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On the stability of the stochastic parabolic Itô equation with delay and Markovian jump

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ABSTRACT

We present a new result concerning the stability of the stochastic parabolic Itô equation subject to homogenous white noise. Our main results state that this system is exponentially stable by means of a new Lyapunov–Krasovskii functional and a linear matrix inequality (LMI). A numerical example is exploited to show the usefulness of the derived LMI-based stability.

1. Introduction

During the last decades, a considerable amount of attention has been paid to studying the stability and control of dynamic systems governed by ordinary differential equations (ODEs) with uncertain constant or time-varying delays (see for example [1–9]). It is well known that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability conditions. Special forms of the Lyapunov–Krasovskii functional have been used for the derivation of simple finite-dimensional conditions in terms of linear matrix inequalities (LMIs). These conditions are either delay independent or delay dependent.

The stability analysis of partial differential equations (PDEs) with delay is essentially more complicated. There are only a few works on Lyapunov-based techniques for PDEs with delay. Wang [10] extended the second Lyapunov method to an abstract nonlinear time-delay system in Banach spaces and applied the result to the delay-independent stability analysis of some scalar heat and wave equations with constant delays of Dirichlet boundary conditions [11].

In real-life models, the functions of actual delayed systems are influenced by unknown disturbances, which may be regarded as stochastic. In order to fix these problems, the system dynamics are suitably approximated by a stochastic linear or nonlinear delayed system. Thus, stochastic delayed PDEs have their own characteristics and it is desirable to obtain stability criteria that make full use of these characteristics.

To the best of our knowledge, only a few works have been done on the stability criterion for exponential stability of the delayed stochastic parabolic Itô equation with reaction–diffusion terms along Markovian jumping parameters, and the LMI approach has never been tackled for any such works. The LMI approach is an open and very challenging problem concerning the stability of delayed and Markovian jumping parameter systems. In this paper, a new criterion for the exponential stability of the system is investigated by constructing the Lyapunov–Krasovskii functional in terms of LMIs, which can be easily calculated by the Matlab toolbox. A numerical example is given to illustrate the effectiveness and less conservativeness of the proposed system.

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2. Model description

In this paper, we analyze the exponential stability of the stochastic parabolic Itô equation of the form

$$\left.\begin{array}{l} \mathrm{d}u(x,t,\omega) = [\mathrm{div}(K(x,t)\circ\nabla u(x,t,\omega)) + A(r(t))g(u(x,t,\omega)) - M(r(t))u(x,t,\omega) \\ \quad + C(r(t))u(x,t-\tau,\omega)]\mathrm{d}t + [g(u(x,t,\omega)) + g(u(x,t-\tau,\omega))]\,\mathrm{d}B(t,\omega) \quad (x,t,\omega)\in \mathrm{D}\times \mathrm{R}^+\times\Omega, \\ u(x,0,\omega) = \phi(x,\omega)\in(u_0,u_1), \quad (x,t)\in \mathrm{D}\times\Omega, \\ \dfrac{\partial u(x,t,\omega)}{\partial n(k)} = 0, \quad (x,t,\omega)\in\partial \mathrm{D}\times \mathrm{R}^+\times\Omega, \end{array}\right\} \tag{1}$$

where $\{r(t), t > 0\}$ is a right-continuous Markov process on the probability space which takes values in the finite space $H = \{1, 2, \ldots, N\}$ with generator $\Gamma = \{\gamma_{ij}\}$ $(i, j \in H)$ (also called the jumping transfer matrix) given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

in which $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$.

Assume that $(B(t,\cdot))_{t\in \mathrm{R}_0^+}$ denotes one-dimensional Brownian motion starting at the origin and defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $(\mathcal{F}_t)_{t\in\mathrm{R}_0^+}$ be a filtration on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $B(t+\eta,\cdot) - B(t,\cdot)$ is independent of $(\mathcal{F}_t)$ for all $t, \eta \in \mathrm{R}_0^+$. Without restricting the generality, we assume that the stochastic basis $(\Omega, F, (\mathcal{F}_t)_{t\in\mathrm{R}_0^+}, \mathrm{P})$ is complete. D denotes an open bounded and connected subset of R^N with a sufficient regular boundary ∂D. Let the notation $\|\cdot\|_p$ denote the usual $L^p(\mathrm{D})$-norm,

$$\|f\|^p_{L^p(\mathrm{D})} = \int_{\mathrm{D}} |f(x)|^p \mathrm{d}x, \quad p \in [1, \infty].$$

g is the composition of the real-valued stochastic process u. $g \in C^2([u_0, u_1])$ and $g(u_0) = g(u_1) = 0$ and $g(u) > 0$ for every $u \in (u_0, u_1)$, and g satisfy the following Lipschitz condition

$$|g(x_1) - g(x_2)| \leq L|(x_1 - x_2)| \quad \forall x_1, x_2 \in \mathrm{R}. \tag{2}$$

$A(r(t)), M(r(t)), C(r(t))$ are known constant matrices with appropriate dimensions. The functional $K(x, t)$ is matrix valued with entries that are Lebesgue measurable in $(x, t) \in \mathrm{D}\times\mathrm{R}^+$. Recall that the Markov process $\{r(t), t > 0\}$ takes values in the finite space $\{H = 1, 2, \ldots, N\}$. For the sake of simplicity, we write $r(t) = i$, $\forall i \in H$

$$A(i) = A_i, \qquad M(i) = M_i, \qquad C(i) = C_i.$$

For the purpose of simplicity, we rewrite Eq. (1) as follows; usually we suppress $u(t)$ instead of $u(x, t, \omega)$ and also we suppress $B(t)$ instead of $B(t, \omega)$. Thus Eq. (1) can be rewritten as

$$\mathrm{d}u(t) = [\mathrm{div}(K(x,t)\nabla u(t)) + A_i g(u(t)) - M_i u(t) + C_i u(t-\tau)]\,\mathrm{d}t + [g(u(t)) + g(u(t-\tau))]\,\mathrm{d}B(t). \tag{3}$$

3. Main result

In this section we state results and definitions that are needed to prove the main theorem.

Lemma 3.1 ([12]). *If $u(t)$ is a solution of system (1), then*

$$\int_D u^T(t)\nabla\cdot(K(x,t)\circ\nabla u(t))\mathrm{d}x = -\int_D (K(x,t)\cdot(\nabla u\circ\nabla u))E\mathrm{d}x.$$

Definition 3.2 ([5]). The trivial solution of system (1) is said to be exponentially stable in mean square if there exist positive constants λ and c such that, for any v_0,

$$E(|v(x,t,t_0,v_0)|^2) \leq c|v_0|^2 \mathrm{e}^{-\lambda(t-t_0)},$$

from which one deduces immediately that

$$\lim_{t\to\infty}\sup \frac{1}{t}\lg(E|v(x,t,t_0,v_o)|) < 0.$$

The above equation can be written as

$$\mathrm{d}u(t) = f(u(t))\mathrm{d}t + \sigma(u(t))\mathrm{d}B(t),$$

where

$$f(u(t)) = \mathrm{div}(K(x,t)\circ\nabla u(t)) + A_i g(u(t)) - M_i u(t) + C_i u(t-\tau),$$
$$\sigma(u(t)) = g(u(t)) + g(u(t-\tau)).$$

Theorem 3.3. *Given positive definite matrices A_i, M_i, C_i and under the given assumption, the solution of system (1) is exponentially stable on norm $\|\cdot\|_2$ in the mean square for a given constant delay τ, provided there exist positive definite matrices P_i, Q and R and a positive constant α such that the following* LMI *holds:*

$$\Xi = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ * & \Omega_{22} \end{bmatrix} < 0, \tag{4}$$

where

$$\Omega_{11} = \alpha P_i + Q - P_i M_i - M_i^T P_i + P_i A_i L + L^T A_i^T P_i + L^T R L + L^T P_i L + \sum_{j=1}^{n} \gamma_{ij} P_j,$$

$$\Omega_{12} = P_i C_i + L^T P_i L, \qquad \Omega_{22} = -\mathrm{e}^{-\alpha\tau} Q - \mathrm{e}^{-\alpha\tau} L^T R L + L^T P_i L.$$

Proof. Define the following Lyapunov–Krasovskii functional:

$$V(u(t), t, i) = \int_D \left\{ \mathrm{e}^{\alpha t} u^T(t) P_i u(t) + \int_{t-\tau}^{t} \mathrm{e}^{\alpha s} u^T(s) Q u(s) \mathrm{d}s + \int_{t-\tau}^{t} \mathrm{e}^{\alpha s} g^T(u(s)) R g(u(s)) \mathrm{d}s \right\} \mathrm{d}x,$$

where $P_i = P_i^T$; $Q = Q^T$; $R = R^T$.

The stability results can be proved using the following way.

By the Itô differential rule, the stochastic derivative of V along trajectories of (3) can be obtained as follows:

$$\mathrm{d}V(u(t), t, i) = \mathcal{L}V(u(t), t, i)\mathrm{d}t + \frac{\partial V(u(t), t, i)}{\partial y} \sigma(u(t)) \mathrm{d}B(t).$$

Integrating on both sides of the above equation over $(0, t)$ with respect to t and taking expectation, we get

$$E\left[V(u(t), t, i)\right] = E\left[V(u(0), 0, i)\right] + E\left[\int_0^t \mathcal{L}V(u(t), t, i)\mathrm{d}t\right]. \tag{5}$$

Define an operator $\mathcal{L}$ associated with the Eq. (3) acting on V by

$$\mathcal{L}V(u(t), t, i) = \frac{\partial V(u(t), t, i)}{\partial t} + \frac{\partial V(u(t), t, i)}{\partial u} f(u(t)) + \frac{1}{2} \mathrm{trace}\left[\sigma^T(u(t)) \frac{\partial^2 V(u(t), t, i)}{\partial u^2} \sigma(u(t))\right] + \sum_{j=1}^{n} \gamma_{ij} V(u(t), t, j).$$

The generator $\mathcal{L}V$ for the evolution of V is given by

$$\begin{aligned} \mathcal{L}V(u(t), t, i) = \int_D \Big\{ & \alpha \mathrm{e}^{\alpha t} u^T(t) P_i u(t) + \mathrm{e}^{\alpha t} u^T(t) Q u(t) - \mathrm{e}^{\alpha(t-\tau)} u^T(t-\tau) Q u(t-\tau) \\ & + \mathrm{e}^{\alpha t} g^T(u(t)) R g(u(t)) - \mathrm{e}^{\alpha(t-\tau)} g^T(u(t-\tau)) R g(u(t-\tau)) \\ & + 2\mathrm{e}^{\alpha t} u^T(t) P_i \nabla \cdot (K(x,t) \circ \nabla u(t)) + 2\mathrm{e}^{\alpha t} u^T(t) P_i A_i g(u(t)) - 2\mathrm{e}^{\alpha t} u^T(t) P_i M_i u(t) \\ & + 2\mathrm{e}^{\alpha t} u^T(t) P_i C_i u(t-\tau) + \mathrm{e}^{\alpha t} \mathrm{trace}\left[\sigma^T(u(t)) P_i \sigma(u(t))\right] + \sum_{j=1}^{n} \gamma_{ij} V(u(t), t, j) \Big\} \mathrm{d}x. \end{aligned}$$

The following equalities hold, since $\sum_{j=1}^{n} \gamma_{ij} = 0$:

$$\begin{aligned} \sum_{j=1}^{n} \gamma_{ij} \left[\int_D \left\{\int_{t-\tau}^{t} \mathrm{e}^{\alpha s} u^T(s) Q u(s) \mathrm{d}s\right\} \mathrm{d}x\right] &= \left(\sum_{j=1}^{n} \gamma_{ij}\right) \int_D \left\{\int_{t-\tau}^{t} \mathrm{e}^{\alpha s} u^T(s) Q u(s) \mathrm{d}s\right\} \mathrm{d}x \\ &= 0; \\ \sum_{j=1}^{n} \gamma_{ij} \left[\int_D \left\{\int_{t-\tau}^{t} \mathrm{e}^{\alpha s} g(u^T(s)) R g(u(s)) \mathrm{d}s\right\} \mathrm{d}x\right] &= \left(\sum_{j=1}^{n} \gamma_{ij}\right) \int_D \left\{\int_{t-\tau}^{t} \mathrm{e}^{\alpha s} g(u^T(s)) R g(u(s)) \mathrm{d}s\right\} \mathrm{d}x \\ &= 0. \end{aligned}$$

By using Lemma 3.1 and Eq. (2), we have

$$\mathcal{L}V(u(t), t, i) \le \mathrm{e}^{\alpha t} \int_D \Big\{ u^T(t) \left[P_i C_i + L^T P_i L\right] u(t-\tau) - (K(x,t) \cdot (\nabla u \circ \nabla u)) E$$

$$+u^T(t)\left[\alpha P_i + Q - P_i M_i - M_i^T P_i + P_i A_i L + L^T A_i^T P_i + L^T R L + L^T P_i L + \sum_{j=1}^{n} \gamma_{ij} P_j\right] u(t) + u^T(t-\tau)\left[-e^{-\alpha\tau} Q - e^{-\alpha\tau} L^T R L + L^T P_i L\right] u(t-\tau)\Bigg\} dx.$$

That is,

$$\mathcal{L}V(u(t), t, i) \le e^{\alpha t} \int_D N^T(t) \Xi N(t) dx,$$

where $N^T(t) = \left[u^T(t), u^T(t-\tau)\right]$.

By using condition (4), we get

$$\mathcal{L}V(u(t), t, i) < 0,$$

we can rewrite Eq. (5) as

$$E\left[V(u(t), t, i)\right] \le E\left[V(u(0), 0, i)\right], \tag{6}$$

where

$$E\left[V(u(0), 0, i)\right] = BE\|\phi\|_2^2,$$

in which

$$B = \lambda_M(P_i) - \frac{e^{-\alpha\tau}\lambda_M(Q)}{\alpha} + \frac{\lambda_M(Q)}{\alpha} - \frac{e^{-\alpha\tau}\lambda_M(L^T R L)}{\alpha} + \frac{\lambda_M(L^T R L)}{\alpha}$$

and $E\|\phi\|_2^2 = \sup_{0<s<\tau} E\|u(s)\|_2^2$.

On the other hand,

$$E\left[V(u(t), t, i)\right] \ge e^{\alpha t}\lambda_M(P_i) E\|u(t)\|_2^2.$$

Therefore, Eq. (6) becomes

$$E\|u(t)\|_2^2 \le e^{-\alpha t}\lambda_M^{-1}(P_i) BE\|\phi\|_2^2.$$

This completes the proof. □

4. Example

In this section, the main result is demonstrated with the following example. Our aim is to examine the exponential stability of a given stochastic parabolic Itô equation. For the sake of simplicity, we consider Eq. (3) with the given parameters

$$A_1 = \begin{bmatrix} 0.45 & 0.35 \\ 0.26 & 0.85 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.12 & 0.75 \\ 0.96 & 0.88 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 8 \\ 1 & 3 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -3 & 4 \\ -5 & 6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -4 & 2 \\ -7 & 3 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} -6 & 7 \\ 6 & -7 \end{bmatrix}.$$

Suppose that g is described by

$$g(u) = \tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Let $\tau = 0.45$ and $\alpha = 0.25$ in Theorem 3.3; then, using Matlab, we solve the LMI. The feasible solutions for Theorem 3.3 are

$$P_1 = \begin{bmatrix} 2.7157 & 0.4403 \\ 0.4403 & 2.6097 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3.0514 & 0.9182 \\ 0.9182 & 4.6348 \end{bmatrix}, \quad Q = \begin{bmatrix} 6.9451 & -3.9349 \\ -3.9349 & 8.2693 \end{bmatrix},$$
$$R = \begin{bmatrix} 6.9451 & -3.9349 \\ -3.9349 & 8.2693 \end{bmatrix}.$$

The above results show that all conditions stated in Theorem 3.3 have been satisfied. Hence the stochastic parabolic Itô equation is exponentially stable for the above given parameters. □

5. Conclusion

In this paper, using a suitable Lyapunov–Krasovskii functional, inequality techniques and LMIs, sufficient conditions have been derived for checking the exponential stability of a stochastic parabolic Itô equation with delay and Markovian jumping parameters. The derived conditions are expressed in terms of LMIs, which have been checked numerically for less conservative results. The main advantages of the LMI-based approach is that LMI stability conditions can be solved numerically using the Matlab LMI toolbox, which implements the state of the art of interior point algorithms.

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