JOURNAL OF ALGEBRA 7, 140-143 (1967)

Do Simple Rings Have Unity Elements?

J. C. ROBSON

Department of Mathematics, The University, Leeds, England

Communicated by A. W. Goldie

Received October 20, 1966

1. It is well known that a simple artinian ring has a unity element, and that a commutative simple ring, being a field, has a unity element. However, it can easily be seen that not every simple ring has a unity element, this being shown in the first proposition of this note. Nevertheless, it could be asked whether a simple right noetherian ring necessarily has a unity element and the answer (which comprises the main result of this note) is a qualified "yes".

2. There are simple integral domains which are not division rings—see, for example, [*2*]. Thus the following result demonstrates that there exist simple integral domains without unity elements.

PROPOSITION. *Let R be a simple integral domain which is not a division ring and let I be a proper nonzero right ideal of R. Then IR is a simple integral domain without a unity element.*

Proof. Let T be a nonzero ideal of IR. Then

$$IR = I(RTIR) = (IR)\,T(IR) \subseteq T,$$

and so $T = IR$ and IR is a simple integral domain.

Say IR has a unity element e. Then e is an idempotent. Thus $e^2r = er$ for all $r \in R$, and hence $er = r$ since R is an integral domain. Therefore

$$R = eR \subseteq IR \cdot R = IR \subseteq I,$$

which contradicts the assumption that I was a proper right ideal of R. So IR does not have a unity element.

3. We will be using the concept of the characteristic of a simple ring. The next proposition, which considers the more general case of a prime ring, clarifies this concept.

PROPOSITION. *Let s be a nonzero element of a prime ring R and suppose that there is a least positive integer m such that $ms = 0$. Then R has characteristic m and m is a prime.*

Proof. Let $X = \{x \in R \mid mx = 0\}$ and let $mR = \{mr \mid r \in R\}$. Evidently X and mR are ideals of R and $XmR = 0$. Now R is prime and $X \neq 0$, since $s \in X$. Thus $mR = 0$, and so R has characteristic m.

Assume that m is not a prime, $m = m_1 m_2$, say. Then $m_1 R m_2 R = mR = 0$. So either $m_1 R = 0$ or $m_2 R = 0$. But this contradicts the minimality of m. Therefore m is a prime.

4. The next lemma is crucial for the proof of the main theorem. First we recall that a simple right noetherian ring R has a right quotient ring Q which is a simple artinian ring (see [1]). Q has a unity element 1 and so has a nonzero center F. Since Q is simple, F is a field.

LEMMA. *Let R be a simple ring with a simple artinian right quotient ring Q, and let F be the center of Q.*

(i) *For each nonzero element $f \in F$, $fR = Rf = R$.*

(ii) *The center of R is either F or zero.*

Proof. (i) $f = ac^{-1}$ for some $a, c \in R$, c regular. Let $X = \{x \in R \mid fx \in R\}$. Then X is a nonzero right ideal of R since $c \in X$. However, $f \in F$, so $X = \{x \in R \mid xf \in R\}$. Hence X is an ideal of R, and therefore $X = R$. Thus $fR \subseteq R$. However $Rf = fR$ and so fR is an ideal of R. But, since $a \in fR$, fR is nonzero. Hence $fR = Rf = R$.

(ii) Since Q is the right quotient ring of R, the center of R is contained in F. Thus, if f' is a nonzero element in the center of R, then $f' \in F$. Now F is a field, so $f'F = F$. But, by (i), $f'F \subseteq R$. So $F \subseteq R$ and thus F is the center of R.

5. Continuing with the notation of the preceding section, it is not difficult to see that R, Q and F must all have the same characteristic. If that characteristic is zero, then F is infinite, since it contains the rational field. Thus the corollary below will be an immediate consequence of the theorem.

THEOREM. *Let R be a simple right noetherian ring whose right quotient ring Q has an infinite center F. Then R has a unity element.*

COROLLARY. *A simple right noetherian ring of characteristic zero has a unity element.*

Proof of Theorem. We will denote by N the subring of F generated by 1;

and we will say that an element $f \in F$ is *integral over* N if there exist n_0, $n_1, \ldots, n_m \in N$, such that

$$f^{m+1} = n_0 + fn_1 + \cdots + f^m n_m$$

for some $m \geqslant 0$. There are two possibilities which must be considered separately.

I. *There is an element* $f \in F$ *which is not integral over* N. We note first that this includes the case when R has characteristic zero. For then N is the rational integers and $\frac{1}{2} \in F$; but it is easily verified that $\frac{1}{2}$ is not integral over the rational integers.

Now let x be any regular element of R, and consider the chain of right ideals of R,

$$xR^1 \subseteq xR^1 + xfR^1 \subseteq xR^1 + xfR^1 + xf^2R^1 \subseteq \cdots,$$

where xR^1 denotes the right ideal of R generated by x. Since R is right noetherian, this chain terminates. So we see that

$$xf^{m+1} \in xR^1 + xfR^1 + \cdots + xf^mR^1$$

for some $m \geqslant 0$. Thus

$$xf^{m+1} = x(r_0 + n_0) + xf(r_1 + n_1) + \cdots + xf^m(r_m + n_m), \quad r_i \in R, \quad n_i \in N.$$

But x is regular and so

$$f^{m+1} = (r_0 + n_0) + f(r_1 + n_1) + \cdots + f^m(r_m + n_m);$$

i.e.,

$$f^{m+1} - (n_0 + fn_1 + \cdots + f^m n_m) = r_0 + fr_1 + \cdots + f^m r_m = g,$$

say. Clearly $g \in F \cap R$, using (i) of the lemma. Also $g \neq 0$ since f is not integral over N. Thus R has a nonzero center. So, by (ii) of the lemma, F is the center of R and $1 \in R$.

II. *Every element of* F *is integral over* N. In this case we know that R has finite characteristic. So N is a finite field. But F is infinite, so there exists $f_1 \in F, f_1 \notin N$. Since f_1 is integral over N, $N[f_1]$ is also a finite field. So there exists $f_2 \in F, f_2 \notin N[f_1]$; and, once again, $N[f_1, f_2]$ is a finite field. In this way, we obtain an infinite sequence $\{f_1, f_2, f_3, \ldots\}$ of elements of F such that $f_{m+1} \notin N[f_1, \ldots, f_m]$.

Consider the chain of right ideals of R,

$$xR^1 \subseteq xR^1 + xf_1R^1 \subseteq xR^1 + xf_1R^1 + xf_2R^1 \subseteq \cdots.$$

Since R is right noetherian, this chain terminates. Therefore

$$xf_{m+1} \in xR^1 + xf_1R^1 + \cdots + xf_mR^1 \quad \text{for some} \quad m.$$

So

$$xf_{m+1} = x(r_0 + n_0) + xf_1(r_1 + n_1) + \cdots + xf_m(r_m + n_m), \quad r_i \in R, \quad n_i \in N.$$

Hence

$$f_{m+1} - (n_0 + f_1 n_1 + \cdots + f_m n_m) = r_0 + f_1 r_1 + \cdots + f_m r_m = g,$$

say. Clearly $g \in F \cap R$; and $g \neq 0$ since $f_{m+1} \notin N[f_1, ..., f_m]$. Hence F is the center of R and $1 \in R$.

It would now seem reasonable to conjecture that every simple right noetherian ring has a unity element. A search for a counter-example has revealed an interesting point—namely, that all the standard examples of simple noetherian rings (which are not artinian) have characteristic zero.

6. As a further consequence of the lemma of Section 4, we obtain the following result.

THEOREM. *A simple ring which satisfies a polynomial identity is artinian.*

Proof. Let the ring be R. By [*3*], R has a right and left quotient ring Q which is finite-dimensional over its center F. But, by the lemma, $RF = R$. So R is itself a finite-dimensional algebra over F.

Let c be any regular element of R. For some integer m, the set $c, c^2, ..., c^m$ is linearly dependent. So, for some integer k, $1 \leqslant k \leqslant m - 1$,

$$c^k = c^{k+1}f_{k+1} + \cdots + c^m f_m, \quad f_{k+1}, ..., f_m \in F.$$

Since c is regular, we see that

$$1 = cf_{k+1} + \cdots + c^{m-k}f_m \in R.$$

It follows that $F \subseteq R$, and hence right ideals of R are F-subspaces of R. So R is right artinian and, by symmetry, left artinian.

REFERENCES

1. GOLDIE, A. W. The structure of prime rings under ascending chain conditions. *Proc. London Math. Soc.* **8** (1958), 589-608.

2. HIRSCH, K. A. A note on non-commutative polynomials. *J. London Math. Soc.* **12** (1937), 264-266.

3. POSNER, E. C. Prime rings satisfying a polynomial identity. *Proc. Am. Math. Soc.* **11** (1960), 180-183.

Printed in Belgium