



Archimedean operator-theoretic Positivstellensätze

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Received 22 November 2010; accepted 1 February 2011

Communicated by D. Voiculescu

Abstract

We prove a general archimedean positivstellensatz for hermitian operator-valued polynomials and show that it implies the multivariate Fejer–Riesz theorem of Dritschel–Rovnyak and positivstellensätze of Ambrozie–Vasilescu and Scherer–Hol. We also obtain several generalizations of these and related results. The proof of the main result depends on an extension of the abstract archimedean positivstellensatz for $*$ -algebras that is interesting in its own right.

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Keywords: Real algebraic geometry; Operator algebras; Moment problems

1. Introduction

We fix $d \in \mathbb{N}$ and write $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_d]$. In real algebraic geometry, a *positivstellensatz* is a theorem which for given polynomials $p_1, \dots, p_m \in \mathbb{R}[x]$ characterizes all polynomials $p \in \mathbb{R}[x]$ which satisfy $p_1(a) \geq 0, \dots, p_m(a) \geq 0 \Rightarrow p(a) > 0$ for every point $a \in \mathbb{R}^d$. A nice survey of them is [11]. The name *archimedean positivstellensatz* is reserved for the following result of Putinar [14, Lemma 4.1]:

Theorem A. *Let $S = \{p_1, \dots, p_m\}$ be a finite subset of $\mathbb{R}[x]$. If the set $M_S := \{c_0 + \sum_{i=1}^m c_i p_i \mid c_0, \dots, c_m \text{ are sums of squares of polynomials from } \mathbb{R}[x]\}$ contains an element g such that the set $\{x \in \mathbb{R}^d \mid g(x) \geq 0\}$ is compact, then for every $p \in \mathbb{R}[x]$ the following are equivalent:*

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- (1) $p(x) > 0$ on $K_S := \{x \in \mathbb{R}^d \mid p_1(x) \geq 0, \dots, p_m(x) \geq 0\}$.
- (2) There exists an $\epsilon > 0$ such that $p - \epsilon \in M_S$.

An important corollary of Theorem A is the following theorem of Putinar and Vasilescu [15, Corollary 4.4]. The case $S = \emptyset$ was first done by Reznick [16, Theorem 3.15], see also [2, Theorem 4.13].

Theorem B. *Notation as in Theorem A. If p_1, \dots, p_m and p are homogeneous of even degree and if $p(x) > 0$ for every nonzero $x \in K_S$, then there exists $\theta \in \mathbb{N}$ such that $(x_1^2 + \dots + x_d^2)^\theta p \in M_S$.*

Another important corollary of Theorem A (take $S = \{1 - x_1^2 - y_1^2, x_1^2 + y_1^2 - 1, \dots, 1 - x_d^2 - y_d^2, x_d^2 + y_d^2 - 1\} \subseteq \mathbb{R}[x_1, y_1, \dots, x_d, y_d]$) is the following multivariate Fejer–Riesz theorem.

Theorem C. *Every element of $\mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_d, \sin \phi_d]$ which is strictly positive for every ϕ_1, \dots, ϕ_d is equal to a sum of squares of elements from $\mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_d, \sin \phi_d]$.*

Note that Theorem C implies neither the classical univariate Fejer–Riesz theorem nor its multivariate extension from [13] which both work for nonnegative trigonometric polynomials.

Various generalizations of Theorems A, B and C have been considered. Theorem D extends Theorems A and C from finite to arbitrary sets S and from algebras $\mathbb{R}[x]$ and $\mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_d, \sin \phi_d]$ to arbitrary algebras of the form $\mathbb{R}[x]/I$. It also implies that Theorem B holds for arbitrary S . It is a special case of Jacobi’s representation theorem and Schmüdgen’s positivstellensatz, see [11, 5.7.2 and 6.1.4]. Generalizations from sums of squares to sums of even powers and from \mathbb{R} to subfields of \mathbb{R} will not be considered here, see [2,8,12].

Theorem D. *Let R be a commutative real algebra and M a quadratic module in R (i.e. $1 \in M \subseteq R, M + M \subseteq M, r^2M \subseteq M$ for all $r \in R$). If M is archimedean (i.e. for every $r \in R$ we have $l \pm r \in M$ for some real $l > 0$) then for every $p \in R$ the following are equivalent:*

- (1) $p \in \epsilon + M$ for some real $\epsilon > 0$,
- (2) $\phi(p) > 0$ for all $\phi \in V_R := \text{Hom}(R, \mathbb{R})$ such that $\phi(M) \geq 0$.

If R is affine then M is archimedean iff it contains an element g such that the set $\{\phi \in V_R \mid \phi(g) \geq 0\}$ is compact in the coarsest topology of V_R for which all evaluations $\phi \mapsto \phi(a), a \in R$, are continuous.

We are interested in generalizations of this theory from usual to hermitian operator-valued polynomials, i.e. from $\mathbb{R}[x]$ to $\mathbb{R}[x] \otimes A_h$ where A is some operator algebra with involution. Below, we will survey known generalizations of Theorems A, B and C and formulate our main result which is a generalization of Theorem D. Such results are of interest in control theory. They fit into the emerging field of noncommutative real algebraic geometry, see [18].

The first result in this direction was the following generalization of Theorem B which was proved by Ambrozie and Vasilescu in [1], see the last part of their Theorem 8. We say that an element a of a C^* -algebra A is *nonnegative* (i.e. $a \geq 0$) if $a = b^*b$ for some $b \in A$ and that it is *strictly positive* (i.e. $a > 0$) if $a - \epsilon \geq 0$ for some real $\epsilon > 0$.

Theorem E. *Let A be a C^* -algebra and let $p \in \mathbb{R}[x] \otimes A_h$ and $p_k \in \mathbb{R}[x] \otimes M_{v_k}(\mathbb{C})_h$, $k = 1, \dots, m$, $v_k \in \mathbb{N}$, be homogeneous polynomials of even degree. Assume that $K_0 := \{t \in S^{d-1} \mid p_1(t) \geq 0, \dots, p_m(t) \geq 0\}$ is nonempty and $p(t) > 0$ for all $t \in K_0$. Then there are homogeneous polynomials $q_j \in \mathbb{R}[x] \otimes A$, $q_{jk} \in \mathbb{R}[x] \otimes M_{v_k \times 1}(A)$, $j \in J$, J finite, and an integer θ such that*

$$(x_1^2 + \dots + x_d^2)^\theta p = \sum_{j \in J} \left(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk} \right).$$

Our interest in this subject stems from the following generalization of Theorem A which is a reformulation of a result of Scherer and Hol. See [17, Corollary 1] for the original result and [10, Theorem 13] for the reformulation and extension to infinite S .

Theorem F. *For a finite subset $S = \{p_1, \dots, p_m\}$ of $M_v(\mathbb{R}[x])_h$, $v \in \mathbb{N}$, write $K_S := \{t \in \mathbb{R}^d \mid p_1(t) \geq 0, \dots, p_m(t) \geq 0\}$ and $M_S := \{\sum_{j \in J} (q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk}) \mid q_j, q_{jk} \in M_v(\mathbb{R}[x]), j \in J, J \text{ finite}\}$. If there is $g \in M_S \cap \mathbb{R}[x]$ such that the set $\{x \in \mathbb{R}^d \mid g(x) \geq 0\}$ is compact (i.e. the quadratic module $M_S \cap \mathbb{R}[x]$ in $\mathbb{R}[x]$ is archimedean) then for every $p \in M_v(\mathbb{R}[x])_h$ such that $p(t) > 0$ on K_S we have that $p \in M_S$.*

Finally, we mention an interesting generalization of Theorem C which was proved by Dritschel and Rovnyak in [6, Theorem 5.1].

Theorem G. *Let A be the $*$ -algebra of all bounded operators on a Hilbert space. If an element*

$$p \in \mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_n, \sin \phi_n] \otimes A_h$$

is strictly positive for every ϕ_1, \dots, ϕ_n then $p = \sum_{j \in J} q_j^ q_j$ for some finite J and $q_j \in \mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_n, \sin \phi_n] \otimes A$.*

The aim of this paper is to prove the following very general operator-theoretic positivstellensatz and show that it implies generalizations of Theorems E, F and G. (They will be extended from finite to arbitrary S , from C^* -algebras to algebraically bounded $*$ -algebras A and from (trigonometric) polynomials to affine commutative real algebras. Theorem F will also be extended from matrices to more general operators.)

Theorem H. *Let R be a commutative real algebra, A a real or complex $*$ -algebra and M a quadratic module (cf. Section 2) in $R \otimes A$. If M is archimedean then for every $p \in R \otimes A_h$ the following are equivalent:*

- (1) $p \in \epsilon + M$ for some real $\epsilon > 0$.
- (2) For every multiplicative state ϕ on R , there exists real $\epsilon_\phi > 0$ such that $(\phi \otimes \text{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \text{id}_A)(M)$.

If A is algebraically bounded (cf. Section 4) and the quadratic module $M \cap R$ in R is archimedean (cf. Theorem D) then M is archimedean.

One of the main differences between the operator case and the scalar case is that in the operator case an element of A_h that is not ≤ 0 is not necessarily > 0 . We would like to give an algebraic characterization of operator-valued polynomials that are not ≤ 0 in every point from a given set. Every theorem of this type is called a *nichtnegativsemidefinitheitsstellensatz*. We will prove variants of Theorems F and G that fit into this context.

Finally, we use our results and the main theorem from [9] to get a generalization of the existence result for operator-valued moment problems from [1] to algebraically bounded $*$ -algebras.

2. Factorizable states

Associative unital algebras with involution will be called $*$ -algebras for short. Let B be a $*$ -algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$ where F always comes with complex conjugation as involution. Write $Z(B)$ for the center of B and write $B_h = \{b \in B \mid b^* = b\}$ for its set of hermitian elements. Note that the set B_h is a real vector space; we assume that it is equipped with the *finest locally convex topology*, i.e. the coarsest topology such that every convex absorbing set in B_h is a neighborhood of zero.

Clearly, every linear functional on B_h is continuous with respect to the finest locally convex topology. In other words, the algebraic and the topological dual of B_h are the same; we will write $(B_h)'$ for both. We assume that $(B_h)'$ is equipped with the weak*-topology, i.e. topology of pointwise convergence. We say that $\omega \in (B_h)'$ is *factorizable* if $\omega(xy) = \omega(x)\omega(y)$ for every $x \in B_h$ and $y \in Z(B)_h$. Clearly, the set of all factorizable linear functionals on B_h is closed in the weak*-topology.

We say that a subset M of B_h is a *quadratic module* if $1 \in M$, $M + M \subseteq M$ and $b^*Mb \subseteq M$ for every $b \in B$. The smallest quadratic module in B is the set $\Sigma^2(B)$ which consists of all finite sums of elements b^*b with $b \in B$. The largest quadratic module in B is the set B_h . A quadratic module M in B is *proper* if $M \neq B_h$ (or equivalently, if $-1 \notin M$). Proper quadratic modules in B exist iff $-1 \notin \Sigma^2(B)$. We say that an element $b \in B_h$ is *bounded* w.r.t. a quadratic module M if there exists a number $l \in \mathbb{N}$ such that $l \pm b \in M$. A quadratic module M is *archimedean* if every element $b \in B_h$ is bounded w.r.t. M (or equivalently, if 1 is an interior point of M).

For every subset M of B_h write M^\vee for the set of all $f \in (B_h)'$ which satisfy $f(1) = 1$ and $f(M) \geq 0$. The set of all extreme points of M^\vee will be denoted by $\text{ex } M^\vee$. Elements of M^\vee will be called *M-positive states* and elements of $\text{ex } M^\vee$ *extreme M-positive states*. A $\Sigma^2(B)$ -positive state will simply be called a *state*. Suppose now that M is an archimedean quadratic module. Applying the Banach–Alaoglu Theorem to $V = (M - 1) \cap (1 - M)$ which is a neighborhood of zero, we see that M^\vee is compact. The Krein–Milman theorem then implies, that M^\vee is equal to the closure of the convex hull of the set $\text{ex } M^\vee$. We will show later (see Corollary 4) that M^\vee is nonempty iff M is proper.

Recall that a (bounded) **-representation* of B is a homomorphism of unital $*$ -algebras from B to the algebra of all bounded operators on some Hilbert space H_π . We say that a $*$ -representation π of B is *M-positive* for a given subset M of B_h if $\pi(m)$ is positive semidefinite for every $m \in M$. For every such π and every $v \in H_\pi$ of norm 1, $\omega_{\pi,v}(x) := \langle \pi(x)v, v \rangle$ belongs to M^\vee . Conversely, if M is an archimedean quadratic module, then every $\omega \in M^\vee$ is of this form by the GNS construction.

The equivalence of (1)–(4) in the following result is sometimes referred to as archimedean positivstellensatz for $*$ -algebras. It originates from the Vidav–Handelmann theory, cf. [7, Section 1] and [20]. Our aim is to add assertions (5) and (6) to this equivalence.

Proposition 1. For every archimedean quadratic module M in B and every element $b \in B_h$ the following are equivalent:

- (1) $b \in M^\circ$ (the interior w.r.t. the finest locally convex topology),
- (2) $b \in \epsilon + M$ for some real $\epsilon > 0$,
- (3) $\pi(b)$ is strictly positive definite for every M -positive $*$ -representation π of B ,
- (4) $f(b) > 0$ for every $f \in \underline{M^\vee}$,
- (5) $f(b) > 0$ for every $f \in \overline{\text{ex } M^\vee}$,
- (6) $f(b) > 0$ for every factorizable $f \in M^\vee$.

Proof. (1) implies (2) because the set $M - b$ is absorbing, hence $-1 \in t(M - b)$ for some $t > 0$. Clearly (2) implies (3). (3) implies (4) because the cyclic $*$ -representation that belongs to f by the GNS construction clearly has the property that $\pi(m)$ is positive semidefinite for every $m \in M$. (4) implies (1) by the separation theorem for convex sets. The details can be found in [4, Theorem 12] or [18, Proposition 15] or [5, Proposition 1.4].

If (5) is true then, by the compactness of $\overline{\text{ex } M^\vee}$, there exists $\epsilon > 0$ such that $f(b) \geq \epsilon$ for every $f \in \overline{\text{ex } M^\vee}$, hence (4) is true by the Krein–Milman theorem. Clearly, (4) implies (6). By Proposition 3 below and the fact that the set of all factorizable M -positive states is closed, (6) implies (5). \square

Similarly, we have the following:

Proposition 2. For every archimedean quadratic module M in B and every element $b \in B_h$ the following are equivalent:

- (1) $b \in \overline{M}$ (the closure w.r.t. the finest locally convex topology),
- (2) $b + \epsilon \in M$ for every $\epsilon > 0$,
- (3) $\pi(b)$ is positive semidefinite for every M -positive $*$ -representation π of B ,
- (4) $f(b) \geq 0$ for every $f \in \underline{M^\vee}$,
- (5) $f(b) \geq 0$ for every $f \in \overline{\text{ex } M^\vee}$,
- (6) $f(b) \geq 0$ for every factorizable $f \in M^\vee$.

The following proposition which extends [19, Ch. IV, Lemma 4.11] was used in the proof of equivalences (4)–(6) in Propositions 1 and 2. Its proof depends on the equivalence of (2) and (3) in Proposition 2.

Proposition 3. If M is an archimedean quadratic module in B then all extreme M -positive states are factorizable.

Proof. Pick any $\omega \in \text{ex } M^\vee$ and $y \in Z(B)_h$. We claim that $\omega(xy) = \omega(x)\omega(y)$ for every $x \in B_h$. Since $y = \frac{1}{4}((1+y)^2 - (1-y)^2)$ and $(1 \pm y)^2 \in M$, we may assume that $y \in M$. Since M is archimedean, we may also assume that $1 - y \in M$.

Claim. If $\omega(y) = 0$, then $\omega(y^2) = 0$. (Equivalently, if $\omega(1 - y) = 0$, then $\omega((1 - y)^2) = 0$.)

Since $y, 2 - y \in M$, it follows that $1 - (1 - y)^2 = \frac{1}{2}(y(2 - y)^2 + (2 - y)y^2) \in M$. Since ω is an M -positive state, it follows that $\omega((1 - y)^2) \leq 1$. On the other hand, $\omega((1 - y)^2)\omega(1^2) \geq$

$|\omega((1 - y) \cdot 1)|^2$ by the Cauchy–Schwartz inequality. Now, $\omega(y) = 0$ implies that $\omega((1 - y)^2) = 1$, hence $\omega(y^2) = 0$.

Case 1: If $\omega(y) = 0$, then $\omega(xy) = 0$ for every $x \in B_h$. (Equivalently, if $\omega(1 - y) = 0$, then $\omega(x(1 - y)) = 0$ for every $x \in B_h$.) Namely, by the Cauchy–Schwartz inequality and the Claim, $|\omega(xy)|^2 \leq \omega(x^2)\omega(y^2) = 0$. It follows that $\omega(xy) = \omega(x)\omega(y)$ if $\omega(y) = 0$ or $\omega(y) = 1$.

Case 2: If $0 < \omega(y) < 1$, then ω_1 and ω_2 defined by

$$\omega_1(x) := \frac{1}{\omega(y)}\omega(xy) \quad \text{and} \quad \omega_2(x) := \frac{1}{\omega(1 - y)}\omega(x(1 - y))$$

($x \in B_h$) are M -positive states on B_h . Namely, for every M -positive $*$ -representation π of B and every $x \in M$, we have that $\pi(xy) = \pi(x)\pi(y)$ is a product of two commuting positive semidefinite bounded operators, hence a positive semidefinite bounded operator. By the equivalence of assertions (2) and (3) in Proposition 2, $xy + \epsilon \in M$ for every $\epsilon > 0$. Since ω is M -positive, it follows that $\omega(xy) \geq 0$ as claimed. Similarly, we prove that ω_2 is M -positive. Clearly, $\omega = \omega(y)\omega_1 + \omega(1 - y)\omega_2$. Since ω is an extreme point of the set of all M -positive states on B_h , it follows that $\omega = \omega_1 = \omega_2$. In particular, $\omega(xy) = \omega(x)\omega(y)$. \square

If we apply Proposition 1 or 2 to $b = -1$, we get the following corollary, parts of which were already mentioned above.

Corollary 4. *For every archimedean quadratic module M in B the following are equivalent:*

- (1) $-1 \notin M$,
- (2) *there exists an M -positive $*$ -representation of B ,*
- (3) *there exists an M -positive state on B ,*
- (4) *there exists an extreme M -positive state on B ,*
- (5) *there exists a factorizable M -positive state on B .*

The following variant of Proposition 1 which follows easily from Corollary 4 was proved in [3, Theorem 5]. We could call it archimedean nichtnegativsemidefinitheitsstellensatz for $*$ -algebras.

Proposition 5. *For every archimedean proper quadratic module M on a real or complex $*$ -algebra B and for every $x \in B_h$, the following are equivalent:*

- (1) *For every M -positive $*$ -representation ψ of B , $\psi(x)$ is not negative semidefinite (i.e. $\langle \psi(x)v, v \rangle > 0$ for some $v \in H_\psi$).*
- (2) *There exist $k \in \mathbb{N}$ and $c_1, \dots, c_k \in B$ such that $\sum_{i=1}^k c_i x c_i^* \in 1 + M$.*

3. Theorems H and F

The aim of this section is to prove Theorem H (see Theorem 6) and show that it implies a generalization of Theorem F to compact operators. We also prove a concrete version of Proposition 5.

Theorem 6. Let R be a commutative real algebra with trivial involution, A a $*$ -algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$ and M an archimedean quadratic module in $B := R \otimes A$. For every element p of $B_h = R \otimes A_h$, the following are equivalent:

- (1) $p \in \epsilon + M$ for some real $\epsilon > 0$.
- (2) For every multiplicative state ϕ on R , there exists real $\epsilon_\phi > 0$ such that $(\phi \otimes \text{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \text{id}_A)(M)$.

The following are also equivalent:

- (1') $p + \epsilon \in M$ for every real $\epsilon > 0$.
- (2') For every multiplicative state ϕ on R and every real $\epsilon > 0$ we have that $(\phi \otimes \text{id}_A)(p) + \epsilon \in (\phi \otimes \text{id}_A)(M)$.

Moreover, the following are equivalent:

- (1'') There exist finitely many $c_i \in B$ such that $\sum_i c_i^* p c_i \in 1 + M$.
- (2'') For every multiplicative state ϕ on R there exist finitely many $d_i \in A$ such that $\sum_i d_i^* (\phi \otimes \text{id}_A)(p) d_i \in 1 + (\phi \otimes \text{id}_A)(M)$.

Proof. Clearly (1) implies (2). We will prove the converse in several steps. Note that for every multiplicative state ϕ on R , the mapping $\phi \otimes \text{id}_A : B \rightarrow A$ is a surjective homomorphism of $*$ -algebras, hence $(\phi \otimes \text{id}_A)(M)$ is an archimedean quadratic module in A . Replacing B , M , f and p in Proposition 1 with A , $(\phi \otimes \text{id}_A)(M)$, σ and $(\phi \otimes \text{id}_A)(p)$, we see that (2) is equivalent to:

- (A) For every multiplicative state ϕ on R and every state σ on A_h such that $\sigma((\phi \otimes \text{id}_A)(M)) \geq 0$ we have that $\sigma((\phi \otimes \text{id}_A)(p)) > 0$.

Note that $(\phi \otimes \sigma)(r \otimes a) = \phi(r)\sigma(a) = \sigma(\phi(r)a) = \sigma((\phi \otimes \text{id}_A)(r \otimes a))$ for every $r \in R$ and $a \in A_h$. It follows that $\phi \otimes \sigma = \sigma \circ (\phi \otimes \text{id}_A)$. Thus, (A) is equivalent to:

- (B) For every M -positive state on $R \otimes A_h$ of the form $\omega = \phi \otimes \sigma$ where ϕ is multiplicative, we have that $\omega(p) > 0$.

Since $R \otimes 1 \subseteq Z(B)$, every factorizable state ω satisfies $\omega(r \otimes a) = \omega(r \otimes 1)\omega(1 \otimes a)$ and $\omega(rs \otimes 1) = \omega(r \otimes 1)\omega(s \otimes 1)$ for any $r, s \in R$ and $a \in A_h$. Hence $\omega = \phi \otimes \sigma$ where ϕ is a multiplicative state on R and σ is a state on A_h . Therefore, (B) implies that:

- (C) $\omega(p) > 0$ for every factorizable $\omega \in M^\vee$.

By Proposition 1, (C) is equivalent to (1).

The equivalence of (1') and (2') can be proved in a similar way using Proposition 2. It can also be easily deduced from the equivalence of (1) and (2).

Clearly (1'') implies (2''). Conversely, if (1'') is false, then $-1 \notin N$ where $N := \{m - \sum c_i^* p c_i \mid m \in M, c_i \in B\}$ is the smallest quadratic module in B which contains M and $-p$. By Corollary 4, there exists a factorizable state $\omega \in N^\vee$. From the proof of (1) \Leftrightarrow (2), we know that $\omega = \phi \otimes \sigma = \sigma \circ (\phi \otimes \text{id}_A)$ for a multiplicative state ϕ on R and a state σ on A .

Since $\sigma((\phi \otimes \text{id}_A)(N)) = \omega(N) \geq 0$, it follows that $-1 \notin (\phi \otimes \text{id}_A)(N)$. Since $(\phi \otimes \text{id}_A)(N) = \{(\phi \otimes \text{id}_A)(m) - \sum_j d_j^*(\phi \otimes \text{id}_A)(p)d_j \mid m \in M, d_j \in A\}$, we get that (2'') is false. \square

For every Hilbert space H we write $B(H)$ for the set of all bounded operators on H , $P(H) = \Sigma^2(B(H))$ for the set of all positive semidefinite operators on H and $K(H)$ for the set of all compact operators on H .

Lemma 7. *Let H be a separable Hilbert space and M a quadratic module in $B(H)$ which is not contained in $P(H)$. Then \overline{M} contains all hermitian compact operators, i.e. $K(H)_h \subseteq \overline{M}$.*

Proof. Let M be a quadratic module in $B(H)$ which is not contained in $P(H)$. Pick an arbitrary operator L in $M \setminus P(H)$ and a vector $v \in H$ such that $\langle v, Lv \rangle < 0$. Write P for the orthogonal projection of H on the span of v . Clearly, $PLP = \lambda P$ where $\lambda < 0$, hence $-P \in M$. If Q is an orthogonal projection of rank 1, then $Q = U^*PU$ for some unitary U , hence $-Q \in M$. Since also $Q = Q^*Q \in M$, M contains all hermitian operators of rank 1. Therefore, M contains all finite rank operators. Pick any $K \in K(H)_h \cap P(H)$ and note that $\sqrt{K} \in K(H)_h \cap P(H)$ as well. Clearly, $-K + \epsilon\sqrt{K} \in M$ for every $\epsilon > 0$ since it is a sum of an element from $K(H)_h \cap P(H)$ and a finite rank operator (check the eigenvalues). It follows that $-K \in \overline{M}$. It is also clear that every element of $K(H)_h$ is a difference of two elements from $K(H)_h \cap P(H)$, hence $K(H)_h \subseteq \overline{M}$. \square

As the first application of Theorem 6 and Lemma 7, we prove the following generalization of Theorem F. By Lemma 11 below, Theorem F corresponds to the case $R = \mathbb{R}[x]$ and H finite-dimensional, i.e. in the finite-dimensional case we can omit the assumption on p .

Theorem 8. *Let R be a commutative real algebra with trivial involution, H a separable Hilbert space, M an archimedean quadratic module in $R \otimes B(H)$ and p an element of $R \otimes B(H)$.*

If for every multiplicative state ϕ on R there exists a real $\eta_\phi > 0$ such that $(\phi \otimes \text{id}_{B(H)})(p) \in \eta_\phi + P(H) + K(H)_h$ (e.g. if $p \in \eta + R \otimes K(H)_h$ for some $\eta > 0$) then the following are equivalent:

- (1) $p \in M^\circ$.
- (2) For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{B(H)})(M) \subseteq P(H)$, there exists $\epsilon_\phi > 0$ such that $(\phi \otimes \text{id}_{B(H)})(p) \in \epsilon_\phi + P(H)$.

If $(\phi \otimes \text{id}_{B(H)})(p) \in P(H) + K(H)_h$ for every multiplicative state ϕ on R (e.g. if $p \in R \otimes K(H)_h$) then the following are equivalent:

- (1') $p \in \overline{M}$.
- (2') For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{B(H)})(M) \subseteq P(H)$, we have that $(\phi \otimes \text{id}_{B(H)})(p) \in P(H)$.

Proof. Suppose that (1) is true, i.e. $p \in \epsilon + M$ for some $\epsilon > 0$. For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{B(H)})(M) \subseteq P(H)$ we have that $(\phi \otimes \text{id}_{B(H)})(p) \in (\phi \otimes \text{id}_{B(H)})(\epsilon + M) \subseteq \epsilon + P(H)$, hence (2) is true. Conversely, suppose that (2) is true. We claim that for every multiplicative state ϕ on R there exists $\epsilon_\phi > 0$ such that $(\phi \otimes \text{id}_{B(H)})(p) \in \epsilon_\phi + (\phi \otimes \text{id}_{B(H)})(M)$. Then it follows by Theorem 6 that (1) is true. If $(\phi \otimes \text{id}_{B(H)})(M) \subseteq P(H)$, then $(\phi \otimes \text{id}_{B(H)})(p) \in$

$\epsilon_\phi + P(H) \subseteq \epsilon_\phi + (\phi \otimes \text{id}_{B(H)})(M)$ by (2) and the fact that $(\phi \otimes \text{id}_{B(H)})(M)$ is a quadratic module in $B(H)$. On the other hand, if $(\phi \otimes \text{id}_{B(H)})(M) \not\subseteq P(H)$, then $K(H)_h \subseteq \overline{(\phi \otimes \text{id}_{B(H)})(M)}$ by Lemma 7. The assumption $(\phi \otimes \text{id}_{B(H)})(p) \in \eta_\phi + P(H) + K(H)_h$ for some $\eta_\phi > 0$ then implies that $(\phi \otimes \text{id}_{B(H)})(p) \in \frac{\eta_\phi}{2} + (\phi \otimes \text{id}_{B(H)})(M)$ as claimed. The proof of the equivalence (1') \Leftrightarrow (2') is similar. \square

In the infinite-dimensional case, the assumption on p cannot be omitted as the following example shows:

Example. Let H be an infinite-dimensional separable Hilbert space, $0 \neq E \in B(H)_h$ an orthogonal projection of finite rank and T an element of $B(H)_h$ such that $T \notin P(H) + K(H)_h$. Since the quadratic module $\Sigma^2(B(H)/K(H))$ is closed, also $P(H) + K(H)_h$ is closed, hence there exists a real $\epsilon > 0$ such that $T + \epsilon \notin P(H) + K(H)_h$. Write $p_1 = -x^2E$, $p_2 = 1 - x^2$ and $p = \epsilon + x^2T$. Let M be the quadratic module in $\mathbb{R}[x] \otimes B(H)$ generated by p_1 and p_2 . Since $p_2 \in M$, it follows from Lemma 11 below that M is archimedean. For every point $a \in \mathbb{R}$ such that $p_1(a) \geq 0$ and $p_2(a) \geq 0$ we have that $a = 0$, hence $p(a) = \epsilon$. Therefore, assertion (2) of Theorem 8 is true for our M and p . Assertion (1), however, fails for our M and p . If it was true then there would exist finitely many $q_i, u_j, v_k \in \mathbb{R}[x] \otimes B(H)$ and a real $\eta > 0$ such that $p = \eta + \sum_i q_i^* q_i + \sum_j u_j^* p_1 u_j + \sum_k v_k^* p_2 v_k$. For $x = 1$, we get $\epsilon + T = \eta + \sum_i q_i(1)^* q_i(1) - \sum_j u_j(1)^* E u_j(1)$. The first two terms belong to $P(H)$ and the last term belongs to $K(H)_h$, a contradiction with the choice of T .

We finish this section with a concrete version of Proposition 5 in the spirit of Theorem F. For $R = \mathbb{R}[x]$, we get [10, Corollary 22].

Theorem 9. *Let R be a commutative real algebra with trivial involution, $v \in \mathbb{N}$, and M an archimedean quadratic module in $M_v(R)$. For every element $p \in M_v(R)_h$, the following are equivalent:*

- (1) *There are finitely many $c_i \in M_v(R)$ such that $\sum_i c_i^* p c_i \in 1 + M$.*
- (2) *For every multiplicative state ϕ on R such that $(\phi \otimes \text{id})(m)$ is positive semidefinite for all $m \in M$, we have that the operator $(\phi \otimes \text{id})(p)$ is not negative semidefinite.*

Proof. Write $A = M_v(\mathbb{R})$. Clearly, a matrix $C \in A_h$ is not negative semidefinite (i.e. it has at least one strictly positive eigenvalue) iff there exist matrices $D_i \in A$ such that $\sum_i D_i^* C D_i - I$ is positive semidefinite. It follows that a quadratic module M in A which is different from $\Sigma^2(A)$ contains $-I$, hence it is equal to A_h . (This also follows from Lemma 7.) Now we use equivalence (1'') \Leftrightarrow (2'') of Theorem 6. \square

4. Theorem E

Recall that a $*$ -algebra A is algebraically bounded if the quadratic module $\Sigma^2(A)$ is archimedean. For an element $a \in A_h$ we say that $a \geq 0$ iff $a + \epsilon \in \Sigma^2(A)$ for all real $\epsilon > 0$ (i.e. iff $a \in \overline{\Sigma^2(A)}$) and that $a > 0$ iff $a \in \epsilon + \Sigma^2(A)$ for some real $\epsilon > 0$ (i.e. iff $a \in \Sigma^2(A)^\circ$). It is well known that every Banach $*$ -algebra is algebraically bounded.

The aim of this section is to deduce the following theorem from Theorem 6 and to show that it implies Theorem E. Other applications of Theorem 6 will be discussed in Section 5.

Theorem 10. *Let R be a commutative real algebra with trivial involution and A an algebraically bounded $*$ -algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$. Let U be an inner product space over F , $\mathcal{L}(U)$ the $*$ -algebra of all adjointable linear operators on U , $\mathcal{L}(U)_+$ its subset of positive semidefinite operators, and M an archimedean quadratic module in $R \otimes \mathcal{L}(U)$.*

Write $B := R \otimes A$ and consider the vector space $B \otimes U$ as left $R \otimes \mathcal{L}(U)$ right B bimodule which is equipped with the biadditive form $\langle \cdot, \cdot \rangle$ defined by $\langle b_1 \otimes u_1, b_2 \otimes u_2 \rangle := b_1^ b_2 \langle u_1, u_2 \rangle_U$. Write M' for the subset of B_h which consists of all finite sums of elements of the form $\langle q, mq \rangle$ where $m \in M$ and $q \in B \otimes U$.*

We claim that the set M' is an archimedean quadratic module and that for every element $p \in R \otimes A_h$ the following are equivalent:

- (1) $p \in \epsilon + M'$ for some real $\epsilon > 0$.
- (2) For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \text{id}_A)(p) > 0$.

Moreover, the following are equivalent:

- (1') $p \in \overline{M'}$.
- (2') For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \text{id}_A)(p) \geq 0$.

Finally, the following are equivalent:

- (1'') There exist finitely many $c_i \in B$ such that $\sum_i c_i^* p c_i \in 1 + M'$.
- (2'') For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, there exist finitely many elements $d_i \in A$ such that $\sum_i d_i^* (\phi \otimes \text{id}_A)(p) d_i - 1 \geq 0$.

We will need the following observation which follows from the fact that the set of bounded elements w.r.t. a given quadratic module is closed for addition and multiplication of commuting elements.

Lemma 11. *Let R be a commutative algebra with trivial involution and A an algebraically bounded $*$ -algebra. A quadratic module N in $R \otimes A$ is archimedean if and only if $N \cap R$ is archimedean in R . If x_1, \dots, x_d are generators of R , then N is archimedean if and only if it contains $K^2 - x_1^2 - \dots - x_d^2$ for some real K .*

Proof of Theorem 10. To prove that (1) implies (2), it suffices to prove:

Claim 1. *For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \text{id}_A)(M') \subseteq \Sigma^2(A)$.*

For every $q \in B \otimes U$ and $m \in M$ we have that $(\phi \otimes \text{id}_A)(\langle q, mq \rangle) = \langle s, (\phi \otimes \text{id}_{\mathcal{L}(U)})(m)s \rangle$ where $s = (\phi \otimes \text{id}_A \otimes \text{id}_U)(q) \in A \otimes U$. If $s = \sum_{i=1}^k a_i \otimes u_i$, then

$$\langle s, (\phi \otimes \text{id}_{\mathcal{L}(U)})(m)s \rangle = [a_1^* \quad \dots \quad a_k^*] T \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

where $T = [\langle u_i, (\phi \otimes \text{id}_{\mathcal{L}(U)})(m)u_j \rangle]_{i,j=1}^k \in M_k(F)$. Since $(\phi \otimes \text{id}_{\mathcal{L}(U)})(m)$ is positive semidefinite for every $m \in M$, T is also positive semidefinite.

To prove that (2) implies (1), consider the following statement:

(3) For every multiplicative state ϕ on R there exists a real $\epsilon_\phi > 0$ such that $(\phi \otimes \text{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \text{id}_A)(M')$.

We claim that (2) implies (3) and (3) implies (1).

Clearly, $b^*(q, mq)b = \langle qb, mqb \rangle \in M'$ for every $m \in M$, $q \in B \otimes U$ and $b \in B$, hence the set M' is a quadratic module in B . Clearly, $M' \cap R$ is archimedean since it contains $M \cap R$. By Lemma 11, M' is also archimedean. Hence (3) implies (1) by Theorem 6.

Suppose that (2) is true and pick a multiplicative state ϕ on R . Clearly, $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M)$ is a quadratic module in $\mathcal{L}(U)$ and $(\phi \otimes \text{id}_A)(M')$ is a quadratic module in A . If $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, then (2) implies that $(\phi \otimes \text{id}_A)(p) \in \epsilon_\phi + \Sigma^2(A) \subseteq \epsilon_\phi + (\phi \otimes \text{id}_A)(M')$ for some real $\epsilon_\phi > 0$, hence (3) is true. If $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \not\subseteq \mathcal{L}(U)_+$ then (3) follows from:

Claim 2. For every multiplicative state ϕ on R such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \not\subseteq \mathcal{L}(U)_+$ we have that $(\phi \otimes \text{id}_A)(M') = A_h$.

We could use Lemma 7 but we prefer to prove this claim from scratch. Pick any $C \in (\phi \otimes \text{id}_{\mathcal{L}(U)})(M) \setminus \mathcal{L}(U)_+$. There exists $u \in U$ of length 1 such that $\langle u, Cu \rangle < 0$. Write P for the orthogonal projection of U on the span of $\{u\}$. Clearly, $P^*CP = -\lambda P$ for some $\lambda > 0$, hence $-P \in (\phi \otimes \text{id}_{\mathcal{L}(U)})(M)$. Also, $P = P^*P \in (\phi \otimes \text{id}_{\mathcal{L}(U)})(M)$. Let $m_\pm \in M$ be such that $(\phi \otimes \text{id}_{\mathcal{L}(U)})(m_\pm) = \pm P$. Pick any $a \in A_h$ and write $q_\pm = 1_R \otimes \frac{1 \pm a}{2} \otimes u$ where $1 = 1_A$. The element $m' = \langle q_+, m_+q_+ \rangle + \langle q_-, m_-q_- \rangle$ belongs to M' and, by the proof of Claim 1, $(\phi \otimes \text{id}_A)(m') = \langle s_+, (\phi \otimes \text{id}_{\mathcal{L}(U)})(m_+)s_+ \rangle + \langle s_-, (\phi \otimes \text{id}_{\mathcal{L}(U)})(m_-)s_- \rangle$ where $s_\pm = (\phi \otimes \text{id}_A \otimes \text{id}_U)(q_\pm) = \frac{1 \pm a}{2} \otimes u$. Therefore, $(\phi \otimes \text{id}_A)(m') = (\frac{1+a}{2})^2 \langle u, Pu \rangle + (\frac{1-a}{2})^2 \langle u, -Pu \rangle = (\frac{1+a}{2})^2 - (\frac{1-a}{2})^2 = a$.

Claim 1 also gives implications (1') \Rightarrow (2') and (1'') \Rightarrow (2'') and Claim 2 gives their converses. Note that assertion (3) must be replaced with suitable assertions (3') and (3'') to which Theorem 6 can be applied. \square

For $U = F^v$, we have that $B \otimes U \cong B^v \cong M_{v \times 1}(B)$, $R \otimes \mathcal{L}(U) \cong M_v(R) \subseteq M_v(B)$ and $\langle q, mq \rangle = q^*mq$ in Theorem 10. However, if also $A = M_v(F)$ (i.e. $B = M_v(R)$), we do not get Theorem F. Combining both theorems, we get that archimedean quadratic modules M and M' in $M_v(R)$ have the same interior and the same closure.

Finally, we would like to show that Theorem 10 implies Theorem E. The proof also works for algebraically bounded $*$ -algebras.

Proof of Theorem E. Write $v = 2 + v_1 + \dots + v_m$, $\|x\| = \sqrt{x_1^2 + \dots + x_d^2}$ and $p_0 = [1 - \|x\|^2] \oplus [\|x\|^2 - 1] \oplus p_1 \oplus \dots \oplus p_m \in M_v(\mathbb{R}[x])$. Clearly, $K_0 = \{t \in \mathbb{R}^d \mid p_0(t) \geq 0\}$. Let M_0 be the quadratic module in $M_v(\mathbb{R}[x])$ generated by p_0 . Since M_0 contains $(1 - \|x\|^2)I_v$, it is archimedean by Lemma 11. By Theorem 10 applied to $U = F^v$, every element $p \in \mathbb{R}[x] \otimes A$ which is strictly positive definite on K_0 belongs to M'_0 . From the definition of M'_0 , we get that $p = \sum_{j \in J} (s_j^*s_j + q_j^*p_0q_j)$ for a finite J , $s_j \in \mathbb{R}[x] \otimes A$ and $q_j \in M_{v \times 1}(\mathbb{R}[x] \otimes A)$, hence

$$p = \sum_{j \in J} \left(s_j^* s_j + w_j^* (1 - \|x\|^2) w_j + z_j^* (\|x\|^2 - 1) z_j + \sum_{k=1}^m s_{jk}^* p_k s_{jk} \right)$$

for a finite J , $s_j, w_j, z_j \in \mathbb{R}[x] \otimes A$ and $s_{jk} \in M_{v_k \times 1}(\mathbb{R}[x] \otimes A)$. Replacing x by $\frac{x}{\|x\|}$ and multiplying with a large power of $\|x\|^2$ we get that

$$\|x\|^{2\theta} p(x) = \sum_{j \in J} \left((u_j(x) + \|x\| v_j(x))^* (u_j(x) + \|x\| v_j(x)) + \sum_{k=1}^m (u_{jk}(x) + \|x\| v_{jk}(x))^* p_k(x) (u_{jk}(x) + \|x\| v_{jk}(x)) \right)$$

where $u_j, v_j \in \mathbb{R}[x] \otimes A$ and $u_{jk}, v_{jk} \in M_{v_k \times 1}(\mathbb{R}[x] \otimes A)$ for every $j \in J$. Finally, we can get rid of the terms containing $\|x\|$ by replacing $\|x\|$ with $-\|x\|$ and adding the old and the new equation. \square

5. Theorem G and moment problems

Our next result, Theorem 12, is a special case of Theorem 10 for $U = F$. The proof can be shortened in this case because both claims become trivial. For $R = \mathbb{R}[\cos \phi_1, \sin \phi_1, \dots, \cos \phi_n, \sin \phi_n]$, $M = \Sigma^2(R)$ and $A = B(H)$ it implies Theorem G. For $R = \mathbb{R}[x]$ and A a finite-dimensional C^* -algebra it implies [17, Theorem 2], a step in the original proof of Theorem F. Both special cases can also be obtained from the original proof of Theorem E, namely, Theorem 3 and Lemma 5 from [1] imply Theorem 12 for $R = \mathbb{R}[x]$ and A a C^* -algebra.

Theorem 12. *Let R be a commutative real algebra with trivial involution, A an algebraically bounded $*$ -algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$ and M an archimedean quadratic module in R . Write $M' = M \cdot \Sigma^2(R \otimes A)$ for the quadratic module in $R \otimes A$ which consists of all finite sums of elements of the form $m q^* q$ with $m \in M$ and $q \in R \otimes A$. For every element $p \in R \otimes A_h$, the following are equivalent:*

- (1) $p \in \epsilon + M'$ for some real $\epsilon > 0$.
- (2) For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, we have that $(\phi \otimes \text{id}_A)(p) > 0$.

If we combine Theorem 12 with a suitable version of the Riesz representation theorem (namely, Theorem 1 in [9]) we get the following existence result for operator-valued moment problems which extends Theorem 3 and Lemma 5 from [1].

Theorem 13. *Let A be an algebraically bounded $*$ -algebra, R a commutative real algebra and M an archimedean quadratic module on R . For every linear functional $L : R \otimes A \rightarrow \mathbb{R}$ such that $L(m q^* q) \geq 0$ for every $m \in M$ and $q \in R \otimes A$, there exists an A' -valued nonnegative measure m on M^\vee such that $L(p) = \int_{M^\vee} (dm, p)$ for every $p \in R \otimes A$. (Note that p defines a function $\phi \mapsto (\phi \otimes \text{id}_A)(p)$ from M^\vee to A .)*

Proof. Recall that the set M^\vee is compact in the weak*-topology. We assume that A is equipped with its natural C^* -seminorm induced by the archimedean quadratic module $\Sigma^2(A)$, see [4,

Section 3], hence it is a locally convex $*$ -algebra. We will write $\mathcal{C}^+(M^\vee, A) := \mathcal{C}(M^\vee, \overline{\Sigma^2(A)})$ for the positive cone of $\mathcal{C}(M^\vee, A)$. Let i be the mapping from $R \otimes A$ to $\mathcal{C}(M^\vee, A)$ defined by $i(p)(\phi) = (\phi \otimes \text{id}_A)(p)$ for every $p \in R \otimes A$ and $\phi \in M^\vee$. By Theorem 12, we have that $\mathcal{C}^+(M^\vee, A) \cap i(R \otimes A) = i(\overline{M'})$ where $M' = M \cdot \Sigma^2(R \otimes A)$. Note that L is an $\overline{M'}$ -positive functional on $R \otimes A$ and that it defines in the natural way an $i(\overline{M'})$ -positive functional L' on $i(R \otimes A)$. By the Riesz extension theorem for positive functionals, L' extends to a $\mathcal{C}^+(M^\vee, A)$ -positive functional on $\mathcal{C}(M^\vee, A)$ which has the required integral representation by Theorem 1 in [9]. Hence L also has the required integral representation. \square

Finally, we would like to prove a nichtnegativsemidefinitheitsstellensatz that corresponds to Theorem 12.

Theorem 14. *Let H be a separable infinite-dimensional complex Hilbert space and R a commutative real algebra with trivial involution. Let M be an archimedean quadratic module in R and $M' = M \cdot \Sigma^2(R \otimes B(H))$. For every $p \in R \otimes B(H)_h$, the following are equivalent:*

- (1) *There are finitely many $c_i \in R \otimes B(H)$ such that $\sum_i c_i^* p c_i \in 1 + M'$.*
- (2) *For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, the operator $(\phi \otimes \text{id}_{B(H)})(p)$ is not the sum of a negative semidefinite and a compact operator.*

Note that for finite-dimensional H , (1) is equivalent to the following: For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, the operator $(\phi \otimes \text{id}_{B(H)})(p)$ is not negative semidefinite; cf. Theorem 9.

Proof. The equivalence (1'') \Leftrightarrow (2'') of Theorem 10 (with $U = \mathbb{C}$ and $A = B(H)$) says that our assertion (1) is equivalent to the following:

- (3) *For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, there exist finitely many operators $D_i \in B(H)$ such that $\sum_i D_i^*(\phi \otimes \text{id}_A)(p)D_i \in 1 + P(H)$.*

Therefore it suffices to prove the following claim:

Claim. *For every operator $C \in B(H)_h$, the following are equivalent:*

- (i) *C is not the sum of a negative semidefinite and a compact operator,*
- (ii) *the positive part of C is not compact,*
- (iii) *there exists an operator D such that $D^*CD = 1$,*
- (iv) *there exist finitely many $D_i \in B(H)$ such that $\sum_i D_i^*CD_i \in 1 + P(H)$.*

The implications (i) \Rightarrow (ii), (iii) \Rightarrow (iv) and (iv) \Rightarrow (i) are clear. To prove that (ii) implies (iii) we first note that $C_+ := E_0C = E_0^*CE_0$ where E_0 is the spectral projection belonging to the interval $[0, \infty)$. Since C_+ is not compact, there exists by the spectral theorem a real number $\gamma > 0$ such that the spectral projection E_γ belonging to the interval $[\gamma, \infty)$ has infinite-dimensional range. The operator $C_\gamma := E_\gamma C_+ = E_\gamma^*CE_\gamma$ has decomposition $C_\gamma = \tilde{C}_\gamma \oplus 0$ with respect to $H = E_\gamma H \oplus (1 - E_\gamma)H$ where $\tilde{C}_\gamma \geq \gamma$. Write $F = (\tilde{C}_\gamma)^{-1/2} \oplus 0$ and note that $(E_\gamma F)^*C(E_\gamma F) = 1 \oplus 0$. Since $E_\gamma H$ is infinite-dimensional, it is isometric to H . If G is an isometry from H onto $E_\gamma H$ then $D := E_\gamma FG$ satisfies (iii). \square

References

- [1] C.-G. Ambrozie, F.-H. Vasilescu, Operator-theoretic Positivstellensätze, *Z. Anal. Anwend.* 22 (2) (2003) 299–314.
- [2] R. Berr, T. Wörmann, Positive polynomials and tame preorderings, *Math. Z.* 236 (4) (2001) 813–840.
- [3] J. Cimprič, Maximal quadratic modules on $*$ -rings, *Algebr. Represent. Theory* 11 (1) (2008) 83–91.
- [4] J. Cimprič, A representation theorem for archimedean quadratic modules on $*$ -rings, *Canad. Math. Bull.* 52 (2009) 39–52.
- [5] J. Cimprič, M. Marshall, T. Netzer, Closures of quadratic modules, *Trans. Amer. Math. Soc.*, in press.
- [6] M.A. Dritschel, J. Rovnyak, The operator Fejer–Riesz theorem, arXiv:0903.3639v1.
- [7] D. Handelman, Rings with involution as partially ordered abelian groups, *Rocky Mountain J. Math.* 11 (3) (1981) 337–381.
- [8] T. Jacobi, A representation theorem for certain partially ordered commutative rings, *Math. Z.* 237 (2) (2001) 259–273.
- [9] G.W. Johnson, The dual of $C(S, F)$, *Math. Ann.* 187 (1970) 1–8.
- [10] I. Klep, M. Schweighofer, Pure states, positive matrix polynomials and sums of hermitian squares, arXiv:0907.2260.
- [11] M. Marshall, Positive polynomials and sums of squares, in: *Math. Surveys Monogr.*, vol. 146, American Mathematical Society, Providence, RI, 2008, xii+187 pp., ISBN 978-0-8218-4402-1; 0-8218-4402-4.
- [12] M. Marshall, A general representation theorem for partially ordered commutative rings, *Math. Z.* 242 (2) (2002) 217–225.
- [13] A. Naftalevich, B. Schreiber, Trigonometric polynomials and sums of squares, in: *Number Theory*, New York, 1983–1984, in: *Lecture Notes in Math.*, vol. 1135, Springer, Berlin, 1985, pp. 225–238.
- [14] M. Putinar, Positive polynomials on compact semi-algebraic sets, *Indiana Univ. Math. J.* 42 (3) (1993) 969–984.
- [15] M. Putinar, F.-H. Vasilescu, Solving moment problems by dimensional extension, *Ann. of Math.* (2) 149 (3) (1999) 1087–1107.
- [16] B. Reznick, Uniform denominators in Hilbert’s seventeenth problem, *Math. Z.* 220 (1) (1995) 75–97.
- [17] C.W. Scherer, C.W.J. Hol, Matrix sum-of-squares relaxations for robust semi-definite programs, *Math. Program.* 107 (1–2 (B)) (2006) 189–211.
- [18] K. Schmüdgen, Noncommutative real algebraic geometry—some basic concepts and first ideas, in: *Emerging Applications of Algebraic Geometry*, in: *IMA Vol. Math. Appl.*, vol. 149, Springer, New York, 2009, pp. 325–350.
- [19] M. Takesaki, *Theory of Operator Algebras, I*, Springer, New York, Heidelberg, ISBN 0-387-90391-7, 1979, vii+415 pp.
- [20] I. Vidav, On some $*$ -regular rings, *Acad. Serbe Sci. Pubi. Inst. Math.* 13 (1959) 73–80.

Further reading

- [21] K. Schmüdgen, The K -moment problem for compact semi-algebraic sets, *Math. Ann.* 289 (2) (1991) 203–206.