K,0}([0, T], \Omega)$ is a solution to (N), then $(\partial_t^M \mathbf{u})(0) = \mathbf{u}_M$ for $0 \leqslant M \leqslant K$. For the later references, we give

LEMMA 3.1. *Let* $\mathbb{B}, \mathbf{u}_0, \mathbf{u}_1$, *and* $\mathbf{f}_\Omega(t)$ *be the same as in Theorem* 1.1. *Then,* $\mathbf{u}_M \in H^{K-M}(\Omega)$ *and* $\|\mathbf{u}_M\|_{K-M} \leqslant C_1(K, \mathbb{B})$ *for* $2 \leqslant M \leqslant K$.

Noting (3.2) and applying Theorem Ap.1 to (3.3), we can prove easily Lemma 3.1 by induction on M. So, we may omit the proof (cf. [9, Appendix 3]).

If $\mathbf{u}(t) \in X^{K,0}([0, T], \Omega)$ is a solution to (N), we see that $\partial_t^M \{ v_l \mathbf{P}^l(t, D^1 \mathbf{u}(t)) + \mathbf{Q}_\Gamma(t, D^1 \mathbf{u}(t)) \}|_{t=0} = (\partial_t^M \mathbf{f}_\Gamma)(0)$ on Γ for $0 \leqslant M \leqslant K-2$. Keeping this in mind, let us define the compatibility condition for (N) as follows. We shall say that $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, and $\mathbf{f}_\Gamma(t)$ satisfy the compatibility condition of order $K-2$ if

$$\begin{aligned} & v_l (\partial_t^M \mathbf{P}^l)(0, (\mathbf{u}_1, D_x^1 \mathbf{u}_0)) + (\partial_t^M \mathbf{Q}_\Gamma)(0, (\mathbf{u}_1, D_x^1 \mathbf{u}_0)) \\ & \quad + \sum_{h=1}^{M} \sum{}' \, [v_l P^{l,M,h}_{\alpha^h, \beta^h}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0) + Q^{M,h}_{\Gamma, \alpha^h, \beta^h}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)] \\ & \quad \times (D_x^1 \mathbf{u}_1)^{\alpha_1^h} \cdots (D_x^1 \mathbf{u}_h)^{\alpha_h^h} (\mathbf{u}_2)^{\beta_1^h} \cdots (\mathbf{u}_{h+1})^{\beta_h^h} = (\partial_t^M \mathbf{f}_\Gamma)(0) \qquad \text{on } \Gamma \end{aligned} \tag{3.4}$$

for $0 \leqslant M \leqslant K-2$.

4. ITERATION SCHEME

Since the full nonlinearity of the boundary condition in (N) causes the derivative loss which breaks down the usual iteration process, we use the following simple reduction of (N) to a "hyperbolic–elliptic" coupled system for unknowns $\mathbf{u}$ and $\partial_t \mathbf{u}$. Differentiate (N) once in t and put $\partial_t \mathbf{u} = \mathbf{v}$ and $U(t) = (\mathbf{v}(t), D_x^1 \mathbf{u}(t))$. Then, using (1.1), we have

$$(\mathrm{H})\begin{cases}\partial_t^2 \mathbf{v}(t)-\partial_i(A^{i0}(t,\cdot,U(t))\,\partial_t \mathbf{v}(t)+A^{ij}(t,\cdot,U(t))\,\partial_j \mathbf{v}(t))-\mathbf{F}_\Omega(t,U(t))\\ \qquad =\partial_t \mathbf{f}_\Omega(t) \quad \text{in} \quad (0,T)\times\Omega;\\ \nu_i A^{ij}(t,\cdot,U(t))\,\partial_j \mathbf{v}(t)+B^i_\Gamma(t,\cdot,U(t))\,\partial_i \mathbf{v}(t)+\bar{B}^0_\Gamma(t,\cdot,U(t))\,\partial_t \mathbf{v}(t)\\ \qquad +\mathbf{F}_\Gamma(t,U(t))=\partial_t \mathbf{f}_\Gamma(t) \quad \text{on} \quad (0,T)\times\Gamma;\\ \mathbf{v}(0)=\mathbf{u}_1 \quad \text{and} \quad (\partial_t \mathbf{v})(0)=\mathbf{u}_2 \quad \text{in } \Omega,\end{cases}$$

where

$$\bar{B}^0_\Gamma(t,\cdot,U(t))=\nu_i A^{i0}(t,\cdot,U(t))+B^0_\Gamma(t,\cdot,U(t)); \tag{4.1a}$$

$$\mathbf{F}_\Omega(t,U(t))=-\mathbf{F}_{\Omega 1}(t,U(t))+\mathbf{F}_{\Omega 2}(t,U(t)); \tag{4.1b}$$

$$\mathbf{F}_{\Omega 1}(t,U(t))=\partial_i(A^{in+1}(t,\cdot,U(t))\,\mathbf{v}(t) + (\partial_t \mathbf{P}^i)(t,U(t)))-(\partial_t \mathbf{Q}_\Omega)(t,U(t)); \tag{4.1c}$$

$$\mathbf{F}_{\Omega 2}(t,U(t))=(d\mathbf{Q}_\Omega)(t,U(t))\,D^1\mathbf{v}(t); \tag{4.1d}$$

$$\mathbf{F}_\Gamma(t,U(t))=\{\nu_i A^{in+1}(t,\cdot,U(t))+B^{n+1}_\Gamma(t,\cdot,U(t))\}\,\mathbf{v}(t) + \nu_i(\partial_t \mathbf{P}^i)(t,U(t))+(\partial_t \mathbf{Q}_\Gamma)(t,U(t)). \tag{4.1e}$$

And also, the original problem (N) can be rewritten as follows:

$$(\mathrm{E})\begin{cases}\partial_t \mathbf{v}(t)-\partial_i(\mathbf{P}^i(t,U(t)))+\mathbf{Q}_\Omega(t,U(t))+\lambda\mathbf{u}(t)\\ \qquad =\mathbf{f}_\Omega(t)+\lambda\left(\mathbf{u}_0+\int_0^t \mathbf{v}(s)\,ds\right) \quad \text{in } \Omega,\\ \nu_i \mathbf{P}^i(t,U(t))+\mathbf{Q}_\Gamma(t,U(t))=\mathbf{f}_\Gamma(t) \quad \text{on } \Gamma\end{cases}$$

for all $t\in[0,T]$, where λ is a constant determined in Theorem 5.3 below.

Below, we shall solve systems (H) and (E) for unknowns $\mathbf{u}(t)$ and $\mathbf{v}(t)$. This simple procedure was first developed by [7, 9]. T. Kato [2] also used this procedure in his abstract framework. In the proofs of [7, 9], problem (H) was reduced to the zero initial data case, because of the compatibility condition. Furthermore, somewhat rough linear theory on hyperbolic mixed problems was used. These are the reasons why the assumption $K\geqslant[n/2]+8$ was needed in the original papers [7, 9] (cf. remark after Theorem 1.1).

Since (E) is still fully nonlinear with respect to $\mathbf{u}(t)$, we shall reduce (E) to an equivalent problem (E)′ as follows. Below, $\mathbf{u}^0(t)$ will always refer to a function in $X^{K,0}(\mathbb{R},\Omega)$ such that

$$(\partial_t^M \mathbf{u}^0)(0)=\mathbf{u}_M \quad \text{in } \Omega \text{ for } 0\leqslant M\leqslant K;$$
$$\|D^K\mathbf{u}^0(t)\|_0\leqslant C_2(K,\mathbb{B}) \quad \text{for all } t\in\mathbb{R}. \tag{4.2}$$

The existence of such a $\mathbf{u}^0(t)$ is assured by Theorem Ap.5. Put $\mathbf{u}(t)=\mathbf{u}^0(t)+\mathbf{w}(t)$ and $U^0(t)=(\mathbf{v}(t), D_x^1\mathbf{u}^0(t))$. Then, noting that $U^0(0)=(\mathbf{u}_1, D_x^1\mathbf{u}_0)$, we can rewrite (E) as equations for unknown $\mathbf{w}(t)$ as follows:

(E)′ $p_{\Omega\lambda}[\mathbf{w}(t)]=\mathbf{g}_\Omega(t)$ in Ω and $p_\Gamma[\mathbf{w}(t)]=\mathbf{g}_\Gamma(t)$ on Γ for all $t\in[0, T]$, where

$$p_{\Omega\lambda}[\mathbf{w}]=-\partial_i(A^{ij}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\,\partial_j\mathbf{w}+A^{in+1}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\mathbf{w})$$
$$+B_\Omega^i(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\,\partial_i\mathbf{w}+B_\Omega^{n+1}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\,\mathbf{w}+\lambda\mathbf{w}; \tag{4.3a}$$

$$p_\Gamma[\mathbf{w}]=\nu_i(A^{ij}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\,\partial_j\mathbf{w}+A^{in+1}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\mathbf{w})$$
$$+B_\Gamma^i(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\,\partial_i\mathbf{w}+B_\Gamma^{n+1}(0,\cdot,\mathbf{u}_1, D_x^1\mathbf{u}_0)\mathbf{w}; \tag{4.3b}$$

$$\mathbf{g}_\Omega(t)=\mathbf{G}_{\Omega1}(t,\mathbf{v}(t))+\mathbf{Q}_{\Omega2}(t,\mathbf{v}(t),\mathbf{w}(t))+\mathbf{G}_{\Omega3}(t,\mathbf{v}(t),\mathbf{w}(t)); \tag{4.4a}$$

$$\mathbf{G}_{\Omega1}(t,\mathbf{v}(t))=\mathbf{f}_\Omega(t)-\partial_t\mathbf{v}(t)+\partial_i(\mathbf{P}^i(t, U^0(t)))-\mathbf{Q}_\Omega(t, U^0(t))$$
$$+\lambda\int_0^t(\mathbf{v}(s)-\partial_s\mathbf{u}^0(s))\,ds$$
$$\left(\text{note that } \mathbf{u}^0(t)=\mathbf{u}_0+\int_0^t\partial_s\mathbf{u}^0(s)\,ds\right); \tag{4.4b}$$

$$\mathbf{G}_{\Omega2}(t,\mathbf{v}(t),\mathbf{w}(t))=\partial_i\{(A^{ij}(t,\cdot,U^0(t))-A^{ij}(0,\cdot,U^0(0)))\,\partial_j\mathbf{w}(t)$$
$$+(A^{in+1}(t,\cdot,U^0(t))-A^{in+1}(0,\cdot,U^0(0)))\,\mathbf{w}(t)\}$$
$$-(B_\Omega^i(t,\cdot,U^0(t))-B_\Omega^i(0,\cdot,U^0(0)))\,\partial_i\mathbf{w}(t)$$
$$-(B_\Omega^{n+1}(t,\cdot,U^0(t))-B_\Omega^{n+1}(0,\cdot,U^0(0)))\,\mathbf{w}(t); \tag{4.4c}$$

$$\mathbf{G}_{\Omega3}(t,\mathbf{v}(t),\mathbf{w}(t))=\partial_i\{\mathbf{P}^i(t, U(t))-\mathbf{P}^i(t, U^0(t))$$
$$-A^{ij}(t,\cdot,U^0(t))\,\partial_j\mathbf{w}(t)-A^{in+1}(t,\cdot,U^0(t))\,\mathbf{w}(t)\}$$
$$-\{\mathbf{Q}_\Omega(t, U(t))-\mathbf{Q}_\Omega(t, U^0(t))$$
$$-B_\Omega^i(t,\cdot,U^0(t))\,\partial_i\mathbf{w}(t)-B_\Omega^{n+1}(t,\cdot,U^0(t))\,\mathbf{w}(t)\}; \tag{4.4d}$$

$$\mathbf{g}_\Gamma(t)=\mathbf{G}_{\Gamma1}(t,\mathbf{v}(t))+\mathbf{G}_{\Gamma2}(t,\mathbf{v}(t),\mathbf{w}(t))+\mathbf{G}_{\Gamma3}(t,\mathbf{v}(t),\mathbf{w}(t)); \tag{4.5a}$$

$$\mathbf{G}_{\Gamma1}(t,\mathbf{v}(t))=\mathbf{f}_\Gamma(t)-\nu_i\mathbf{P}^i(t, U^0(t))-\mathbf{Q}_\Gamma(t, U^0(t)); \tag{4.5b}$$

$$\begin{aligned}\mathbf{G}_{\Gamma 2}(t, \mathbf{v}(t), \mathbf{w}(t)) = v_i\{&(A^{ij}(t, \cdot, U^0(t)) - A^{ij}(0, \cdot, U^0(0)))\,\partial_j \mathbf{w}(t) \\ &+ (A^{in+1}(t, \cdot, U^0(t)) - A^{in+1}(0, \cdot, U^0(0)))\,\mathbf{w}(t)\} \\ &+ (B^i_\Gamma(t, \cdot, U^0(t)) - B^i_\Gamma(0, \cdot, U^0(0)))\,\partial_t \mathbf{w}(t) \\ &+ (B^{n+1}_\Gamma(t, \cdot, U^0(t)) - B^{n+1}_\Gamma(0, \cdot, U^0(0)))\,\mathbf{w}(t); \qquad (4.5c)\end{aligned}$$

$$\begin{aligned}\mathbf{G}_{\Gamma 3}(t, \mathbf{v}(t), \mathbf{w}(t)) = v_i\{&\mathbf{P}^i(t, U(t)) - \mathbf{P}^i(t, U^0(t)) \\ &- A^{ij}(t, \cdot, U^0(t))\,\partial_j \mathbf{w}(t) - A^{in+1}(t, \cdot, U^0(t))\,\mathbf{w}(t)\} \\ &+ \{\mathbf{Q}_\Gamma(t, U(t)) - \mathbf{Q}_\Gamma(t, U^0(t)) \\ &- B^i_\Gamma(t, \cdot, U^0(t))\,\partial_t \mathbf{w}(t) - B^{n+1}_\Gamma(t, \cdot, U^0(t))\,\mathbf{w}(t)\}.\end{aligned}$$

By using the method of successive approximations, we shall solve systems (H) and (E)′. To do this, first let us introduce the spaces Z and Z_c in which all the functions in our iteration scheme belong. The space Z is the set of all pairs $(\mathbf{v}(t), \mathbf{w}(t)) \in Y^{K-1,0}([0, T], \Omega) \times Y^{K-2,2}([0, T], \Omega)$ such that

$$\partial_t^M \mathbf{w}(0) = 0 \quad \text{for } 0 \leqslant M \leqslant K-3; \qquad \partial_t^M \mathbf{v}(0) = \mathbf{u}_{M+1} \quad \text{for } 0 \leqslant M \leqslant K-2; \tag{4.6a}$$

$$|\mathbf{v}|_{K-1,0,[0,T]} \leqslant A_H; \qquad |\mathbf{w}|_{K-2,2,[0,T]} \leqslant A_E; \qquad |\mathbf{w}|_{K-3,2,[0,T]} \leqslant \varepsilon_E; \tag{4.7}$$

$$(\mathbf{v}(t), D_x^1 \mathbf{u}_0(t)) \quad \text{and} \quad (\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t))) \in H^{\infty,1}(\bar{\Omega}, D(U_1)) \quad \text{for} \quad t \in [0, T]. \tag{4.8}$$

Here and hereafter, $T, A_H\, A_E$, and ε_E are constants determined below which depend only on K and $\mathbb{B}$ essentially; U_1 is a constant $\in (0, U_0)$ determined below. We assume that

(AS.1) $0 < T < \min(1, T_0)$ and $0 < \varepsilon_E < 1$.

The space Z_c is the set of all pairs $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$ such that $\mathbf{v}(t) \in X^{K-1,0}([0, T], \Omega)$, $\mathbf{w}(t) \in X^{K-2,2}([0, T], \Omega)$, and

$$\partial_t^M \mathbf{w}(0) = 0 \quad \text{for } 0 \leqslant M \leqslant K-2; \qquad \partial_t^M \mathbf{v}(0) = \mathbf{u}_{M+1} \quad \text{for } 0 \leqslant M \leqslant K-1. \tag{4.6b}$$

Our iteration scheme is defined as follows: For given $p \geqslant 2$ and $(\mathbf{v}^{p-1}(t), \mathbf{w}^{p-1}(t)) \in Z_c$, let us define $\mathbf{v}^p(t)$ by a solution to the following linear problem:

$$(\mathrm{H})_p \begin{cases} \partial_t^2 \mathbf{v}^p(t) - \partial_i(A^{i0}(t, \cdot, U^{p-1}(t))\, \partial_t \mathbf{v}^p(t) + A^{ij}(t, \cdot, U^{p-1}(t))\, \partial_j \mathbf{v}^0(t)) \\ \qquad = \partial_t \mathbf{f}_\Omega(t) + \mathbf{F}_\Omega(t, U^{p-1}(t)) \quad \text{in} \quad (0, T) \times \Omega; \\ v_i A^{ij}(t, \cdot, U^{p-1}(t))\, \partial_j \mathbf{v}^p(t) + B_\Gamma^j(t, \cdot, U^{p-1}(t))\, \partial_j \mathbf{v}^p(t) \\ \qquad + \bar{B}_\Gamma^0(t, \cdot, U^{p-1}(t))\, \partial_t \mathbf{v}^p(t) \\ \qquad = \partial_t \mathbf{f}_\Gamma(t) - \mathbf{F}_\Gamma(t, U^{p-1}(t)) \quad \text{on} \quad (0, T) \times \Gamma; \\ \mathbf{v}^p(0) = \mathbf{u}_1 \quad \text{and} \quad (\partial_t \mathbf{v}^p)(0) = \mathbf{u}_2 \quad \text{in } \Omega, \end{cases}$$

where $U^{p-1}(t) = (\mathbf{v}^{p-1}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}^{p-1}(t)))$. And, let us define $\mathbf{w}^p(t)$ by a solution to the linear problem:

$(\mathrm{E})_p$ $\;p_{\Omega\lambda}[\mathbf{w}^p(t)] = \mathbf{g}_\Omega^p(t)$ in Ω and $p_\Gamma[\mathbf{w}^p(t)] = \mathbf{g}_\Gamma^p(t)$ on Γ, where $\mathbf{g}_V^p(t) = \mathbf{G}_{V1}(t, \mathbf{v}^p(t)) + \mathbf{G}_{V2}(t, \mathbf{v}^p(t), \mathbf{w}^{p-1}(t)) + \mathbf{G}_{V3}(t, \mathbf{v}^p(t), \mathbf{w}^{p-1}(t))$.

To prove the convergence of our iteration scheme, we must prove that there exist Λ_H, Λ_E, ε_E, and T such that

$$Z_c \text{ is not empty;} \tag{4.9}$$

$$(\mathbf{v}^p(t), \mathbf{w}^p(t)) \in Z_c; \tag{4.10}$$

$$|\mathbf{v}^p - \mathbf{v}^{p-1}|_{1,0,[0,T]} + |\mathbf{w}^p - \mathbf{w}^{p-1}|_{0,2,[0,T]} \leqslant \tfrac{1}{2}\{|\mathbf{v}^{p-1} - \mathbf{v}^{p-2}|_{1,0,[0,T]} + |\mathbf{w}^{p-1} - \mathbf{w}^{p-2}|_{0,2,[0,T]}\}. \tag{4.11}$$

We conclude this section by proving (4.9). Since $D_x^1 \mathbf{u}_0, \mathbf{u}_1 \in H^{K-1}(\Omega)$ and $K - 1 \geqslant [n/2] + 2$, by Sobolev's imbedding theorem we know that $|D_x^1(\mathbf{u}_1(x), D_x^1 \mathbf{u}_0(x))| \to 0$ as $|x| \to \infty$. By this and (1.5), we see that there exists a $U_2 \in (0, U_0)$ such that

$$\|(\mathbf{u}_1, D_x^1 \mathbf{u}_0)\|_{\infty,1} \leqslant U_2. \tag{4.12}$$

Let $(\mathbf{v}(t), \mathbf{w}(t))$ satisfy (4.6a) and (4.7). Put $U(t) = (\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t)))$. By Theorem Ap.7 with $F(t, x, U) = U$, (4.2), (4.6a), (4.7), and (4.12), we have

$$\begin{aligned} \|U(t)\|_{\infty,1} &\leqslant \|(\mathbf{u}_1, D_x^1 \mathbf{u}_0)\|_{\infty,1} + CT^\varepsilon |U|_{K-2,1,[0,T]} \\ &\leqslant U_2 + CT^\varepsilon(C_2(K, \mathbb{B}) + \Lambda_H + \Lambda_E) \quad \text{for} \quad t \in [0, T]. \end{aligned}$$

Here and hereafter, ε always refers to a fixed constant $\in (0, [n/2] + 1 - (n/2))$. Let $U_1 \in (0, U_0)$ and choose T so that

(As.2) $\quad U_2 + CT^\varepsilon(C_2(K, \mathbb{B}) + \Lambda_H + \Lambda_E) < U_1$.

Then, $(\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t))) \in H^{\infty,1}(\bar{\Omega}, D(U_1))$ for $t \in [0, T]$. In the same way, we see that $(\mathbf{v}(t), D_x^1 \mathbf{u}^0(t)) \in H^{\infty,1}(\bar{\Omega}, D(U_1))$ for $t \in [0, T]$ provided that (As.2) is valid. Namely, under assumption (As.2), combining (4.6a) (or (4.6b)) and (4.7) implies (4.8). Since $|\partial_t \mathbf{u}^0|_{K-1,0,[0,T]} \leqslant C_2(K, \mathbb{B})$ as follows from (4.2), if we choose Λ_H so that

(As.3) $\Lambda_H \geqslant C_2(K, \mathbb{B})$,

then $(\partial_t \mathbf{u}^0(t), 0) \in Z_c$. From this, we have (4.9).

5. Preparations for Solving Problem $(\mathrm{E})_p$

First, we shall state a unique existence theorem of solutions to the following problem:

$$q_{\Omega\lambda}[\mathbf{w}] = \mathbf{h}_\Omega \text{ in } \Omega \text{ and } q_\Gamma[\mathbf{w}] = \mathbf{h}_\Gamma \text{ on } \Gamma,$$

where

$$\begin{aligned} q_{\Omega\lambda}[\mathbf{w}] &= -\partial_i(q^{ij}\partial_j \mathbf{w} + q^i \mathbf{w}) + q^i_\Omega \partial_i \mathbf{w} + q^{n+1}_\Omega \mathbf{w} + \lambda \mathbf{w}; \\ q_\Gamma[\mathbf{w}] &= \nu_i(q^{ij}\partial_j \mathbf{w} + q^i \mathbf{w}) + q^i_\Gamma \partial_i \mathbf{w} + q^{n+1}_\Gamma \mathbf{w}. \end{aligned} \tag{5.1}$$

Here, λ is a constant; $q^{ij} = q^{ij}(x)$, $q^i = q^i(x)$, $q^i_V = q^i_V(x)$, and $q^{n+1}_V = q^{n+1}_V(x)$ are $m \times m$ matrices of functions satisfying the following four conditions:

(A.5.1) The q^{ij}, q^i, q^i_V, and q^{n+1}_V are decomposed as follows: $q^{ij} = q^{ij}_\infty + q^{ij}_S$; $q^i = q^i_\infty + q^i_S$; $q^i_V = q^i_{V\infty} + q^i_{VS}$; $q^{n+1}_V = q^{n+1}_{V\infty} + q^{n+1}_{VS}$ where $q^{ij}_\infty, q^i_\infty, q^i_{\Gamma\infty}, q^{n+1}_{\Gamma\infty} \in \mathscr{B}^{K-1}(\bar{\Omega})$; $q^i_{\Omega\infty}, q^{n+1}_{\Omega\infty} \in \mathscr{B}^{K-2}(\bar{\Omega})$; $q^{ij}_S, q^i_S, q^i_{\Gamma S}, q^{n+1}_{\Gamma S} \in H^{K-1}(\Omega)$; $q^i_{\Omega S}, q^{n+1}_{\Omega S} \in H^{K-2}(\Omega)$.

(A.5.2) ${}^t q^{ij} = q^{ji}$ in Ω.

(A.5.3) $(q^{ij}\partial_j w, \partial_i w) + \langle q^i_\Gamma \partial_i \mathbf{w}, \mathbf{w}\rangle \geqslant \delta_1 \|\mathbf{w}\|_1^2 - \delta_0 \|\mathbf{w}\|_0^2$ for $\mathbf{w} \in H^2(\Omega)$.

(A.5.4) $\nu_i q^i_\Gamma = 0$ on Γ.

First, let us discuss the uniqueness of solutions in $H^2(\Omega)$ and the existence of weak solutions in $H^1(\Omega)$. Multyplying (5.1) by $\mathbf{v}$ and integrating the resulting formulas over Ω and Γ, by integration by parts we have that $(q_{\Omega\lambda}[\mathbf{w}], \mathbf{v}) + \langle q_\Gamma[\mathbf{w}], \mathbf{v}\rangle = q_\lambda[\mathbf{w}, \mathbf{v}]$ where $\mathbf{q}_\lambda[\mathbf{w}, \mathbf{v}] = (q^{ij}\partial_j \mathbf{w}, \partial_i \mathbf{v}) + S_1(q)[\mathbf{w}, \mathbf{v}] + S_2(q)[\mathbf{w}, \mathbf{v}] + (q^i_\Omega \partial_i \mathbf{w} + q^{n+1}_\Omega \mathbf{w}, \mathbf{v}) + (q^i \mathbf{w}, \partial_i \mathbf{v}) + \langle q^{n+1}_\Gamma \mathbf{w}, \mathbf{v}\rangle$. Here, $S_l(q)$ $(l = 1, 2)$ are the bilinear forms defined by (2.5) with $R = q = (q^1_\Gamma, \ldots, q^n_\Gamma)$ and we have used (2.3). Applying Theorem Ap.4(2) with $\varepsilon = \min(\delta_1/4\, \|q^{n+1}_\Gamma\|_{\infty,0}, 1)$, we have that $|\langle q^{n+1}_\Gamma \mathbf{w}, \mathbf{w}\rangle| \leqslant (\delta_1/4)\, \|\mathbf{w}\|_1^2 + C(\partial_1, \|q^{n+1}_\Gamma\|_{\infty,0}, \Gamma)\, \|\mathbf{w}\|_0^2$. Let γ_∞ be a constant such that

$$\sum_{i=1}^{n} \|q^i\|_{\infty,0} + \sum_{k=1}^{n+1} \|q^k_\Omega\|_{\infty,0} + \|q^{n+1}_\Gamma\|_{\infty,0} \leqslant \gamma_\infty. \tag{5.2a}$$

Then, using Schwarz's inequality and (A.5.3), we see easily that there exists a $\mu_0 = C(\delta_1, \Gamma, \gamma_\infty)$ such that

$$(q_{\Omega\lambda}[\mathbf{w}], \mathbf{w}) + \langle q_\Gamma[\mathbf{w}], \mathbf{w}\rangle \geqslant (\delta_1/2)\, \|\mathbf{w}\|_1^2 + (\lambda - \mu_0 - \delta_0)\, \|\mathbf{w}\|_0^2 \qquad \text{for} \quad \mathbf{w} \in H^2(\Omega). \tag{5.3}$$

If we choose $\lambda > 0$ so that $\lambda \geqslant \mu_0 + \delta_0$, then the uniqueness of solutions in $H^2(\Omega)$ to (5.1) is valid. By Schwarz's inequality and (2.6), we see that $q_\lambda[\mathbf{w}, \mathbf{v}]$ is a continuous bilinear form on $H^1(\Omega) \times H^1(\Omega)$. By (5.3), we see that $q_\lambda[\mathbf{w}, \mathbf{w}] \geqslant (\delta_1/2)\, \|\mathbf{w}\|_1^2$ provided that $\lambda \geqslant \mu_0 + \delta_0$. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, it follows from this inequality that q_λ is a coercive bilinear form on $H^1(\Omega) \times H^1(\Omega)$. Hence, by the Lax–Milgram theorem, we have the existence of weak solutions in $H^1(\Omega)$ to (5.1). Furthermore, by the usual method (cf. [5, Sect. 3]), we get the regularities of weak solutions. Namely, we have

THEOREM 5.1. *Assume that* (A.5.1)–(A.5.4) *are valid. Let L be an integer* $\in [2, K]$. *Let* γ_K *be a constant such that*

$$\sum_{i,j=1}^{n} (\|q_\infty^{ij}\|_{\infty,K-1} + \|q_S^{ij}\|_{K-1}) + \sum_{i=1}^{n} (\|q_\infty^{i}\|_{\infty,K-1} + \|q_S^{i}\|_{K-1}) + \sum_{k=1}^{n+1} (\|q_{\Omega\infty}^k\|_{\infty,K-2} + \|q_{\Gamma\infty}^k\|_{\infty,K-1} + \|q_{\Omega S}^k\|_{K-2} + \|q_{\Gamma S}^k\|_{K-1}) \leqslant \gamma_K. \tag{5.2b}$$

Then, there exists a $\lambda_0 > 0$ *depending only on* $\lambda_K, \delta_0, \delta_1$, *and* Γ *essentially such that for any* $\lambda \geqslant \lambda_0$ *and given* $\mathbf{h}_\Omega \in H^{L-2}(\Omega)$ *and* $\mathbf{h}_\Gamma \in H^{L-(3/2)}(\Gamma)$, (5.1) *admits a unique solution* $\mathbf{w} \in H^L(\Omega)$ *satisfying the estimate*:

$$\|\mathbf{w}\|_L \leqslant C(K, \gamma_K, \Gamma, \delta_0, \delta_1, n, m, \lambda)\{\|\mathbf{h}_\Omega\|_{L-2} + \langle\!\langle \mathbf{h}_\Gamma \rangle\!\rangle_{L-(3/2)}\}. \tag{5.4}$$

Remark. The detailed proof of Theorem 5.1 was given in [8, Sect. 3]. In [8, Sect. 3], it was assumed that $q_\Gamma^k \in H^{K-(3/2)}(\Gamma)$. Since $q_\Gamma^k = q_{\Gamma\infty}^k + q_{\Gamma S}^k \in \mathscr{B}^{K-1}(\bar{\Omega}) + H^{K-1}(\Omega)$ in the present case, noting Theorem Ap.4(1) and the fact that Γ is compact, we have that $\langle\!\langle q_\Gamma^k \rangle\!\rangle_{K-(3/2)} \leqslant C\{\|q_{\Gamma\infty}^k\|_{\infty,K-1} + \|q_{\Gamma S}^k\|_{K-1}\}$. From this, we can apply the result in [8, Sect. 3] to the present case.

When the right members $\mathbf{h}_\Omega$ and $\mathbf{h}_\Gamma$ depend on t, we use the following.

THEOREM 5.2. *Assume that* (A.5.1)–(A.5.4) *are valid. Let* λ *be the same as in Theorem* 5.1. *Let* $T > 0$ *and put* $J = [0, T]$. *If* $\mathbf{h}_\Omega(t) \in X^{K-2,0}(J, \Omega)$ *and*

$\mathbf{h}_\Gamma(t) \in X^{K-2,1/2}(J, \Gamma)$, *then there exists a unique* $\mathbf{w}(t) \in X^{K-2,2}(J, \Omega)$ *satisfying the equations*:

$$q_{\Omega\lambda}[\mathbf{w}(t)] = \mathbf{h}_\Omega(t) \quad \text{in } \Omega \qquad \text{and} \qquad q_\Gamma[\mathbf{w}(t)] = \mathbf{h}_\Gamma(t) \quad \text{on } \Gamma \text{ for every } t \in J. \tag{5.5}$$

The main task in the proof of Theorem 5.2 is to show the dependence on t of solutions $\mathbf{w}(t)$. Since the coefficients of $q_{\Omega\lambda}$ and q_Γ are independent of t and since $q_{\Omega\lambda}$ and q_Γ are linear operators, by using Theorem 5.1, we can get the dependence on t of solutions $\mathbf{w}(t)$ easily. So, we may omit the detailed proof of Theorem 5.2 (cf. [8, Sect. 3]).

Our goal of this section is to prove

THEOREM 5.3. *Assume that* (A.1)–(A.4) *are valid. Let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *be the same as in Theorem* 1.1 (1) *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z_c$ *and let* $p_{\Omega\lambda}, p_\Gamma, \mathbf{g}_\Omega(t)$, *and* $\mathbf{g}_\Gamma(t)$ *be the same as in* (4.3a), (4.3b), (4.4a), *and* (4.5a), *respectively. Then, there exists a* λ *depending only on* K *and* $\mathbb{B}$ *such that there exists a unique* $\mathbf{z}(t) \in X^{K-2,2}([0, T], \Omega)$ *satisfying the equations*:

$$p_{\Omega\lambda}[\mathbf{z}(t)] = \mathbf{g}_\Omega(t) \quad \text{in } \Omega \qquad \text{and} \qquad p_\Gamma[\mathbf{z}(t)] = \mathbf{g}_\Gamma(t) \quad \text{on } \Gamma \text{ for every } t \in [0, T] \tag{5.6}$$

and the properties: $(\partial_t^M \mathbf{z})(0) = 0$ *for* $0 \leqslant M \leqslant K-2$. *Furthermore, there exist* T, Λ_E, *and* ε_E *depending only on* K, $\mathbb{B}$, *and* Λ_H *such that*

$$|\mathbf{z}|_{K-2,2,[0,T]} \leqslant \Lambda_E \qquad \text{and} \qquad |\mathbf{z}|_{K-3,2,[0,T]} \leqslant \varepsilon_E. \tag{5.7}$$

(2) *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$. *Then, there exists a* T *depending only on* K, $\mathbb{B}$, Λ_H, *and* Λ_E *such that for the present* λ, *the inequality*

$$\begin{aligned}(A^{ij}(t, \cdot, U(t))\, \partial_j \mathbf{z}, \partial_i \mathbf{z}) &+ \langle B_\Gamma^i(t, \cdot, U(t))\, \partial_i \mathbf{z}, \mathbf{z} \rangle \\ &+ \lambda \|\mathbf{z}\|_0^2 + \langle B_\Gamma^{n+1}(t, \cdot, U(t))\, \mathbf{z}, \mathbf{z} \rangle + (({}^t A^{in+1}(t, \cdot, U(t)) \\ &+ B_\Omega^i(t, \cdot, U(t)))\, \partial_i \mathbf{z} + B_\Omega^{n+1}(t, \cdot, U(t))\, \mathbf{z}, \mathbf{z}) \geqslant (\delta_1/2)\, \|\mathbf{z}\|_1^2, \end{aligned} \tag{5.8}$$

is valid for $t \in [0, T]$ *and* $\mathbf{z} \in H^2(\Omega)$, *where* $U(t) = (\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t)))$.

To prove Theorem 5.3, we begin with

LEMMA 5.4. *Assume that* (A.1)–(A.5) *are valid. Let* $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *be the same as in Theorem* 1.1. *Put* $U^0 = (\mathbf{u}_1, D_x^1 \mathbf{u}_0)$ *and for* $k = 1, \ldots, n+1$, $i = 1, \ldots, n$, $V = \Omega$, *and* Γ, *set*

$$
\begin{aligned}
&q^{ij}_{\infty} = A^{ij}(0, \cdot, 0); \qquad q^{ij}_S = (A^{ij})_1(0, \cdot, U^0);\\
&q^{i}_{\infty} = A^{in+1}(0, \cdot, 0); \qquad q^{i}_S = (A^{in+1})_1(0, \cdot, U^0);\\
&q^{k}_{V\infty} = B^{k}_{V}(0, \cdot, 0); \qquad q^{k}_{VS} = (B^{k}_{V})_1(0, \cdot, U^0);\\
&q^{ij} = q^{ij}_{\infty} + q^{ij}_S; \qquad q^{i} = q^{i}_{\infty} + q^{i}_S;\\
&q^{k}_{V} = q^{k}_{V\infty} + q^{k}_{VS}.
\end{aligned}
\tag{5.9}
$$

Then, the present q^{ij}, q^{i}, *and* q^k_V $(k=1, ..., n+1)$ *satisfy* (A.5.1)–(A.5.4). *Furthermore,*

$$
\begin{aligned}
&\sum_{i,j=1}^{n} (\|q^{ij}_{\infty}\|_{\infty, K-1} + \|q^{ij}_S\|_{K-1})\\
&\quad + \sum_{i=1}^{n} (\|q^{i}_{\infty}\|_{\infty, K-1} + \|q^{i}_S\|_{K-1})\\
&\quad + \sum_{k=1}^{n+1} (\|q^{k}_{\Omega\infty}\|_{\infty, K-2} + \|q^{k}_{\Omega S}\|_{K-2}\\
&\quad + \|q^{k}_{\Gamma\infty}\|_{\infty, K-1} + \|q^{k}_{\Gamma S}\|_{K-1}) \leqslant C_3(K, \mathbb{B}).
\end{aligned}
\tag{5.10}
$$

Proof. Noting (1.5), we see easily that (A.5.*l*) follows from (A.*l*) for $l=2, 3$, and 4. Noting (1.5), (1.6), and (2.1), by Theorem Ap.3 we have (A.5.1) and (5.10), which completes the proof.

Now, we shall estimate the right-hand side of (5.6).

LEMMA 5.5. *Assume that* (A.1) *is valid. Let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *be the same as in Theorem* 1.1. *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z_c$. *Let* $\mathbf{g}_\Omega(t)$ *and* $\mathbf{g}_\Gamma(t)$ *be the same as in* (4.4a) *and* (4.5a), *respectively. Then, the following two assertions are valid.*

(1) $(\partial_t^M \mathbf{g}_\Omega)(0) = 0$ *on* Ω *and* $(\partial_t^M \mathbf{g}_\Gamma)(0) = 0$ *on* Γ *for* $0 \leqslant M \leqslant K-2$.

(2) $\mathbf{g}_\Omega(t) \in X^{K-2,0}([0, T], \Omega)$, $\mathbf{g}_\Gamma(t) \in X^{K-2,1/2}([0, T], \Gamma)$, *and*

$$
\begin{aligned}
&|\mathbf{g}_\Omega|_{K-2,0,[0,T]} + \langle \mathbf{g}_\Gamma \rangle_{K-2,1/2,[0,T]}\\
&\quad \leqslant C_1(K, \mathbb{B}, \Lambda_H) + C_2(K, \mathbb{B}, \Lambda_H)\, T\Lambda_E + C_1(K, \mathbb{B}, \Lambda_H, \Lambda_E)\varepsilon_E.
\end{aligned}
\tag{5.11}
$$

Proof. First, we proof (1). Since $(\partial_t^M \mathbf{u}^0)(0) = \mathbf{u}_M$ for $0 \leqslant M \leqslant K$ and $(\partial_t^M \mathbf{v})(0) = \mathbf{u}_{M+1}$ for $0 \leqslant M \leqslant K-1$ as follows from (4.2) and (4.6b), by (3.1) and (3.3) we have

$$
\begin{aligned}
&\partial_t^M \{\mathbf{f}_\Omega(t) - \partial_t \mathbf{v}(t) + \partial_i(\mathbf{P}^i(t, U(t))) - \mathbf{Q}_\Omega(t, U(t))\}|_{t=0} = 0\\
&\text{on } \Omega \text{ for } 0 \leqslant M \leqslant K-2,
\end{aligned}
$$

where $U(t)=(\mathbf{v}(t), D_x^1\mathbf{u}^0(t))$. We have also that $\partial_t^M\{\int_0^t(\mathbf{v}(s)-\partial_s\mathbf{u}^0(s))\,ds\}|_{t=0}=0$ on Ω for $0\leqslant M\leqslant K-2$. In the same way, it follows from (3.1), (3.4), (4.2), and (4.6b) that $\partial_t^M\mathbf{G}_{\Gamma 1}(t, \mathbf{v}(t))|_{t=0}=0$ on Γ for $0\leqslant M\leqslant K-2$ (cf. (4.5b)). Since $(\partial_t^M\mathbf{w})(0)=0$ for $0\leqslant M\leqslant K-2$ (cf. (4.6b)), we see easily that $\partial_t^M\mathbf{G}_{V2}(t, \mathbf{v}(t), \mathbf{w}(t))|_{t=0}=0$ on Ω for $0\leqslant M\leqslant K-2$ if we just look at (4.4c) and (4.5c). Applying Taylor expansion to (4.4d) and (4.5d), we can write

$$\mathbf{G}_{\Omega 3}(t, \mathbf{v}(t), \mathbf{w}(t))=\partial_i\left\{\int_0^1 (d^2\mathbf{P}^i)(t, U(\theta))(D_x^1\mathbf{w}(t), D_x^1\mathbf{w}(t))\,d\theta\right\}$$

$$-\int_0^1 (d^2\mathbf{Q}_\Omega)(t, U(\theta))(D_x^1\mathbf{w}(t), D_x^1\mathbf{w}(t))\,d\theta; \tag{5.12a}$$

$$\mathbf{G}_{\Gamma 3}(t, \mathbf{v}(t), \mathbf{w}(t))=v_i\left\{\int_0^1 (d^2\mathbf{P}^i)(t, U(\theta))(D_x^1\mathbf{w}(t), D_x^1\mathbf{w}(t))\,d\theta\right\}$$

$$+\int_0^1 (d^2\mathbf{Q}_\Gamma)(t, U(\theta))(D_x^1\mathbf{w}(t), D_x^1\mathbf{w}(t))\,d\theta, \tag{5.12b}$$

where $U(\theta)=(\mathbf{v}(t), D_x^1(\mathbf{u}^0(t)+\theta\mathbf{w}(t)))$. Thus, by the fact that $(\partial_t^M\mathbf{w})(0)=0$ on Ω for $0\leqslant M\leqslant K-2$, we see easily that $\partial_t^M\mathbf{G}_{V3}(t, \mathbf{v}(t), \mathbf{w}(t))|_{t=0}=0$ on Ω for $0\leqslant M\leqslant K-2$. Thus, we have (1).

Now, we prove (2). Applying Theorem Ap.4(1), we see that $\langle v_i\mathbf{P}^i+\mathbf{Q}_\Gamma\rangle_{K-2,1/2,[0,T]}\leqslant C|v_i\mathbf{P}^i+\mathbf{Q}_\Gamma|_{K-2,1,[0,T]}$. Hence, by Theorem Ap.3, (1.5), (1.6), (4.2), and (4.7) we see easily that $|\mathbf{G}_{\Omega 1}(\cdot, \mathbf{v})|_{K-2,0,[0,T]}+\langle\mathbf{G}_{\Gamma 1}(\cdot, \mathbf{v})\rangle_{K-2,1/2,[0,T]}\leqslant C(K, \mathbb{B}, \Lambda_H)$. Applying (Ap.2) with $u(t)=(\mathbf{v}(t), D_x^1\mathbf{u}^0(t))$ and $v(t)=\partial_j\mathbf{w}(t)$ and so on and using (4.2) and (4.7), we have that $|\mathbf{G}_{\Omega 2}(\cdot, \mathbf{v}, \mathbf{w})|_{K-2,0,[0,T]}+|\mathbf{G}_{\Gamma 2}(\cdot, \mathbf{v}, \mathbf{w})|_{K-2,1,[0,T]}\leqslant C(K, \mathbb{B}, \Lambda_H)\{T\Lambda_E+\varepsilon_E\}$. Applying (Ap.4) with $u(t)=U(\theta)$ and $v(t)=D_x^1\mathbf{w}(t)$ to (5.12) and using (4.2) and (4.7), we have that $|\mathbf{G}_{\Omega 3}(\cdot, \mathbf{v}, \mathbf{w})|_{K-2,0,[0,T]}+|\mathbf{G}_{\Gamma 3}(\cdot, \mathbf{v}, \mathbf{w})|_{K-2,1,[0,T]}\leqslant C(K, \mathbb{B}, \Lambda_H, \Lambda_E)\varepsilon_E$. Noting Theorem Ap.4(1) and combining these estimations, we have (5.11), which completes the proof.

Now, we prove Theorem 5.3. First, we choose $\lambda>0$ so that (2) is valid. Put $U(t)=(\mathbf{v}(t), D_x^1(\mathbf{u}^0(t)+\mathbf{w}(t)))$ for $(\mathbf{v}(t), \mathbf{w}(t))\in Z$. By (Ap.9), (4.2), and (4.7), we see that

$$\sum_{i=1}^{n}\|A^{in+1}(t, \cdot, U(t))\|_{\infty,0}+\sum_{k=1}^{n+1}\|B_\Omega^k(t, \cdot, U(t))\|_{\infty,0}+\|B_\Gamma^{n+1}(t, \cdot, U(t))\|_{\infty,0}$$

$$\leqslant C_1\{1+t|U|_{K-2,1,[0,T]}\}\leqslant C_1\{1+T(C_2(K, \mathbb{B})+\Lambda_H+\Lambda_E)\} \tag{5.13}$$

for $t\in[0, T]$. Choose $T>0$ so that

(As.4) $\quad T\{C_2(K, \mathbb{B})+\Lambda_H+\Lambda_E\}\leqslant 1.$

If we put $q^i = A^{m+1}(t, \cdot, U(t))$ and $q_V^k = B_V^k(t, \cdot, U(t))$ $(k = 1, ..., n+1)$, then in the present case (5.2a) is valid for $t \in [0, T]$ with $\gamma_\infty = 2C_1$ (cf. (5.13) and (As.4)). Hence, we can choose the constant μ_0 appearing in (5.3) so that μ_0 is independent of K, $\mathbb{B}$, Λ_H, Λ_E, ε_E, and T. If $\lambda \leqslant \mu_0 + \delta_0$, then by (5.3) we have (5.8).

Below, μ_0 will always refer to the constant determined just now. Now, we shall prove (1). Let q^{ij}, q^i, and q_V^k $(k = 1, ..., n+1)$ be the same as in (5.9). By (5.10), we put $\gamma_K = C_3(K, \mathbb{B})$ in the present case (cf. (5.2b)). In view of Lemma 5.4, we can apply Theorems 5.1 and 5.2 with $\gamma_K = C_3(K, \mathbb{B})$. Then, by (2) of Lemma 5.5 and Theorems 5.1 and 5.2, we can choose a $\lambda \geqslant \mu_0 + \delta_0$ depending only on K and $\mathbb{B}$ such that there exists a unique $\mathbf{z}(t) \in X^{K-2,2}([0, T], \Omega)$ satisfying (5.6) for every $t \in [0, T]$. Furthermore, by (1) of Lemma 5.5, we see that $p_{\Omega\lambda}[\partial_t^M \mathbf{z}(0)] = 0$ in Ω and $p_\Gamma[\partial_t^M \mathbf{z}(0)] = 0$ on Γ for $0 \leqslant M \leqslant K-2$. Hence, by (5.3), we have that $\partial_t^M \mathbf{z}(0) = 0$ for $0 \leqslant M \leqslant K-2$.

Finally, we prove (5.7). Differentiating (5.6) M-times in t and applying (5.4) with $L = K - M$, we have

$$\|\partial_t^M \mathbf{z}(t)\|_{K-M} \leqslant C_4(K, \mathbb{B})\{\|\partial_t^M \mathbf{g}_\Omega(t)\|_{K-2-M} + \langle\!\langle \partial_t^M \mathbf{g}_\Gamma(t) \rangle\!\rangle_{K-(3/2)-M}\} \tag{5.14}$$

for $t \in [0, T]$ and $0 \leqslant M \leqslant K-2$, where we have used the facts that the present γ_K and λ depend on K and $\mathbb{B}$ only. Combining (5.14) and (5.11), we have that $|\mathbf{z}|_{K-2,2,[0,T]} \leqslant C_4(K, \mathbb{B})\{C_1(K, \mathbb{B}, \Lambda_H) + C_2(K, \mathbb{B}, \Lambda_H) T\Lambda_E + C_1(K, \mathbb{B}, \Lambda_H, \Lambda_E)\varepsilon_E\}$. If we choose Λ_E, ε_E, and T so that

(As.5) $\Lambda_E = C_4(K, \mathbb{B})\{C_1(K, \mathbb{B}, \Lambda_H) + C_2(K, \mathbb{B}, \Lambda_H) + 1\}$;

(As.6) $C_1(K, \mathbb{B}, \Lambda_H, \Lambda_E)\varepsilon_E \leqslant 1$;

(As.7) $T\Lambda_E \leqslant \varepsilon_E \leqslant 1$,

then we have $|\mathbf{z}|_{K-2,2,[0,T]} \leqslant \Lambda_E$. Since $\partial_t^M \mathbf{z}(t) = \int_0^t \partial_s^{M+1} \mathbf{z}(s)\, ds$ for $0 \leqslant M \leqslant K-3$, we have $\|\partial_t^M \mathbf{z}(t)\|_{K-1-M} \leqslant \int_0^t \|\partial_s^{M+1} \mathbf{z}(s)\|_{K-1-M}\, ds$. From this, it follows that $|\mathbf{z}|_{K-3,2,[0,T]} \leqslant T|\mathbf{z}|_{K-2,2,[0,T]} \leqslant T\Lambda_E \leqslant \varepsilon_E$ (cf. (As.7)). Hence, we have (5.7), which completes the proof of Theorem 5.3.

6. Preparations for Solving Problem $(\mathrm{H})_p$

First of all, we give a unique existence theorem and energy inequalities of solutions to a mixed problem corresponding to $(\mathrm{H})_p$. Let us consider the equations:

$$R_\Omega(t)[\mathbf{v}(t)] = \partial_t^2 \mathbf{v}(t) - \partial_i (R^{i0}(t)\, \partial_t \mathbf{v}(t) + R^{ij}(t)\, \partial_j \mathbf{v}(t))$$
$$= \mathbf{h}_\Omega(t) \qquad \text{in} \quad (0, T) \times \Omega;$$

$$R_\Gamma(t)[\mathbf{v}(t)] = v_i R^{ij}(t)\,\partial_j \mathbf{v}(t) + R^i(t)\,\partial_i \mathbf{v}(t) + R^0(t)\,\partial_t \mathbf{v}(t) \tag{6.1}$$
$$= \mathbf{h}_\Gamma(t) \qquad \text{on} \quad (0, T)\times\Gamma;$$
$$\mathbf{v}(0) = \mathbf{v}_0 \text{ and } (\partial_t \mathbf{v})(0) = \mathbf{v}_1 \qquad \text{in } \Omega.$$

Here, $R^{ik}(t) = R^{ik}(t, x)$ and $R^k(t) = R^k(t, x)$ $(k = 0, 1, ..., n)$ are $m\times m$ matrices of functions satisfying the following conditions (A.6.1)–(A.6.5):

(A.6.1) The R^{ik} and R^k are decomposed as follows: $R^{ik} = R^{ik}_\infty + R^{ik}_S$ and $R^k = R^k_\infty + R^k_S$ where R^{ik}_∞ and $R^k_\infty \in \mathscr{B}^{K-1}([-T_1, T_1]\times\bar{\Omega})$ and R^{ik}_S and $R^k_S \in Y^{K-2,1}([-T_1, T_1], \Omega)$ with some $T_1 \in (T, T_0]$.

(A.6.2) ${}^tR^{i0} = R^{i0}$ and ${}^tR^{ij} = R^{ji}$ on $[-T_1, T_1]\times\bar{\Omega}$; ${}^tR^0 = R^0$ and ${}^tR^i + R^i = 0$ on $[-T_1, T_1]\times\Gamma$.

(A.6.3) $(R^{ij}(t)\,\partial_j \mathbf{w}, \partial_i \mathbf{w}) + \langle R^i(t)\,\partial_i \mathbf{w}, \mathbf{w}\rangle \geqslant \delta_1 \|\mathbf{w}\|_1^2 - \delta_0 \|\mathbf{w}\|_0^2$ for $\mathbf{w}\in H^2(\Omega)$ and $t\in[-T_1, T_1]$.

(A.6.4) $v_i(x)\,R^i(t, x) = 0$ for $(t, x)\in[-T_1, T_1]\times\Gamma$.

(A.6.5) $(-v_i(x)\,R^{i0}(t, x) + 2R^0(t, x))\,\xi\cdot\xi \geqslant 0$ for $(t, x)\in[-T_1, T_1]\times\Gamma$ and $\xi\in\mathbb{R}^m$.

Following [8], let us define the energy norm $E(R(t))[\mathbf{v}(t)]$ by

$$E(R(t))[\mathbf{v}(t)] = \|\partial_t \mathbf{v}(t)\|_0^2 + (R^{ij}(t)\,\partial_j \mathbf{v}(t), \partial_i \mathbf{v}(t))$$
$$+ S_1(R_\Gamma(t))[\mathbf{v}(t), \mathbf{v}(t)] + d\,\|\mathbf{v}(t)\|_0^2. \tag{6.2}$$

Here, $S_1(R_\Gamma(t))$ is the bilinear form on $H^1(\Omega)\times H^1(\Omega)$ defined by (2.5a) with $R = R_\Gamma(t) = (R^1(t), ..., R^n(t))$ and d is a constant determined as follows. Let $S_2(R_\Gamma(t))$ be the bilinear form on $H^1(\Omega)\times L^2(\Omega)$ defined by (2.5b) with $R = R_\Gamma(t)$. Let $M(K, T_1)$ be a constant such that

$$\sum_{k=0}^{n}\left\{\sum_{i=1}^{n}(|R^{ik}_\infty|_{\infty,K-1,T_1} + |R^{ij}_S|_{K-2,1,[-T_1,T_1]})\right.$$
$$\left.+ |R^k_\infty|_{\infty,K-1,T_1} + |R^k_S|_{K-2,1,[-T_1,T_1]}\right\} \leqslant M(K, T_1). \tag{6.3}$$

By (2.3) and (A.6.3), we have

$$E(R(t))[\mathbf{v}(t)] + S_2(R_\Gamma(t))[\mathbf{v}(t), \mathbf{v}(t)]$$
$$\geqslant \|\partial_t \mathbf{v}(t)\|_0^2 + \delta_1\|\mathbf{v}(t)\|_1^2 + (d-\delta_0)\,\|\mathbf{v}(t)\|_0^2. \tag{6.4}$$

Thus, by (2.6b), Sobolev's imbedding theorem, and (6.3), we have

$$E(R(t))[\mathbf{v}(t)] \geqslant \|\partial_t \mathbf{v}(t)\|_0^2 + (\delta_1/2)\,\|\mathbf{v}(t)\|_1^2, \tag{6.5}$$

if we take $d=\delta_0+\{CM(K, T_1)\}^2/2\delta_1$ with some constant C. This is the manner of choosing the constant d.

To state an existence theorem of solutions to (6.1), we must define the compatibility condition for (6.1). Let $\mathbf{v}_{M+2}=\mathbf{v}_{M+2}(x)$ $(0\leqslant M\leqslant K-3)$ be functions defined successively by

$$\mathbf{v}_{M+2}=\partial_t^M \mathbf{h}_\Omega(0)+\sum_{k=0}^{M}\binom{M}{k}\partial_i\{(\partial_t^k R^{i0})(0)\,\mathbf{v}_{M+1-k}+(\partial_t^k R^{ij})(0)\,\partial_j \mathbf{v}_{M-k}\}. \tag{6.6}$$

If $\mathbf{v}(t)\in X^{K-1,0}([0,T],\Omega)$ is a solution to (6.1), obviously $\partial_t^M \mathbf{v}(0)=\mathbf{v}_M$ for $0\leqslant M\leqslant K-1$. And also, $\partial_t^M R_\Gamma(t)[\mathbf{v}(t)]|_{t=0}=(\partial_t^M \mathbf{h}_\Gamma)(0)$ on Γ for $0\leqslant M\leqslant K-3$. Hence, we shall say that $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{h}_\Omega(t)$, and $\mathbf{h}_\Gamma(t)$ satisfy the compatibility condition of order $K-3$ for (6.1) if

$$\sum_{k=0}^{M}\binom{M}{k}\{v_i(\partial_t^k R^{ij})(0)\,\partial_j \mathbf{v}_{M-k}+(\partial_t^k R^{i})(0)\,\partial_i \mathbf{v}_{M-k} + (\partial_t^k R^0)(0)\,\mathbf{v}_{M+1-k}\}=\partial_t^M \mathbf{h}_\Gamma(0) \quad \text{on } \Gamma \tag{6.7}$$

for $0\leqslant M\leqslant K-3$. The following theorem is a key of solving $(\mathrm{H})_p$.

THEOREM 6.1. *Assume that* (A.6.1)–(A.6.5) *are valid and let* $T\in(0,T_1)$. (1) *Let* $\mathbf{v}_0\in H^{K-1}(\Omega)$, $\mathbf{v}_1\in H^{K-2}(\Omega)$, $\mathbf{h}_\Omega(t)\in X^{K-3,0}([0,T],\Omega)$, *and* $\mathbf{h}_\Gamma(t)\in X^{K-3,1/2}([0,T],\Gamma)$. *Assume that*

$$\partial_t^{K-3}\mathbf{h}_\Omega(t)\in \mathrm{Lip}([0,T],L^2(\Omega)); \qquad \partial_t^{K-3}\mathbf{h}_\Gamma(t)\in \mathrm{Lip}([0,T],H^{1/2}(\Gamma)); \tag{6.8}$$

$\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{h}_\Omega(t)$, *and* $\mathbf{h}_\Gamma(t)$ *satisfy the compatibility condition of order* $K-3$ *for* (6.1). (6.9)

Then, (6.1) *admits a solution* $\mathbf{v}(t)\in X^{K-1,0}([0,T],\Omega)$ *satisfying the properties*: $\partial_t^M \mathbf{v}(0)=\mathbf{v}_M$ *for* $2\leqslant M\leqslant K-1$.

(2) *Let* $\mathbf{v}(t)\in X^{2,0}([0,T],\Omega)$ *and put* $\mathbf{h}_\Omega(t)=R_\Omega(t)[\mathbf{v}(t)]$ *and* $\mathbf{h}_\Gamma(t)=R_\Gamma(t)[\mathbf{v}(t)]$. *Then,*

$$\|D^1\mathbf{v}(t)\|_0^2\leqslant C\left\{\|(D^1\mathbf{v})(0)\|_0^2+\int_0^t(\|\mathbf{h}_\Omega(s)\|_0^2 + \langle\!\langle \mathbf{h}_\Gamma(s)\rangle\!\rangle_{1/2}^2)\,ds\right\} \quad \textit{for} \quad t\in[0,T], \tag{6.10}$$

where $C=C(T_1, M(K,T_1), \delta_0, \delta_1, n, m, \Gamma)$.

(3) *In addition to* (2), *we assume that* $\mathbf{v}(t) \in X^{K-1,0}([0, T], \Omega)$ *and that* $\mathbf{h}_\Omega(t)$ *and* $\mathbf{h}_\Gamma(t)$ *satisfy* (6.8). *Then,*

$$E(R(t))[\partial_t^{K-2}\mathbf{v}(t)] \leqslant e^{Ct}\{E(R(t))[\partial_t^{K-2}\mathbf{v}(t)]|_{t=0} + Ct^{1/2}F(t)\} \quad \text{for} \quad t \in [0, T], \tag{6.11}$$

where $C = C(T_1, M(K, T_1), \delta_0, \delta_1, n, m, \Gamma)$ *and*

$$F(t) = \|(D^{K-1}\mathbf{v})(0)\|_0^2 + |\mathbf{h}_\Omega|^2_{K-3,0,[0,t]} + \langle \mathbf{h}_\Gamma \rangle^2_{K-3,1/2,[0,t]} + \operatorname*{ess\,sup}_{0 \leqslant s \leqslant t} \|\partial_s^{K-2}\mathbf{h}_\Omega(s)\|_0^2 + \operatorname*{ess\,sup}_{0 \leqslant s \leqslant t} \langle\!\langle \partial_s^{K-2}\mathbf{h}_\Gamma(s) \rangle\!\rangle^2_{1/2}.$$

Remark. Theorem 6.1 was proved by the first auther [8]. In the proof of [8], essentially all the coefficients of the operators $R_\Omega(t)$ and $R_\Gamma(t)$ were defined for all $t \in [-T_1, T_1]$ containing $[0, T]$ strictly and (A.6.1)–(A.6.5) were valid for all $t \in [-T_1, T_1]$.

Our goal in this section is to prove

THEOREM 6.2. *Assume that* (A.1)–(A.5) *are valid. Let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *be the same as in Theorem* 1.1. *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$ *and put* $U(t) = (\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t)))$. *Let us consider the linear problem*:

$$\partial_t^2\mathbf{z}(t) - \partial_i(A^{i0}(t, \cdot, U(t))\,\partial_t\mathbf{z}(t) + A^{ij}(t, \cdot, U(t))\,\partial_j\mathbf{z}(t)) = \partial_t\mathbf{f}_\Omega(t) + \mathbf{F}_\Omega(t, U(t)) \quad in \quad (0, T) \times \Omega; \tag{6.12a}$$

$$\nu_i A^{ij}(t, \cdot, U(t))\,\partial_j\mathbf{z}(t) + B'_\Gamma(t, \cdot, U(t))\,\partial_t\mathbf{z}(t) + \bar{B}^0_\Gamma(t, \cdot, U(t))\,\partial_t\mathbf{z}(t) = \partial_t\mathbf{f}_\Gamma(t) - \mathbf{F}_\Gamma(t, U(t)) \quad on \quad (0, T) \times \Gamma; \tag{6.12b}$$

$$\mathbf{z}(0) = \mathbf{u}_1 \text{ and } (\partial_t\mathbf{z})(0) = \mathbf{u}_2 \quad in\ \Omega, \tag{6.12c}$$

where we have used the notations defined in (4.1). *Then, the following assertions are valid.* (1) *There exists a* $T_1 \in (0, T_0]$ *depending only on* K, $\mathbb{B}$, Λ_H, *and* Λ_E *such that for any* $T \in (0, T_1)$, (6.12) *admits a unique solution* $\mathbf{z}(t) \in X^{K-1,0}([0, T], \Omega)$ *satisfying the properties*:

$$(\partial_t^M\mathbf{z})(0) = \mathbf{u}_{M+1} \quad for \quad 0 \leqslant M \leqslant K-1. \tag{6.13}$$

(2) *If* $\mathbf{z}_l(t) \in X^{2,0}([0, T], \Omega)$ $(l = 1, 2)$ *satisfy* (6.12), *then* $\mathbf{z}_1(t) = \mathbf{z}_2(t)$ *for* $t \in [0, T]$.

(3) *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z_c$. *Then, there exist* T *and* Λ_H *depending only on* K and $\mathbb{B}$ *such that the solution* $\mathbf{z}(t)$ *to* (6.12) *satisfies the estimate*:

$$|\mathbf{z}|_{K-1,0,[0,T]} \leqslant \Lambda_H. \tag{6.14}$$

Now, we shall prove Theorem 6.2 by using Theorem 6.1. To do this, we must extend the coefficients of the operators in (6.12) to functions defined on a wider interval than $[0, T]$. As will be seen in Theorem Ap.6, there exist $\mathbf{V}(t) \in Y^{K-1,0}(\mathbb{R}, \Omega)$ and $\mathbf{W}(t) \in Y^{K-2,2}(\mathbb{R}, \Omega)$ such that

$$\mathbf{v}(t) = \mathbf{V}(t) \qquad \text{and} \qquad \mathbf{w}(t) = \mathbf{W}(t) \qquad \text{for} \quad t \in [0, T]; \tag{6.15}$$

$$\begin{aligned} |\mathbf{V}|_{K-1,0,\mathbb{R}} &\leqslant C(K) \left\{ |\mathbf{v}|_{K-1,0,[0,T]} + \sum_{L=0}^{K-2} \|(\partial_t^L \mathbf{v})(0)\|_{K-1-L} \right\} \\ &\leqslant C(K)\{\Lambda_H + C_1(K, \mathbb{B})\}; \end{aligned} \tag{6.16a}$$

$$\begin{aligned} |\mathbf{W}|_{K-2,2,\mathbb{R}} &\leqslant C(K) \left\{ |\mathbf{w}|_{K-2,2,[0,T]} + \sum_{L=0}^{K-3} \|(\partial_t^L \mathbf{w})(0)\|_{K-L} \right\} \\ &\leqslant C(K)\Lambda_E, \end{aligned} \tag{6.16b}$$

where we have used Lemma 3.1, (4.6a), and (4.7). Since we want to substitue $(\mathbf{V}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{W}(t)))$ into nonlinear functions defined on $\{|U| < U_0\}$, let us choose $T_1 > 0$ depending only on K, $\mathbb{B}$, Λ_H, and Λ_E such that

$$\|(\mathbf{V}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{W}(t)))\|_{\infty,1} < U_1 \quad (< U_0) \qquad \text{for} \quad t \in [-T_1, T_1], \tag{6.17}$$

where U_1 is the same as in (As.2). In fact, it suffices to choose T_1 as follows. In the same manner as in the arguments before (As.2), by (6.15), (6.16), (4.2), (4.6a), and (4.12), we have $\|(\mathbf{V}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{W}(t)))\|_{\infty,1} \leqslant U_2 + |t|^\varepsilon C_2(K, \mathbb{B}, \Lambda_H, \Lambda_E)$. Hence, if we choose $T_1 \in (0, T_0]$ so that

(As.8) $\quad U_2 + (T_1)^\varepsilon C_2(K, \mathbb{B}, \Lambda_H, \Lambda_E) < U_1 \quad (< U_0)$,

we have (6.17).

From now on, we use the followidng notations in the proof of Theorem 6.2:

$$\begin{aligned} &U(t) = (\mathbf{v}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}(t))); \\ &U'(t) = (\mathbf{V}(t), D_x^1(\mathbf{u}^0(t) + \mathbf{W}(t))); \\ &R^{ik}(t) = A^{ik}(t, \cdot, U'(t)); \\ &R^i(t) = B_\Gamma^i(t, \cdot, U'(t)); \qquad R^0(t) = \bar{B}_\Gamma^0(t, \cdot, U'(t)); \\ &\mathbf{v}_0 = \mathbf{u}_1; \qquad \mathbf{v}_1 = \mathbf{u}_2; \qquad \mathbf{h}_\Omega(t) = \partial_t \mathbf{f}_\Omega(t) + \mathbf{F}_\Omega(t, U(t)); \\ &\mathbf{h}_\Gamma(t) = \partial_t \mathbf{f}_\Gamma(t) + \mathbf{F}_\Gamma(t, U(t)) \end{aligned} \tag{6.18}$$

$(k = 0, 1, ..., n)$. Using these notations, we can describe (6.12) by (6.1).

Let us check the conditions (A.6.1)–(A.6.5).

LEMMA 6.3. *Assume that* (A.1)–(A.5) *are valid. Let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$ *and* $\mathbf{f}_\Gamma(t)$ *be the same as in Theorem* 1.1. *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$ *and let* $R^{ik}(t)$ *and* $R^k(t)$ $(k = 0, 1, ..., n)$ *be the same as in* (6.18). *Then, the present* $R^{ik}(t)$ *and* $R^k(t)$ *satisfy* (A.6.2)–(A.6.5). *Furthermore, if we put*

$$R^{ik}_\infty(t) = A^{ik}(t, \cdot, 0); \qquad R^{ik}_S(t) = (A^{ik})_1(t, \cdot, U'(t)); \qquad R^i_\infty(t) = B^i_\Gamma(t, \cdot, 0);$$
$$R^i_S(t) = (B^i_\Gamma)_1(t, \cdot, U'(t)); \qquad R^0_\infty(t) = v_i A^{i0}(t, \cdot, 0) + B^0_\Gamma(t, \cdot, 0); \qquad (6.19)$$
$$R^0_S(t) = v_i (A^{i0})_1(t, \cdot, U'(t)) + (B^0_\Gamma)_1(t, \cdot, U'(t)) \qquad (\text{cf. } (4.1\text{a})),$$

then (A.6.1) *is valid and*

$$\sum_{k=0}^{n} \left\{ \sum_{i=1}^{n} (|R^{ik}_\infty|_{K-1,\infty,T_1} + |R^{ik}_S|_{K-2,1,[-T_1,T_1]}) + |R^k_\infty|_{K-1,\infty,T_1} + |R^k_S|_{K-2,1,[-T_1,T_1]} \right\} \leqslant C_3(K, \mathbb{B}, \Lambda_H, \Lambda_E). \qquad (6.20)$$

Proof. Since (6.17) is valid, (A.6.l) follows from (A.l) for $l = 2, 3, 4$, and 5. Applying Theorem Ap.3 to (6.19), we have (6.20) and (A.6.1) easily, which completes the proof.

Now, we shall show that the present $\mathbf{v}_0, \mathbf{v}_1, \mathbf{h}_\Omega(t)$, and $\mathbf{h}_\Gamma(t)$ satisfy all the conditions in Theorem 6.1.

LEMMA 6.4. *Assume that* (A.1) *is valid. Let* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{f}_\Omega(t)$, *and* $\mathbf{f}_\Gamma(t)$ *be the same as in Theorem* 1.1. *Let* $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$ *and let* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{h}_\Omega(t)$, *and* $\mathbf{h}_\Gamma(t)$ *be the same as in* (6.18). *Then,* $\mathbf{v}_0 \in H^{K-1}(\Omega)$, $\mathbf{v}_1 \in H^{K-2}(\Omega)$, $\mathbf{h}_\Omega(t) \in X^{K-3,0}([0, T], \Omega)$, $\mathbf{h}_\Gamma(t) \in X^{K-3,1/2}([0, T], \Gamma)$ *and* (6.8) *and* (6.9) *are valid. Furthermore,*

$$\mathbf{v}_M = \mathbf{u}_{M+1} \qquad \textit{for} \quad 2 \leqslant M \leqslant K-1, \qquad (6.21)$$

where $\mathbf{v}_M$ *are the functions successively defined by* (6.6).

Proof. By (1.2) and Lemma 3.1, we know that $\mathbf{v}_0 = \mathbf{u}_1 \in H^{K-1}(\Omega)$ and $\mathbf{v}_1 = \mathbf{u}_2 \in H^{K-2}(\Omega)$. For notational simplicity, we write $Y^{L,M} = Y^{L,M}([0, T], \Omega)$. Let us prove that

$$\mathbf{F}_\Omega(t, U(t)) \in Y^{K-2,0} \qquad \text{and} \qquad \mathbf{F}_\Gamma(t, U(t)) \in Y^{K-2,1}. \qquad (6.22)$$

If we get (6.22), by Theorem Ap.4(1) and (1.3), we see that $\mathbf{h}_\Omega(t)$ and $\mathbf{h}_\Gamma(t)$ satisfy the desired properties except for (6.9). Recall the notations defined in (6.18) and (4.1). Since $U(t) \in Y^{K-2,1}$, applying (Ap.1) with $N = K-2$ and $M = 1$ and Theorem Ap.3, we have that $\mathbf{F}_{\Omega 1}(t, U(t)) \in Y^{K-2,0}$ and $\mathbf{F}_\Gamma(t, U(t)) \in Y^{K-2,1}$. Since $D^1\mathbf{v}(t) \in Y^{K-2,0}$, by (Ap.1) with $N = K-2$ and

$M=0$, we have also $\mathbf{F}_{\Omega 2}(t, U(t)) \in Y^{K-2,0}$. Combining these facts, we have (6.22).

Now, we shall prove (6.21) and (6.9). Put $\bar{R}^{ik}(t)=A^{ik}(t, \cdot, D^1\mathbf{u}^0(t))$, $\bar{R}^i(t)=B^i_\Gamma(t, \cdot, D^1\mathbf{u}^0(t))$, and $\bar{R}^0(t)=\nu_i A^{i0}(t, \cdot, D^1\mathbf{u}^0(t))+B^0_\Gamma(t, \cdot, D^1\mathbf{u}^0(t))$ $(k=0, 1, ..., n)$. By (1.1) and (4.1), we have

$$\partial_t\{\partial_i(\mathbf{P}^i(t, D^1\mathbf{u}^0(t)))-\mathbf{Q}_\Omega(t, D^1\mathbf{u}^0(t))\}$$
$$=\partial_i(\bar{R}^{i0}(t)\,\partial_t^2\mathbf{u}^0(t)+\bar{R}^{ij}(t)\,\partial_j\partial_t\mathbf{u}^0(t))-\mathbf{F}_\Omega(t, D^1\mathbf{u}^0(t)); \quad (6.23\text{a})$$

$$\partial_t\{\nu_i\mathbf{P}^i(t, D^1u^0(t))+\mathbf{Q}_\Gamma(t, D^1u^0(t))\}$$
$$=\nu_i\bar{R}^{ij}(t)\,\partial_j\partial_t\mathbf{u}^0(t)+\bar{R}^i(t)\,\partial_i\partial_t\mathbf{u}^0(t)+\bar{R}^0(t)\,\partial_t^2\mathbf{u}^0(t)+\mathbf{F}_\Gamma(t, D^1\mathbf{u}^0(t)). \quad (6.23\text{b})$$

On the other hand, since $\partial_t^M U'(0)=\partial_t^M U(0)=(\mathbf{u}_{M+1}, D_x^1\mathbf{u}_M)$ for $0\leqslant M\leqslant K-3$ as follows from (6.18), (6.15), (4.2), and (4.6a), we have

$$\begin{aligned}(\partial_t^M R^{ik})(0)&=(\partial_t^M\bar{R}^{ik})(0); \qquad (\partial_t^M R^k)(0)=(\partial_t^M\bar{R}^k)(0);\\ \partial_t^M\mathbf{F}_V(t, U(t))|_{t=0}&=\partial_t^M\mathbf{F}_V(t, D^1\mathbf{u}^0(t))|_{t=0}\end{aligned} \quad (6.24)$$

for $0\leqslant M\leqslant K-3$. Differentiating both sides of (6.23a) M times in t $(0\leqslant M\leqslant K-3)$, letting $t=0$, and using (6.6), (3.1), (3.3), and (6.24), we have (6.21) easily. Furthermore, differentiating both sides of (6.23b) M times in t $(0\leqslant M\leqslant K-3)$, letting $t=0$, and using (6.24), (6.21), (3.1), and (3.4), we see easily that (6.7) is valid, which implies that (6.9) is valid in the present case. This completes the proof.

In view of Lemmas 6.3 and 6.4, we can apply Theorem 6.1(1) for any $T\in(0, T_1)$. And then, we have Theorem 6.2(1). Since $Y^{K-1,0}([0, Y], \Omega)\subset X^{K-2,0}([0, T], \Omega)\subset X^{2,0}([0, T], \Omega)$ as follows from the fact that $K-2\geqslant[n/2]+1\geqslant 2$, Theorem 6.2(2) follows from Theorem 6.1(2) immediately.

Now, we shall prove (6.14). To do this, we shall prove that

$$\begin{aligned}|\mathbf{z}|^2_{K-1,0,[0,T]}\leqslant{}& C_5(K, \mathbb{B})+T^\varepsilon C_4(K, \mathbb{B}, \Lambda_H, \Lambda_E)\\ &+T^\varepsilon C_5(K, \mathbb{B}, \Lambda_H, \Lambda_E)\,|\mathbf{z}|^2_{K-1,0,[0,T]}.\end{aligned} \quad (6.25)$$

If we get (6.25), we choose T and Λ_H so that

(As.9) $T^\varepsilon C_4(K, \mathbb{B}, \Lambda_H, \Lambda_E)\leqslant 1$; $\quad T^\varepsilon C_5(K, \mathbb{B}, \Lambda_H, \Lambda_E)\leqslant\frac{1}{2}$;

(As.10) $(\Lambda_H)^2\geqslant 2\{C_5(K, \mathbb{B})+1\}$.

Then, we have (6.14).

Below, we assume that $(\mathbf{v}(t), \mathbf{w}(t)) \in Z_c$. Note that $M(K, T_1) = C_3(K, \mathbb{B}, \Lambda_H, \Lambda_E)$ in the present case (cf. (6.20), (6.3)) and that T_1 depends only on K, $\mathbb{B}$, Λ_H, and Λ_E (cf. (As.8)) in the present case. Applying (6.11) to (6.12), we have

$$\begin{aligned} E(R(t))[\partial_t^{K-2}\mathbf{z}(t)] \leqslant (\exp C_6 t)\{&E(R(t))[\partial_t^{K-2}\mathbf{z}(t)]|_{t=0} \\ &+ C_7 T^{1/2}(|\mathbf{F}_\Omega(\cdot, U)|^2_{K-2,0,[0,T]} \\ &+ |\mathbf{F}_\Gamma(\cdot, U)|^2_{K-2,1,[0,T]} + \mathbb{B}^2)\} \qquad \text{for} \quad t \in [0, T], \end{aligned} \tag{6.26}$$

where $C_l = C_l(K, \mathbb{B}, \Lambda_H, \Lambda_E)$ for $l = 6$ and 7 and we have used Theorem Ap.4(1). In the same way as in the proof of (6.22), by (Ap.1) and Theorem Ap.3, we see easily that

$$|\mathbf{F}_\Omega(\cdot, U)|^2_{K-2,0,[0,T]} + |\mathbf{F}_\Gamma(\cdot, U)|^2_{K-2,1,[0,T]} \leqslant C_8(K, \mathbb{B}, \Lambda_H, \Lambda_E). \tag{6.27}$$

Here and hereafter, we use the fact that

$$|U|_{K-2,1,[0,T]} \leqslant C_2(K, \mathbb{B}) + \Lambda_H + \Lambda_E \qquad \text{(cf. (4.2), (4.7)).} \tag{6.28a}$$

Thus, subsituting (6.27) into (6.26) and using (6.2) and (6.4), we have

$$\begin{aligned} &\|\partial_t^{K-1}\mathbf{z}(t)\|_0^2 + \delta_1 \|\partial_t^{K-2}\mathbf{z}(t)\|_1^2 \\ &\quad \leqslant \sum_{k=1}^{3} I_k(t) + \delta_0 \|\partial_t^{K-2}\mathbf{z}(t)\|_0^2 + (\exp C_6 T)\, C_7 T^{1/2}\{C_8 + \mathbb{B}^2\}, \end{aligned} \tag{6.29}$$

where

$$\begin{aligned} I_1(t) &= d\{(\exp C_6 t)\, \|\partial_t^{K-2}\mathbf{z}(0)\|_0^2 - \|\partial_t^{K-2}\mathbf{z}(t)\|_0^2\}; \\ I_2(t) &= S_2(R_\Gamma(t))[\partial_t^{K-2}\mathbf{z}(t), \partial_t^{K-2}\mathbf{z}(t)]; \\ I_3(t) &= (\exp C_6 t)\{(R^{ij}(0)\, \partial_j \partial_t^{K-2}\mathbf{z}(t), \partial_i \partial_t^{K-2}\mathbf{z}(0)) \\ &\qquad + S_1(R_\Gamma(0))[\partial_t^{K-2}\mathbf{z}(0), \partial_t^{K-2}\mathbf{z}(0)]\}; \\ d &= \delta_0 + \{CM(K, T_1)\}^2/2\delta_1 \\ &= C_9(K, \mathbb{B}, \Lambda_H, \Lambda_E) \qquad \text{(cf. (6.3), (6.5), (6.20))}; \\ C_l &= C_l(K, \mathbb{B}, \Lambda_H, \Lambda_E) \qquad \text{for} \quad l = 6, 7, \text{and } 8. \end{aligned}$$

We shall estimate $I_1(t)$ and $\|\partial_t^{K-2}\mathbf{z}(t)\|_0^2$. First, note that

$$\|(D^{K-1}\mathbf{z})(0)\|_0^2 \leqslant C_6(K, \mathbb{B}) \qquad \text{(cf. (6.13) and Lemma 3.1).} \tag{6.28b}$$

Since $(\exp C_6 t) - 1 \leqslant C_6 t(\exp C_6 t)$ and since

$$\begin{aligned} &| \, \|\partial_t^{K-2}\mathbf{z}(t)\|_0^2 - \|(\partial_t^{K-2}\mathbf{z})(0)\|_0^2 \, | \\ &\leqslant \left| \int_0^t \frac{d}{ds} \|\partial_s^{K-2}\mathbf{z}(s)\|_0^2 \, ds \right| \leqslant T \, |\mathbf{z}|_{K-1,0,[0,T]}^2, \end{aligned} \tag{6.30}$$

we have easily that $I_1(t) \leqslant TC_6 C_9 (\exp C_6 T) \, C_6(K, \mathbb{B}) + C_9 T \, |\mathbf{z}|_{K-1,0,[0,T]}^2$. Hence, if we choose T so that

(As.11) $C_6(K, \mathbb{B}, \Lambda_H, \Lambda_E) \, C_9(K, \mathbb{B}, \Lambda_H, \Lambda_E) T \leqslant 1$; $C_6(K, \mathbb{B}, \Lambda_H, \Lambda_E) T \leqslant 1$,

we have

$$I_1(t) \leqslant M_1 + M_2 T \, |\mathbf{z}|_{K-1,0,[0,T]}^2. \tag{6.31}$$

Here and hereafter, for notational simplicity, we use the letter M_1 (resp. M_2) to denote various constants depending only on K and $\mathbb{B}$ (resp. K, $\mathbb{B}$, Λ_H, and Λ_E). In a similar manner to the proof of (6.30), by (6.28b) we see that

$$|\mathbf{z}|_{K-2,0,[0,T]}^2 \leqslant M_1 + T |\mathbf{z}|_{K-1,0,[0,T]}^2. \tag{6.28c}$$

Now, we evaluate $I_k(t)$ $(k = 2, 3)$. Note (2.6b) and the fact that $R^i(t) = B_\Gamma^i(t, \cdot, U(t))$ in the present case. Since $\|R^i(t) - R^i(0)\|_{\infty,1} \leqslant M_2 \{T + T^\varepsilon\}$ as follows from Theorem Ap.7 and since $\|R^i(0)\|_{\infty,1} = \|B_\Gamma^i(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\|_{\infty,1} \leqslant M_1$, we have that $\|R^i(t)\|_{\infty,1} \leqslant M_1 + M_2 T^\varepsilon$. Here and hereafter, we use the fact that $T^\varepsilon \geqslant T^p$ for any $p \geqslant \varepsilon$ (because $0 < T < 1$). Substituting this estimate into (2.6b) and using (6.28c), we have

$$I_2(t) \leqslant (\delta_1/2) \, \|\partial_t^{K-2}\mathbf{z}(t)\|_1^2 + M_1 + T^{2\varepsilon} M_2 \, |\mathbf{z}|_{K-1,0,[0,T]}^2 + T^{2\varepsilon} M_2. \tag{6.32}$$

Since $R^{ij}(0) = A^{ij}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)$ and $R^i(0) = B_\Gamma^i(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)$, by (1.5), (2.6a), Schwarz's inequality, (As.11), and (6.28b), we have

$$I_3(t) \leqslant M_1. \tag{6.33}$$

Since, without loss of generality, we may assume that $0 < \varepsilon < \frac{1}{2}$, combining (6.29), (As.11), (6.31), (6.28c), (6.32), and (6.33), we have

$$\begin{aligned} &\|\partial_t^{K-1}\mathbf{z}(t)\|_0^2 + (\delta_1/2) \, \|\partial_t^{K-2}\mathbf{z}(t)\|_1^2 \\ &\leqslant M_1 + T^\varepsilon M_2 + T^\varepsilon M_2 \, |\mathbf{z}|_{K-1,0,[0,T]}^2. \end{aligned} \tag{6.34}$$

Now, we shall evaluate $\|\partial_t^M \mathbf{z}(t)\|_{K-1-M}$ for $0 \leqslant M \leqslant K-3$ by using (5.4). To do this, we rewrite (6.12) as

$$-\partial_i(A^{ij}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\, \partial_j \mathbf{z}(t)) + \mu \mathbf{z}(t) = \partial_t \mathbf{f}_\Omega(t) - \partial_t^2 \mathbf{z}(t) + \mu \mathbf{z}(t) + \mathbf{H}_\Omega(t) \qquad \text{in } \Omega, \tag{6.35a}$$

$$\nu_i A^{ij}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\, \partial_j \mathbf{z}(t) + B_\Gamma^i(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\, \partial_i \mathbf{z}(t) = \partial_t \mathbf{f}_\Gamma(t) + \mathbf{H}_\Gamma(t) \qquad \text{on } \Gamma, \tag{6.35b}$$

for every $t \in [0, T]$ where μ is a constant determined below;

$$\mathbf{H}_\Omega(t) = \mathbf{F}_\Omega(t, U(t)) + \mathbf{H}_{\Omega 1}(t) + \mathbf{H}_{\Omega 2}(t);$$

$$\mathbf{H}_{\Omega 1}(t) = \partial_i(A^{i0}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\, \partial_t \mathbf{z}(t));$$

$$\mathbf{H}_{\Omega 2}(t) = \sum_{k=0}^{n} \partial_i((A^{ik}(t, \cdot, U(t)) - A^{ik}(0, \cdot, U(0)))\, \partial_k \mathbf{z}(t))$$

$$(\partial_0 = \partial_t, \qquad U(0) = (\mathbf{u}_1, D_x^1 \mathbf{u}_0));$$

$$\mathbf{H}_\Gamma(t) = \mathbf{F}_\Gamma(t, U(t)) + \mathbf{H}_{\Gamma 1}(t) + \mathbf{H}_{\Gamma 2}(t);$$

$$\mathbf{H}_{\Gamma 1}(t) = \{\nu_i A^{i0}(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0) + B_\Gamma^0(0, \cdot, \mathbf{u}_1, D_x^1 \mathbf{u}_0)\}\, \partial_t \mathbf{z}(t);$$

$$\mathbf{H}_{\Gamma 2}(t) = \sum_{k=0}^{n} \{\nu_i (A^{ik}(t, \cdot, U(t)) - A^{ik}(0, \cdot, U(0))) + B_\Gamma^k(t, \cdot, U(t)) - B_\Gamma^k(0, \cdot, U(0))\}\, \partial_k \mathbf{z}(t).$$

If we define q^{ij} and q_Γ^i by the same formulas as in (5.9) and $q_\Omega^k = 0$ $(k = 1, ..., n+1)$, $qi = q_\Gamma^{n+1} = 0$, in the present case (6.35) is also described by (5.1). By Theorem 5.1 and Lemma 5.4, we see that there exists a μ depending only on K and $\mathbb{B}$ such that (5.14) is also valid in the present case. Thus, we have

$$\begin{aligned}\|\partial_t^M \mathbf{z}(t)\|_{K-1-M} \leqslant M_1 \{ &\|\partial_t^{M+1} \mathbf{f}_\Omega(t)\|_{K-3-M} + \langle\!\langle \partial_t^{M+1} \mathbf{f}_\Gamma(t) \rangle\!\rangle_{K-(5/2)-M} \\ &+ \|\partial_t^{M+2} \mathbf{z}(t)\|_{K-3-M} + \|\partial_t^M \mathbf{z}(t)\|_{K-3-M} \\ &+ \|\partial_t^M \mathbf{F}_\Omega(t, U(t))\|_{K-3-M} \\ &+ \|\partial_t^M \mathbf{F}_\Gamma(t, U(t))\|_{K-2-M} + \sum_{k=1}^{2} (\|\partial_t^M \mathbf{H}_{\Omega k}(t)\|_{K-3-M} \\ &+ \|\partial_t^M \mathbf{H}_{\Gamma k}(t)\|_{K-2-M})\},\end{aligned} \tag{6.36}$$

where we have used Theorem Ap.4(1). The right-hand side of (6.35) is estimated as follows:

$$\|\partial_t^M \mathbf{F}_\Omega(t, U(t))\|_{K-3-M} \leqslant M_1 + TM_2;$$
$$\|\partial_t^M \mathbf{F}_\Gamma(t, U(t))\|_{K-2-M} \leqslant M_1 + TM_2; \tag{6.37a}$$
$$\|\partial_t^M \mathbf{H}_{\Omega 1}(t)\|_{K-3-M} \leqslant M_1 \|\partial_t^{M+1}\mathbf{z}(t)\|_{K-2-M};$$
$$\|\partial_t^M \mathbf{H}_{\Gamma 1}(t)\|_{K-2-M} \leqslant M_1 \|\partial_t^{M+1}\mathbf{z}(t)\|_{K-2-M}; \tag{6.37b}$$
$$\|\partial_t^M \mathbf{H}_{\Omega 2}(t)\|_{K-3-M} \leqslant M_1 + TM_2 |\mathbf{z}|_{K-1,0,[0,T]};$$
$$\|\partial_t^2 \mathbf{H}_{\Gamma 2}(t)\|_{K-2-M} \leqslant M_1 + TM_2 |\mathbf{z}|_{K-1,0,[0,T]}. \tag{6.37c}$$

In fact, note that

$$\|\partial_t^M \mathbf{F}_\Omega(t)\|_{K-3-M} \leqslant \|(\partial_t^M \mathbf{F}_\Omega)(0)\|_{K-3-M} + \int_0^t \|\partial_s^{M+1}\mathbf{F}_\Omega(s)\|_{K-3-M}\, ds.$$

By (Ap.1) we see easily that $\|(\partial_t^M \mathbf{F}_\Omega)(0)\|_{K-3-M} = \|\partial_t^M \mathbf{F}_\Omega(t)\|_{K-3-M}|_{t=0} \leqslant M_1$. Thus, by (6.27) we have the first part of (6.37a). In the same way, we have the second part of (6.37a). Let $F = (A'^0)_1$ or $(B_\Gamma^0)_1$. Applying Theorem Ap.1 with $k=2, S=K-2, L=K-2-M, r_1=0$, and $r_2=M$, and using Theorem Ap.3, we have

$$\|F(0, \cdot, \mathbf{u}_1, D_x^1\mathbf{u}_0)\, \partial_t^{M+1}\mathbf{z}(t)\|_{K-2-M}$$
$$\leqslant C(K, \|\mathbf{u}_1\|_{K-2}, \|D_x^1\mathbf{u}_0\|_{K-2})\, \|\partial_t^{M+1}\mathbf{z}(t)\|_{K-2-M}.$$

From this, (6.37b) follows immediately. Note (6.28c) and the fact that $\|(D^{K-2}U)(0)\|_0 \leqslant M_1$. Applying (Ap.3) with $u=U$ and $v=\partial_k \mathbf{z}$ and using (6.28a) we have (6.37c).

Substituting (6.37) into (6.36) and using (6.28c), we have

$$\|\partial_t^M \mathbf{z}(t)\|_{K-1-M} \leqslant M_1 + M_1\{\|\partial_t^{M+2}\mathbf{z}(t)\|_{K-3-M} + \|\partial_t^{M+1}\mathbf{z}(t)\|_{K-2-M}\}$$
$$+ TM_2 + TM_2 |\mathbf{z}|_{K-1,0,[0,T]} \qquad \text{for} \quad 0 \leqslant M \leqslant K-3.$$

Repeated use of (6.38) implies that

$$\sum_{M=0}^{K-1} \|\partial_t^M \mathbf{z}(t)\|_{K-1-M}^2 \leqslant M_1 + M_1\{\|\partial_t^{K-1}\mathbf{z}(t)\|_0^2 + \|\partial_t^{K-2}\mathbf{z}(t)\|_1^2\}$$
$$+ T^2M_2 + T^2M_2 |\mathbf{z}|_{K-1,0,[0,T]}^2. \tag{6.39}$$

Substituting (6.34) into (6.39), we have

$$\sum_{M=0}^{K-1} |\partial_t^M \mathbf{z}|_{0,K-1-M,[0,T]}^2 \leqslant M_1 + T^\varepsilon M_2 + T^\varepsilon M_2 |z|_{K-1,0,[0,T]}^2. \tag{6.40}$$

Since $\mathbf{z} \in X^{K-1,0}([0, T], \Omega)$, we have

$$|\mathbf{z}|^2_{K-1,0,[0,T]} = \sum_{M=0}^{K-1} |\partial_t^M \mathbf{z}|^2_{0,K-1-M,[0,T]} + |\mathbf{z}|^2_{K-2,0,[0,T]}.$$

Hence, recalling that $M_1 = C(K, \mathbb{B})$ and $M_2 = C(K, \mathbb{B}, \Lambda_H, \Lambda_E)$ and combining (6.40) and (6.28c), we have (6.25), which completes the proof of Theorem 6.2.

7. A Proof of Theorem 1.1

First of all, we review the way of determining Λ_H, Λ_E, ε_E, T_1, and T. First, we choose $\Lambda_H > 0$ so that (As.3) and (As.10) are valid. Hence, Λ_H depends only on K and $\mathbb{B}$. Second, Λ_E is chosen so that (As.5) is valid. Hence, Λ_E also depends on K and $\mathbb{B}$ only. Third, T_1 is chosen so that (As.8) is valid. Obviously, T_1 depends on K and $\mathbb{B}$ only. Because, Λ_H and Λ_E have been chosen so that they depend on K and $\mathbb{B}$ only. Finally, ε_E and T are chosen so that $0 < T < T_1$ and (As.1), (As.2), (As.4), (As.6), (As.7), (As.9), and (As.11) are valid. Obviously, ε_E and T depend only on K and $\mathbb{B}$.

Now, we shall prove (4.10) and (4.11). Noting the discussions in the final part of Section 4 (cf. (As.2)) and using Theorems 5.3 and 6.2, we have that there exists a pair $(\mathbf{v}^p(t), \mathbf{w}^p(t)) \in Z_c$ satisfying $(\mathrm{H})_p$ and $(\mathrm{E})_p$, which shows (4.10).

Now, we shall prove that there exist T and ε_E depending only on K and $\mathbb{B}$ such that (4.11) is valid. Below, λ will always refer to the number determined in Theorem 5.3 and for notational simplicity, we use the same letter M to denote various constants depending on K, $\mathbb{B}$, Λ_H, and Λ_E, except for determining T and ε_E. Since we already know that Λ_H and Λ_E depend on K and $\mathbb{B}$ only, note that M also depends on K and $\mathbb{B}$ only. In the course of the proof of (4.11), we use the following notations:

$$U^p(t) = (\mathbf{v}^p(t), D_x^1(\mathbf{u}^0(t) + \mathbf{w}^p(t)));$$

$$\mathbf{z}^{p,p-1}(t) = \mathbf{z}^p(t) - \mathbf{z}^{p-1}(t) \qquad \text{for} \quad \mathbf{z} = \mathbf{v} \text{ and } \mathbf{w}.$$

First, by using (6.10) we shall prove that

$$|\mathbf{v}^{p,p-1}|_{1,0,[0,T]} \leqslant MT\{|\mathbf{v}^{p-1,p-2}|_{1,0,[0,T]} + |\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]}\}. \quad (7.1)$$

By $(\mathrm{H})_p$ and $(\mathrm{H})_{p-1}$, we have

$$\partial_t^2 \mathbf{v}^{p,p-1}(t) - \partial_i\left(\sum_{k=0}^{n} A^{ik}(t, \cdot, U^{p-1}(t))\, \partial_k \mathbf{v}^{p,p-1}(t)\right)$$
$$= \mathbf{h}_\Omega^p(t) \quad \text{in} \quad (0, T)\times\Omega; \tag{7.2a}$$
$$\nu_i A^{ij}(t, \cdot, U^{p-1}(t))\, \partial_j \mathbf{v}^{p,p-1}(t) + B_\Gamma^i(t, \cdot, U^{p-1}(t))\, \partial_i \mathbf{v}^{p,p-1}(t)$$
$$+ \bar{B}_\Gamma^0(t, \cdot, U^{p-1}(t))\, \partial_t \mathbf{v}^{p,p-1}(t) = \mathbf{h}_\Gamma^p(t) \quad \text{on} \quad (0, T)\times\Gamma; \tag{7.2b}$$
$$\mathbf{v}^{p,p-1}(0) = \partial_t \mathbf{v}^{p,p-1}(0) = 0 \quad \text{in } \Omega, \tag{7.2c}$$

where

$$\mathbf{h}_\Omega^p(t) = \mathbf{F}_\Omega(t, U^{p-1}(t)) - \mathbf{F}_\Omega(t, U^{p-2}(t))$$
$$+ \sum_{k=0}^{n} [(A^{ik}(t, \cdot, U^{p-1}(t)) - A^{ik}(t, \cdot, U^{p-2}(t)))\, \partial_k \mathbf{v}^{p-1}(t)];$$
$$\mathbf{h}_\Gamma^p(t) = \mathbf{F}_\Gamma(t, U^{p-1}(t)) - \mathbf{F}_\Gamma(t, U^{p-2}(t))$$
$$+ \sum_{k=0}^{n} [\nu_i(A^{ik}(t, \cdot, U^{p-1}(t)) - A^{ik}(t, \cdot, U^{p-2}(t)))$$
$$+ (B_\Gamma^k(t, \cdot, U^{p-1}(t)) - B_\Gamma^k(t, \cdot, U^{p-2}(t)))]\, \partial_k \mathbf{v}^{p-1}(t).$$

Extending the coefficients of the operators in (7.2a) and (7.2b) to the functions defined on $[-T_1, T_1]$ in the same way as in the proof of Theorem 6.2 (cf. (6.15), (6.16), (6.17), (6.18)) and applying (6.10) of Theorem 6.1(2), we have

$$|\mathbf{v}^{p,p-1}|_{1,0,[0,T]} \leqslant MT\{|\mathbf{h}_\Omega^p|_{0,0,[0,T]} + |\mathbf{h}_\Gamma^p|_{0,1,[0,T]}\}, \tag{7.3}$$

where we have used the fact that $\langle \mathbf{h}_\Gamma^p \rangle_{0,1/2,[0,T]} \leqslant C|\mathbf{h}_\Gamma^p|_{0,1,[0,T]}$ (cf. Theorem Ap.4(1)). On the other hand, applying (Ap.5) and (Ap.6) with $u_l = U^{p-l}(t)$ and $v_l = \mathbf{v}^{p-l}(t)$ and so on to the first terms of $\mathbf{h}_V^p(t)$, and (Ap.6) with $N=1$, $u_l = U^{p-l}(t)$ $(l=1, 2)$, and $v_1 = v_2 = \partial_k \mathbf{v}^{p-1}(t)$ to the second terms of $\mathbf{h}_V^p(t)$, we have easily that

$$|\mathbf{h}_\Omega^p|_{0,0,[0,T]} + |\mathbf{h}_\Gamma^p|_{0,1,[0,T]}$$
$$\leqslant M\{|\mathbf{v}^{p-1,p-2}|_{1,0,[0,T]} + |\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]}\}. \tag{7.4}$$

Combining (7.3) and (7.4) implies (7.1).

Now, we shall prove that

$$|\mathbf{w}^{p,p-1}|_{0,2,[0,T]} \leqslant M\{|\mathbf{v}^{p,p-1}|_{1,0,[0,T]} + (T+\varepsilon_E)|\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]}\}. \tag{7.5}$$

Considering the equations which $\mathbf{w}^{p,p-1}(t)$ satisfies and applying (5.4) with $L=2$, we see easily that

$$\|\mathbf{w}^{p,p-1}(t)\|_2 \leqslant M \sum_{k=1}^{3} \{I_{\Omega k}(t) + I_{\Gamma k}(t)\}, \tag{7.7}$$

where

$$I_{V1}(t) = \|\mathbf{G}_{V1}(t, \mathbf{v}^p(t)) - \mathbf{G}_{V1}(t, \mathbf{v}^{p-1}(t))\|_{J(V)};$$
$$I_{Vl}(t) = \|\mathbf{G}_{Vl}(t, \mathbf{v}^p(t), \mathbf{w}^{p-1}(t)) - \mathbf{G}_{Vl}(t, \mathbf{v}^{p-1}(t), \mathbf{w}^{p-2}(t))\|_{J(V)} \quad \text{for} \quad l=2 \text{ and } 3.$$

Here, we have put $J(V)=0$ for $V=\Omega$ and $=1$ for $V=\Gamma$, and we have used Theorem Ap.4(1). We shall prove that for all $t \in [0, T]$

$$I_{V1}(t) \leqslant M|\mathbf{v}^{p,p-1}|_{0,1,[0,T]}; \tag{7.7a}$$
$$I_{V2}(t) \leqslant M\{T|\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]} + \varepsilon_E |\mathbf{v}^{p,p-1}|_{0,1,[0,T]}\}; \tag{7.7b}$$
$$I_{V3}(t) \leqslant M\varepsilon_E\{|\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]} + |\mathbf{v}^{p,p-1}|_{1,0,[0,T]}\}. \tag{7.7c}$$

If we get (7.7), substituting (7.7) into (7.6), we have (7.5). Since $\mathbf{P}'(t, x, 0) = \mathbf{Q}_V(t, x, 0) = 0$, applying (Ap.5) and recalling that λ depends on K and $\mathbb{B}$ only, we have (7.7a). Since $|D_x^1 \mathbf{w}^{p-2}|_{0,K-2,[0,T]} \leqslant |\mathbf{w}^{p-2}|_{K-3,2,[0,T]} \leqslant \varepsilon_E$ as follows from (4.7), by (Ap.7) we have easily (7.7b). Applying (Ap.8) with $u_l = (\mathbf{v}^{p+1-l}(t), D_x^1(\mathbf{u}^0(t) + \theta \mathbf{w}^{p-l}(t)))$ and $v_l = D_x^1 \mathbf{w}^{p-l}(t)$ $(l=1, 2)$ and $\Delta = \varepsilon_E$ to (5.12), we have (7.7c).

Combining (7.1) and (7.5), we have

$$|\mathbf{v}^{p,p-1}|_{1,0,[0,T]} + |\mathbf{w}^{p,p-1}|_{0,2,[0,T]} \leqslant \{C_{10} T |\mathbf{v}^{p-1,p-2}|_{1,0,[0,T]} + C_{11}(T+\varepsilon_E) |\mathbf{w}^{p-1,p-2}|_{0,2,[0,T]}\},$$

where $C_l = C_l(K, \mathbb{B}, \Lambda_H, \Lambda_E)$ for $l=10$ and 11. If we choose T and ε_E so that

(As.12) $\quad C_{10}(K, \mathbb{B}, \Lambda_H, \Lambda_E)\, T \leqslant \frac{1}{2},\ C_{11}(K, \mathbb{B}, \Lambda_H, \Lambda_E)(T+\varepsilon_E) \leqslant \frac{1}{2},$

then we have (4.11).

Now, using (4.10) and (4.11), we shall prove the existence of a pair $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$ satisfying (H) and (E)'. By (4.11), we see easily that the sequences $\{\mathbf{v}^p\}$ and $\{\mathbf{w}^p\}$ are Cauchy in $X^{1,0}([0, T], \Omega)$ and $X^{0,2}([0, T], \Omega)$, respectively. We can prove that these sequences are Cauchy in $X^{K-2,0}([0, T], \Omega)$ and $X^{K-3,2}([0, T], \Omega)$, respectively, by (4.10) and the following lemma.

LEMMA 7.1 (cf. [6, Lemma 2.2.7]). *If* $u \in X^{N,0}([0, T], \Omega)$ *(N being an integer* $\geqslant 1$*), then*

$$|D^M u|_{0,0,[0,T]} \leqslant C(N, T)\, |u|_{0,0,[0,T]}^{1-(M/N)}\, |u|_{N,0,[0,T]}^{(M/N)}$$

for any integer $M \in [0, N]$.

In fact, by Lemma 7.1 and (4.7), we have

$$|D^M D_x^2(\mathbf{w}^p - \mathbf{w}^{p'})|_{0,0,[0,T]} \leqslant C|D_x^2(\mathbf{w}^p - \mathbf{w}^{p'})|_{0,0,[0,T]}^{1-(M/K-2)}(2A_E)^{(M/K-2)} \qquad \text{for} \quad 0 \leqslant M \leqslant K-3.$$

Since $\{\mathbf{w}^p\}$ is Cauchy in $X^{0,2}([0, T], \Omega)$, $\{\mathbf{w}^p\}$ is also Cauchy in $X^{K-3,2}([0, T], \Omega)$. In the same manner, we see that $\{\mathbf{v}^p\}$is Cauchy in $X^{K-2,0}([0, T], \Omega)$. As a result, there exist $\mathbf{v}(t) \in X^{K-2,0}([0, T], \Omega)$ and $\mathbf{w}(t) \in X^{K-3,2}([0, T], \Omega)$ such that

$$\lim_{p \to \infty} |\mathbf{v}^p - \mathbf{v}|_{K-2,0,[0,T]} = \lim_{p \to \infty} |\mathbf{w}^p - \mathbf{w}|_{K-3,2,[0,T]} = 0. \tag{7.8}$$

Our next task is to prove that the present pair $(\mathbf{v}(t), \mathbf{w}(t))$ belongs to Z. To see this, we need

LEMMA 7.2. *Let J be a compact interval of* $\mathbb{R}$. *Let L and M be integers such that* $L \geqslant 1$ *and* $M \geqslant 0$. *Let the sequence* $\{v^p\}$ *be bounded in* $Y^{L,M}(J, \Omega)$ *and Cauchy in* $X^{L-1,M}(J, \Omega)$. *Let* A *and* v *be a number and an element in* $X^{L-1,M}(J, \Omega)$ *such that*

$$\lim_{p \to \infty} |v^p - v|_{L-1,M,J} = 0 \qquad \textit{and} \qquad |v^p|_{L,M,J} \leqslant A \qquad \textit{for all } p. \tag{7.9}$$

Then, $v \in Y^{L,M}(J, \Omega)$ *and* $|v|_{L,M,J} \leqslant A$.

Proof. In the same manner as in [4, p. 40] or [8, the proof of Lemma 5.4], we see that (7.9) implies that $\partial_t^k v^p(t) \to \partial_t^k v(t)$ weakly in $H^{L+M-k}(\Omega)$ as $p \to \infty$ for $0 \leqslant k \leqslant L-1$ and $t \in J$, and that $\partial_t^k v(t)$ is continuous in $t \in J$ in the weak topology of $H^{L+M-k}(\Omega)$ for $0 \leqslant k \leqslant L-1$. From these facts, it follows immediately that $\partial_t^k v(t) \in L^\infty(J, H^{L+M-k}(\Omega)) \cap \mathrm{Lip}(J, H^{L+M-k-1}(\Omega))$ for $0 \leqslant k \leqslant L-1$. Furthermore, we have

$$|v|_{L,M,J} \leqslant \liminf_{p \to \infty} |v^p|_{L,M,J}.$$

Hence, we have proved the lemma.

Since (4.7) is valid for every $\mathbf{v}^p(t)$ and $\mathbf{w}^p(t)$, by (7.8) and Lemma 7.2, we have that $\mathbf{v}(t)\in Y^{K-1,0}([0,T],\Omega)$, $\mathbf{w}(t)\in Y^{K-2,2}([0,T],\Omega)$, $|\mathbf{v}|_{K-1,0,[0,T]}\leqslant A_H$, $|\mathbf{w}|_{K-2,2,[0,T]}\leqslant A_E$, and $|\mathbf{w}|_{K-3,2,[0,T]}\leqslant \varepsilon_E$. Since (4.6a) is valid for every $\mathbf{v}^p(t)$ and $\mathbf{w}^p(t)$, by (7.8) we see that $\partial_t^M\mathbf{w}(0)=0$ for $0\leqslant M\leqslant K-3$ and $\partial_t^M\mathbf{v}(0)=\mathbf{u}_{M+1}$ for $0\leqslant M\leqslant K-2$. Since (4.8) follows from (4.6a), (4.7), and (As.2), we obtain that $(\mathbf{v}(t),\mathbf{w}(t))\in Z$.

Furthermore, letting $p\to\infty$ in $(\mathrm{H})_p$ and $(\mathrm{E})_p$ and using (7.8), (Ap.5), (Ap.6), (Ap.7), and (Ap.8), we see easily that the present $\mathbf{v}(t)$ and $\mathbf{w}(t)$ satisfy (H) and (E)′. If we put $\mathbf{u}(t)=\mathbf{u}^0(t)+\mathbf{w}(t)$, from the manner of deriving (E)′ from (E) we see that $\mathbf{v}(t)$ and $\mathbf{u}(t)$ satisfy (H) and (E).

Now, we shall prove that $\partial_t\mathbf{u}(t)=\mathbf{v}(t)$ for all $t\in[0,T]$. Since $(\mathbf{v}(t), D_x^1\mathbf{u}(t))\in Y^{K-2,1}([0,T],\Omega)$, by Theorem Ap.3, we see that $\mathbf{P}^i(t,U(t))$, $\mathbf{Q}_\Omega(t,U(t))\in Y^{K-2,0}([0,T],\Omega)\subset X^{K-3,0}([0,T],\Omega)$ and $\mathbf{Q}_\Gamma(t,U(t))\in Y^{K-2,1}([0,T],\Omega)\subset X^{K-3,1}([0,T],\Omega)$ where $U(t)=(\mathbf{v}(t),D_x^1\mathbf{u}(t))$. Since $K-3\geqslant[n/2]\geqslant 1$, differentiating (E) once in t, combining the resulting equations and (H), and putting $\mathbf{z}(t)=\partial_t\mathbf{u}(t)-\mathbf{v}(t)$, we have

$$-\partial_i(A^{ij}(t,\cdot,U(t))\,\partial_j\mathbf{z}(t)+A^{in+1}(t,\cdot,U(t))\,\mathbf{z}(t))$$
$$+B_\Omega^i(t,\cdot,U(t))\,\partial_i\mathbf{z}(t)+B_\Omega^{n+1}(t,\cdot,U(t))\,\mathbf{z}(t)+\lambda\mathbf{z}(t)=0\quad\text{in }\Omega;\quad(7.10\text{a})$$
$$\nu_i(A^{ij}(t,\cdot,U(t))\,\partial_j\mathbf{z}(t)+A^{in+1}(t,\cdot,U(t))\,\mathbf{z}(t))$$
$$+B_\Gamma^i(t,\cdot,U(t))\,\partial_i\mathbf{z}(t)+B_\Gamma^{n+1}(t,\cdot,U(t))\,\mathbf{z}(t)=0\quad\text{on }\Gamma,\quad(7.10\text{b})$$

for every $t\in[0,T]$. Since $\mathbf{z}(t)\in H^2(\Omega)$ for all $t\in[0,T]$, multiplying (7.10) by $\mathbf{z}(t)$ and integrating the resulting equations over Ω and Γ, by integration by parts, we have that the left-hand side of (5.8) in Theorem 5.3 equals zero. Hence, we have that $\|\mathbf{z}(t)\|_1^2=0$ for $t\in[0,T]$, which implies that $\partial_t\mathbf{u}(t)=\mathbf{v}(t)$ for $t\in[0,T]$. In particular, substituting $\partial_t\mathbf{u}(t)=\mathbf{v}(t)$ into (E), we see that $\mathbf{u}(t)$ satisfies the original problem (N).

Finally, we shall prove that $u(t)\in X^{K,0}([0,T],\Omega)$. Since $\mathbf{v}(t)$ can be regarded as a solution in $X^{2,0}([0,T],\Omega)$ to linear equations (6.12) and since $(\mathbf{v}(t),\mathbf{w}(t))\in Z$, by Theorem 6.2(1) and (2) we see that $\mathbf{v}(t)\in X^{K-1,0}([0,T],\Omega)$ (the uniqueness of solutions in $X^{2,0}([0,T],\Omega)$ follows from (2)). Since $\partial_t\mathbf{u}(t)=\mathbf{v}(t)$, to get that $\mathbf{u}(t)\in X^{K,0}([0,T],\Omega)$, it suffices to prove that $\mathbf{u}(t)\in C^0([0,T],H^K(\Omega))$. Let t and s be any points in $[0,T]$ such that $t\neq s$. Putting $U(t)=(\mathbf{v}(t),D_x^1\mathbf{u}(t))$ and $V(\theta)=\theta U(t)+(1-\theta)U(s)$ and applying Taylor expansion to (N) we can write

$$-\partial_i(q^{ij}\partial_j(\mathbf{u}(t)-\mathbf{u}(s)))+\mu(\mathbf{u}(t)-\mathbf{u}(s))=\mathbf{h}_\Omega\quad\text{in }\Omega;\quad(7.11\text{a})$$
$$\nu_i q^{ij}\partial_j(\mathbf{u}(t)-\mathbf{u}(s))+q_\Gamma^i\partial_i(\mathbf{u}(t)-\mathbf{u}(s))=\mathbf{h}_\Gamma\quad\text{on }\Gamma,\quad(7.11\text{b})$$

where μ is a constant determined below, $q^{ij}=q_\infty^{ij}+q_S^{ij}$, $q_\Gamma^i=q_{\Gamma\infty}^i+q_{\Gamma S}^i$,

$q^{ij}_{\infty} = A^{ij}(s, \cdot, 0)$, $q^{i}_{\Gamma\infty} = B^{i}_{\Gamma}(s, \cdot, 0)$, $q^{ij}_{S} = \int_0^1 (A^{ij})_1 (s, \cdot, V(\theta))\, d\theta$, $q^{i}_{\Gamma S} = \int_0^1 (B^{i}_{\Gamma})_1 (s, \cdot, V(\theta))\, d\theta$, and

$$\mathbf{h}_{\Omega} = \mathbf{f}_{\Omega}(t) - \mathbf{f}_{\Omega}(s) - (\partial_t \mathbf{v}(t) - \partial_t \mathbf{v}(s)) + \mu(\mathbf{u}(t) - \mathbf{u}(s)) + I_1 + I_2 + I_3,$$

$$I_1 = \partial_t(\mathbf{P}^i(t, \cdot, U(t)) - \mathbf{P}^i(s, \cdot, U(t))) + \mathbf{Q}_{\Omega}(s, \cdot, U(t)) - \mathbf{Q}_{\Omega}(t, \cdot, U(t)),$$

$$I_2 = \partial_t \left(\int_0^1 A^{i0}(t, \cdot, V(\theta))(\mathbf{v}(t) - \mathbf{v}(s))\, d\theta \right) + \int_0^1 A^{in+1}(t, \cdot, V(\theta))(\mathbf{u}(t) - \mathbf{u}(s))\, d\theta),$$

$$I_3 = -\int_0^1 (d\mathbf{Q}_{\Omega})(s, \cdot, V(\theta))(U(t) - U(s))\, d\theta,$$

$$\mathbf{h}_{\Gamma} = \mathbf{f}_{\Gamma}(t) - \mathbf{f}_{\Gamma}(s) + I_4 + I_5,$$

$$I_4 = v_i(\mathbf{P}^i(s, \cdot, U(t)) - \mathbf{P}^i(t, \cdot, U(t))) + \mathbf{Q}_{\Gamma}(s, \cdot, U(t)) - \mathbf{Q}_{\Gamma}(t, \cdot, U(t)),$$

$$I_5 = \left[v_i \int_0^1 A^{in+1}(s, \cdot, V(\theta))\, d\theta + \int_0^1 B^{n+1}_{\Gamma}(s, \cdot, V(\theta))\, d\theta \right] (\mathbf{u}(s) - \mathbf{u}(t)) + \left[v_i \int_0^1 A^{i0}(s, \cdot, V(\theta))\, d\theta + \int_0^1 B^{0}_{\Gamma}(s, \cdot, V(\theta))\, d\theta \right] (\mathbf{v}(s) - \mathbf{v}(t)).$$

If we put $q^i = q^k_{\Omega} = q^{n+1}_{\Gamma} = 0$ $(k = 1, ..., n+1)$, (7.11) is described by (5.1). By Theorem Ap.3, (4.2), and (4.7) we have

$$\sum_{i,j=1}^{n} (\|q^{ij}_{\infty}\|_{\infty, K-1} + \|q^{ij}_{S}\|_{K-1}) + \sum_{i=1}^{n} (\|q^{i}_{\Gamma\infty}\|_{\infty, K-1} + \|q^{i}_{\Gamma S}\|_{K-1}) \leqslant C_{12}(K, \mathbb{B}, \Lambda_H, \Lambda_E)$$

for all t and $s \in [0, T]$. Hence, in the present case, $\gamma_{\infty} = 0$ and $\gamma_K = C_{12}(K, \mathbb{B}, \Lambda_H, \Lambda_E)$ for all $t, s \in [0, T]$ (cf. (5.2a) and (5.2b). Thus, there exists a μ depending only on K, $\mathbb{B}$, Λ_H, and Λ_E and independent of t and $S \in [0, T]$ such that we can apply (5.4) with $L = K$ to (7.11). And then, we have

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_K \leqslant M\{\|\mathbf{h}_{\Omega}\|_{K-2} + \langle\!\langle \mathbf{h}_{\Gamma} \rangle\!\rangle_{K-(3/2)}\} \quad \text{for any } t \text{ and } s \in [0, T]. \tag{7.12}$$

Let us estimate the right-hand side of (7.12). Since $(\mathbf{v}(t), \mathbf{w}(t)) \in Z$, we have that $\|V(\theta)\|_{K-1}$, $\|U(t)\|_{K-1} \leqslant M$ for all $t, s \in [0, T]$ and $0 \leqslant \theta \leqslant 1$. Noting (A.1)(*), by the mean value theorem we have that $\|I_1\|_{K-2}$,

$\|I_4\|_{K-1} \leqslant M|t-s|$. By (Ap.1) with $N=0$ and $M=K-1$, we have that $\|I_2\|_{K-2}, \|I_5\|_{K-1} \leqslant M\{\|\mathbf{v}(t)-\mathbf{v}(s)\|_{K-1} + \|\mathbf{u}(t)-\mathbf{u}(s)\|_{K-1}\}$. And also, by (Ap.1) with $N=0$ and $M=K-2$, we have that $\|I_3\|_{K-2} \leqslant M\{\|\mathbf{v}(t)-\mathbf{v}(s)\|_{K-2} + \|\mathbf{u}(t)-\mathbf{u}(s)\|_{K-1}\}$. Noting Theorem Ap.4(1) and substituting these estimations into (7.12), we have

$$\begin{aligned}\|\mathbf{u}(t)-\mathbf{u}(s)\|_K \leqslant M\{&\|\mathbf{f}_\Omega(t)-\mathbf{f}_\Omega(s)\|_{K-2} \\ &+ \langle\!\langle \mathbf{f}_\Gamma(t)-\mathbf{f}_\Gamma(s)\rangle\!\rangle_{K-(3/2)} + |t-s| \\ &+ \|\partial_t\mathbf{v}(t)-\partial_t\mathbf{v}(s)\|_{K-2} \\ &+ \|\mathbf{v}(t)-\mathbf{v}(s)\|_{K-1} + \|\mathbf{u}(t)-\mathbf{u}(s)\|_{K-1}\}\end{aligned}$$

for all $t, s \in [0, T]$. Since $\mathbf{v}(t) \in X^{K-1,0}([0, T], \Omega)$ and $\mathbf{u}(t) \in Y^{K-2,2}([0, T], \Omega) \subset C^0([0, T], H^{K-1}(\Omega))$, by (1.2) and (7.13) we see that $\mathbf{u}(t) \in C^0([0, T], H^K(\Omega))$, which completes the proof of Theorem 1.1.

8. Applications of Theorem 1.1

8.1. *An Application to Nonlinear Wave Equations.* Let us consider the following scalar equations:

$$\partial_t^2 u(t) - \partial_i(a^i(t, D^1u(t))) + b_\Omega(t, D^1u(t)) = f_\Omega(t) \quad \text{in} \quad (0, T)\times\Omega; \quad (8.1\text{a})$$

$$\nu_i a^i(t, D^1u(t)) + b_\Gamma(t, \partial_t u(t), u(t)) = f_\Gamma(t) \quad \text{on} \quad (0, T)\times\Gamma; \quad (8.1\text{b})$$

$$u(0) = u_0, \qquad \partial_t u(0) = u_1 \qquad \text{in } \Omega. \quad (8.1\text{c})$$

Let all the functions considered in this paragraph be scalar valued. Let w_0, w_i, and w_{n+1} be independent variables corresponding to functions $\partial_t u, \partial_i u$, and u, respectively. Put $W=(w_0, w_i, w_{n+1})$, $W'=(w_0, w_{n+1})$, $D(U_0)=\{W\in\mathbb{R}^{n+2} \mid |W|<U_0\}$, and $D'(U_0)=\{W'\in\mathbb{R}^2 \mid |W'|<U_0\}$. We make the following assumptions:

(A.8.1) The nonlinear functions $a^i = a^i(t, x, W)$ and $b_\Omega = b_\Omega(t, x, W)$ are in $\mathscr{B}^\infty([-T_0, T_0]\times\bar{\Omega}\times D(U_0))$ and $b_\Gamma = b_\Gamma(t, x, W')$ is in $\mathscr{B}^\infty([-T_0, T_0]\times\bar{\Omega}\times D'(U_0))$. Furthermore, $a^i(t, x, 0) = b_V(t, x, 0) = 0$ for $(t, x) \in [-T_0, T_0]\times\bar{\Omega}$.

(A.8.2) $(\partial a^i/\partial w_j)(t, x, W) = (\partial a^j/\partial w_i)(t, x, W)$ for $(t, x, W) \in [-T_0, T_0] \times\bar{\Omega}\times D(U_0)$.

(A.8.3) There exists a constant $\delta>0$ such that $\sum_{i,j=1}^n (\partial a^i/\partial w_j)(t, x, W)\,\xi_i\xi_j \geqslant \delta|\xi|^2$ for $(t, x, W)\in[-T_0, T_0]\times\bar{\Omega}\times D(U_0)$ and $\xi=(\xi_1, \ldots, \xi_n)\in\mathbb{R}^n$.

(A.8.4) $-v_t(x)(\partial a^t/\partial w_0)(t, x, W) + 2(\partial b_\Gamma/\partial w_0)(t, x, W') \geqslant 0$ for $(t, x) \in [-T_0, T_0] \times \Gamma$, $|W| < U_0$, and $|W'| < U_0$.

Remark. (1) Assumption (A.8.2) can be replaced by the condition that there exists a function $p(t, x, W) \in \mathscr{B}^\infty([-T_0, T_0] \times \bar{\Omega} \times D(U_0))$ such that $a^t = \partial p/\partial w_t$.

(2) In the scalar operators case, (A.4) implies that $\mathbf{Q}_\Gamma$ does not contain the derivatives $\partial_1 \mathbf{u}, ..., \partial_n \mathbf{u}$ of $\mathbf{u}$. Thus, in the present case, it is assumed that b_Γ depends only on t, x, u, and $\partial_t u$.

Since we can easily check that the original assumptions (A.1)–(A.5) are valid under (A.8.1)–(A.8.4), applying Theorem 1.1 implies

THEOREM 8.1. *Assume that* (A.8.1)–(A.8.4) *are valid. Let K be an integer* $\geqslant [n/2] + 3$. *If* $u_0, u_1, f_\Omega(t)$, *and* $f_\Gamma(t)$ *satisfy* (1.2)–(1.6), *then there exist T and* $\Lambda > 0$ *depending on K and* $\mathbb{B}$ *only such that* (8.1) *admits a unique solution* $u(t) \in X^{K,0}([0, T], \Omega)$ *satisfying the conditions*: $|u|_{K,0,[0,T]} \leqslant \Lambda$ *and* $\|D^1 u(t)\|_{\infty,1} < U_0$ *for* $t \in [0, T]$.

If we put $a^t = \partial_t u/\sqrt{1 + |\nabla_x u|^2}$, $b_\Omega = 0$, and $b_\Gamma = a(u)$, (N.W) can be described by (8.1). We can easily check that the present a^t, b_Ω, and b_Γ satisfy (A.8.1)–(A.8.4). Thus, (N.W) admits a local solution. As another important boundary condition in (N.W), we can consider the case $b_\Gamma = a(u) + c(\partial_t u)$, where $c(0) = 0$ and c is a nondecreasing function in $\partial_t u$. This boundary condition describes the effect of the dissipation on the boundary.

8.2. *An Application to Three-Dimensional Elastodynamics.* If the undeformed state Ω of a three-dimensional, homogeneous, isotropic, hyperelastic material has not any stress in it, the equation of motion describing its small displacement $\mathbf{u}(t, x) = {}^t(u_1(t, x), u_2(t, x), u_3(t, x))$ under the action of the body force $\mathbf{b} = {}^t(b_1(t, x, \mathbf{u}), b_2(t, x, \mathbf{u}), b_3(t, x, \mathbf{u}))$ and the pressure is described by (N) with $n = m = 3$. And then, $\mathbf{P}^i, \mathbf{Q}_V, \mathbf{f}_\Omega(t)$, and $\mathbf{f}_\Gamma(t)$ are defined as follows (cf. [1, Chap. 1]):

$$(\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3) = \rho^{-1} \nabla_x \boldsymbol{\phi} \cdot \Sigma(E); \qquad \mathbf{Q}_\Omega = \mathbf{b}(t, x, \mathbf{u}) - \mathbf{b}(t, x, 0);$$

$$\mathbf{Q}_\Gamma = -p[(\det \nabla_x \boldsymbol{\phi})(\nabla_x \boldsymbol{\phi})^* \, \mathbf{v}(x) - (\det \nabla_x \mathbf{x})(\nabla_x \mathbf{x})^* \, \mathbf{v}(x)] + \mathbf{D}(t, x, \partial_t \mathbf{u}); \tag{8.2}$$

$$\mathbf{f}_\Omega(t) = \mathbf{b}(t, x, 0); \qquad \mathbf{f}_\Gamma(t) = p(\det \nabla_x \mathbf{x})(\nabla_x \mathbf{x})^* \, \mathbf{v}(x).$$

Here, $\mathbf{x} = {}^t(x_1, x_2, x_3) = {}^t x$; $\mathbf{v}(x) = {}^t(v_1(x), v_2(x), v_3(x)) = {}^t v(x)$; $\boldsymbol{\phi} = \mathbf{x} + \mathbf{u}$; p is a constant describing the pressure density of Γ; ρ is a positive constant describing the mass density of Ω; $M^* = {}^t(M^{-1})$; $\mathbf{D}(t, x, \partial_t \mathbf{u}) = {}^t(D_1(t, x, \partial_t \mathbf{u}), D_2(t, x, \partial_t \mathbf{u}), D_3(t, x, \partial_t \mathbf{u}))$ is a 3-vector of functions in $\mathscr{B}^\infty([-T_0, T_0] \times \bar{\Omega} \times \{U' = (u_{01}, u_{02}, u_{03}) \in \mathbb{R}^3 \mid |U'| < U_0\})$ such that $(\partial D_a / \partial u_{0b})$ is a 3×3 nonnegative definite matrix and $\mathbf{D}(t, x, 0) = 0$, which describes the effect of the dissipation on Γ;

$$\nabla_x \boldsymbol{\phi} = \begin{bmatrix} \partial_1 \phi_1, \partial_2 \phi_1, \partial_3 \phi_1 \\ \partial_1 \phi_2, \partial_2 \phi_2, \partial_3 \phi_2 \\ \partial_1 \phi_3, \partial_2 \phi_3, \partial_3 \phi_3 \end{bmatrix} \qquad \text{for} \quad \boldsymbol{\phi} = {}^t(\phi_1, \phi_2, \phi_3);$$

$E = (E_{ij}) = \frac{1}{2}({}^t(\nabla_x \boldsymbol{\phi}) \cdot (\nabla_x \boldsymbol{\phi}) - I_3)$ (I_3 being the 3×3 identity matrix); $\Sigma(E) = (\Sigma^{ij}(E))$ is a 3×3 symmetric matrix called the second Piola–Kirchhoff stress tensor having the following properties:

$$\Sigma(E) = \lambda(\text{trace } E) I_3 + 2\mu E + o(E) \qquad \text{as} \quad |E| \to 0; \tag{8.3a}$$

λ and μ are Lamé constants satisfying the conditions: $\mu > 0$ and $\lambda + \mu > 0$; (8.3b)

there exists a stored energy function $w(E)$ such that $\Sigma^{ij}(E) = (\partial w / \partial E_{ij})(E)$. (8.3c)

Note that E is a real symmetric matrix. As was seen in [9, Sect. 3], we have easily from (8.3c)

$$A^{ij}_{ab} = \partial P^i_a / \partial u_{jb} = \delta_{ab} (\partial w / \partial E_{ij})(E) + \sum_{k,h=1}^{3} (\partial^2 w / \partial E_{ik} \partial E_{jh})(E) \, \phi_{ka} \phi_{hb}, \tag{8.4}$$

where $\mathbf{P}^i = {}^t(P^i_1, P^i_2, P^i_3)$, $\phi_{ka} = \partial_k \phi_a$, and δ_{ab} are Kronecker's delta symbols, i.e., $\delta_{aa} = 1$ and $\delta_{ab} = 0$ for $a \neq b$. Since $E_{ij} = E_{ij}$, if we put $A^{ij} = (A^{ij}_{ab})$, we see by (8.4) that ${}^t A^{ij} = A^{ji}$. Substituting (8.3a) into (8.4) and using (8.3c), we have

$$\sum_{i,j,a,b=1}^{3} A^{ij}_{ab}(0) \, \partial_j v_b \partial_i v_a = \lambda \left(\sum_{i=1}^{3} \varepsilon_{ii} \right)^2 + 2\mu \sum_{i,j=1}^{3} (\varepsilon_{ij})^2 \tag{8.5}$$

where $\varepsilon_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. According to the result due to Simpson and Spector [10, Corollary of Theorem 6], it follows from (8.3b) and (8.5) that there exist positive constants c_0 and c_1 such that

$$\sum_{i,j,a,b=1}^{3} \int_\Omega A^{ij}_{ab}(0) \, \partial_j v_b(x) \, \partial_i v_a(x) \, dx \geqslant c_1 \|\mathbf{v}\|_1^2 - c_0 \|\mathbf{v}\|_0^2 \qquad \text{for} \quad \mathbf{v} \in H^1(\Omega).$$

Thus, by the mean value theorem we see that there exist positive numbers δ_1, δ_0, and U_0 such that

$$\sum_{i,j,a,b=1}^{3} \int_\Omega A^{ij}_{ab}(U(x))\, \partial_j v_b(x)\, \partial_i v_a(x)\, dx \geqslant \delta_1 \|\mathbf{v}\|_1^2 - \delta_0 \|\mathbf{v}\|_0^2 \qquad (8.6)$$

for $\mathbf{v} \in H^1(\Omega)$ and $U \in H^{\infty,1}(\bar{\Omega}, D(U_0))$.

If we put $(\det \nabla_x \boldsymbol{\phi})(\nabla_x \boldsymbol{\phi})^* \mathbf{v}(x) = {}^t(R_1, R_2, R_3)$, we see easily that

$$R_1 = v_1(\phi_{22}\phi_{33} - \phi_{32}\phi_{23}) + v_2(\phi_{32}\phi_{13} - \phi_{12}\phi_{33}) + v_3(\phi_{12}\phi_{23} - \phi_{22}\phi_{13}),$$
$$R_2 = v_1(\phi_{23}\phi_{31} - \phi_{33}\phi_{21}) + v_2(\phi_{33}\phi_{11} - \phi_{13}\phi_{31}) + v_3(\phi_{13}\phi_{21} - \phi_{23}\phi_{11}),$$
$$R_3 = v_1(\phi_{21}\phi_{32} - \phi_{31}\phi_{22}) + v_2(\phi_{31}\phi_{12} - \phi_{11}\phi_{32}) + v_3(\phi_{11}\phi_{22} - \phi_{21}\phi_{12}),$$

where $\phi_{ij} = \delta_{ij} + u_{ij}$. If we put $B^i_{\Gamma ab} = \partial Q_{\Gamma a}/\partial u_{ib}$, then $B^i_{\Gamma ab} = -p\partial R_a/\partial \phi_{ib}$. From this we can easily check that $B^i_{\Gamma ab} + B^i_{\Gamma ba} = 0$ and $v_i B^i_{\Gamma ab} = 0$ for $a, b = 1, 2, 3$. Since $(\partial D_a/\partial u_{0b})$ is nonnegative definite, the rest of the conditions of (A.2) and (A.5) are valid. If $|p|$ is very small, then from (8.6) and similar arguments from the last part of Section 2, it follows with positive constants δ'_0 and δ'_1 that

$$\sum_{i,j,a,b=1}^{3} (A^{ij}_{ab}(U(\cdot))\, \partial_j v_b, \partial_i v_a) + \sum_{i,a,b=1}^{3} \langle B^i_{\Gamma ab}(\cdot, U(\cdot))\, \partial_i v_b, v_a \rangle$$
$$\geqslant \delta'_1 \|\mathbf{v}\|_1^2 - \delta'_0 \|\mathbf{v}\|_0^2 \qquad \text{for} \quad \mathbf{v} \in H^2(\Omega) \text{ and } U(x) \in H^{\infty,1}(\bar{\Omega}, D(U_0)).$$

These facts imply that (A.1)–(A.5) are valid if $|p|$ and U_0 are small. According to Theorem 1.1, we can conclude that, if the pressure and the displacement are very small in the initial state, we get a local existence theorem of solutions to the three-dimensional elastodynamics when the applied surface force is the pressure. Unfortunately, for another important traction boundary condition which is not dead load, i.e., $\mathbf{Q}_\Gamma = (\det \nabla_x \boldsymbol{\phi})\, |(\nabla_x \boldsymbol{\phi})^* \mathbf{v}|\, \mathbf{b}_\Gamma(\boldsymbol{\phi})$ (cf. [1, p. 21]), it seems that our theorem cannot be applied, because the condition ${}^t B^i_\Gamma + B^i_\Gamma = 0$ is not satisfied in this case.

APPENDIX 1: SOME ESTIMATES OF NONLINEAR TERMS

In this appendix, we summarize the estimations of a product of functions and composed functions by Sobolev norms. In the same way as in the proof of Theorem 7.1 of Mizohata [5], by using Sobolev's imbedding theorem we have

THEOREM Ap.1. *Let $r_1, \dots, r_k$ $(k \geq 2)$ and S be nonnegative real numbers and L a nonnegative integer such that $S > n/2$ and $S \geq r_1 + \cdots + r_k + L$. If $u_l \in H^{S-r_l}(\Omega)$, then the product $\prod u_l$ of $u_1, \dots, u_k$ belongs to $H^L(\Omega)$ and $\|\prod u_l\|_L \leq C(k, L) \prod \|u_l\|_{S-r_l}$.*

Applying Theorem Ap.1, we easily have the following two theorems.

THEOREM Ap.2. *Let J be an interval of $\mathbb{R}$. Let L and M be integers such that $L, M \geq 0$ and $L + M > n/2$. If $u_l(t) \in Z^{L,M}(J, \Omega)$ $(l = 1, \dots, k$ and $Z = X$ or $Y)$, then their product $\prod u_l \in Z^{L,M}(J, \Omega)$. Furthermore, when $Z = X$, $\|D^L \prod u_l(t)\|_M \leq C(k, L, M) \prod \|D^L u_l(t)\|_M$ for $t \in J$.*

THEOREM Ap.3. *Let J, L, and M be the same as in Theorem Ap.2. Let $F(t, x, u) \in \mathscr{B}^\infty(J \times \bar{\Omega} \times \{|u| \leq u_0\})$ such that $F(t, x, 0) = 0$. Let $u(t, x) \in Z^{L,M}(J, \Omega)$ $(Z = X$ or $Y)$ such that $\|u(t)\|_{\infty,0} \leq u_0$ for all $t \in J$. Then, $F(t, x, u(t, x)) \in Z^{L,M}(J, \Omega)$. Furthermore, when $Z = X$, $\|D^L F(t, x, u(t))\|_M \leq C(L, M, F)\{1 + \|D^L u(t)\|_M\}^{L+M-1} \|D^L u(t)\|_M$.*

Remark. When u_l, u, and F do not depend on t in Theorems Ap.2 and Ap.3, all the assertions are valid if we put $L = 0$ and $Z^{L,M}(J, \Omega)$ is replaced by $H^M(\Omega)$.

Now, we give several estimations of nonlinear functions used in the text. Below, $J = [0, T]$, $G(t, x, u) \in \mathscr{B}^\infty(J \times \bar{\Omega} \times \{|u| \leq u_0\})$, and $H(x, u) \in \mathscr{B}^\infty(\bar{\Omega} \times \{|u| \leq u_0\})$.

(Ap.1) Let M and N be integers ≥ 0 such that $K - 2 \leq N + M \leq K - 1$. If $u(t) \in Z^{N,M}(J, \Omega)$, $v(t) \in Z^{N,M}(J, \Omega)$ $(Z = X$ or $Y)$, and $\|u(t)\|_{\infty,0} \leq u_0$ for $t \in J$, then $G(t, \cdot, u(t))\, v(t) \in Z^{N,M}(J, \Omega)$. Furthermore, when $Z = X$,

$$\|D^N(G(t, \cdot, u(t))\, v(t))\|_M \leq$$
$$C(M, N)\{1 + \|D^N u(t)\|_M\}^{N+M-1} \|D^N u(t)\|_M \|D^N v(t)\|_M$$

for $t \in J$.

(Ap.2) Let $u(t) \in X^{K-2,1}(J, \Omega)$ such that $\|u(t)\|_{\infty,0} \leq u_0$ for all $t \in J$ and $v(t) \in X^{K-2,1}(J, \Omega)$. Put $I(t) = \{G(t, \cdot, u(t)) - G(0, \cdot, u(0))\}\, v(t)$. Then, $I(t) \in X^{K-2,1}(J, \Omega)$ and $|I|_{K-2,1,J} \leq C(K, |u|_{K-2,1,J})\{T|v|_{K-2,1,J} + |v|_{K-3,1,J}\}$.

(Ap.3) Let $v(t) \in X^{K-2,0}(J, \Omega)$ and let $u(t)$ and $I(t)$ be the same as in (Ap.2). Then, $I(t) \in X^{K-2,0}(J, \Omega)$ and

$$|I|_{K-3,1,J} \leq C(K, |u|_{K-2,1,J})\, T|v|_{K-2,0,J}$$
$$+ C(K, \|(D^{K-2}u)(0)\|_1)\, |v|_{K-3,0,J}.$$

(Ap.4) Let $u(t)$ and $v(t)$ be the same as in (Ap.2). Put $I(t)=G(t,\cdot,u(t))\,v(t)\,v(t)$. Then, $I(t)\in X^{K-2,1}(J,\Omega)$ and $|I|_{K-2,1,J}\leqslant C(K,|u|_{K-2,1,J})\,|v|_{K-2,1,J}\,|v|_{K-3,1,J}$.

(Ap.5) Let $N=0$ or 1. Assume that $H(x,0)=0$. If $u_l\in H^{K-2}(\Omega)$ and $\|u_l\|_{\infty,0}\leqslant u_0$ for $l=1,2$, then $\|H(\cdot,u_1)-H(\cdot,u_2)\|_N\leqslant C(K,\|u_1\|_{K-2},\|u_2\|_{K-2})\,\|u_1-u_2\|_N$.

(Ap.6) Let $N=0$ or 1. If $u_l, v_l\in H^{K-2}(\Omega)$ and $\|u_l\|_{\infty,0}\leqslant u_0$ for $l=1,2$, then

$$\|H(\cdot,u_1)v_1-H(\cdot,u_2)v_2\|_N\leqslant C(K,\|u_1\|_{K-2},\|u_2\|_{K-2})\\ \times\{\|v_1-v_2\|_N+\|v_2\|_{K-2}\,\|u_1-u_2\|_N\}.$$

(Ap.7) Let $u_l(t)$ and $v_l(t)\in X^{K-2,1}(J,\Omega)$ for $l=1,2$. Assume that $\|u_l(t)\|_{\infty,0}\leqslant u_0$ for $t\in J$ and $l=1,2$ and that $u_1(0)=u_2(0)$. Put $I(t)=I_1(t)\,v_1(t)-I_2(t)\,v_2(t)$ where $I_l(t)=G(t,\cdot,u_l(t))-G(0,\cdot,u_l(0))$. Then,

$$|I|_{0,1,J}\leqslant C(K,|u_1|_{K-2,1,J},|u_2|_{K-2,1,J})\\ \times\{T|v_1-v_2|_{0,1,J}+|v_2|_{0,K-2,J}\,|u_1-u_2|_{0,1,J}\}.$$

(Ap.8) If $u_l, v_l\in H^{K-2}(\Omega)$, $\|u_l\|_{\infty,0}\leqslant u_0$, and $\|v_l\|_{K-2}\leqslant \Delta<1$ for $l=1,2$, then

$$\|H(\cdot,u_1)\,v_1\,v_1-H(\cdot,u_2)\,v_2v_2\|_1\\ \leqslant C(K,\|u_1\|_{K-2},\|u_2\|_{K-2})\,\Delta\{\|u_1-u_2\|_1+\|v_1-v_2\|_1\}.$$

(Ap.9) Let $u(t)$ be the same as in (Ap.2). Then, $\|G(t,\cdot,u(t))\|_{\infty,0}\leqslant C_1+C_2T\,|u|_{K-2,1,J}$ for $t\in J$, where $C_1=\sup\{|G(t,x,u)|\mid (t,x)\in\bar{J}\times\bar{\Omega}, |u|\leqslant u_0\}$ and $C_2=\sup\{|\partial_tG(t,x,u)|+|dG(t,x,u)|\mid (t,x)\in\bar{J}\times\bar{\Omega}, |u|\leqslant u_0\}$.

All the assertions can be easily checked by using Theorems Ap.1, Ap.2, and Ap.3. So, we may omit their proofs.

APPENDIX 2: Supplement to the Text

Here, we summarize several facts which play important roles in the text.

Theorem Ap.4. (1) *There exists a constant* $C=C(\Gamma)>0$ *such that* $\langle\!\langle u\rangle\!\rangle_{1/2}\leqslant C\,\|u\|_1$ *for all* $u\in H^1(\Omega)$.

(2) *For any* $\varepsilon>0$, *there exists a constant* $C(\varepsilon,\Gamma)$ *such that* $\langle\!\langle u\rangle\!\rangle_0^2\leqslant \varepsilon\,\|u\|_1^2+C(\varepsilon,\Gamma)\,\|u\|_0^2$ *for* $u\in H^1(\Omega)$.

Proof. When $\Omega=\{x=(x_1,...,x_n)\in\mathbb{R}^n \mid x_n>0\}$, we know Theorem Ap.4 very well (cf. [5, Proposition 3.6]). Hence, using the partition of unity near Γ, we have Theorem Ap.4 immediately.

THEOREM Ap.5. *If $u_M\in H^{K-M}(\Omega)$ for $0\leqslant M\leqslant K$, then there exists a $v(t)\in X^{K,0}(\mathbb{R},\Omega)$ such that $(\partial_t^M v)(0)=u_M$ in Ω for $0\leqslant M\leqslant K$ and*

$$\|D^K v(t)\|\leqslant C(K)\sum_{M=0}^{K}\|u_M\|_{K-M}\qquad \textit{for } t\in\mathbb{R}.$$

Proof. Using the Fourier transform, we shall prove the theorem. Using Lions' well-known method of extending functions defined on Ω to whole $\mathbb{R}^n$, we know that there exist $v_M\in H^{K-M}(\mathbb{R})$ such that $v_M(x)=u_M(x)$ on Ω and $\|v_M\|'_{K-M}\leqslant C\|u_M\|_{K-M}$ where C is independent of u_M and v_M and $\|\cdot\|'_r$ are the norms of Sobolev spaces of order r over $\mathbb{R}^n$. Put

$$\hat{v}(t,\xi)=\sum_{L,N=0}^{K-1}(\exp\sqrt{-1}\,(L+1)(1+|\xi|^2)^{1/2}t)\,a_{LN}\hat{v}_N(\xi)(1+|\xi|^2)^{-N/2},$$

where $\hat{v}_N$ are the Fourier transforms of v_N; the a_{LN} are constants satisfying the linear algebraic equations:

$$\sum_{L=1}^{K-1}(\sqrt{-1}(L+1))^M a_{LN}=\delta_{MN}\qquad\text{for}\quad M,N=0,1,...,K-1$$

$$(\delta_{MM}=1\text{ and }\delta_{MN}=0\text{ for }M\neq N).$$

Obviously, $(\partial_t^M\hat{v})(0,\xi)=\hat{v}_M(\xi)$. Put $v(t,x)=$ real part of the inverse Fourier transform of $\hat{v}(t,\xi)$ with respect to ξ. By Parseval's formula we see easily that the $v(t,x)$ has the desired properties,which completes the proof of Theorem Ap.5.

THEOREM Ap.6. *Let $T>0$ and let L and M be nonnegative integers. Let $u(t)\in Y^{L,M}([0,T],\Omega)$. Then, there exists a $v(t)\in Y^{L,M}(\mathbb{R},\Omega)$ such that $v(t)=u(t)$ for $t\in[0,T]$ and*

$$|v|_{L,M,\mathbb{R}}\leqslant C(L,M)\{|u|_{L,M,[0,T]}+\sum_{N=0}^{L-1}\|\partial_t^N u(0)\|_{L+M-N}\}.$$

Proof. By our definition of $L^\infty(J,X)$ (cf. Sect. 2), $\partial_t^N u(0)$ exist and $\in H^{L+M-N}(\Omega)$ for $0\leqslant N\leqslant L-1$. By employing the same arguments as in the proof of Theorem Ap.5, we can find $z(t)\in X^{L+M,0}(\mathbb{R},\Omega)$ such that $\partial_t^N z(0)=\partial_t^N u(0)$ for $0\leqslant N\leqslant L-1$ and

$$\|D^L z(t)\|_M\leqslant C(L)\sum_{N=0}^{L-1}\|\partial_t^N u(0)\|_{L+M-N}\qquad\text{for}\quad t\in\mathbb{R}.$$

Put $z_1(t)=u(t)$ for $0\leqslant t\leqslant T$ and $=z(t)$ for $t<0$. Then, we see easily that $z_1(t)\in Y^{L,M}((-\infty, T], \Omega)$ and

$$|z_1|_{L,M,(-\infty,T]}\leqslant |u|_{L,M,[0,T]}+|z|_{L,M,(-\infty,T]}$$
$$\leqslant |u|_{L,M,[0,T]}+C(L)\sum_{N=0}^{L-1}\|\partial_t^N u(0)\|_{L+M-N}.$$

Put $a_k=2^k$ and choose b_k so that $\sum_{k=0}^{L-1}(-a_k)^l b_k=1$ for $l=0, 1, ..., L-1$. If we put

$$v(t)=z_1(t)\quad \text{for } t\leqslant T \qquad \text{and}$$
$$=\sum_{k=0}^{L-1} b_k z_1(T-a_k(t-T)\quad \text{for } t>T,$$

then we see easily that $v(t)$ has the desired properties, which completes the proof.

THEOREM Ap.7. *Let $F(t, x, U)\in \mathscr{B}^\infty([0, T]\times\bar{\Omega}\times\{|U|<U_0\})$ and let $u(t)\in Y^{K-2,1}([0, T], \Omega)$ such that $\|u(t)\|_{\infty,0}<U_0$ for all $t\in[0, T]$. Then,*

$$\|F(t,\cdot,u(t))-F(0,\cdot,u(0))\|_{\infty,1}\leqslant C(K, |u|_{K-2,1,[0,T]})\{T+C(\varepsilon)T^\varepsilon\}$$

for $t\in[0, T]$, where ε is a constant in $(0, [n/2]+1-(n/2))$.

Proof. Since $F(t,\cdot,u(t))-F(0,\cdot,u(0))=F(t,\cdot,u(t))-F(0,\cdot,u(t))+F(0,\cdot,u(t))-F(0,\cdot,u(0))$, we have

$$\|F(t,\cdot,u(t))-F(0,\cdot,u(0))\|_{\infty,1}$$
$$\leqslant C(F)\{t(1+\|u(t)\|_{\infty,1})+\|u(t)-u(0)\|_{\infty,1}\}.$$

Let σ be a number such that $\varepsilon=1+[n/2]-(n/2)-\sigma>0$. Since $\varepsilon(1+[n/2])+(1-\varepsilon)(2+[n/2])=1+(n/2)+\sigma$, by Sobolev's imbedding theorem and a classical interpolation inequality, we have

$$\|u(t)-u(0)\|_{\infty,1}\leqslant C\|u(t)-u(0)\|_{(n/2)+1+\sigma}$$
$$\leqslant C\|u(t)-u(0)\|_{[n/2]+1}^{\varepsilon}\,\|u(t)-u(0)\|_{[n/2]+2}^{1-\varepsilon}$$
$$\leqslant Ct^\varepsilon\,|u|_{1,[n/2]+1,[0,T]}^{\varepsilon}(2|u|_{0,[n/2]+2,[0,T]})^{1-\varepsilon}$$
$$\leqslant Ct^\varepsilon\,|u|_{K-2,1,[0,T]};$$
$$\|u(t,\cdot)\|_{\infty,1}\leqslant C|u|_{0,[n/2]+2,[0,T]}\leqslant C|u|_{K-2,1,[0,T]}.$$

Here, we have used the fact that $K\geqslant[n/2]+3$. Combining these estimations, we have the theorem immediately.

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