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AN α-DISCONNECTED SPACE HAS NO PROPER MONIC PREIMAGE

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All spaces are compact Hausdorff. α is an uncountable cardinal or the symbol ∞. A continuous map $\tau: X \to Y$ is called an α-SpFi morphism if $\tau^{-1}(G)$ is dense in X whenever G is a dense α-cozero set of Y. We thus have a category α-SpFi (spaces with the α-filter) which, like any category, has its monomorphisms; these need not be one-to-one. For general α, we cannot say what the α-SpFi monics are, but we show, and R.G. Woods showed, that ∞-SpFi monic means range-irreducible. The main theorem here is: X has no proper α-SpFi monic preimage if and only if X is α-disconnected. This generalizes (by putting in $\alpha = \infty$) the well-known fact: X has no proper irreducible preimage if and only if X is extremally disconnected. If, in our theorem, we restrict to Boolean spaces and apply Stone duality, we have the theorem of R. Lagrange, that in Boolean α-algebras, epimorphisms are surjective.

The theory of spaces with filters has a lot of connections with ordered algebra—Boolean algebras of course, but also lattice-ordered groups and frames. This paper is a contribution to the development of this topological theory.

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α-complete Boolean algebra
α-disconnected space
spaces with filters

1. Spaces with filters

X, Y and Z will denote compact Hausdorff spaces α will denote an uncountable cardinal number or the symbol ∞; the meaning of $\alpha = \infty$ will be clear from, or explained in, the context.

Definition 1.1. The category of spaces with filters, denoted SpFi, has for objects pairs $(X, \mathcal{F})$ where $\mathcal{F}$ is a filter base of dense subsets of X. A morphism $\tau: (X, \mathcal{F}) \to (Y, \mathcal{H})$ is a continuous function from X into Y that inversely preserves the elements of the filter bases, i.e., $\tau^{-1}(H) \in \mathcal{F}$ for all $H \in \mathcal{H}$.

Definition 1.2. Let $C(X)$ be the real-valued continuous functions on X. Let $\mathrm{Coz}(X)=\{f^{-1}(\mathbb{R}\setminus\{0\}): f\in C(X)\}$ be the set of cozero sets of X. A subset $V\subset X$ is said to be an *α-cozero set* if $V=\bigcup\{U_i: i\in I, |I|<\alpha, U_i\in \mathrm{Coz}(X)\}$. Note that an ω_1-cozero set is a cozero set. By "$|I|<\infty$" we mean "$|I|$ is unrestricted", so that every open set is an ∞-cozero set. We denote the collection of α-cozero sets by $\mathrm{Coz}_\alpha(X)$. Let $\mathscr{I}_\alpha(X)$ be the filter base of the dense members of $\mathrm{Coz}_\alpha(X)$; thus $\mathscr{I}_\infty(X)$ will denote the filter base of the dense open sets. Finally let α-SpFi be the full subcategory of SpFi that has for objects pairs of the form $(X, \mathscr{I}_\alpha(X))$.

Observe that a continuous function $\tau: X\to Y$ is a morphism in α-SpFi if and only if $\tau^{-1}(G)$ is dense in X for all $G\in\mathscr{I}_\alpha(Y)$.

Definition 1.3. Let $(X,\mathscr{F})\in|\mathrm{SpFi}|$ and let E be a closed subset of X. Let $\mathscr{F}\cap E=\{F\cap E: F\in\mathscr{F}\}$. This collection may or may not consist of dense subsets of E. If it does, we call E a SpFi *subset* of $(X,\mathscr{F})$. This means that $(E,\mathscr{F}\cap E)\in|\mathrm{SpFi}|$ and the inclusion, $(E,\mathscr{F}\cap E)\hookrightarrow(X,\mathscr{F})$, is a SpFi morphism. If E is a SpFi subset of $(X,\mathscr{I}_\alpha(X))$, we simply say E is an α-SpFi subset of X, and write $E\subset^\alpha X$.

From this definition the following proposition is immediate.

Proposition 1.4. *The following are equivalent*:

(a) $E\subset^\infty X$;

(b) *E is a* SpFi *subset of* $(X,\mathscr{F})$ *for all* $\mathscr{F}$;

(c) *E is a regular closed subset of* X.

Also it has recently come to our attention that what we call an ω_1-SpFi subset of X, Veksler in [17], calls a P'-set of X.

Proposition 1.5. *Let* $\tau:(X,\mathscr{F})\to(Y,\mathscr{H})$ *be a* SpFi *morphism and let E be a* SpFi *subset of* $(X,\mathscr{F})$, *then*

(a) $\tau[E]$ *is a* SpFi *subset of* $(Y,\mathscr{H})$.

(b) *The injective ∘ surjective factorization of* τ *is a factorization in* SpFi, *as*

$$(X,\mathscr{F})\twoheadrightarrow(\tau[X],\mathscr{H}\cap\tau[X])\hookrightarrow(Y,\mathscr{H}).$$

Proof. (a) Suppose $\tau:(X,\mathscr{F})\to(Y,\mathscr{H})$ is a SpFi morphism. Let E be a SpFi subset of $(X,\mathscr{F})$ and let $H\in\mathscr{H}$. Then $\tau^{-1}(H)\in\mathscr{F}$ so $E\cap\tau^{-1}(H)$ is dense in E. It follows that $\tau[E\cap\tau^{-1}(H)]$ is dense in $\tau[E]$; and since $\tau[E\cap\tau^{-1}(H)]=\tau[E]\cap H$, we have that $\tau[E]\cap H$ is also dense in $\tau[E]$. Hence $\tau[E]$ is a SpFi subset of $(Y,\mathscr{H})$.

(b) The proof of (b) follows directly from (a) and the definition of a SpFi subset. □

We do not know if the converse to Proposition 1.5(a) is true in SpFi. However it is true in α-SpFi.

Proposition 1.6. *A continuous function $\tau: X \to Y$ is an α-*SpFi *morphism if and only if $E \subset^{\alpha} X$ implies $\tau[E] \subset^{\alpha} Y$.*

Proof. The necessity is provided by Proposition 1.5. On the other hand suppose τ is not an α-SpFi morphism. Then there is a $G \in \mathcal{I}_{\alpha}(Y)$ such that $\overline{\tau^{-1}(G)}$ is not dense in X. It follows that there is a regular closed set, U, of X with $U \subset X \setminus \overline{\tau^{-1}(G)}$. Now $U \subset^{\alpha} X$, but $\tau[U] \cap G = \emptyset$, so $\tau[U] \not\subset^{\alpha} Y$. □

The issue of factorizations in α-SpFi as in Proposition 1.5(b) is more complicated, and is treated in the next section.

2. α-disconnected spaces

Definition 2.1. X is said to be *α-disconnected* if $\bar{F}$ is open for all $F \in \mathrm{Coz}_{\alpha}(X)$ (see Definition 1.2).

Observe that ω_1-disconnected and ∞-disconnected, are respectively, the usual definitions of basically and extremally disconnected [5].

Definition 2.2. A closed set $E \subset X$ is called a *P_{α}-set* if whenever $\{V_i : i \in I, |I| < \alpha\}$ is a family of neighborhoods of E, then $\bigcap_{i \in I} V_i$ is also a neighborhood of E. The family of P_{α}-sets in X is denoted $P_{\alpha}(X)$.

The definition of P_{ω_1}-set is the usual definition of P-set that appears in the literature (see [15, 16]).

We now show that the α-SpFi subsets of an α-disconnected space are exactly the P_{α}-sets of X. We begin with a few preliminary propositions. The proofs of some of these are straightforward and are omitted.

Proposition 2.3. *Let K be closed in X. Then $E \in P_{\alpha}(X)$ implies $E \cap K \in P_{\alpha}(K)$.*

Proposition 2.4. *Let C be clopen in X. Then $E \in P_{\alpha}(X)$ implies $E \cap C \in P_{\alpha}(X)$.*

Lemma 2.5. *Let X be zero-dimensional, $E \in P_{\alpha}(X)$, and let L be a dense F_{α}-subset of X (i.e., $L = \bigcup_{i \in I} K_i$ where the K_i are closed in X and $|I| < \alpha$)). Then $E \cap L$ is dense in E.*

Proof. Let C be clopen in X and suppose $C \cap E \neq \emptyset$. By Proposition 2.4 $C \cap E \in P_{\alpha}(X)$. We claim $L \cap C \cap E \neq \emptyset$. If $L \cap C \cap E = \emptyset$, then $C \cap E \subset \bigcap_{i \in I} X \setminus K_i = X \setminus L$. Since $C \cap E \in P_{\alpha}(X)$, then $X \setminus L$ is a neighborhood of $C \cap E$, but this contradicts the density of L. □

Proposition 2.6. *Let X be a α-disconnected. Then $E \subset^{\alpha} X$ if and only if $E \in P_{\alpha}(X)$.*

Proof. Let $E \in P_{\alpha}(X)$. If $G \in \mathcal{I}_{\alpha}(X)$, then G is a dense F_{α}-set so Lemma 2.5 implies that $E \cap G$ is dense in E. Hence $E \subset^{\alpha} X$.

Now suppose $E \notin P_{\alpha}(X)$. Then, without loss of generality, we may assume that there is a family, $\{Z_i : i \in I, |I| < \alpha\}$, of zero-set neighborhoods of E such that $\bigcap_{i \in I} Z_i$ is not a neighborhood of E. Since X is α-disconnected $U = [\bigcap_{i \in I} Z_i]^{\circ}$ is clopen. It follows that $G = [\bigcup_{i \in I} X \backslash Z_i] \cup U \in \mathcal{I}_{\alpha}(X)$. Furthermore, $E \cap X \backslash U$ is a nonempty open set in E with $(E \cap X \backslash U) \cap (E \cap G) = \emptyset$. Hence, $E \cap G$ is not dense in E, i.e., $E \not\subset^{\alpha} X$. □

Lemma 2.7. *Let X be α-disconnected, $E \subset^{\alpha} X$, and $U \in \mathrm{Coz}_{\alpha}(X)$. Then $\overline{U \cap E}^{E} = \bar{U}^{X} \cap E$.*

Proof. Suppose $\overline{U \cap E}^{E} \neq \bar{U}^{X} \cap E$, then $\overline{U \cap E}^{E} \subsetneq \bar{U}^{X} \cap E$. Since $\bar{U}^{X}$ is open in X, then $V = \bar{U}^{X} \cap E \backslash \overline{U \cap E}^{E}$ is a nonempty open set in E. Now $G = U \cup (X \backslash \bar{U}^{X})$ is a member of $\mathcal{I}_{\alpha}(X)$. It follows that $G \cap E \in \mathcal{I}_{\alpha}(E)$, but $V \cap G \cap E = \emptyset$. This contradicts the density of $G \cap E$ in E. Hence $\overline{U \cap E}^{E} = \bar{U}^{X} \cap E$. □

Corollary 2.8. *Let X be α-disconnected and let $E \in P_{\alpha}(X)$, then:*
(a) *E is also α-disconnected;*
(b) *$\mathcal{I}_{\alpha}(E) = \mathcal{I}_{\alpha}(X) \cap E$.*

Proof. (a) By Proposition 2.6, $E \subset^{\alpha} X$. Let $V \in \mathrm{Coz}_{\alpha}(E)$; then $V = U \cap E$ for some $U \in \mathrm{Coz}_{\alpha}(X)$. By Lemma 2.7 we have $\bar{V}^{E} = \bar{U}^{X} \cap E$, which is open in E since $\bar{U}^{X}$ is open in X.

(b) Again since $E \subset^{\alpha} X$ we have $\mathcal{I}_{\alpha}(X) \cap E \subset \mathcal{I}_{\alpha}(E)$. Let $G \in \mathcal{I}_{\alpha}(E)$, then $G = U \cap E$ for some $U \in \mathrm{Coz}_{\alpha}(X)$. By the density of G in E and Lemma 2.7 we have $E = \bar{G}^{E} = \bar{U}^{X} \cap E$. It follows that $G = E \cap [U \cup (X \backslash \bar{U}^{X})]$. Since $U \cup (X \backslash \bar{U}^{X}) \in \mathcal{I}_{\alpha}(X)$ we have $\mathcal{I}_{\alpha}(E) \subset \mathcal{I}_{\alpha}(X) \cap E$. □

For $\alpha = \omega_1$, Corollary 2.8(a) was known to both Tzeng [15] and Veksler [16], and Corollary 2.8(b) to Tzeng.

Proposition 2.9. *The category* ∞-SpFi *has injective* $\circ$ *surjective factorizations.*

Proof. Let $\tau : X \to Y$ be an ∞-SpFi morphism. We claim that its injective $\circ$ surjective factorization $\tau = i \circ s$ (as in Proposition 1.5(b)) is in ∞-SpFi. By Proposition 1.6, $i : \tau[X] \hookrightarrow Y$ is an ∞-SpFi morphism. Also $s : (X, \mathcal{I}_{\infty}(X)) \twoheadrightarrow (\tau[X], \mathcal{I}_{\infty}(Y) \cap \tau[X])$ is an ∞-SpFi morphism; that is, by Proposition 1.4, $\tau[X]$ is a regular closed set in Y so $\mathcal{I}_{\infty}(\tau[X]) = \mathcal{I}_{\infty}(Y) \cap \tau[X]$. □

Unfortunately, for $\alpha < \infty$, an α-SpFi morphism need not have its injective ∘ surjective factorization in α-SpFi:

Example 2.10. There is an ω_1-SpFi morphism which does not have its injective ∘ surjective factorization in ω_1-SpFi.

The space $\beta\mathbb{N}\backslash\mathbb{N} = \mathbb{N}^*$ has the following properties (see [19, 3.23 and 3.28]):

(i) $\mathcal{I}_{\omega_1}(\mathbb{N}^*) = \{\mathbb{N}^*\}$.

(ii) There is a countable and relatively discrete set $F \subset \mathbb{N}^*$, and $\bar{F}$ is homeomorphic to $\beta\mathbb{N}$.

Pick a point $p \in \bar{F} - F$ and let $Y = \beta\mathbb{N}_1 \oplus \beta\mathbb{N}_2$ be the disjoint union of two copies of $\beta\mathbb{N}$. Define $\tau: Y \to \mathbb{N}^*$ by letting τ map $\beta\mathbb{N}_1$ homeomorphically onto $\bar{F}$ and $\beta\mathbb{N}_2$ onto $\{p\}$. Since $\mathcal{I}_{\omega_1}(\mathbb{N}^*) = \{\mathbb{N}^*\}$, then τ is an ω_1-SpFi morphism. We claim that $s: Y \twoheadrightarrow \bar{F}$ is not an ω_1-SpFi morphism; for $F \in \mathcal{I}_{\omega_1}(\bar{F})$, but $s^{-1}(F) = \beta\mathbb{N}_1 \notin \mathcal{I}_{\omega_1}(Y)$.

The crux of Example 2.10 is that the codomain $\beta\mathbb{N}\backslash\mathbb{N}$ is not ω_1-disconnected. For:

Proposition 2.11. *An* α-SpFi *morphism with an α-disconnected codomain has its injective ∘ surjective factorization in* α-SpFi.

Proof. Let $\tau: X \to Y$ be an α-SpFi morphism and let Y be α-disconnected. The inclusion, $i: \tau[X] \hookrightarrow Y$, is an α-SpFi morphism by Proposition 1.5. Also, since $\tau[X] \subset^{\alpha} Y$ and Y is α-disconnected, Proposition 2.6 implies $\tau[X] \in P_\alpha(Y)$. So by Corollary 2.8(b), $\mathcal{I}_\alpha(\tau[X]) = \mathcal{I}_\alpha(Y) \cap \tau[X]$. Hence, $s: (X, \mathcal{I}_\alpha(X)) \twoheadrightarrow (\tau[X], \mathcal{I}_\alpha(Y) \cap \tau[X])$ is an α-SpFi morphism. □

There is, of course, much more to the issue of factorizations in α-SpFi. For example, it is shown in [20] that ∞-SpFi has monic ∘ extremal epic factorizations, and in [11] that an α-SpFi morphism with an α-disconnected codomain has a monic ∘ extremal epic factorization.

The following proposition is germane to this paper, but not needed for the main result.

Proposition 2.12. *Let X be α-disconnected and let $E_1, E_2 \in P_\alpha(X)$. Then $E_1 \cap E_2 \in P_\alpha(X)$.*

Proof. By Corollary 2.8(a), E_2 is α-disconnected and by Proposition 2.3, $E_1 \cap E_2 \in P_\alpha(E_2)$. Then by Proposition 2.6, $E_1 \cap E_2 \subset^{\alpha} E_2$. Since $E_2 \in P_\alpha(X)$, Proposition 2.6 also implies $E_2 \subset^{\alpha} X$. It follows by transitivity that $E_1 \cap E_2 \subset^{\alpha} X$. Applying Proposition 2.6 yet again gives $E_1 \cap E_2 \in P_\alpha(X)$. □

In [16] Veksler points out that if X is ω_1-disconnected, then $P_{\omega_1}(X)$ forms a complete lattice. Actually, if X is α-disconnected, then $P_\alpha(X)$ forms a complete distributive lattice; actually, for any X, $\{E: E \subset^{\alpha} X\}$ forms a complete distributive lattice: a frame. This will be explored further in a forthcoming publication.

3. Monomorphic preimages

In this section we prove the main result of the paper, Theorem 3.3 below.

In general, recall that in a category, a morphism, τ, is monic if it is left cancellable, i.e., $\tau \circ \varphi_1 = \tau \circ \varphi_2$ implies $\varphi_1 = \varphi_2$. As indicated in the introduction, an α-SpFi monic need not be one-to-one. This is visible in the proof of Theorem 3.3, but clearly a one-to-one α-SpFi morphism is an α-SpFi monic for all α.

The following proposition gives a condition sufficient but not necessary for monicity in SpFi. However, we show in Section 5 that this condition is necessary in ∞-SpFi, but for $\alpha < \infty$ the situation remains unclear.

Proposition 3.1. *Let $\tau : (X, \mathcal{F}) \to (Y, \mathcal{H})$ be a morphism in* SpFi. *Suppose for all x_1, $x_2 \in X$ with $x_1 \neq x_2$ there exist neighborhoods U_1, U_2 of x_1 and x_2 respectively and an $F \in \mathcal{F}$ such that $\tau[U_1 \cap F] \cap \tau[U_2 \cap F] = \emptyset$. Then τ is* SpFi *monic.*

Proof. Suppose τ is not monic. Then there is a $(Z, \mathcal{L}) \in$ SpFi, and, there are morphisms φ_1, φ_2 from $(Z, \mathcal{L})$ to $(X, \mathcal{F})$ such that $\tau\varphi_1 = \tau\varphi_2$ and $\varphi_1 \neq \varphi_2$. So there is a $z \in Z$ with $\varphi_1(z) \neq \varphi_2(z)$. Let U_i be a neighborhood of $\varphi_i(z)$. Let $F \in \mathcal{F}$. We claim that $\tau[U_1 \cap F] \cap \tau[U_2 \cap F] \neq \emptyset$. Suppose not. Let $V = \varphi_1^{-1}(U_1) \cap \varphi_2^{-1}(U_2)$. Then V is a neighborhood of z. Let $L = \varphi_1^{-1}(F) \cap \varphi_2^{-1}(F)$. Then $L \in \mathcal{L}$. Since L is dense there is a $z' \in L \cap V$ and $\varphi_i(z') \in U_i \cap F$ so then $\tau\varphi_1(z') \neq \tau\varphi_2(z')$ since $\tau[U_1 \cap F] \cap \tau[U_2 \cap F] = \emptyset$. This contradicts $\tau\varphi_1 = \tau\varphi_2$. □

Lemma 3.2. *Let Y be α-disconnected and let $\tau : X \to Y$ be an α-*SpFi *morphism. If U_1 and U_2 are regular closed sets in X such that $\tau[U_1] \cap \tau[U_2] \neq \emptyset$, then $\tau[U_1 \cap G] \cap \tau[U_2] \neq \emptyset$ for all $G \in \mathcal{I}_\alpha(X)$.*

Proof. Since $U_i \subset^\alpha X$, Proposition 1.6 implies $\tau[U_i] \subset^\alpha Y$ and by Proposition 2.6, $\tau[U_i] \in P_\alpha(Y)$. Hence, by Proposition 2.3, $E = \tau[U_1] \cap \tau[U_2] \in P_\alpha(\tau[U_1])$. Let $G \in \mathcal{I}_\alpha(X)$. G is an F_α-set, and $U_1 \cap G$ is dense in U_1, so $U_1 \cap G$ is a dense F_α-set in U_1. It follows that $\tau[U_1 \cap G]$ is a dense F_α in $\tau[U_1]$. So by Lemma 2.5, $\tau[U_1 \cap G] \cap E \neq \emptyset$ (actually $\tau[U_1 \cap G] \cap E$ is dense in $\tau[U_1 \cap G]$) and $\tau[U_1 \cap G] \cap E \subset \tau[U_1 \cap G] \cap \tau[U_2]$. □

Theorem 3.3. *X is α-disconnected if and only if X has no proper (i.e., not one-to-one) α-*SpFi *monic preimage.*

Proof. Suppose X is not α-disconnected. Then there is a $U \in \mathrm{Coz}_\alpha(X)$ with $\bar{U}$ not open. Consider the topological sum, $Y = \bar{U} \oplus \overline{X - \bar{U}}$, of $\bar{U}$ and $\overline{X - \bar{U}}$. Let $\tau : Y \twoheadrightarrow X$ be the inclusion of each summand into X. This construction is used in [12, 18]. τ is irreducible, hence τ is an α-SpFi morphism (see Corollary 5.3). We claim that τ is also an α-SpFi monic. Since $U \in \mathrm{Coz}_\alpha(X)$, then $U \in \mathrm{Coz}_\alpha(Y)$ and clearly $G = U \cup (X \backslash \bar{U}) \in \mathcal{I}_\alpha(Y)$. For all y_1, y_2 there are neighborhoods V_1 and V_2 of y_1 and y_2

respectively, so that $\tau(V_1 \cap G) \cap \tau(V_2 \cap G) = \emptyset$. We can now use the argument employed in Proposition 3.1 (with G in the role of F) to conclude that τ is an α-SpFi monic.

On the other hand, we claim that if $\tau: Y \to X$ is not one-to-one and X is α-disconnected, then τ is not an α-SpFi monic.

Suppose τ is not one-to-one. Let $Z = \{(y_1, y_2) \in Y \times Y: \tau(y_1) = \tau(y_2)\}$. Clearly the restricted projections $\pi_i | Z: Z \to Y$ have $\tau \circ \pi_1 | Z = \tau \circ \pi_2 | Z$, but $\pi_1 | Z \neq \pi_2 | Z$. So the claim will be true if the $\pi_i | Z$ are α-SpFi morphisms. This will be the case if $\pi_i^{-1}(G) \cap Z$ is dense in Z for all $G \in \mathscr{I}_\alpha(Y)$. We verify this for $i = 1$.

Let $V \subset Z$ be open. There are regular closed sets, U_1 and U_2, of Y with $(U_1 \times U_2) \cap Z \subset V$. Hence there are $y_1 \in U_1$ and $y_2 \in U_2$ such that $\tau(y_1) = \tau(y_2)$. Obviously $\tau[U_1] \cap \tau[U_2] \neq \emptyset$; so by Lemma 3.2, $\tau[U_1 \cap G] \cap \tau[U_2] \neq \emptyset$ for all $G \in \mathscr{I}_\alpha(Y)$. It follows that there are $y_1' \in U_1 \cap G$ and $y_2' \in U_2$ such that $(y_1', y_2') \in Z$ and $(y_1', y_2') \in \pi_1^{-1}(G) \cap (U_1 \times U_2)$. Hence, $(y_1', y_2') \in V \cap Z \cap \pi_2^{-1}(G)$ and so $\pi_1^{-1}(G) \cap Z$ is dense in Z. □

Corollary 3.4. *An* α-SpFi *monic into an* α*-disconnected space is one-to-one.*

Proof. Let $\tau: Y \to X$ be an α-SpFi monic and let X be α-disconnected. Then, by Proposition 1.6, $\tau[Y] \subset^\alpha X$; so by Proposition 2.6 and Corollary 2.8(a), $\tau[Y]$ is also α-disconnected. Now, by Proposition 2.11, the factorization, $\tau = i \circ s$, is in α-SpFi; so s is an α-SpFi monic. Then by Theorem 3.3, s, and therefore τ, is one-to-one. □

4. Lagrange's theorem

We explain how Theorem 3.3 is a generalization of a rather important theorem about Boolean algebras, Corollary 4.3(b) below. Our discussion, however, goes beyond a proof of Corollary 4.3(b).

Let α-BA stand for the category whose objects are Boolean algebras, whose morphisms are the Boolean homomorphisms which preserve the existing suprema of sets of cardinal $<\alpha$. Let α-BA(α) be the full subcategory of α-BA whose objects are α-complete, i.e., *have* suprema existing for each set of cardinal $<\alpha$. (In our notation, ω-BA $= \omega$-BA(ω) $=$ BA.)

Let BS be the full subcategory of compact Hausdorff spaces whose objects are Boolean spaces, i.e., zero-dimensional (and compact). We have, of course, the functors

$$\mathrm{BA} \underset{\mathrm{Clop}}{\overset{S}{\rightleftarrows}} \mathrm{BS}$$

of Stone duality [14].

Let α-SpFi $\cap$ BS be the full subcategory of α-SpFi whose objects are Boolean spaces.

Proposition 4.1. (a) [14, 22.7]. *$\mathscr{A} \in |\mathrm{BA}|$ is α-complete if and only if $S(\mathscr{A})$ is α-disconnected.*

(b) (from [14, 22.5]). *Stone duality restricts to a duality of* α-BA *with* α-SpFi $\cap$ BS.

Corollary 4.2. (a) *A Boolean homomorphism φ is an epic of* α-BA *if and only if its Stone dual $S(\varphi)$ is a monic of* α-SpFi.

(b) *A Boolean algebra has no proper* α-BA-*epic extension if and only if its Stone space has no proper* α-SpFi-*monic preimage.*

Proof. (a) Proposition 4.1(b) implies that φ is α-BA epic if and only if $S(\varphi)$ is α-SpFi $\cap$ BS monic. It remains to see that a monic f, of α-SpFi $\cap$ BS is actually α-SpFi monic. Suppose g, $h \in \alpha$-SpFi have $f \circ g = f \circ h$. As discussed in the next section: Let (E, π) be the absolute of dom g = dom h; π is ∞-SpFi, hence α-SpFi. We thus have $f \circ g \circ \pi = f \circ h \circ \pi$ in α-SpFi, whence $g \circ \pi = h \circ \pi$. But π is surjective hence epic (wherever), so $g = h$.

(b) Stone duality interchanges embeddings with surjections. Apply (a). □

We finally use Theorem 3.3:

Corollary 4.3. (a) *A Boolean algebra $\mathscr{A}$ is α-complete if and only if $\mathscr{A}$ has no proper* α-BA-*epic extension.*

(b) *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be epic in* α-BA, *with $\mathscr{A}$ α-complete. Then φ is surjective.*

(c) [10]. *In* α-BA(α), *epic means surjective.*

Proof. (a) Corollary 4.2(b), Theorem 3.3, and Proposition 4.1(a).

(b) Corollary 4.2(a), Proposition 4.1(a), and Corollary 3.4.

(c) Epics are surjective, by (a). The converse holds even at the level of sets. □

Note that hiding in the proof of Corollary 4.3(b) is the α-BA-dual of Proposition 2.11: If $\varphi \in \alpha$-BA has α-complete domain, then the injective $\circ$ surjective factorization of φ has both factors in α-BA. (The dual of Example 2.10 shows the need for the α-completeness of the domain.)

5. The ∞-SpFi monics

We now characterize ∞-SpFi monics. For $\alpha < \infty$, there are serious complications and the situation is not clear.

Definition 5.1. A continuous map $\tau : X \to Y$ is said to be *range-irreducible* (see [20]) if whenever K is a proper closed set of X, then $\tau[K] \neq \tau[X]$. τ is called irreducible if τ is range-irreducible and surjective.

The following Proposition 5.2 and Theorem 5.5 are well known. (See, e.g. [6] or [7].)

Proposition 5.2. *Let $\tau: X \to Y$ be irreducible and let H be a dense subset of Y, then $\tau^{-1}(H)$ is dense in X. Moreover, if U is an open subset of X, then there is an open subset, V, of Y such that $\tau^{-1}(V) \subset U$ and $\overline{\tau^{-1}(V)} = \bar{U}$.*

Corollary 5.3. *If a map $\tau: X \to Y$ is irreducible, then τ is an α-*SpFi *morphism for all α, i.e., τ is an ∞-*SpFi *morphism.*

Proposition 5.4 [13]. *Let $\tau: X \to Y$ be continuous and let $\gamma: Y \to Z$ be irreducible. Suppose further that $\gamma \circ \tau$ is an ∞-*SpFi *morphism. Then, τ is also an ∞-*SpFi *morphism.*

Proof. Let $G \in \mathscr{I}_\infty(Y)$. Since γ is irreducible, Proposition 5.2 implies there is a $H \in \mathscr{I}_\infty(X)$ such that $\gamma^{-1}(H) \subset G$. Now since $\gamma \circ \tau$ is an ∞-SpFi morphism $\tau^{-1}(\gamma^{-1}(H))$ is dense in X and clearly $\tau^{-1}(\gamma^{-1}(H)) \subset \tau^{-1}(G)$. Hence $\tau^{-1}(G)$ is dense in X and τ is an ∞-SpFi morphism. □

Theorem 5.5 (Gleason [6]). *The ∞-disconnected (i.e., extremally disconnected) spaces are the projectives in the category of compact Hausdorff spaces.*

For each X there is an ∞-disconnected space, EX, and an irreducible map $\varphi_X: EX \twoheadrightarrow X$. The pair (EX, φ_X) is essentially unique (and projective resolution, or absolute, of X).

The following was obtained independently by Woods in [20].

Proposition 5.6. *An ∞-*SpFi *morphism $\tau: X \to Y$ is ∞-*SpFi *monic if and only if τ is range-irreducible.*

Proof. Suppose τ is not ∞-SpFi monic. Then there exists a space Z and ∞-SpFi morphisms, α_1 and α_2, with $\alpha_1 \neq \alpha_2$ such that $\alpha_i: Z \to X$ and $\tau \circ \alpha_1 = \tau \circ \alpha_2$. Since $\alpha_1 \neq \alpha_2$ there is a $z \in Z$ with $\alpha_1(z) \neq \alpha_2(z)$. It follows that there is a regular closed set, U, of z with $z \in U$ and $\alpha_1[U] \cap \alpha_2[U] = \emptyset$. Now by Proposition 1.6, $\alpha_1[U] \subset^\infty X$, i.e., $\alpha_1[U]$ is a regular closed set in X, but then $\tau[X \backslash (\alpha_1[U])^\circ] = \tau[X]$. So τ is not range-irreducible.

To prove the other implication we use the theory of the absolute, (EX, φ_X), of X. Let $\tau: X \to Y$ be an ∞-SpFi monic. Then by Corollary 5.3 and the sufficiency of Proposition 5.6, $\tau \circ \varphi_X: EX \to Y$ is also an ∞-SpFi monic. By projectivity (Theorem 5.5) there is a map $\gamma: EX \to EY$ such that $\tau \circ \varphi_X = \varphi_Y \circ \gamma$. By Proposition 5.4, γ is an ∞-SpFi morphism; moreover γ is an ∞-SpFi monic by being the first factor of an ∞-SpFi monic. Now by Corollary 3.4, γ must be one-to-one. Hence, γ is range-irreducible. It follows that $\tau \circ \varphi_X = \varphi_Y \circ \gamma$ is range-irreducible, and thereby, it follows that τ is range-irreducible. □

The following also appears in [20], but seems to be originally from [13].

Corollary 5.7. *The* ∞*-disconnected spaces form a monocoreflective subcategory of* ∞-SpFi.

Proof. Proposition 5.6 and Theorem 5.5 imply that (EX, φ_X) is the ∞-SpFi monocoreflection of X. □

Remark 5.8. We show in [11] that the α-disconnected spaces form a monocoreflective subcategory of α-SpFi. This result was known implicitly for α-SpFi $\cap$ BS. If $Z \in |\mathrm{BS}|$, then the α-SpFi monocoreflection of Z, denoted $(m_\alpha Z, \varphi_z^\alpha)$, "is" the Stone dual of the free α-regular extension of the Boolean algebra Clop(Z). See [14, 21] and Section 4 here.

Proposition 5.9. *Let* $\tau: X \to Y$ *be an* ∞-SpFi *morphism, then the following are equivalent*:

(i) τ *is monic.*

(ii) *For* $x_1, x_2 \in X$ *with* $x_1 \neq x_2$ *there exist neighborhoods* U_1 *and* U_2 *of* x_1 *and* x_2 *respectively in* X *and a* $G \in \mathcal{I}_\infty(X)$ *such that* $\tau[U_1 \cap G] \cap \tau[U_2 \cap G] = \emptyset$.

Proof. (ii)⇒(i), Proposition 3.1.

(i)⇒(ii) By Proposition 5.6, τ is range-irreducible. Choose open neighborhoods, U_1 and U_2, of x_1 and x_2 respectively, with the closures of the U_i disjoint. By Proposition 5.2 there are V_i open in Y such that $\bar{U}_i = \overline{\tau^{-1}(V_i)}$. It follows that the V_i are disjoint. Choose an open set, V, in Y disjoint from $V_1 \cup V_2$ such that $V \cup V_1 \cup V_2$ is dense in Y. So $\tau^{-1}(V)$ is disjoint from $\tau^{-1}(V_i)$, hence from its closure which contains U_i. Now let $G = \tau^{-1}(V \cup V_1 \cup V_2)$. $G \in \mathcal{I}_\infty(X)$ and

$$\begin{aligned}\tau[U_i \cap G] &= \tau[U_i \cap \tau^{-1}(V) \cup \tau^{-1}(V_1) \cup \tau^{-1}(V_2)] \\ &= \tau[U_i \cap \tau^{-1}(V_i)] \subset V_i.\end{aligned}$$

Hence $\tau[U_1 \cap G] \cap \tau[U_2 \cap G] = \emptyset$. □

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