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The Elliptic Gamma Function and $SL(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$

Giovanni Felder

Departement Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland
E-mail: felder@math.ethz.ch

and

Alexander Varchenko[1]

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599-3250
E-mail: av@math.unc.edu

The elliptic gamma function is a generalization of the Euler gamma function and is associated to an elliptic curve. Its trigonometric and rational degenerations are the Jackson q-gamma function and the Euler gamma function, respectively. The elliptic gamma function appears in Baxter's formula for the free energy of the eight-vertex model and in the hypergeometric solutions of the elliptic qKZB equations. In this paper, the properties of this function are studied. In particular we show that elliptic gamma functions are generalizations of automorphic forms of $G = SL(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$ associated to a non-trivial class in $H^3(G, \mathbb{Z})$. © 2000 Academic Press

1. INTRODUCTION

This paper deals with the properties of the *elliptic gamma function*, an elliptic generalization of the Euler gamma function. It is the meromorphic function of three complex variables z, τ, σ, with $\operatorname{Im} \tau$, $\operatorname{Im} \sigma > 0$ defined by the convergent infinite product

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}}.$$

It is the unique solution of a functional equation involving a Jacobi theta function: Let for $z, \tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$, θ_0 denote the theta function

$$\theta_0(z, \tau) = \prod_{j=0}^{\infty} (1 - e^{2\pi i((j+1)\tau - z)})(1 - e^{2\pi i(j\tau + z)}).$$

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Then the elliptic gamma function may be characterized as follows.

THEOREM 1.1. *Suppose that* τ, σ *are complex numbers with positive imaginary part. Then* $u(z) = \Gamma(z, \tau, \sigma)$ *is the unique meromorphic solution of the difference equation*

$$u(z+\sigma) = \theta_0(z, \tau)\, u(z)$$

such that:

(i) $u(z)$ *obeys* $u(z+1) = u(z)$ *and is holomorphic on the upper half plane* $\operatorname{Im} z > 0$,

(ii) $u((\tau+\sigma)/2) = 1$.

This theorem is proved in 3.1 below.

The elliptic gamma function was introduced by Ruijsenaars [R]. Similar double products appeared earlier in statistical mechanics. Probably the first (implicit) appearance of the elliptic gamma function in this context is in Baxter's formula [B] for the free energy of the eight-vertex model. This model has four parameters, which can be taken to be c_B, x_B, q_B, z_B in Baxter's notation (to which we add a subscript B to avoid conflicts with our notations); see [B, Eqs. (D1)–(D8)]. If we set $c_B = c$, $q_B = e^{2\pi i\tau}$, $x_B = e^{2\pi i\sigma}$, $z_B = e^{2\pi i u}$, then the free energy (times the inverse temperature) is $f(c, u, \tau, \sigma) = -\ln c - \ln Z(u, \tau, \sigma)$ where Z can be expressed in terms of theta and elliptic gamma functions:

$$Z(u, \tau, \sigma) = \frac{\theta_0(\sigma-u, 2\tau)\, \theta_0(\sigma+u, 2\tau)\, \Gamma(\sigma-u, \tau, 4\sigma)\, \Gamma(\sigma+u, \tau, 4\sigma)}{\theta_0(\tau, 2\tau)\, \theta_0(2\sigma, 2\tau)\, \Gamma(3\sigma-u, \tau, 4\sigma)\, \Gamma(3\sigma+u, \tau, 4\sigma)}.$$

The elliptic gamma function and similar double and even triple infinite products appear in correlation functions of the eight-vertex model [JMN, JKKMW] and boundary spontaneous magnetization at corners of the Ising model [DP].

Our own interest in the elliptic gamma functions arose from the study of hypergeometric solutions of elliptic qKZB difference equations [FTV, FV]. In these solutions the role of powers of linear functions appearing in the Gauss hypergeometric function is played by ratios of elliptic gamma functions.

In this paper, after reviewing some well-known properties of the theta function θ_0, we derive several identities for elliptic gamma functions. The most remarkable identities are the "modular" three-term relations of

Theorem 4.1, which connect values of Γ at points related by $\mathrm{SL}(3,\mathbb{Z})$ acting on the periods (τ,σ) by fractional linear transformations. For example, there is a polynomial $Q(z;\tau,\sigma)$ of degree three in z whose coefficients are rational functions of τ,σ such that

$$\Gamma(z/\sigma,\tau/\sigma,-1/\sigma)=e^{i\pi Q(z;\tau,\sigma)}\,\Gamma((z-\sigma)/\tau,-1/\tau,-\sigma/\tau)\,\Gamma(z,\tau,\sigma).$$

These identities have an interpretation in terms of a generalization of Jacobi modular forms: Γ may be interpreted as the value of generators of an "automorphic form of degree 1": just as the theta function θ_0 is a "degree 0" automorphic form associated to a 1-cocycle in $H^1(G,M)$ where $G=SL(2,\mathbb{Z})\ltimes\mathbb{Z}^2$ an $M=\exp 2\pi i\mathbb{Q}(\tau)[z]$, the elliptic gamma function defines a "degree 1" automorphic form associated to a 2-cocycle in $H^2(G,M)$ with $G=\mathrm{SL}(3,\mathbb{Z})\ltimes\mathbb{Z}^3$ and $M=\exp 2\pi i\mathbb{Q}(\tau,\sigma)[z]$. See Section 7 for a more precise statement.

The modular identities have interesting degenerations: limiting versions relate the gamma functions at points where the periods are linearly dependent over the rationals to the Euler dilogarithm function (Theorems 5.2, 5.4, 5.5). On the other hand, in the semiclassical limit $\varepsilon\to 0$ the "phase function" $\Gamma(z+\varepsilon,\tau,2\varepsilon/\beta)/\Gamma(z-\varepsilon,\tau,2\varepsilon/\beta)$ tends to $\theta_0(z,\tau)^\beta$, and the modular identities reduce to the Jacobi modular properties of theta functions, see Section 6.

One intriguing aspect of the elliptic gamma function, which is a priori defined for periods τ,σ in the upper half plane, is that it may be extended, by a simple reflection, to a function of $\tau,\sigma\in\mathbb{C}-\mathbb{R}$ in such a way that all identities remain true. Moreover the elliptic gamma function also has a limit as τ or σ (but not both) approach, from either side, a subset X of full Lebesgue measure of the real axis, see Theorem 3.5. This subset contains all irrational algebraic real numbers.

The paper is organized as follows. In Section 2 we review some well-known properties of the odd Jacobi theta function. Then in Section 3 we introduce the elliptic gamma function: after giving its definition and its elementary properties, we study its trigonometric and rational degenerations. Then we derive a summation formula that allows us to study the limit as one approaches the real axis. The modular properties of the elliptic gamma function are given in Section 4. The properties of the elliptic gamma function at special values of its arguments are studied in Section 5. There the relation to dilogarithms appears. In Section 6 we study the semiclassical limit, in which the identities of elliptic gamma functions reduce to differential and difference equations obeyed by powers of theta functions. In the last section we introduce the notion of automorphic forms of degree 1 and relate the modular identities satisfied by gamma functions to these automorphic forms and to the cohomology of $\mathrm{SL}(3,\mathbb{Z})\times\mathbb{Z}^3$.

2. THETA FUNCTIONS

2.1. *The theta function.* Jacobi's first theta function is defined by the series

$$\theta(z,\tau) = -\sum_{j\in\mathbb{Z}} e^{i\pi\tau(j+1/2)^2 + 2\pi i(j+1/2)(z+1/2)}, \qquad z,\tau\in\mathbb{C}, \qquad \operatorname{Im}\tau > 0.$$

It is an entire holomorphic odd function such that

$$\theta(z+n+m\tau,\tau) = (-1)^{m+n}\, e^{-\pi i m^2\tau - 2\pi i m z}\,\theta(z,\tau), \qquad m,n\in\mathbb{Z}, \tag{1}$$

and obeys the heat equation

$$4\pi i\frac{\partial}{\partial\tau}\theta(z,\tau) = \theta''(z,\tau). \tag{2}$$

Its transformation properties with respect to $SL(2,\mathbb{Z})$ are described in terms of generators by the identities:

$$\theta(-z,\tau) = -\theta(z,\tau), \qquad \theta(z,\tau+1) = e^{(i\pi/4)}\,\theta(z,\tau),$$
$$\theta\left(\frac{z}{\tau},-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\,e^{i\pi z^2\tau}\theta(z,\tau).$$

The square root is the one in the right half plane.

2.2. *Infinite products.* Let $x, q\in\mathbb{C}$ with $|q|<1$. The function

$$(x;q) = \prod_{j=0}^{\infty}(1-xq^j)$$

is a solution of the functional equation

$$(qx;q) = \frac{1}{1-x}(x;q).$$

Using the identity

$$1-y = \exp\left(-\sum_{j=1}^{\infty} y^j/j\right), \qquad |y|<1, \tag{3}$$

and summing the geometric series yields the summation formula

$$(x;q) = \exp\left(-\sum_{j=1}^{\infty}\frac{x^j}{j(1-q^j)}\right), \qquad |x|<1, \qquad |q|<1. \tag{4}$$

2.3. *Product representation of theta functions.* Let $x=e^{2\pi iz}$ and $q=e^{2\pi i\tau}$. Then we have the Jacobi triple product identity

$$\theta(z,\tau)=ie^{\pi i(\tau/4-z)}(x;q)(q/x;q)(q;q). \tag{5}$$

In particular we have $\theta'(0,\tau)=2\pi\eta(\tau)^3$, with $\eta(\tau)=e^{\pi i\tau/12}(q;q)$ the Dedekind function.

From the modular properties of θ, we deduce the modular properties of the Dedekind function: $\eta(\tau+1)=e^{\pi i/12}\eta(\tau)$ and $\eta(-1/\tau)=(-i\tau)^{1/2}\,\eta(\tau)$ (up to a third root of unity, which is 1, as one sees by setting $\tau=i$).

We will also need

$$\theta_0(z,\tau)=(x;q)(q/x;q)=-i\,\frac{e^{\pi i(z-\tau/4)}}{(q;q)}\,\theta(z,\tau).$$

This function obeys

$$\begin{aligned}\theta_0(z+1,\tau)&=\theta_0(z,\tau),\\ \theta_0(z+\tau,\tau)&=-e^{-2\pi iz}\theta_0(z,\tau),\\ \theta_0(\tau-z,\tau)&=\theta_0(z,\tau).\end{aligned} \tag{6}$$

Its modular properties follow from those of θ and η: they are $\theta_0(z,\tau+1)=\theta_0(z,\tau)$, and if $z'=z/\tau$, $\tau'=-1/\tau$,

$$e^{\pi i(\tau/6-z)}\theta_0(z,\tau)=ie^{\pi i(-zz'+\tau'/6-z')}\,\theta_0(z',\tau'). \tag{7}$$

The summation formula (4) implies

$$\theta_0(z,\tau)=\exp\left(-i\sum_{j=1}^{\infty}\frac{\cos\pi j(2z-\tau)}{j\sin\pi j\tau}\right),\qquad 0<\operatorname{Im}z<\operatorname{Im}\tau. \tag{8}$$

3. ELLIPTIC GAMMA FUNCTIONS

3.1. *Definitions and elementary properties.* Here we consider two parameters τ and σ in the upper half plane, and set $q=e^{2\pi i\tau}$, $r=e^{2\pi i\sigma}$, and consider the function of $x=e^{2\pi iz}$,

$$(x;q,r)=\prod_{j,k=0}^{\infty}(1-xq^jr^k)=(x;r,q)$$

It is a solution of the functional equations

$$(qx; q, r) = \frac{(x; q, r)}{(x; r)}, \qquad (rx; q, r) = \frac{(x; q, r)}{(x; q)}. \tag{9}$$

For $|x| < 1$, we have the formula

$$(x; q, r) = \exp\left(- \sum_{l=1}^{\infty} x^l / l(1-q^l)(1-r^l) \right). \tag{10}$$

It is obtained as (4) by expanding the logarithm in a Taylor series in x and then summing the resulting geometric series.

The *elliptic gamma function* is

$$\Gamma(z, \tau, \sigma) = \frac{(qr/x; q, r)}{(x; q, r)}.$$

THEOREM 3.1. *The elliptic gamma function obeys the identities*

$$\Gamma(z, \tau, \sigma) = \Gamma(z, \sigma, \tau), \tag{11}$$

$$\Gamma(z+1, \tau, \sigma) = \Gamma(z, \tau, \sigma), \tag{12}$$

$$\Gamma(z+\sigma, \tau, \sigma) = \theta_0(z, \tau)\, \Gamma(z, \tau, \sigma), \tag{13}$$

$$\Gamma(z+\tau, \tau, \sigma) = \theta_0(z, \sigma)\, \Gamma(z, \tau, \sigma), \tag{14}$$

and is normalized by $\Gamma((\tau+\sigma)/2, \tau, \sigma) = 1$. *As a function of* z, $\Gamma(z, \tau, \sigma)$ *is a meromorphic function whose zeros and poles are all simple. The zeros are at* $z = (j+1)\,\tau + (k+1)\,\sigma + l$, *and the poles are at* $z = -j\tau - k\sigma + l$, *where* j, k *run over nonnegative integers and* l *over all integers.*

Proof. It is obvious that Γ is symmetric under interchange of τ and σ and is 1-periodic. The remaining identities follow from 9. The zeros of $(x; q, r)$ are at $x = q^{-j} r^{-k}$, $j, k = 0, 1, 2, \ldots$. This implies the statement about zeros and poles. ∎

Proof of Theorem 1.1. It remains to prove uniqueness. The point is that $u(z) = \Gamma(z, \tau, \sigma)$ has no zeros in the strip $0 < \operatorname{Im} z < \operatorname{Im} \sigma + \varepsilon$, for some $\varepsilon > 0$. If $v(z)$ is another 1-periodic solution, holomorphic in the upper half plane, then $v(z)/u(z)$ is a doubly periodic function with periods 1 and σ. It is holomorphic in the same strip and thus, by periodicity, a bounded entire function. By Liouville's theorem, v/u is thus constant, which implies our claim. ∎

Finally, we mention some elementary identities, which may be thought of as the analogues of the classical formula $\Gamma(1-z)\,\Gamma(z)=\pi/\sin(\pi z)$:

PROPOSITION 3.2.

$$\Gamma(z,\tau,\sigma)\,\Gamma(\sigma-z,\tau,\sigma)=\frac{1}{\theta_0(z,\sigma)},$$

$$\Gamma(z,\tau,\sigma)\,\Gamma(\tau-z,\tau,\sigma)=\frac{1}{\theta_0(z,\tau)},$$

$$\Gamma(z,\tau,\sigma)\,\Gamma(\tau+\sigma-z,\tau,\sigma)=1.$$

3.2. *Trigonometric and rational limit.* We have the trigonometric and rational limit of the theta function:

$$\frac{\theta_0(\sigma s,\tau)}{\theta_0(\sigma,\tau)}\xrightarrow{\tau\to i\infty}\frac{1-e^{2\pi i\sigma s}}{1-e^{2\pi i\sigma}}\xrightarrow{\sigma\to 0} s.$$

Let $\bar{\Gamma}$ be the function

$$\bar{\Gamma}(s,\tau,\sigma)=\frac{(r;r)}{(q;q)}\,\theta_0(\sigma,\tau)^{1-s}\,\Gamma(\sigma s,\tau,\sigma),\qquad q=e^{2\pi i\tau},\qquad r=e^{2\pi i\sigma}.$$

Then $u(s)=\bar{\Gamma}(s,\tau,\sigma)$ is a solution of the functional equation

$$u(s+1)=\frac{\theta_0(\sigma s,\tau)}{\theta_0(\sigma,\tau)}\,u(s).$$

The normalization was chosen here so that $u(1)=1$. As $\tau\to i\infty$ we recover F. H. Jackson's q-gamma function,

$$\Gamma_{\mathrm{trig}}(s,\sigma)=\lim_{\tau\to i\infty}\bar{\Gamma}(s,\tau,\sigma)=(1-r)^{1-s}\,\frac{(r;r)}{(r^s;r)}.$$

This function obeys the functional equation $\Gamma_{\mathrm{trig}}(s+1,\sigma)=\frac{1-e^{2\pi i\sigma s}}{1-e^{2\pi i\sigma}}\,\Gamma_{\mathrm{trig}}(s,\sigma)$, and degenerates to the Euler gamma function $\Gamma_{\mathrm{Euler}}(s)=\int_0^\infty t^{s-1}e^{-t}\,dt$:

$$\lim_{\sigma\to 0}\Gamma_{\mathrm{trig}}(s,\sigma)=\Gamma_{\mathrm{Euler}}(s).$$

See [A] for an account of the properties of the q-gamma function.

3.3. *The summation formula.* From (10) we get

$$\Gamma(z,\tau,\sigma)=\exp\left(-\frac{i}{2}\sum_{j=1}^{\infty}\frac{\sin(\pi j(2z-\tau-\sigma))}{j\sin(\pi j\tau)\sin(\pi j\sigma)}\right). \tag{15}$$

The region of absolute convergence of this series also include points where τ or σ do not have positive imaginary part. If $\tau,\sigma\in\mathbb{C}-\mathbb{R}$, then the series converges absolutely if and only if

$$|\mathrm{Im}(2z-\tau-\sigma)|<|\mathrm{Im}(\tau)|+|\mathrm{Im}(\sigma)|. \tag{16}$$

Whenever both sides of the equations are in the region of convergence of this series we then clearly have

$$\Gamma(z,-\tau,\sigma)=\Gamma(\sigma-z,\tau,\sigma),\qquad \Gamma(z,\tau,-\sigma)=\Gamma(\tau-z,\tau,\sigma).$$

These formulae can be used to extend the definition of the elliptic gamma function for $\sigma,\tau\in\mathbb{C}-\mathbb{R}$, as we do in the next subsection.

3.4. *Extending the range of parameters.* Since many operations we perform do not preserve the upper half plane, it is important to extend the range of values τ and σ can take. We set

$$(x;q^{-1})=\frac{1}{(qx;q)},\qquad (x;q^{-1},r)=\frac{1}{(qx;q,r)}\qquad (x;q,r^{-1})=\frac{1}{(rx;q,r)}. \tag{17}$$

These formulae define a unique extension of the functions $(x;q)$, $(x;q,r)$ to meromorphic functions on $\{(x,q,r)\mid |q|\neq 1\neq |r|\}$ obeying 17. It is clear that the functional relations

$$(qx,q)=\frac{1}{1-x}(x,q),\qquad (qx;q,r)=\frac{1}{(x;r)}(x;q,r),$$

still hold in this larger domain. Correspondingly, we extend the definition of θ_0 and the elliptic gamma function by using the same formulae in terms of the infinite products. We obtain:

$$\theta_0(z,-\tau)=\frac{1}{\theta_0(z+\tau,\tau)},$$

$$\Gamma(z,-\tau,\sigma)=\frac{1}{\Gamma(z+\tau,\tau,\sigma)},$$

$$\Gamma(z,\tau,-\sigma)=\frac{1}{\Gamma(z+\sigma,\tau,\sigma)}.$$

A straightforward check gives the following result:

THEOREM 3.3. *The identities* (6) *for* θ_0 *hold for all* $\tau \in \mathbb{C} - \mathbb{R}$. *The identities of Theorem* 1.1 *and Prop.* 3.2 *for* Γ *and* θ_0 *hold for all* $z \in \mathbb{C}$, $\tau, \sigma \in \mathbb{C} - \mathbb{R}$ *whenever both sides are defined. The summation formula* (15) *is valid for all* $z \in \mathbb{C}$, $\tau, \sigma \in \mathbb{C} - \mathbb{R}$ *such that the sum converges absolutely.*

However, the statements about the position of zeros and poles are no longer valid.

3.5. *Approaching the real axis.* Here we notice that the series (15) actually also converges for certain real values of τ or σ. Indeed let, for any $\alpha > 1$, X_α denote the set of real numbers τ such that $\min_{k \in \mathbb{Z}} |j\tau - k| > j^{-\alpha}$ for all but finitely many integers $j > 0$. By Khintchin's theorem (see, e.g., [C, Chapt. VII]), X_α, and therefore also $X = \bigcup_{\alpha > 1} X_\alpha$, is the complement in $\mathbb{R}$ of a set of Lebesgue measure zero. The set X contains in particular all irrational algebraic numbers. For $\tau \in X$, one has, for some $\alpha > 1$, the bound $|\sin(\pi j \tau)| \geqslant j^{-\alpha}$ for all sufficiently large j. Therefore, the series is also absolutely convergent if

$$\tau \in X, \qquad |\mathrm{Im}(2z - \sigma)| < |\mathrm{Im}\, \sigma|,$$

or, by symmetry, if

$$\sigma \in X, \qquad |\mathrm{Im}(2z - \tau)| < |\mathrm{Im}\, \tau|.$$

These results may be summarized as follows.

PROPOSITION 3.4. *Let* $X = \bigcup_{\alpha > 1} \{\tau \in \mathbb{R} \mid \min_{k \in \mathbb{Z}} |j\tau - k| > j^{-\alpha}, \forall j \gg 1\}$. *Then* $N = \mathbb{R} - X$ *has Lebesgue measure zero and the series appearing in* (15) *converges absolutely for all* $\tau, \sigma \in \mathbb{C} - N$ *and* $z \in \mathbb{C}$ *obeying*

$$|\mathrm{Im}(2z - \tau - \sigma)| < |\mathrm{Im}(\tau)| + |\mathrm{Im}(\sigma)|.$$

A more precise result is the following "wall crossing theorem", which shows that the values of $\Gamma(z, \tau, \sigma)$ for real τ are obtained as suitable limits from either side.

THEOREM 3.5. *Let* X *be the subset of the real line of Prop.* 3.4. *and suppose that* $\tau \in X$ *and* $|\mathrm{Im}(2z - \sigma)| < |\mathrm{Im}\, \sigma|$. *Then, as a function of* $\varepsilon \in \mathbb{R}$, $\Gamma(z, \tau + i\varepsilon, \sigma)$, *as given by the convergent series* (15) *is continuous at* $\varepsilon = 0$.

Proof. We need to estimate the terms in the sum (15) uniformly in ε. Let us assume for definiteness that $\varepsilon \geqslant 0$. The case $\varepsilon \leqslant 0$ is treated in the same way. If $|\mathrm{Im}\, x| \geqslant \delta > 0$, we have $c(\delta) \exp(|\mathrm{Im}\, x|) \leqslant |\sin x| \leqslant \exp(|\mathrm{Im}\, x|)$, with $0 < c(\delta) = 1 - e^{-2\delta} < 1$. This implies the bound on the jth term of (15):

$$\left| \frac{\sin(\pi j(2z - \tau - i\varepsilon - \sigma))}{j \sin(\pi j(\tau + i\varepsilon)) \sin(\pi j \sigma)} \right| \leqslant c(\pi\, |\mathrm{Im}\, \sigma|)^{-1} \frac{e^{\pi j(|\mathrm{Im}(2z - \sigma) - \varepsilon| - |\mathrm{Im}\, \sigma|)}}{j \sin \pi j(\tau + i\varepsilon)}.$$

Let α, N be such that $\min_{k \in \mathbb{Z}} |j\tau - k| > j^{-\alpha}$, $\forall j \geqslant N$. The next step is to find a lower bound for $\sin \pi j(\tau + i\varepsilon)$ for $j \geqslant N$. This is done in two different ways depending on whether $j\varepsilon$ is small or large.

(a) If $\sinh \pi j\varepsilon \leqslant j^{-\alpha} e^{\pi\varepsilon j}$, the triangle inequality can be used in the form

$$\begin{aligned} |\sin \pi j(\tau + i\varepsilon)| &= \tfrac{1}{2} |e^{\pi i\tau j - \pi\varepsilon j} - e^{-\pi i\tau j + \pi\varepsilon j}| \\ &\geqslant \tfrac{1}{2} |e^{\pi i\tau j} - e^{-\pi i\tau j}|\, e^{\pi\varepsilon j} - \tfrac{1}{2} |e^{\pi\varepsilon j} - e^{-\pi\varepsilon j}| \\ &= |\sin \pi\tau j|\, e^{\pi\varepsilon j} - \sinh \pi\varepsilon j. \end{aligned} \tag{18}$$

Let $k \in \mathbb{Z}$ so that $|\tau j - k| \leqslant 1/2$. By using $|\sin \pi x| \geqslant 2\,|x|$ for $-1/2 \leqslant x \leqslant 1/2$, we get $|\sin \pi\tau j| = |\sin \pi(\tau j - k)| \geqslant 2j^{-\alpha}$. Thus

$$|\sin \pi j(\tau + i\varepsilon)| \geqslant 2j^{-\alpha} e^{\varepsilon\pi j} - \sinh \pi\varepsilon j \geqslant j^{-\alpha}\, e^{\varepsilon\pi j}.$$

(b) If $\sinh \pi j\varepsilon \geqslant j^{-\alpha} e^{\pi\varepsilon j}$, the triangle inequality implies

$$\begin{aligned} |\sin \pi j(\tau + i\varepsilon)| &= \tfrac{1}{2} |e^{\pi i\tau j - \pi\varepsilon j} - e^{-\pi i\tau j + \pi\varepsilon j}| \\ &\geqslant \tfrac{1}{2} (|e^{-\pi i\tau j + \pi\varepsilon j}| - |e^{\pi i\tau j - \pi\varepsilon j}|) \\ &= \sinh \pi\varepsilon j \geqslant j^{-\alpha} e^{\pi\varepsilon j} \end{aligned}$$

In both cases (a) and (b) we get the lower bound

$$|\sin \pi j(\tau + i\varepsilon)| \geqslant j^{-\alpha} e^{\pi\varepsilon j},$$

for all $j \geqslant N$. Therefore we have the uniform bound on the jth term of the series

$$\left| \frac{\sin(\pi j(2z - \tau - i\varepsilon - \sigma))}{j \sin(\pi j(\tau + i\varepsilon)) \sin(\pi j \sigma)} \right| \leqslant c(\pi\, \mathrm{Im}\, \sigma)^{-1}\, j^{\alpha - 1} e^{\pi j(|\mathrm{Im}(2z - \sigma)| - |\mathrm{Im}\, \sigma|)}$$

for all $j \geqslant N$. The sum over j of this expression converges if $|\mathrm{Im}(2z - \sigma)| \leqslant |\mathrm{Im}\, \sigma|$. So our series is bounded, for all ε, by a single absolutely convergent series. It follows that the sum is a continuous function of ε.

4. MODULAR PROPERTIES

We consider the transformation properties of the elliptic gamma function under modular transformations of σ and τ. We have the identities

THEOREM 4.1. *Suppose that* $\tau, \sigma, \sigma/\tau, \tau+\sigma \in \mathbb{C}-\mathbb{R}$. *Let*

$$Q(z;\tau,\sigma)=\frac{z^3}{3\tau\sigma}-\frac{\tau+\sigma-1}{2\tau\sigma}z^2+\frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{6\tau\sigma}z$$

$$+\frac{1}{12}(\tau+\sigma-1)(\tau^{-1}+\sigma^{-1}-1).$$

Then

$$\Gamma(z,\tau+1,\sigma)=\Gamma(z,\tau,\sigma+1)=\Gamma(z,\tau,\sigma), \tag{18}$$

$$\Gamma(z,\tau+\sigma,\sigma)=\frac{\Gamma(z,\tau,\sigma)}{\Gamma(z+\tau,\tau,\sigma+\tau)}, \tag{19}$$

$$\Gamma(z/\sigma,\tau/\sigma,-1/\sigma)=e^{i\pi Q(z;\tau,\sigma)}\Gamma((z-\sigma)/\tau,-1/\tau,-\sigma/\tau)\,\Gamma(z,\tau,\sigma), \tag{20}$$

$$\Gamma(z/\tau,-1/\tau,\sigma/\tau)=e^{i\pi Q(z;\tau,\sigma)}\Gamma((z-\tau)/\sigma,-\tau/\sigma,-1/\sigma)\,\Gamma(z,\tau,\sigma). \tag{21}$$

Proof. We give the proof of these identities in the domain where the second and third arguments of all gamma functions have positive imaginary part, so that the gamma functions are defined by the product formula. The general case is reduced to this case by inserting the definitions of 3.4, as a straightforward check shows. The first two identities are obvious, the third follows from the identity

$$(x;qr,r)(qx;q,qr)=(x;q,r),$$

which is easy to check. The last identity is obtained from (21) by exchanging τ and σ and using the symmetry (11).

To prove (21), we show that the ratio between the two sides of the equation is a triply periodic meromorphic function and is therefore constant, and determine the constant by evaluating the ratio at a special value.

Let $A(z;\tau,\sigma)$ be the ratio

$$\frac{\Gamma\left(\frac{z}{\sigma},\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)}{\Gamma\left(\frac{z-\sigma}{\tau},-\frac{1}{\tau},-\frac{\sigma}{\tau}\right)\Gamma(z,\tau,\sigma)}.$$

We have

$$\frac{A(z-1;\tau,\sigma)}{A(z;\tau,\sigma)}=\frac{\theta_0\left(\dfrac{z}{\sigma},\dfrac{\tau}{\sigma}\right)}{\theta_0\left(\dfrac{z-\sigma}{\tau},-\dfrac{\sigma}{\tau}\right)}$$

$$=ie^{\pi i(-\tau/6\sigma+z/\sigma-\sigma/6\tau-z/\tau-z^2/\tau\sigma)}\frac{\theta_0\left(\dfrac{z}{\tau},-\dfrac{\sigma}{\tau}\right)}{\theta_0\left(\dfrac{z-\sigma}{\tau},-\dfrac{\sigma}{\tau}\right)}$$

$$=\exp(\pi iP(z;\tau,\sigma)),$$

with

$$P(z,\tau,\sigma)=-\frac{z^2}{\tau\sigma}+z(\tau^{-1}+\sigma^{-1})-\frac{\sigma}{6\tau}-\frac{\tau}{6\sigma}-\frac{1}{2}=P(z,\sigma,\tau).$$

Similarly, we find

$$\frac{A(z+\tau;\tau,\sigma)}{A(z;\tau,\sigma)}=\exp(\pi iP(z;-1,\sigma)),$$

$$\frac{A(z+\sigma;\tau,\sigma)}{A(z;\tau,\sigma)}=\exp(\pi iP(z;\tau,-1)).$$

The polynomial Q is designed to compensate for these terms. This is most easily seen by setting $Q(z;\tau,\sigma,\rho)=Q(-z/\rho;-\tau/\rho,-\sigma/\rho)$ which is symmetric in τ,σ,ρ and obeys

$$Q(z+\tau;\tau,\sigma,\rho)=Q(z;\tau,\sigma,\rho)+P(z;\sigma,\rho).$$

By using this identity for permutations of τ,σ,ρ, with $\rho=-1$, we deduce that $Ae^{-\pi iQ}$ is a triply periodic meromorphic function of z and is thus constant. To compute this constant, we set $z=(\tau+\sigma-1)/2$. Then all gamma functions are equal to one, and $Q(z;\tau,\sigma)=0$. Thus the constant is one. ∎

5. SPECIAL VALUES

Here we consider the degeneration of our three term relations when the periods τ,σ are linearly dependent. The simplest case is when $\tau=\sigma$. Then we can write the gamma function as

$$\Gamma(z,\tau,\tau)=\prod_{j=0}^{\infty}\left(\frac{1-e^{-2\pi iz}q^{j+2}}{1-e^{2\pi iz}q^{j}}\right)^{j+1},\qquad q=e^{2\pi i\tau}.$$

To express the result we need to recall a simple property of the dilogarithm function.

PROPOSITION 5.1. *Let* $\mathrm{Li}_2(x) = \sum_{j=1}^{\infty} (x^j/j^2)$ *be the dilogarithm and let for* $\mathrm{Im}\, t < 0$,

$$\psi(t) = \exp\left(t \ln(1 - e^{-2\pi i t}) - \frac{1}{2\pi i} \mathrm{Li}_2(e^{-2\pi i t}) \right),$$

where the branch of the logarithm is determined by $\ln(1-x) = -\sum_1^{\infty} x^j/j$, $(|x| < 1)$. *Then* $\psi(t)$ *has an analytic continuation to a meromorphic function on the complex plane. It has a zero of order* n *at* $t = n$ *and a pole of order* n *at* $t = -n$ $(n = 1, 2, ...)$ *and no other zeros or poles. Moreover* ψ *obeys the functional equation*

$$\psi(t+1) = (1 - e^{-2\pi i t})\, \psi(t),$$

and the estimate

$$\psi(t) = 1 + 0(|\mathrm{Im}\, t|\, e^{-2\pi\, |\mathrm{Im}\, t|}),$$

as $\mathrm{Im}\, t \to -\infty$.

Proof. It is clear that the Taylor series defines a holomorphic function on the lower half plane obeying the functional equation. The singularities on the real axis can be studied using the integral representation of the dilogarithm:

$$\psi(t) = \exp\left(2\pi i \int_t^{-i\infty} \frac{s\, ds}{e^{2\pi i s} - 1} \right).$$

This well-known formula may be checked by expanding the geometric series in the integrand and integrating term by term. From this formula we see that the only potential singularities of the argument of the exponential function are at integer values of t. At $t = 0$, however, the function is regular ($\psi(0) = \exp i\pi/12$). The functional equation then implies the statement about zeros and poles. In particular ψ is single-valued at the singularities, so that the integral representation defines a meromorphic function with no other zeros or poles.

The estimate follows from the inequalities $|\ln(1-x)| \leqslant 2\, |x|$, if $|x|$ is sufficiently small and $|\mathrm{Li}_2(x)| \leqslant |x| \sum_1^{\infty} 1/j^2$ if $|x| \leqslant 1$. ▮

THEOREM 5.2. *Let* $\operatorname{Im}\tau > 0$ *and* $z \in \mathbb{C} - (\mathbb{Z} + \tau\mathbb{Z})$. *Then*

$$\Gamma(z, \tau, \tau) = \frac{e^{-\pi i Q(z;\, \tau,\, \tau)}}{\theta_0\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)} \prod_{k=0}^{\infty} \frac{\psi\left(\frac{k+1+z}{\tau}\right)}{\psi\left(\frac{k-z}{\tau}\right)}.$$

The infinite product is convergent thanks to the estimate of Prop. 5.1

The following calculation is not a completely rigorous proof of this theorem, but it is more transparent than the correct proof, which consists of showing that the ratio between left and right-hand side is an entire doubly periodic meromorphic function taking the value 1 at a special point.

We start from the three term relation for Γ.

$$\Gamma(z, \tau, \sigma) = e^{-\pi i Q(z;\, \tau,\, \sigma)} \frac{\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{z-\tau}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)} = \frac{e^{-\pi i Q(z;\, \tau,\, \sigma)}}{\theta_0\left(\frac{z}{\sigma}, -\frac{1}{\sigma}\right)} \frac{\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{z}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)}.$$

Let us take the limit $\sigma \to \tau$. The limit of the ratio of gamma function is delicate. Set $\sigma = \tau(1+\varepsilon)$ and introduce multiplicative variables:

$$q_1 = e^{-2\pi i/\tau}, \quad q_2 = e^{-2\pi i/\sigma}, \quad r_1 = e^{2\pi i \sigma/\tau},$$
$$r_2 = e^{-2\pi i \tau/\sigma}, \quad x_1 = e^{2\pi i z/\tau}, \quad x_2 = e^{2\pi i z/\sigma}.$$

Then, by the summation formula,

$$\ln \frac{\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{z}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)} = -\sum_{j=1}^{\infty} \frac{(q_1 r_1/x_1)^j - x_1^j}{j(1-q_1^j)(1-r_1^j)}$$
$$+ \sum_{j=1}^{\infty} \frac{(q_2 r_2/x_2)^j - x_2^j}{j(1-q_2^j)(1-r_2^j)}.$$

We now expand the various terms around $\varepsilon = 0$. We get

$$q_2 = q_1\left(1 + \frac{2\pi i}{\tau}\varepsilon\right) + O(\varepsilon^2), \qquad r_1 = 1 + 2\pi i \varepsilon + O(\varepsilon^2),$$

$$r_2 = 1 + 2\pi i\varepsilon + O(\varepsilon^2), \qquad x_2 = e^{2\pi i z/\tau}\left(1 - 2\pi i \frac{z}{\tau}\varepsilon\right).$$

The singular terms have the expansion

$$\frac{1}{1-r_1^j} = -\frac{1}{2\pi ij\varepsilon} + \frac{1}{2} + O(\varepsilon)\,,$$
$$\frac{1}{1-r_2^j} = -\frac{1}{2\pi ij\varepsilon} - \frac{1}{2\pi ij} + \frac{1}{2} + O(\varepsilon)\,.$$

Inserting this in the summation formula yields

$$\ln \frac{\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{z}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)} = -\sum_1^\infty \frac{q_1^j\, x_1^{-j}\left(\frac{1}{\tau}+\frac{z}{\tau}\right)}{j(1-q_1^j)} - \sum_1^\infty \frac{x_1^j}{j(1-q_1^j)}\cdot\frac{z}{\tau}$$
$$-\sum_1^\infty \frac{q_1^j x_1^{-j} - x_1^j}{2\pi i j^2(1-q_1^j)} - \sum_1^\infty \frac{q_1^j\, x_1^{-j} - x_1^j}{j(1-q_1^j)^2}\cdot\frac{q_1^j}{\tau} + O(\varepsilon).$$

After expanding the denominators into geometric series and exchanging the summations, the sums over j become Taylor series for (di)logarithms. The result is the formula of the theorem.

COROLLARY 5.3. *Let* $\tau \to 0$ *on a ray* $\{s\tau_0 \mid s>0\}$ *with* $\operatorname{Im}(\tau_0)>0$ *and let* $z=u+v\tau_0$ *with* $-1<u<0$, $v\in\mathbb{R}$ *be fixed. Then*

$$\Gamma(z,\tau,\tau) = e^{-\pi i Q(z;\,\tau,\,\tau)}(1+\mathcal{O}(e^{-c/\operatorname{Im}\tau}))$$

for some $c>0$ *depending on* z, τ_0.

Remark. As $\tau\to 0$ along this ray, the zeros and poles of $\Gamma(z,\tau,\tau)$ as a function of z accumulate on the lines $n+s\tau_0$, $n\in\mathbb{Z}$, $s\in\mathbb{R}$. The assumption on z means that z lies between two lines. One can relate this case to the more general case of z between any two other lines by using the fact that $\Gamma(z,\tau,\tau)$ is 1-periodic.

Proof of Corollary 5.3. The assumption on z implies that the arguments of the ψ functions in Theorem 5.2 obey

$$\operatorname{Im}\frac{k+1+z}{\tau} = (k+1+u)\operatorname{Im}\frac{1}{\tau}, \qquad \operatorname{Im}\frac{k-z}{\tau} = (k-u)\operatorname{Im}\frac{1}{\tau},$$

Since $\operatorname{Im}(1/\tau)\to -\infty$ and $k+1+u, k-u>0$ for all $k=0,1,2,\ldots$, we can use the estimate of Prop. 5.1 to show that the product of ratios of ψ functions tends to 1 as $\tau\to 0$ with an error term that is smaller than $c_1\,e^{-c/\operatorname{Im}\tau}$ for some constants $c_1, c>0$. Similarly θ_0 tends to one, and we are left with the exponential of Q. ∎

More generally, one can find similar formulae when the periods τ, σ are linearly dependent over the rationals. The following two theorems reduce (in different ways) this computation to the case studied above.

THEOREM 5.4. *Let a, b be positive integers. Then*

$$\Gamma(z, a\tau, b\tau) = \prod_{r=0}^{b-1} \prod_{s=0}^{a-1} \Gamma(z + (ar + bs)\,\tau, ab\tau, ab\tau).$$

Proof. Let $q = e^{2\pi i \tau}$, $x = e^{2\pi i z}$. Then

$$\Gamma(z, a\tau, b\tau) = \frac{(q^{a+b}x^{-1}; q^a, q^b)}{(x; q^a, q^b)}.$$

We first prove an identity for double products

$$\begin{aligned}(x; q^a, q^b) &= \prod_{j,k=0}^{\infty} (1 - xq^{aj+bk}) \\ &= \prod_{r=0}^{b-1} \prod_{s=0}^{a-1} \prod_{j,k=0}^{\infty} (1 - xq^{a(r+bj)+b(s+ak)}) \\ &= \prod_{r=0}^{b-1} \prod_{s=0}^{a-1} (xq^{ar+bs}; q^{ab}, q^{ab}).\end{aligned}$$

If we replace x by $q^{a+b}x^{-1}$ in this identity and change variables $r \to b-1-r$, $s \to a-1-s$, we obtain

$$(q^{a+b}x^{-1}; q^a, q^b) = \prod_{r=0}^{b-1} \prod_{s=0}^{a-1} (x^{-1}q^{2ab-ar-bs}; q^{ab}, q^{ab}).$$

Taking the ratio we get the desired identity for gamma functions. ∎

EXAMPLES.

1. $\Gamma(z, \tau, 3\tau) = \Gamma(z, 3\tau, 3\tau)\, \Gamma(z + \tau, 3\tau, 3\tau)\, \Gamma(z + 2\tau, 3\tau, 3\tau)$.
2. $\Gamma(z, 2\tau, 3\tau) = \prod_{j \in \{0, 2, 3, 4, 5, 7\}} \Gamma(z + j\tau, 6\tau, 6\tau)$
3. Setting $a = b$ and rescaling τ we get

$$\Gamma(z, \tau, \tau) = \prod_{j=0}^{2(a-1)} \Gamma(z + j\tau, a\tau, a\tau)^{a - |j-a+1|}.$$

Inserting the formula of Theorem 5.2 into the formula of Theorem 5.4 yields an expression for $\Gamma(z, a\tau, b\tau)$ in terms of dilogarithms. As an application we can compute from Corollary 5.3, the asymptotics of the infinite products

$$\Gamma(z, a\tau, b\tau) = \prod_{j=0}^{\infty} \left(\frac{1 - q^{j+a+b} e^{-2\pi i z}}{1 - q^j e^{2\pi i z}} \right)^{N_{a,b}(j)}.$$

Here $N_{a,b}(j)$ denotes the number of ways j can be written as $j = ar + bs$ with r, s nonnegative integers. We have, as $\tau \to 0$ as in Corollary 5.3,

$$\Gamma(z, a\tau, b\tau) = e^{-\pi i \sum_{r=0}^{b-1} \sum_{s=0}^{a-1} Q(z + (ar+bs)\tau;\, ab\tau,\, ab\tau)} (1 + \mathcal{O}(e^{-c/\operatorname{Im} \tau})).$$

THEOREM 5.5. *Let the greatest common divisor of natural numbers a, b be* 1. *Consider the function*

$$\Gamma(z, a\tau, b\tau) = \prod_{j=0}^{\infty} \left(\frac{1 - q^{j+a+b} e^{-2\pi i z}}{1 - q^j e^{2\pi i z}} \right)^{N_{a,b}(j)},$$

where $N_{a,b}(j)$ denotes the number of ways j can be written as $j = ar + bs$ with r, s nonnegative integers. Then

$$\Gamma(z, a\tau, b\tau)^{ab} = \Gamma(z, \tau, \tau) \prod_{k=0}^{ab-1} \theta_0(z + k\tau, ab\tau)^{\alpha_k},$$

where $\alpha_k = -ab + k + 1$ if $k = ar + bs$ for some integers $r, s \geqslant 0$ and $\alpha_k = k + 1$ if k cannot be represented in this form.

EXAMPLE.

$$\Gamma(z, 2\tau, 3\tau)^6 = \Gamma(z, \tau, \tau)\, \theta_0(z, 6\tau)^{-5}\, \theta_0(z+\tau, 6\tau)^2\, \theta_0(z + 2\tau, 6\tau)^{-3} \times \theta_0(z + 3\tau, 6\tau)^{-2}\, \theta_0\, (z + 4\tau, 6\tau)^{-1}.$$

To prove the Theorem we shall use the following Lemma.

LEMMA 5.6. *Let $k \in \{0, ..., ab - 1\}$.*

1. *If $k = ar + bs$ for some integers $r, s \geqslant 0$, then $ab - k$ cannot have the form $ab - k = a(i+1) + b(j+1)$ for some integers $i, j \geqslant 0$.*

2. *If k cannot be represented in the form $k = ar + bs$ for some integers $r, s \geqslant 0$, then $ab - k = a(i+1) + b(j+1)$ for some integers $i, j \geqslant 0$.*

Proof of the lemma. If $k = ar + bs$ and $ab - k = a(i+1) + b(j+1)$, then $ab = a(r+i+1) + b(s+j+1)$. Since a, b are relatively prime, this leads to a contradiction. Part 1 is proved.

If k cannot be represented in the form $k = ar + bs$ for some integers $r, s \geqslant 0$, then k can be represented in the form $k = ar' + bs'$ where $0 < r' < b$, $s' < 0$. This gives the desired representation for $ab - k$, $ab - k = a(b - r') - bs'$. Part 2 is proved.

Proof of Theorem 5.5. We have

$$\Gamma(z, a\tau, b\tau)^{ab} = \prod_{k=0}^{ab-1} \prod_{s=0}^{\infty} \frac{(1 - q^{ab(s+1)} q^{-k} e^{-2\pi i z})^{ab\beta_{k,s}}}{(1 - q^{abs} q^{k} e^{2\pi i z})^{ab\gamma_{k,s}}},$$

where $\beta_{k,s}$ is the number of ways $ab(s+1) - k$ can be written as $ab(s+1) - k = a(i+1) + b(j+1)$ with nonnegative integers i, j and $\gamma_{k,s}$ is the number of ways $abs + k$ can be written as $abs + k = ai + bj$ with nonnegative integers i, j. It is easy to see that $\beta_{k,s} = s + \beta_{k,0}$ and $\gamma_{k,s} = s + \gamma_{k,0}$. By the Lemma, $\beta_{k,0} + \gamma_{k,0} = 1$. Notice also that $\alpha_k = k + 1 - ab\gamma_{k,0}$.

Thus we have

$$\begin{aligned}
\Gamma(z, a\tau, b\tau)^{ab} &= \prod_{k=0}^{ab-1} \prod_{s=0}^{\infty} \frac{(1 - q^{ab(s+1)} q^{-k} e^{-2\pi i z})^{ab(s+1) - ab\gamma_{k,0}}}{(1 - q^{abs} q^{k} e^{2\pi i z})^{abs + ab\gamma_{k,0}}} \\
&= \prod_{k=0}^{ab-1} \left(\prod_{s=0}^{\infty} (1 - q^{ab(s+1)} q^{-k} e^{-2\pi i z})(1 - q^{abs} q^{k} e^{2\pi i z}) \right)^{k+1-ab\gamma_{k,0}} \\
&\quad \times \prod_{k=0}^{ab-1} \prod_{s=0}^{\infty} \frac{(1 - q^{ab(s+1)} q^{-k} e^{-2\pi i z})^{ab(s+1) - k - 1}}{(1 - q^{abs} q^{k} e^{2\pi i z})^{abs + k + 1}} \\
&= \Gamma(z, \tau, \tau) \prod_{k=0}^{ab-1} \theta_0(z + k\tau, ab\tau)^{\alpha_k}.
\end{aligned}$$

The theorem is proved.

6. THE PHASE FUNCTION AND THE SEMICLASSICAL LIMIT

Here we introduce the “phase function”, which is the ratio of elliptic gamma functions appearing in hypergeometric integrals [FTV]. It obeys identities which are direct consequences of the identities for elliptic gamma functions. We discuss the phase function here since it has a semiclassical limit in which the identities reduce to more familiar differential equations and modular properties of theta functions.

We keep the notation of 3.1 and introduce a new variable a, setting $\alpha = e^{2\pi i a}$. The phase function is defined as the ratio

$$\Omega_a(z, \tau, \sigma) = \frac{\Gamma(z+a, \tau, \sigma)}{\Gamma(z-a, \tau, \sigma)} = \frac{(qr/x\alpha; q, r)(x/\alpha; q, r)}{(x\alpha; q, r)(qr\alpha/x; q, r)}. \tag{22}$$

The following identities are direct translations of identities for gamma functions:

$$\Omega_a(z+\sigma, \tau, \sigma) = \frac{\theta_0(z+a, \tau)}{\theta_0(z-a, \tau)}\, \Omega_a(z, \tau, \sigma), \tag{23}$$

$$\Omega_a(z+\tau, \tau, \sigma) = \frac{\theta_0(z+a, \sigma)}{\theta_0(z-a, \sigma)}\, \Omega_a(z, \tau, \sigma), \tag{24}$$

$$\Omega_a(z+1, \tau, \sigma) = \Omega_a(z, \tau, \sigma). \tag{25}$$

The modular properties of this function also follow directly from those of the gamma function. In particular

$$\Omega_a(z, \tau+\sigma, \sigma) = \frac{\Omega_a(z, \tau, \sigma)}{\Omega_a(z+\tau, \tau, \sigma+\tau)}, \tag{26}$$

$$\Omega_a(z, \tau+1, \sigma) = \Omega_a(z, \tau, \sigma), \tag{27}$$

$$\Omega_{a/\tau}\left(\frac{z}{\tau}, \frac{\sigma}{\tau}, -\frac{1}{\tau}\right) = e^{\pi i (Q(z+a;\, \tau, \sigma) - Q(z-a;\, \tau, \sigma))}\, \Omega_{a/\sigma}\left(\frac{z-\tau}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) \times \Omega_a(z, \tau, \sigma). \tag{28}$$

The argument of the exponential function is the polynomial

$$R_a(z, \tau, \sigma) = \frac{\pi i a}{3\tau\sigma}\left(6z^2 - 6(\tau+\sigma-1)\, z + 2a^2 + \tau^2 + \sigma^2 + 3\tau\sigma - 3\tau - 3\sigma + 1\right). \tag{29}$$

The summation formula (15) for Γ implies

$$\Omega_a(z, \tau, \sigma) = \exp\left(-i \sum_{\ell=1}^{\infty} \frac{\cos(\pi\ell(2z-\tau-\sigma))\sin(2\pi\ell a)}{\ell \sin(\pi\ell\tau)\sin(\pi\ell\sigma)}\right). \tag{30}$$

By using this formula we can compute the *semiclassical limit* $\sigma \to 0$, $a \to 0$ with $\beta = 2a/\sigma$ fixed: if $0 < \operatorname{Im} z < \operatorname{Im} \tau$ we are in the region of convergence of the series (30) and we get

$$
\begin{aligned}
u(z,\tau) &= \lim_{\varepsilon\to 0} \Omega_\varepsilon(z,\tau,2\varepsilon/\beta)\\
&= \exp\left(-i\sum_{\ell=1}^{\infty}\frac{\cos(\pi\ell(2z-\tau))}{\ell\sin(\pi\ell\tau)}\beta\right)\\
&= \theta_0(z,\tau)^\beta, \qquad 0<\operatorname{Im} z<\operatorname{Im}\tau;
\end{aligned}
$$

cf. (8). To avoid discussing cuts we assume here that β is an integer.

Now the identities for Ω become differential and difference equations for the limit $u(z,\tau)$. Eq. (23) is the obvious differential equation $u'(z,\tau)=\beta(\theta'_0(z,\tau)/\theta_0(z,\tau))\,u(z,\tau)$, where the derivative with respect to z is denoted by a prime. By (7) we have in the semiclassical limit

$$
\frac{\theta_0(z+\varepsilon,\sigma)}{\theta_0(z-\varepsilon,\sigma)}\to e^{-\pi i\beta(2z-1)}.
$$

Then from (24), (25) we see that the semiclassical limit exists for almost all z and has the theta function property

$$
u(z+\tau,\tau)=-e^{-\pi i\beta(2z-1)}u(z,\tau), \qquad u(z+1,\tau)=u(z,\tau).
$$

Therefore the limit is θ_0^β for almost all z,τ. Let us now consider the semi-classical limit of (26). Expanding (30) yields $\Omega_\varepsilon(z+\tau,\tau,\tau+2\varepsilon/\beta)=1-2\pi i\varepsilon\, r(z,\tau)+O(\varepsilon^2)$, with

$$
r(z,\tau)=\sum_{j=1}^{\infty}\frac{\cos(2\pi jz)}{\sin^2(\pi j\tau)}.
$$

Therefore both sides of (27) tend to the same limit. But the terms of order ε reduce to the differential equation

$$
\frac{\partial u}{\partial\tau}=\pi i\beta r(z,\tau)\,u.
$$

The identity (27) becomes $u(z,\tau+1)=u(z,\tau)$. We now turn to (28). Let us assume that $\operatorname{Im}\sigma$, $\operatorname{Im}\sigma/\tau>0$, $\operatorname{Im}\tau>0$. Then all factors in the infinite products (22) tend to one in the semiclassical limit and we get

$$
\Omega_{a/\sigma}\left(\frac{z-\tau}{\sigma},\ -\frac{1}{\sigma},\ -\frac{\tau}{\sigma}\right)\to 1.
$$

Therefore, (28) implies in the limit the modular transformation properties of u:

$$u(z/\tau, -1/\tau) = e^{R_0(z,\tau)} u(z, \tau).$$

The expression R_0 is (see (29))

$$R_0(z, \tau) = \lim_{\varepsilon \to 0} R_\varepsilon(z, \tau, 2\varepsilon\beta^{-1}) = \pi i \beta \left(\frac{z^2}{\tau} - z + \frac{z}{\tau} + \frac{\tau}{6} - \frac{1}{2} + \frac{1}{6\tau} \right)$$

in agreement with (7).

7. A COHOMOLOGICAL INTERPRETATION

Here we give an interpretation of the modular identities obeyed by elliptic gamma functions. These identities may be formulated in terms of the cohomology of $\mathrm{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$ and automorphic forms "of degree 1".

7.1. *Automorphic forms.* Let us first review the well-known theory in degree zero. First some notational preliminaries. We write all groups multiplicatively unless stated otherwise. We denote by $[u]$ the equivalence class of an element u in a quotient of abelian groups. If G is a group, a G-module A is an abelian group with a group homomorphism ρ: $G \to \mathrm{Aut}(A)$. The group $C^j(G, A)$ of j-cochains is the group of maps ϕ: $G^j \to A$, such that $\phi(g_1, ..., g_j) = 1$ if some $g_i = 1$. One sets $C^0(G, A) = A$. The differential $\delta = \delta_j$: $C^j(G, A) \to C^{j+1}(G, A)$ is defined by

$$\delta\phi(g_1, ..., g_{j+1}) = [\rho(g_1)\, \phi(g_2, ..., g_{j+1}) \\ \times \prod_{i=1}^{j} \phi(g_1, ..., g_i g_{i+1}, ..., g_{j+1})^{(-1)^i}]^{(-1)^{j+1}} \phi(g_1, ..., g_j),$$

for $j \geqslant 1$ and $\delta_0 \phi(g) = \phi/\rho(g)\, \phi$ for $j = 0$. The jth cohomology group of G with coefficients A is then $H^j(G, A) = \mathrm{Ker}\, \delta_j / \mathrm{Im}\, \delta_{j-1}$ for $j \geqslant 1$ and $\mathrm{Ker}\, \delta_0$ for $j = 0$.

Suppose now that X is a connected complex manifold with nice action of a group G, so that X/G is a complex manifold. Let N be the multiplicative group of the field of meromorphic functions on X and M be the subgroup of nowhere vanishing holomorphic functions. These groups are G-modules, i.e., we have homomorphisms $G \to \mathrm{Aut}(N)$, $G \to \mathrm{Aut}(M)$: $g \in G$ is mapped to the automorphism $u \mapsto u(g^{-1} \cdot)$. The group of invariants

$H^0(G, N) = N^G$ is identified with the group of non-zero meromorphic functions on X/G. Now let ϕ: $G \to M$ be a 1-cocycle with coefficients M.

We define[2] *automorphic forms of type* ϕ to be functions $u \in N$ so that

$$u(x) = \phi(g, x)\, u(g^{-1}x)$$

or $\phi = \delta u$ in $C^1(G, N)$. The 1-cocycle ϕ is called the factor of automorphy of the automorphic form u.

Automorphic forms corresponding to cocycles ϕ_1, $\phi_2 = \phi_1\, \delta\psi$, ($\psi \in C^0(G, M)$) in the same cohomology class in $H^1(G, M)$ are in one-to-one correspondence via $u \mapsto \psi u$. Thus it is convenient to consider equivalence classes of automorphic forms modulo M, which are associated to a cohomology class of 1-cocycles: for each $[\phi] \in H^1(G, M)$, an *automorphic class* of type $[\phi]$ is a class $[u] \in (N/M)^G = H^0(G, N/M)$ so that $[\delta u] = [\phi]$.

The basic properties of automorphic classes can be expressed as follows: to any short exact sequence $1 \to M \to N \to N/M \to 1$ of G-modules is associated the long exact sequence of cohomology groups

$$\cdots \to H^j(G, M) \xrightarrow{i_*} H^j(G, N) \xrightarrow{p_*} H^j(G, N/M) \xrightarrow{\delta_*} H^{j+1}(G, M) \xrightarrow{i_*} \cdots,$$

The image of a class $[u]$ by the connecting homomorphism δ_* is obtained by viewing any representative u as a cochain $u \in C^j(G, N)$, and setting $\delta_*[u] = [\delta u]$,

In our case, with $j = 0$, the set of automorphic classes of type $[\phi] \in H^1(G, M)$ is $\delta_*^{-1}[\phi] \subset H^0(G, N/M)$. Exactness at $H^1(G, M)$ tells us that the factors of automorphy $[\phi]$ for which there exist automorphic forms are those in the kernel of i_* and exactness at $H^0(G, N/M)$ implies that the group $H^0(G, N)$ of non-zero meromorphic functions on X acts transitively on $\delta_*^{-1}[\phi]$.

For example, take G to be the free abelian group on two generators t_1, t_2 acting on $X = \mathbb{C}$ by $t_1 z = z + 1$, $t_2 z = z + \tau$. Then $H^0(G, N)$ is the group of non-zero elliptic functions, θ_0 represents a class of $H^0(G, N/M)$ and $[\phi] = \delta_*[\theta_0]$ is the class of the 1-cocycle

$$\phi(t_1^l t_2^m, z) = \frac{\theta_0(z)}{\theta_0(z - l - m\tau)} = e^{-\pi i m(2z + 1 - (m+1)\tau)}.$$

More generally, let H be the upper half plane and let $G = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$, the "Jacobi group", act on $X = \mathbb{C} \times H$ by $((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), \vec{n})(z, \tau) = ((z + n_1 + n_2\tau)/(c\tau + d), (a\tau + b)/(c\tau + d))$. Then one has

$$\theta_0(y) = \phi(g, y)\, \theta_0(g^{-1}y), \qquad y \in \mathbb{C} \times H, \qquad g \in G.$$

[2] We consider meromorphic invertible automorphic forms. In particular 0 is not considered as an automorphic form and the additive structure is disregarded.

The factor of automorphy is defined on generators t_1, t_2 of $\mathbb{Z}^2$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{Z})$ by

$$\phi(t_2, z, \tau) = e^{-\pi i(2z - 2\tau + 1)}, \qquad \phi(S, z, \tau) = e^{\pi i(-z^2/\tau + z - z/\tau - \tau/6 - 1/6\tau + 1/2)},$$

and $\phi(t_1, z, \tau) = \phi(T, z, \tau) = 1$. The value of ϕ on arbitrary group elements is then uniquely determined by the cocycle relation.

Thus θ_0 is a representative of an automorphic class of type $[\phi] \in H^1(G, M)$.

Let us review the proof that $[\phi]$ (and thus $[\theta_0] \in H^0(G, N/M)$) is a non-trivial cohomology class by showing that the corresponding first Chern class is non-trivial. The first Chern class $c_1([\phi])$ is the image of $[\phi]$ under the connecting homomorphism $H^1(G, M) \to H^2(G, 2\pi i\mathbb{Z})$ in the long exact sequence associated to $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}(\mathbb{C} \times H) \xrightarrow{\exp} M \to 1$. Here $\mathcal{O}(X)$ is the additive group of holomorphic functions on X. Then one checks that $c_1([\phi])$ is non-trivial by showing that it is sent to a generator of $H^2(\mathbb{Z}^2, 2\pi i\mathbb{Z}) \simeq \mathbb{Z}$ by the map $H^2(G, 2\pi i\mathbb{Z}) \to H^2(\mathbb{Z}^2, 2\pi i\mathbb{Z})$ induced by inclusion. We do this calculation explicitly in degree 1 below.

7.2. *Automorphic forms of degree* 1. There is an obvious generalization of these constructions one degree higher: so for G-modules $M \subset N$ consisting of certain classes of functions on an G-set X we consider the piece

$$H^1(G, N) \xrightarrow{p_*} H^1(G, N/M) \xrightarrow{\delta_*} H^2(G, M) \xrightarrow{i_*} H^2(G, N)$$

of the long exact sequence. A *degree* 1 *automorphic class* of type $[\phi] \in \mathrm{Ker}(i_*: H^2(G, M) \to H^2(G, N))$ is then a class in $\delta_*^{-1}[\phi]$. Elements of these equivalence classes we call *degree* 1 *automorphic forms*. Degree 1 automorphic classes of type $[\phi]$ are acted upon transitively by $H^1(G, N)$.

We wish to show that θ_0, Γ are values on generators of a degree 1 automorphic form for certain modules M, N over $\mathrm{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$. As above let us start by considering the translation subgroup $\mathbb{Z}^3$.

Proposition 7.1. *Fix σ, τ in the upper half plane. Let G be the free abelian group on three generators t_1, t_2, t_3 acting on $\mathbb{C}$ by $t_1 z = z + 1$, $t_2 z = z + \sigma$, $t_3 z = z + \tau$. Let N be the group of non-zero meromorphic functions on $\mathbb{C}$, and M the subgroup on nowhere vanishing holomorphic functions. Then*

$$u(t_1^l t_2^m t_3^n, z) = \prod_{j=1}^{m} \theta_0(z - j\sigma, \tau)$$

represents a non-trivial class $[u]$ in $H^1(G, N/M)$. It corresponds to the class $\delta_*[u] \in H^2(G, M)$ of the cocycle

$$\phi(t_1^l t_2^m t_3^n, t_1^{l'} t_2^{m'} t_3^{n'}, z) = e^{\pi i(nm'(2z+1) - m'n(n+1)\tau - nm'(m'+1+2m)\sigma)}.$$

Proof. We need to compute

$$\delta u(t_1^l t_2^m t_3^n, t_1^{l'} t_2^{m'} t_3^{n'}, z) = \frac{\prod_{j=1}^{m'} \theta_0(z-l-m\sigma-n\tau-j\sigma,\tau)\prod_{j=1}^{m}\theta_0(z-j\sigma,\tau)}{\prod_{j=1}^{m+m'}\theta_0(z-j\sigma,\tau)}.$$

Using the transformation properties of θ_0 under translations by $\mathbb{Z}+\tau\mathbb{Z}$ we see that this coboundary is in M, so that u represents a class in $H^1(G, N/M)$. The class of δu in $H^2(G, M)$ is $\delta_*[u]$ and is easily computed to give the above expression.

To show that the class of u is non-trivial it is sufficient to show that $[\phi]$ is non-trivial. To this purpose we compute the analogue of the first Chern class: let $\mathcal{O}(\mathbb{C})$ denote the additive group of holomorphic functions on $\mathbb{C}$. To the short exact sequence $0\to 2\pi i\mathbb{Z}\to\mathcal{O}(\mathbb{C})\xrightarrow{\exp} M\to 1$ of G-modules (with trivial action on $2\pi i\mathbb{Z}$) there corresponds a long exact sequence and in particular a connecting homomorphism

$$c_1\colon H^2(G, M)\to H^3(G, 2\pi i\mathbb{Z})\simeq\mathbb{Z}.$$

We claim that $[\phi]$ is mapped to a generator under this homomorphism and thus is non-trivial. The calculation goes as follows: if we write $\phi(g_1, g_2, z)=\exp(R(g_1, g_2, z))$, with R the polynomial appearing in the exponential function above, then, according to the rules of homological algebra,

$$\begin{aligned} c_1(\phi)(g_1, g_2, g_3) = {} & -R(g_2, g_3, g_1^{-1}z)+R(g_1g_2, g_3, z)\\ & -R(g_1, g_2g_3, z)+R(g_1, g_2, z).\end{aligned}$$

If $g_j=t_1^{l_j}t_2^{m_j}t_3^{n_j}$, we then obtain $c_1(\phi)(g_1, g_2, g_3)=2\pi i l_1 n_2 m_3$ which is indeed the class of a generator of $H^3(G, 2\pi i\mathbb{Z})$ (see [MacL] Section VI.6). ∎

Let us now give a similar interpretation for the elliptic gamma function. By an *invertible analytic function* on a complex manifold X we mean an equivalence class of pairs (f, D) where $D\subset X$ is a dense open subset and f is a holomorphic, nowhere vanishing function on D. Two pairs (f_1, D_1), (f_2, D_2) are equivalent if $f_1=f_2$ on $D_1\cap D_2$. Invertible analytic functions form a group with respect to the pointwise product $(f_1, D_1)\cdot(f_2, D_2)=(f_1f_2, D_1\cap D_2)$.

It is convenient to pass from affine coordinates τ, σ to homogeneous coordinates x_1, x_2, x_3. Let $G=\mathrm{SL}(3,\mathbb{Z})\ltimes\mathbb{Z}^3$ act on $X=\mathbb{C}\times\mathbb{C}^3$ by $(A,\vec{n})(z,\vec{x})=(z+\vec{n}\cdot\vec{x}, A\vec{x})$, $A\in\mathrm{SL}(3,\mathbb{Z})$, $\vec{n}=(n_1, n_2, n_3)\in\mathbb{Z}^3$, $\vec{x}\in\mathbb{C}^3$, $z\in\mathbb{C}$.

The group G has generators $e_{i,j}$, $1 \leqslant i, j \leqslant 3, i \neq j$ and t_i, $1 \leqslant i \leqslant 3$. The elementary matrix $e_{i,j}$ is the element of $\mathrm{SL}(3, \mathbb{Z})$ which differ from the identity matrix by having the i, j matrix element equal to 1. The t_i are the canonical generators of the $\mathbb{Z}^3$ subgroup.

THEOREM 7.2. *Let $G = \mathrm{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$ act on $X = \mathbb{C} \times \mathbb{C}^3$ as above. Let N be the G-module of invertible analytic functions on X such that $f(\lambda z, \lambda \vec{x}) = f(z, \vec{x})$ for all $\lambda \in \mathbb{C} - \{0\}$. Let M be the submodule of functions of the form* $\exp 2\pi i f$ *with $f \in \mathbb{Q}(x_1/x_3, x_2/x_3)[z/x_3]$ a polynomial in z with coefficients in the rational functions of $\vec{x}$. Then the classes in N/M of the functions*

$$u(e_{1,2}, z, \vec{x}) = \Gamma\left(\frac{z - x_2}{x_3}, \frac{x_1 - x_2}{x_3}, -\frac{x_1}{x_3}\right)^{-1},$$

$$u(e_{3,2}, z, \vec{x}) = \Gamma\left(\frac{z}{x_1}, \frac{x_2 - x_3}{x_1}, \frac{x_3}{x_1}\right),$$

$$u(e_{i,j}, z, \vec{x}) = 1, \qquad j \neq 2,$$

$$u(t_2, z, \vec{x}) = \theta_0\left(\frac{z - x_2}{x_1}, \frac{x_3}{x_1}\right),$$

$$u(t_j, z, \vec{x}) = 1, \qquad j \neq 2,$$

extend to a 1-*cocycle $u: G \to N/M$. Its cohomology class $[u] \in H^1(G, N/M)$ is independent of the choice of extension and is non-trivial. The corresponding cohomology class $[\phi] = \delta_*[u]$ is represented by a function $\phi: G^2 \to M$ whose restriction to $(\mathbb{Z}^3)^2$ is given by*

$$\phi(t_1^l t_2^m t_3^n, t_1^{l'} t_2^{m'} t_3^{n'}, z, \vec{x}) = e^{\pi i (nm'(2z/x_1 + 1) - m'n(n+1)\, x_3/x_1 - nm'(m' + 1 + 2m)\, x_2/x_1)}. \tag{31}$$

Proof. The proof is based a the presentation of G by generators and relations. The $\mathrm{SL}(3, \mathbb{Z})$ subgroup is generated by the elementary matrices $e_{i,j}$, $(i \neq j)$. The relations can be chosen [M] to be

$$e_{i,j}\, e_{k,l} = e_{k,l}\, e_{i,j}, \qquad i \neq l, \qquad j \neq k,$$

$$e_{i,j}\, e_{j,k} = e_{i,k}\, e_{jk}\, e_{i,j},$$

$$(e_{1,3}\, e_{3,1}^{-1}\, e_{1,3})^4 = 1.$$

The relations of the generators of the $\mathbb{Z}^3$ subgroup are $t_i t_j = t_j t_i$ and the relations between $e_{i,j}$, t_k are

$$e_{i,j}\, t_k = t_k\, e_{i,j}, \qquad i \neq k,$$

$$e_{i,j}\, t_i = t_i t_j^{-1}\, e_{i,j}.$$

The cocycle condition uniquely determines a 1-cocycle in terms of its values on generators. For any functions $u_{i,\,j}, u_k \in N/M$ there exists a unique 1-cocycle of the free group on generators $e_{i,\,j}$, e_k such that $u_{i,\,j}=u(e_{i,\,j})$ and $u_i=u(t_i)$. This cocycle defines a 1-cocycle of G if and only if the relations are sent to 1.

This can be checked using the identities of the functions Γ, θ_0. Let us consider some non-trivial examples.

The relation $t_2\, e_{3,\,2} t_3=t_3\, e_{3,\,2}$ translates to the condition

$$u_2(z,\vec{x})\, u_{3,\,2}(z-x_2,\vec{x})\, u_3(z-x_2, e_{3,\,2}^{-1}\vec{x})=u_3(z,\vec{x})\, u_{3,\,2}(z-x_3,\vec{x}),$$

which reduces to the defining functional equation for Γ:

$$\theta_0\left(\frac{z-x_2}{x_1},\frac{x_3}{x_1}\right)\Gamma\left(\frac{z-x_2}{x_1},\frac{x_2-x_3}{x_1},\frac{x_3}{x_1}\right)=\Gamma\left(\frac{z-x_3}{x_1},\frac{x_2-x_3}{x_1},\frac{x_3}{x_1}\right).$$

The relation $e_{1,\,3}\, e_{3,\,2}=e_{1,\,2}\, e_{3,\,2}\, e_{1,\,3}$ translates to the condition

$$\begin{aligned}&u_{1,\,3}(z,\vec{x})\, u_{3,\,2}(z, e_{1,\,3}^{-1}\vec{x})\\&\quad=u_{1,\,2}(z,\vec{x})\, u_{3,\,2}(z, e_{1,\,2}^{-1}\vec{x})\, u_{1,\,3}(z, e_{1,\,2}^{-1}\, e_{1,\,3}^{-1}\vec{x}) \qquad \bmod M.\end{aligned}$$

By inserting the given expressions for $u_{i,\,j}$, we see that the condition is

$$\Gamma\left(\frac{z}{x_1-x_3},\frac{x_2-x_3}{x_1-x_3},\frac{x_3}{x_1-x_3}\right)=\frac{\Gamma\left(\frac{z}{x_1-x_2},\frac{x_2-x_3}{x_1-x_2},\frac{x_3}{x_1-x_2}\right)}{\Gamma\left(\frac{z-x_2}{x_3},\frac{x_1-x_2}{x_3},-\frac{x_1}{x_3}\right)} \qquad \bmod M. \tag{32}$$

Using the fact that Γ is periodic with period 1 in all its arguments, and setting $Z=(z-x_1+x_3)/(x_1-x_3)$, $\sigma=x_3/(x_1-x_3)$, $\tau=(x_2-x_1)/(x_1-x_3)$, one sees that this identity reduces to

$$\Gamma(Z,\tau,\sigma)=\frac{\Gamma\left(-\frac{Z+1}{\tau},-\frac{1}{\tau},-\frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{Z-\tau}{\sigma},-\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)} \qquad \bmod M.$$

By the last identity of Proposition 3.2 and the rules for changing signs before Theorem 3.3, this identity reduces to one of the three-term relation in Theorem 4.1.

To show that the class of $[\phi]=\delta_*[u]$ is non-trivial, one notices that the restriction of $[\phi]$ to $\mathbb{Z}^3$ is given by the formula (31). So its image under the map $c_1: H^1(\mathbb{Z}^3, M) \to H^3(\mathbb{Z}^3, 2\pi i\mathbb{Z})$, which comes from the short exact sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow 2\pi i\mathbb{Q}(x_1/x_3, x_2/x_3)[z/x_3] \xrightarrow{\exp} M \longrightarrow 1$$

is calculated as in the proof of Prop. 7.1 and is non-trivial. ∎

7.3. *Explicit description of the 2-cocycle.* To describe the cohomology class $[\phi]=\delta_*[u] \in H^2(G, M)$ of $G=\mathrm{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$ with coefficients M arising in Theorem 7.2, it is convenient to use the isomorphism of $H^2(G, M)$ with the set $\mathscr{E}(G, M)$ of equivalence classes of group extensions

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

of the G-module M. The isomorphism assigns to any such extension its *characteristic class* in $H^2(G, M)$. It is defined as follows: choose a map $\sigma: G \to E$ such that $p \circ \sigma = \mathrm{id}_G$. Then $\sigma(g)\,\sigma(h)=i(\phi(g, h))\,\sigma(gh)$ for some 2-cocycle $\phi \in C^2(G, M)$ whose class, the characteristic class of the extension, is independent of the choice of σ.

In our case the extension can be described in terms of generators and relations.

Let us introduce a set of elements $\phi_j^k=\exp(i\pi L_j^k)$, $\phi_{j,k}^l=\exp(i\pi L_{j,k}^l)$, $\phi_{j,k}^{l,m}=\exp(i\pi L_{j,k}^{l,m})$, $(1 \leqslant j, k, l, m \leqslant 3)$ of M with

$$L_{1,2}^2=\frac{x_2(6z^2-6(x_3+2x_2)\,z+x_2x_1-x_1{}^2+6x_2{}^2+6x_2x_3+x_3{}^2)}{6x_3x_1(x_2-x_1)},$$

$$L_{1,2}^1=\frac{\begin{bmatrix} -6z^2+6(x_1+x_3+2x_2)\,z-x_3{}^2-6x_2x_1 \\ +3x_1x_3-6x_2x_3-6x_2{}^2-x_1{}^2 \end{bmatrix}}{6x_1x_3},$$

$$L_{1,3}^2=\frac{6z^2-6(x_3+2x_2)\,z+6x_2x_3+5x_1{}^2+6x_2{}^2+x_3{}^2-5x_1x_3}{6(x_3-x_1)\,x_1},$$

$$L_2^3=\frac{2z-2x_2-2x_3+x_1}{x_1},$$

$$L_{1,3}^{3,2}=\frac{(2z-x_2)(2z^2-2zx_2-x_1{}^2+x_1x_3-x_3{}^2+x_2x_1)}{12(x_3-x_1)\,x_3(x_2-x_1)},$$

$$L_{3,1}^{1,2}=\frac{(2z-x_2)(2z^2-2zx_2+x_2x_3-x_1{}^2+x_1x_3-x_3{}^2)}{12x_1(x_2-x_3)(x_3-x_1)},$$

$$L_{1,2}^{3,2}=\frac{x_2(2z-x_2)(2z^2-2zx_2+x_2x_3-x_3{}^2-x_1{}^2+x_2x_1)}{12x_1(x_2-x_1)\,x_3(x_2-x_3)},$$

$$L_{3,2}^{1,2}=-L_{1,2}^{3,2}, \qquad L_3^2=-L_2^3,$$

and all other $L^{\cdots}_{\cdots}$ are zero. Let E be the group generated by $\hat{e}_{i,j}$, $1\leqslant i\neq j\leqslant 3$, $\hat{t}_i$, $1\leqslant i\leqslant 3$ and the elements of M subject to the following defining relations:

(a) The product of elements of M in E is the product in M.

(b) Relations with M:

$$\hat{e}_{i,j}u=\rho(e_{i,j})(u)\,\hat{e}_{i,j},\qquad \hat{t}_iu=\rho(t_i)(u)\,\hat{t}_i,$$

$u\in M$.

(c) Relations among the $\hat{e}_{i,j}$:

$$\hat{e}_{i,j}\,\hat{e}_{k,l}=\phi^{k,l}_{i,j}\,\hat{e}_{k,l}\,\hat{e}_{i,j},\qquad i\neq l,\qquad j\neq k,$$

$$\hat{e}_{i,j}\,\hat{e}_{j,k}=\phi^{j,k}_{i,j}\,\hat{e}_{i,k}\,\hat{e}_{j,k}\,\hat{e}_{i,j},$$

$$(\hat{e}_{1,3}\,\hat{e}^{-1}_{3,1}\,\hat{e}_{1,3})^4=1.$$

(d) Relations among the $\hat{t}_i$: $\hat{t}_i\hat{t}_j=\phi^j_i\hat{t}_j\hat{t}_i$

(e) Relations between $\hat{e}_{i.j}$ and $\hat{t}_i$:

$$\hat{e}_{i,j}\hat{t}_k=\phi^k_{i,j}\hat{t}_k\,\hat{e}_{i,j},\qquad i\neq k,$$

$$\hat{t}_j\,\hat{e}_{i,j}\hat{t}_i=\phi^i_{i,j}\hat{t}_i\,\hat{e}_{i,j}.$$

This group comes with a natural homomorphism $M\to E$.

THEOREM 7.3. *Let G, M, $[\phi]$ be as in Theorem 7.2. The map $M\to E$ fits into a group extension*

$$1\to M\to E\to G\to 1,$$

of the G-module M, whose characteristic class in $H^2(G,M)$ is $[\phi]=\delta_*[u]$ of Theorem 7.2.

To prove this theorem we need to recall some facts about the description of extensions by generators and relations. Let $G=F/R$ with R a normal subgroup of a free group F with generators $(e_i)_{i\in I}$. The canonical projection $F\to G$ will be denoted by $x\mapsto\bar{x}$. Let M be a G-module and ρ: $G\to\mathrm{Aut}(M)$ the corresponding homomorphism.

Suppose that a cohomology class $[\phi]\in H^2(G,M)$ is given. We want to describe the middle group of the corresponding extension by generators and relations. The relations are written in terms of a map ψ: $R\to M$ built out of $[\phi]$. We proceed to explain how to construct an extension associated to a map ψ with certain properties, and how to construct ψ given the characteristic class $[\phi]$ of the extension.

Abstractly, the relation between ψ and $[\phi]$ is that ψ is any inverse image of $[\phi]$ by the surjective homomorphism $H^0(G, \mathrm{Hom}(R_{\mathrm{ab}}, M)) \to H^2(G, M)$ described in [MacL], Section VIII.9. Here R_{ab} denotes the abelianization $R/[R, R]$ of R,

This homomorphism can be described explicitly as follows.

An element of $H^0(G, \mathrm{Hom}(R_{\mathrm{ab}}, M))$ is, by definition, a map $\psi: R \to M$ such that

$$\psi(rs) = \psi(r)\,\psi(s) \qquad \text{and} \qquad \psi(xrx^{-1}) = \rho(\bar{x})\,\psi(r), \tag{33}$$

for all $r, s \in R$, $x \in F$. The semidirect product $\hat{E} = F \times_\rho M$ is the cartesian product with group multiplication rule $(x, u)(y, v) = (xy, u\rho(x)\,v)$. Then the properties of ψ imply that $\hat{R} = \{(r, \psi(r)^{-1}) \mid r \in R\}$ is a normal subgroup of $\hat{E}$. Let $E = \hat{E}/\hat{R}$. Then it is easy to show that we have an extension $1 \to M \to E \to G \to 1$, with the obvious maps. If R is generated by relations r_j, $j \in J$, then E is the group with generators $(e_i)_{i \in I}$, M and defining relations:

1. the product of elements of M in E is the product in M,
2. $e_i u = (\rho(e_i)\,u)\,e_i$,
3. $r_j = \psi(r_j)$.

$i \in I$, $j \in J$, $u, v \in M$. The characteristic class of this extension is the class of $\psi \circ \mu$, where $\mu(g, h)$ is defined by $\lambda(g)\,\lambda(h) = \mu(g, h)\,\lambda(gh)$, for any section $\lambda: G \to F$, as can be seen by choosing the section $\sigma(g) = [(\lambda(g), 1)] \in E$.

Conversely, given a 2-cocycle $\phi \in C^2(G, M)$ we may find a $\psi \in H^0(G, \mathrm{Hom}(R_{\mathrm{ab}}, M))$ mapping to $[\phi]$ as follows. Let $\psi: F \to M$ be the unique map such that

$$\begin{aligned} &\psi(e_i^{\pm 1}) = 1, \qquad i = 1, ..., n, \\ &\psi(xy) = \phi(\bar{x}, \bar{y})\,\psi(x)\,\rho(\bar{x})\,\psi(y), \qquad \forall x, y \in F. \end{aligned} \tag{34}$$

Then the restriction of ψ to R obeys (33). For any section $\lambda: G \to F$ we have

$$\begin{aligned} \psi(\mu(g, h))\,\psi(\lambda(gh)) &= \psi(\mu(g, h)\,\lambda(gh)) \\ &= \psi(\lambda(g)\,\lambda(h)) \\ &= \phi(g, h)\,\psi(\lambda(g))\,\rho(g)\,\psi(\lambda(h)). \end{aligned}$$

Thus the 2-cocycle $\psi \circ \mu$ is indeed in the same cohomology class as ϕ.

Let us apply this construction to our case. Let $[u]\in H^1(G, N/M)$ be the cohomology class described in the Theorem. Let us choose a representative $u: G\to N$, which on the generators coincides with the functions given in the claim. Then a representative of the class $[\phi]=\delta_*[u]\in H^2(G, M)$ is given by $\phi(g,h)=u(gh)/(u(g)\,\rho(g)\,u(h))$. The generators e_i are here $e_{i,j}, t_i$. We need to compute the value of ψ on relations.

Every element $\neq 1$ of F is can uniquely be written as a reduced word $x_1\cdots x_k$, with $x_j\in\{e_i^{\pm 1}, i\in I\}$. Reduced means that $x_j\neq x_{j+1}^{-1}$, $j=1, ..., k-1$. The function $\psi: F\to M$ is then given according to the rule (34) by the formula

$$\psi(x_1\cdots x_k)=\frac{u(\bar x_1)\prod_{i=2}^k \rho(\bar x_1\cdots\bar x_{i-1})\,u(\bar x_i)}{u(\bar x_1\cdots\bar x_k)},$$

for $x_i\in\{e_j, e_j^{-1}\}$. If $x_1\cdots x_k$ is a relation, the denominator is equal to 1, so to compute $\psi(r)$ we only need the value of u on generators and their inverses. The value of u on inverses of generators obeys, by the cocycle condition, $u(\bar x^{-1})=(\rho(\bar x^{-1})\,u(\bar x))^{-1}$ modulo M. Since we have no generators such that $\bar x=\bar x^{-1}$ we may choose u so that this relation holds for representatives, not just modulo M.

We explain the calculation in the case of the relation $r=e_{1,3}\,e_{3,2}(e_{1,2}\,e_{3,2}\,e_{1,3})^{-1}$. For the other relations the calculation is similar. The equation for $\phi_{1,3}^{3,2}=\psi(r)$ can be written as

$$u_{1,3}(z,\vec x)\,u_{3,2}(z, e_{1,3}^{-1}\vec x)=\psi(r,z,\vec x)\,u_{1,2}(z,\vec x)\,u_{3,2}(z, e_{1,2}^{-1}\vec x)\,u_{1,3}(z, e_{1,2}^{-1}e_{1,3}^{-1}\vec x).$$

with $u_{i,j}=u(e_{i,j})$.

$$\Gamma\left(\frac{z}{x_1-x_3},\frac{x_2-x_3}{x_1-x_3},\frac{x_3}{x_1-x_3}\right)=\psi(r,z,\vec x)\,\frac{\Gamma\left(\frac{z}{x_1-x_2},\frac{x_2-x_3}{x_1-x_2},\frac{x_3}{x_1-x_2}\right)}{\Gamma\left(\frac{z-x_2}{x_3},\frac{x_1-x_2}{x_3},-\frac{x_1}{x_3}\right)}.$$

Comparing with Theorem 4.1, we see that (cf. the calculation following (32))

$$\psi(r,z,\vec x)=\exp\left(-\pi i Q\left(\frac{z-x_1+x_3}{x_1-x_3};\frac{x_2-x_1}{x_1-x_3},\frac{x_3}{x_1-x_3}\right)\right).$$

This expression is easily checked to be identical to $\exp(\pi i L_{1,3}^{3,2})$.

Proceeding in the same way with the other relations, we find that the non-trivial values of ϕ on relations are $\phi_A^B=\exp(\pi i L_A^B)$ with

$$L^2_{1,2} = -2\frac{z-x_2}{x_3} + 1 + F\left(\frac{z-x_2}{x_1}, \frac{x_3}{x_1}\right) - F\left(\frac{z-x_1}{x_1-x_2}, \frac{x_3}{x_1-x_2}\right),$$

$$L^1_{1,2} = 2\frac{z-x_2}{x_3} + 1 - F\left(\frac{z-x_2}{x_1}, \frac{x_3}{x_1}\right), \qquad L^2_{1,3} = -F\left(\frac{z-x_2}{x_1-x_3}, \frac{x_1}{x_1-x_3}\right),$$

$$L^3_2 = 2\frac{z-x_2-x_3}{x_1} + 1, \qquad L^{1,2}_{3,1} = Q\left(\frac{z-x_1}{x_1}; \frac{x_2-x_3}{x_1}, \frac{x_3-x_1}{x_1}\right),$$

$$L^{3,2}_{1,2} = Q\left(\frac{z-x_1}{x_1}; \frac{x_2-x_3}{x_1}, \frac{x_3-x_1}{x_1}\right) + Q\left(\frac{z-x_1+x_3}{x_1-x_3}; \frac{x_3}{x_1-x_3}, \frac{x_2-x_1}{x_1-x_3}\right).$$

Here F is the polynomial appearing in the modular transformation properties of θ_0:

$$F(z, \tau) = \frac{z^2}{\tau} + z\left(\frac{1}{\tau} - 1\right) + \frac{\tau}{6} + \frac{1}{2} + \frac{1}{6\tau}$$

These expressions may be (preferably with a computer) simplified to give the claim of the theorem. The proof is complete.

7.4. *The restriction of the 2-cocycle to* SL(3, $\mathbb{Z}$). Let G, M, $[\phi]$ be as in Theorem 7.2. We show here that the restriction $[\bar{\phi}] \in H^2(\mathrm{SL}(3, \mathbb{Z}), M)$ to $\mathrm{SL}(3, \mathbb{Z}) \subset G$ is non-trivial (See [S] for a description of the cohomology of SL(3, $\mathbb{Z}$)). This is proved by showing that the restriction to a D_4 subgroup is non-trivial. This D_4 subgroup is generated by $a = (e_{2,1}^{-1} e_{1,2} e_{2,1}^{-1})^2$: $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$ and $b = e_{3,1} e_{1,3}^{-1} e_{1,3}$: $(x_1, x_2, x_3) \mapsto (-x_3, x_2, x_1)$. The defining relations of D_4 are $a^2 = b^4 = 1$, $bab = a$. The restriction of $[\phi]$ to this subgroup is the characteristic class of the extension $1 \to M \to \hat{D}_4 \to D_4 \to 1$, where $\hat{D}_4$ is the inverse image of D_4 by the projection $E \to G$. A presentation of $\hat{D}_4$ is obtained by choosing lifts of generators

$$\hat{a} = e^{\pi i/2(z^2/x_1x_3 - 2z/x_3 + 1 + x_1/6x_3 + x_3/6x_1)}(\hat{e}_{2,1}^{-1} \hat{e}_{1,2} \hat{e}_{2,1}^{-1})^2$$

$$\hat{b} = \hat{e}_{3,1} \hat{e}_{1,3}^{-1} \hat{e}_{1,3}.$$

The relations between these generators are computable from the presentation of E above, with the result

$$\hat{a}^2 = \hat{b}^4 = 1, \qquad \hat{b}\hat{a}\hat{b} = i\hat{a}, \qquad i = e^{2\pi i/4} \in M.$$

PROPOSITION 7.4. *The pull-back $i^*[\phi]$ of the class $[\phi] \in H^2(G, M)$ of Theorem 7.2 by the inclusion i: $D_4 \to SL(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$ is the characteristic class in $H^2(D_4, M)$ of the extension $1 \to M \to \hat{D}_4 \to D_4 \to 1$. It is a non-trivial cohomology class.*

Proof. It remains to prove that the class is non-trivial. If the characteristic class were trivial, the exact sequence of the extension would split. This would mean that, for suitable homogeneous functions $A, B \in \mathbb{Q}(x_1, x_2, x_3)[z]$, $e^{2\pi i A}\hat{a}$ and $e^{2\pi i B}\hat{b}$ obey the relations of D_4. Suppose, by contradiction, that such functions exist. Then A, B obey

$$A(z, \vec{x}) + A(z, a^{-1}\vec{x}) = r,$$
$$B(z, \vec{x}) + B(z, b^{-1}\vec{x}) + B(z, b^{-2}\vec{x}) + B(z, b^{-3}\vec{x}) = s,$$
$$B(z, \vec{x}) + A(z, b^{-1}\vec{x}) + B(z, (ba)^{-1}\,\vec{x}) = -\tfrac{1}{4} + A(z, \vec{x}) + t.$$

for some integers r, s, t. Let $\bar{A} = A - r/2$, $\bar{B} = B - s/4$, then $\bar{A}$, $\bar{B}$ obey

$$\bar{A}(z, \vec{x}) + \bar{A}(z, a^{-1}\vec{x}) = 0,$$
$$\bar{B}(z, \vec{x}) + \bar{B}(z, b^{-1}\vec{x}) + \bar{B}(z, b^{-2}\vec{x}) + \bar{B}(z, b^{-3}\vec{x}) = 0,$$
$$\bar{B}(z, \vec{x}) + \bar{A}(z, b^{-1}\vec{x}) + \bar{B}(z, (ba)^{-1}\,\vec{x}) = -\frac{1}{4} + \bar{A}(z, \vec{x}) + t - \frac{s}{2}.$$

If we view $\mathbb{Q}(x_1, x_2, x_3)[z]$ as a module over the group ring $\mathbb{Z}D_4$, the first two equations can be written as $(1 + a)\,\bar{A} = 0$, $(1 + b + b^2 + b^3)\,\bar{B} = 0$. This implies that $\bar{A}$, $\bar{B}$ are annihilated by the idempotent

$$P = \tfrac{1}{8} \sum_{g \in D_4} g = \tfrac{1}{8}(1 + b + b^2 + b^3)(1 + a) = \tfrac{1}{8}(1 + a)(1 + b + b^2 + b^3).$$

Applying P to the third equation, we get $0 = -1/4 + t - s/2$, $t, s \in \mathbb{Z}$, a contradiction. ∎

REFERENCES

[A] R. Askey, The q-gamma and q-beta functions, *Appl. Anal.* **8**, No. 2 (1978/79), 125–141.

[B] R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Phys.* **70** (1972), 193–228.

[C] J. W. S. Cassels, "An Introduction to Diophantine Approximation," Cambridge University Press, Cambridge, UK, 1957.

[DP] B. Davies and I. Peschel, A unified treatment of Ising model magnetizations, *Ann. Physik* **6** (1997), 187–214.

[FTV] G. Felder, V. Tarasov, and A. Varchenko, Solutions of the elliptic qKZB equations and Bethe ansatz I, *Amer. Math. Soc. Transl.* **180** (1997), 45–75; Monodromy of solutions of the elliptic Knizhnik–Zamolodchikov–Bernard difference equations, q-alg/9705017.

[FV] G. Felder and A. Varchenko, Quantum KZB heat equation, modular transformations and $SL(3, \mathbb{Z})$ I, II, preprints.

[JMN] M. Jimbo, T. Miwa and A. Nakayashiki, Difference equations for the correlation functions of the eight-vertex model, *J. Phys. A: Math. Gen.* **26** (1993), 2199–2209.

[JKKMW] M. Jimbo, R. Kedem, H. Konno, T. Miwa, and R. Weston, Difference equations in spin chains with a boundary, *Nucl. Phys. B* **448** (1995), 429–456.

[MacL] S. Mac Lane, "Homology," Springer-Verlag, Berlin/New York, 1975.

[M] J. Milnor, "Introduction to Algebraic K-Theory," Annals of Mathematics Studies, No. 72, Princeton University Press, Princeton; University of Tokyo Press, Tokyo, 1971.

[R] S. N. M. Ruijsenaars, First order analytic difference equations and integrable quantum systems, *J. Math. Phys.* **38** (1997), 1069–1146.

[S] C. Soulé, The cohomology of $SL_3(Z)$, *Topology* **17** (1978), 1–22.