ON THE PROBLEM OF FINDING SMALL SUBDIVISION AND HOMOMORPHISM BASES FOR CLASSES OF COUNTABLE GRAPHS

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Let $\mathcal{G}$ be a class of countable graphs given by a set $\Gamma$ of forbidden configurations. We consider the following problem: for which $\Gamma$ is $\mathcal{G}$ well characterized by the simplicial decompositions of its members into prime graphs, that is for which $\mathcal{G}$ is it possible to find a small subset $\mathcal{A}$ of $\mathcal{G}$ such that all graphs of $\mathcal{G}$ can be constructed from elements of $\mathcal{A}$ by successive amalgamations identifying complete subgraphs?

Introduction

Let $G$ be a graph, $\sigma > 0$ an ordinal, and $G_\lambda$ induced subgraphs of $G$ for each $\lambda < \sigma$. The family $(G_\lambda)_{\lambda < \sigma}$ is said to form a (reduced) simplicial decomposition of $G$ if

(i) $G = \bigcup_{\lambda < \sigma} G_\lambda$
(ii) $(\bigcup_{\lambda < \sigma} G_\lambda) \cap G_\tau =: S_\tau$ is complete for every $\tau, 0 < \tau < \sigma$
(iii) no $S_\tau$ contains $G_\lambda$ or any other $G_\gamma, 0 < \lambda < \tau < \sigma$.

The $S_\tau$'s are called simplices of attachment, but also any other complete graph may later be referred to as a simplex. A graph is called prime if it cannot be decomposed in this way into more than one subgraph, i.e. if it has no separating simplex, a proof of this easy equivalence is found in [3], and a prime decomposition is a simplicial decomposition each member of which is prime.

The theory of simplicial decompositions is largely due to R. Halin (see [3, 6-8] for surveys); a thorough account of most of the results known is found in [5]. Halin proved that every graph that contains no infinite simplex has a prime decomposition, a result which we shall use implicitly at several points.

Let $G$ be a graph. Then a subdivision $T =: TG$ of $G$ is any graph arising from $G$ by replacing the edges of $G$ with independent paths; the original vertices of $G$ are then called the branchvertices of $T$. If $\Gamma$ is a set of graphs then $T\Gamma$ denotes the class of all subdivisions $TG$ of graphs $G \in \Gamma$.

Similarly, if the vertex set $V$ of a graph $H$ is partitioned into finite subsets $V_i, i \in I$, such that each $V_i$ spans a connected subgraph in $H$, and if $G$ is the graph defined on the vertex set $\{V_i \mid i \in I\}$ by joining $V_i$ to $V_j$ if and only if $H$ contains a
V_i-V_j edge, then we say that G is a contraction of H, write H=HG (H for ‘homomorphism’), and call the vertices of G the branch sets of HG. If H is a subgraph of H' then G is also called a subcontraction of H' and we write G ≈ H'. Finally, if Γ is a set of graphs we shall use HΓ to denote {HG' | G' = G for some G ∈ Γ}.

For a set Γ of finite graphs, let $G(Γ)$ denote the class of all countable (i.e., finite or countably infinite) graphs that contain no subgraph isomorphic to an element of $Γ$. Note that each $G ∈ G(Γ)$ can be extended to a maximal graph $G^* ∈ G(Γ)$ by adding edges, i.e., for every $G = (V, E) ∈ G$ there exists some $G^* = (V, E^*) ∈ G$ such that $E(G^*) = E(G)$ and $G' = (V, E') ∈ G$, whenever $E' ⊆ E^*$, $E' ≠ E^*$.

Consider a class $G$ of the form $G(Γ)$ or $G(HT)$ for some $Γ$. Since $G$ is monotone decreasing (i.e., $G' ⊆ G ∈ G$ implies $G' ∈ G$), we know a lot about $G$ once we can tell what the maximal elements of $G$ look like. It is therefore of interest to determine, if possible, the class of all those graphs in $G$ that occur in a prime decomposition of a maximal element of $G$. This class is called the (subdivision or homomorphism) base of $G$ (depending on whether $G = G(Γ)$ or $G = G(HT)$) and will be denoted by $B(Γ)$ (or $B(HT)$, respectively).

The idea to use subdivision or homomorphism bases in this way is due to K. Wagner, who proved his well-known Equivalence Theorem essentially by determining the homomorphism base of $G(1K^5)$ [10]. A more recent example of how a class of graphs can be characterized by its base is found in [2].

However, it is not always easy to determine the base of a class $G$, and most results known so far are cases where $Γ$ contains just one relatively simple graph (for detailed references see [5]).

This paper aims at giving some account for this difficulty by approaching the problem of determining bases from the negative side. It will be shown that as soon as $Γ$ contains any relatively dense graph the bases of $G(Γ)$ and $G(HT)$ are likely to be uncountable, and therefore hardly easier to assess than $G$ itself.

In Section 1 we shall look at a few preliminary results, one of which relates our problem to that of the existence of a universal graph in the class $G$ considered. Section 2 is independent of Section 1; it contains the main result of this paper, a sufficient condition for $G$ to have an uncountable base. Section 3 gives applications of this result to a number of classes $G(Γ)$ and $G(HT)$, as well as to classes $G(TX)$ and $G(HX)$ for a range of finite graphs $X$.

Unless otherwise specified, the notation used is that of [1]. In particular, $K^n$ denotes the complete graph on $n$ vertices, $E^n$ its complement, and $K(n_1, \ldots, n_r)$ the complete $r$-partite graph with vertex classes of order $n_1 ≥ \cdots ≥ n_r$. Finally, we put $|x - y|_{\text{mod } k} := \min\{z ∈ \N | x - y \equiv z (\text{mod } k) \text{ or } y - x \equiv z (\text{mod } k)\}$, for any $k, x, y ∈ \N$.

1. Preliminaries

Let $G$ be any class of graphs. An element $G^*$ of $G$ is called universal in $G$ if it contains a copy of every $G ∈ G$ as a subgraph (Rado [9]).
The following theorem shows that each of the results in Sections 2 and 3 below translates into a negative assertion about the existence of a universal graph for the class \( \mathcal{G} \) considered.

**Theorem 1.1.** Let \( \Gamma \) be a set of finite graphs, and \( \mathcal{G} = \mathcal{G}(TT) \) or \( \mathcal{G} = \mathcal{G}(HG) \). If the base \( \mathcal{B} \) of \( \mathcal{G} \) is uncountable then \( \mathcal{G} \) contains no universal element.

**Proof.** Suppose the contrary, and let \( G^* \) be universal in \( \mathcal{G} \). Let \( (P^*_\mu)_{\mu < \sigma} \) be a prime decomposition of \( G^* \). We shall show that for every \( P \in \mathcal{B} \) that is not a simplex there exists some \( \rho < \tau \) such that \( P = P^*_\rho \).

Let such a \( P \in \mathcal{B} \) be given, and let \( G \) be maximal in \( \mathcal{G} \) such that \( G \) has a prime decomposition \( (P^*_\mu)_{\mu < \sigma} \) with \( P = P^*_\mu \). Consider any embedding \( \phi \) of \( G \) into \( G^* \) and identify \( G \) with its image under \( \phi \). By \( G^* \in \mathcal{G} \) and the maximality of \( G \) in \( \mathcal{G} \), \( P \) must be an induced subgraph of \( G^* \), so \( P \) is contained in one member of \( (P^*_\mu)_{\mu < \sigma} \), say \( P = P^*_\rho \).

Suppose \( P^* \setminus P \neq \emptyset \), and let \( x \) be a vertex of \( P^* \setminus P \).

If \( x \in G^* \setminus G \), then, by the maximality of \( G \) in \( \mathcal{G} \), those vertices of \( G \) that lie on a \( G-x \) path in \( G^* \) span a simplex \( S \) in \( G^* \) (for any two vertex sets \( V_1, V_2 \subseteq V(G^*) \) a \( V_1-V_2 \) path in \( G^* \) is one with an endvertex in each of \( V_1, V_2 \) and its interior in \( G^* \setminus (V_1 \cup V_2) \)). Because \( P \) is no simplex, \( P \setminus S \) is not empty, so \( S \cap P^* \) separates \( P^* \), contradicting the fact that \( P^* \) is prime.

Therefore \( x \) is a vertex of \( G \), i.e. \( x \in G \setminus P \). By (iii) of the definition of simplicial decompositions there exists a vertex \( y \in P \) and a simplex of attachment \( S \) of \( G \) such that \( y \notin S \) and \( S \) separates \( x \) from \( y \) in \( G \). But as \( G^* \in \mathcal{G} \) and \( G \) is maximal in \( \mathcal{G} \), \( S \) separates \( x \) and \( y \) also in \( G^* \), again contradicting the fact that \( P^* \) is prime. \( \Box \)

The converse of Theorem 1.1 does not hold in general. An example of a class \( \mathcal{G} \) which has no universal element although its base is finite is given in [4], which also contains a few positive results on universal graphs.

The most straightforward way to show that the subdivision or homomorphism base of some \( \mathcal{G} \) is uncountable is usually to exhibit uncountably many prime and maximal elements of \( \mathcal{G} \). In fact even though this is apparently stronger—base elements need not in general be maximal themselves—no example is known of an uncountable base with only countably many maximal elements.

The following observation helps to broaden the impact of specific results proved in Section 3.

**Proposition 1.2.** Let \( X \) be a finite graph, and \( \mathcal{G} = \mathcal{G}(TX) \) or \( \mathcal{G} = \mathcal{G}(HX) \). Then if \( \mathcal{G} \) has uncountably many prime and maximal elements, so has \( \mathcal{G}' = \mathcal{G}(T(X+E^n)) \) (or \( \mathcal{G}' = \mathcal{G}(H(X+E^n)) \), respectively), for every \( n \in \mathbb{N} \).

**Proof.** We prove the assertion for \( \mathcal{G} = \mathcal{G}(HX) \); the proof for the subdivision case is analogous.
Let \( \mathcal{B} \) be an uncountable set of prime and maximal elements of \( \mathcal{G} \) such that no \( B \in \mathcal{B} \) is a simplex. Put \( \mathcal{B}' := \{ B + E \mid B \in \mathcal{B} \} \) for a fixed graph \( E = E' \).

In order to show \( \mathcal{B}' \subset \mathcal{G}' \), suppose that some \( B' = B + E, B \in \mathcal{B} \), contains an \( H(Y + Y') \) with \( Y = X \) and \( Y' = E' \). Let \( \mathcal{Y} := \{ V \in V(Y) \mid V \cap V(E) \neq \emptyset \} \) and \( \mathcal{Y}' := \{ V' \in V(Y') \mid V' \cap V(E) \neq \emptyset \} \). Then \( |\mathcal{Y}| + |\mathcal{Y}'| < n \), so we have \( \mathcal{V}' \subset V(B) \) for at least \( k = |V| \) branch sets \( V' \in \mathcal{Y}' \). Let these take the places of the sets \( V \in \mathcal{V} \) in \( Y \) to obtain \( B_1 \equiv Y = X \), contradicting \( B \in \mathcal{B} \subset \mathcal{G} \).

Let us turn each \( R' = R + F \in \mathcal{B}' \) into a maximal graph \( R^* := R + F^* \) of \( \mathcal{G} \) by adding edges to its subgraph \( E \). This can clearly be done, because the \( B \)'s are maximal in \( \mathcal{G} \). Also, the \( B^* \)'s are all prime since every \( B \) is prime but no simplex.

It remains to be shown that uncountably many \( B^* \)'s are pairwise non-isomorphic. Suppose not, and let \( D^*_n = D_n + E^*_n \in \mathcal{G}^* \), where \( \mathcal{G}^* \) is an uncountable isomorphism class of \( \{ B^* \mid B \in \mathcal{B} \} \). For each \( B^* \in \mathcal{G}^* \) let \( \sigma_{B^*} : B^* \to B_0^* \) be an isomorphism. Since \( B_0^* \) has only countably many \( n \)-tuples of vertices, two of these \( \sigma \)'s must agree on \( V(E) \), say \( \sigma_{B^*_1} \) and \( \sigma_{B^*_2} \). But then \( \sigma := \sigma_{B^*_1}^{-1} \circ \sigma_{B^*_2} \) is an isomorphism from \( B^*_1 \) to \( B^*_2 \) that fixes \( V(E) \), so \( \sigma \) maps \( B_1 \) onto \( B_2 \), contradicting our assumption that \( B_1 \) and \( B_2 \) were non-isomorphic elements of \( \mathcal{B} \). □

**Corollary 1.3.** If \( \mathcal{G}(TX) \) (or \( \mathcal{G}(HX) \)) has uncountably many prime and maximal elements for some finite graph \( X \), then so has \( \mathcal{G}(T(X + K^n)) \) (or \( \mathcal{G}(H(X + K^n)) \)), resp., for every \( n \in \mathbb{N} \).

**Proof.** Apply induction on \( n \) using Proposition 1.2 for \( E_1 \). □

The following result is one in the vein of those in Section 3, but it has a straightforward proof.

**Proposition 1.4.** Let \( n \geq 3 \) and \( \mathcal{G} = \{ G \mid \Delta(G) \leq n \} \) where \( \Delta \) denotes the maximal degree. Then \( \mathcal{G} \) has uncountably many prime and maximal elements.

**2. A general sufficient condition for uncountable bases**

We shall give a sufficient condition for classes \( \mathcal{G} = \mathcal{G}(TT) \) or \( \mathcal{G} = \mathcal{G}(HT) \) to contain uncountably many maximal and prime graphs, one for every \( 0 \leq \alpha \leq 1 \) sequence \( \alpha : \mathbb{N} \to \{0, 1\} \). These graphs will essentially consist of two parts glued together, of which one, the ‘head’, depends only on \( \mathcal{G} \) while the other, the ‘tail’, depends only on \( \alpha \). However, if \( G \) is such a head and \( H_\alpha \) are tails, one for each \( 0 \leq \alpha \leq 1 \) sequence \( \alpha \), then the amalgamations \( G_\alpha = G \# H_\alpha \) of the two cannot in general be expected to be maximal in \( \mathcal{G} \), as \( H_\alpha \) is independent of the class \( \mathcal{G} \) considered. The graphs we are seeking will therefore be maximal extensions \( G_\alpha^* \) of these \( G_\alpha \)'s, which will also have to be prime and pairwise non-isomorphic. This requires, of course, that we retain some control over each \( G_\alpha \) when we turn it into \( G_\alpha^* \), i.e. in order to resemble the features we gave to \( G_\alpha \) it must not acquire too many new edges.
As regards $G_0$, this can be done by choosing $G$ itself almost maximal in $\mathcal{G}$, i.e., by keeping it just far enough from maximality that attaching $H_0$ does not create a forbidden subgraph. This will be reflected in the condition mentioned, which will essentially require the existence of a finite prime graph $G \in \mathcal{G}$ with this property.

The tail $H_0$, however, will be the same for all classes $\mathcal{G}$, so the only way to keep it sparse when turning $G_0$ into $G^*$, is to glue it to $G$ in such a way that any additional edge in $H_0$ creates a forbidden subgraph that lies essentially in $G$. This will be reflected in the definition of the amalgamation of $G$ and $H_0$.

Before we define the $H_0$'s and state the condition on $G$, let us introduce a few handy definitions.

If $G$ is a graph and $H$ is its complement, let us call the edges of $H$ the non-edges of $G$ and write $N(G)$ for $E(H)$. Let $\mathcal{G}$ be a class of graphs and $G \in \mathcal{G}$. We say that a set $N \subseteq N(G)$ of non-edges of $G$ is admissible to $G$ (with respect to $\mathcal{G}$), if $G \cup N \in \mathcal{G}$ (for $N = \{e\}$ we shall also call $e$ itself admissible); the set of admissible subsets of $N(G)$ will be denoted $\mathcal{A}(G)$. Two sets $N_1, N_2 \subseteq N(G)$ are said to clash (w.r.t. $\mathcal{G}$), if $G \cup N_1 \cup N_2 \notin \mathcal{G}$ (again, if $N_1 = \{e_1\}$ or $N_2 = \{e_2\}$ we shall identify $N_i$ with $e_i$ so that we may speak of clashing non-edges etc.). Note in particular that when a non-edge is not admissible it clashes with every non-edge of $G$, including itself. Finally, if $\mathcal{N} \subseteq \mathcal{A}(G)$ is non-empty and monotone decreasing (i.e., if $N' \subseteq N \in \mathcal{N}$ implies $N' \in \mathcal{N}$), let us call $G$ $\mathcal{N}$-maximal if every non-edge of $G$ clashes with some $N \in \mathcal{N}$. Note that $\{\emptyset\}$-maximality coincides with ordinary maximality.

Similarly to extending a graph $G = (V, E) \in \mathcal{G}$ to a maximal graph $G^*$ by adding edges, we have the following for $\mathcal{N}$-maximality. Let $\mathcal{N}$ be any non-empty subset of $\mathcal{A}(G)$. Denote by $\max_{\mathcal{N}}(G)$ the (not necessarily unique) graph $G' = (V, E')$, $E' \supseteq E$, that is maximal with the property that $\mathcal{N} \cap \mathcal{P}(N(G')) \subseteq \mathcal{A}(G')$, where $\mathcal{P}(N(G'))$ is the set of sets of non-edges of $G'$. Then $\max_{\mathcal{N}}(G)$ is an $\mathcal{N}$-maximal extension of $G$ with $\mathcal{N}' \subseteq \mathcal{N}$. Since $\mathcal{N}$ was assumed to be monotone decreasing, $\max_{\mathcal{N}}(G)$ is simply a graph obtained by successively adding edges to $G$ (and as many as possible) that do not clash with any set $N \in \mathcal{N}$.

Let $P = u_0 u_1 \ldots$ be a one-way infinite path, $C = w_1 \ldots w_5$ a cycle of length 5, and $H = P \times C$, the cartesian product of $P$ and $C$. Denote by $C_i$ the $5$-cycle in $H$ whose vertices have $u_i$ as their first component, $i = 0, 1, \ldots$, and by $P_i$ the path spanned by the vertices having $w_j$ as their second component, $j = 1, \ldots, 5$. Also, let $v_{ij}$ be the vertex in $C_i \cap P_j$, and $P^k_i$ the path of length $k$ induced by $v_{0i}, \ldots, v_{ki}$.

Note that $H$ has the following property.

**Lemma 2.1.** Let $x_1 y_1$ and $x_2 y_2$ be non-edges of $C_0 \subseteq H$, and let $H_1$ and $H_2$ be connected subgraphs of $H$ such that $x_i, y_i \in H_0$, $i = 1, 2$. Then $H_1 \cap H_2 \neq \emptyset$.

For every $0$-$1$ sequence $\alpha : \mathbb{N} \to \{0, 1\}$, let $H_{\alpha}$ be the graph obtained from $H$ by contracting $v_{0,4}v_{0,5}$, all edges $v_{i,4}v_{i,5}$ where $i \neq 0 \pmod{10}$, and the edges $v_{10i,4}v_{10i,5}$ whenever $\alpha(i) = 0$. If $\alpha(i) = 1$ for all $i \in \mathbb{N}$, we put $H_{\alpha} = H$. Vertices of $H_{\alpha}$ arising from contracted edges $v_{i,4}v_{i,5}$ will be denoted $v_{i,4}$, and all other notation defined
for $H$ (such as $C_n$, $P_n$, $P_i^*$ and $v_0$) will again be used for $H_n$ as induced by the contraction.

Since the $H_n$'s are contractions of $H$, the assertion made for $H$ in Lemma 2.1 carries over to every $H_n$. The next lemma shows that the $H_n$'s are in fact almost maximal with this property.

**Lemma 2.2.** Let $\alpha : \mathbb{N} \to \{0, 1\}$ be an 0–1 sequence, and $e = v_0v_{i'}$ a non-edge of $H_n$ such that $H_n \cup e$ has the property asserted for $H$ in Lemma 2.1. Then $|i - i'| = 1$, and $|i - j'| \mod 4 = 1$ if $j, j' \in \{1, \ldots, 4\}$.

**Proof.** Suppose the contrary is true. We shall define a $v_{0,1} - v_{0,3}$ path $P$ and a $v_{0,2} - v_{0,4}$ path $P'$ in $H_n$ such that $P \cap P' = \emptyset$.

Assume that $i' \geq i$. Let $v_{0,k}$ be the vertex of $P_i \cap C_0$ (i.e. $k = j$ or, in the case of $j = 5$, $k = 4$), and let $k'$ be such that $v_{0,k}$ and $v_{0,k'}$ are not adjacent in $C_0$. Put $P := P_i \cup C_i \cup P_{i'}^*$, where $C_i$ is a $v_{i'} - v_{k'}$ path in $C_i$. Let $v_{0,1}$ and $v_{0,4}$ be the remaining two vertices of $C_0$. Since $v_{0,1}$ and $v_{0,4}$ are again non-adjacent in $C_0$, we can define $P' := P_{i'}^* \cup C_{i+1} \cup P_{i+1}^*$, where $C_{i+1}$ is a $v_{i+1,1} - v_{i+1,4}$ path in $C_{i+1}$ that is disjoint from $P$. \(\square\)

Let $\Gamma$ be a set of finite graphs, and $\mathcal{G} = \mathcal{G}(T)$ or $\mathcal{G} = \mathcal{G}(HT)$. We shall define a set of conditions on finite elements of, $\mathcal{G}$ such that the existence of a graph in $\mathcal{G}$ satisfying these conditions will be sufficient for $\mathcal{G}$ to contain uncountably many prime and maximal graphs.

Let $G$ be a finite graph in $\mathcal{G}$ with specified independent non-edges $v_1v_3$ and $v_2v_4$. Put $T := G[v_1, \ldots, v_4]$ and write $e_{ij}$ for $v_iv_j \in E(T) \cup N(T)$. For any 0–1 sequence $\alpha : \mathbb{N} \to \{0, 1\}$ denote by $G_\alpha := G \# H_\alpha$ the graph obtained from the disjoint union of $G$ and $H_\alpha$ by identifying vertices $v_i$ with $v_{0,i}$ for $i = 1, \ldots, 4$ and deleting all edges between these vertices that are not in $T$.

Finally, define a set $\mathcal{N}_G$ as follows. If $\mathcal{G} = \mathcal{G}(T)$ let $\mathcal{N}_G := \{\{e\} | e \in N(T)\}$ \cup $\{e_{1,2}, e_{3,4}, e_{2,3}, e_{1,4}\}$ \cup $\mathcal{P}(N(T))$, and if $\mathcal{G} = \mathcal{G}(HT)$ put $\mathcal{N}_G := \mathcal{P}(N(T)\{e_{1,3}\}$ \cup $\mathcal{P}(N(T) \{e_{2,4}\})$. Then in either case $\mathcal{N}_G$ contains exactly those sets of non-edges of $G$ that can be 'realized' in $G_\alpha$. More precisely: for any $\alpha$ and any given set $N \in \mathcal{N}_G$—and only for these—we can, in the case of $\mathcal{G} = \mathcal{G}(T)$, find a set of independent $G$–$G$ paths in $G_\alpha$ one for each $e \in N$ with the vertices of $e$ as its endpoints, while in the case of $\mathcal{G} = \mathcal{G}(HT)$ we can turn all vertices $v_i \in T$ into branch sets $V_i \subset V(G_\alpha)$. $V_i \cap V(G) = \{v_i\}$, such that $G_\alpha$ contains a $V_i - V_j$ edge if and only if $v_i, v_j \in N$ or $v_i, v_j \in E(T)$ (Fig. 1).

From now on we shall use the notation of $T$, $e_{ij}$ and $\mathcal{N}_G$ freely whenever vertices $v_1, v_2, v_3, v_4$ are specified for some graph, and we may also abbreviate $\max_{\mathcal{N}_G}(G)$ to $\max(G)$.

We can now state and prove the main result of this paper.

**Theorem 2.3.** Let $\Gamma$ be a set of finite graphs, and $\mathcal{G} = \mathcal{G}(T)$ or $\mathcal{G} = \mathcal{G}(HT)$. Suppose
that $\mathcal{G}$ contains a finite graph $G$ with non-edges $v_1v_3$ and $v_2v_4$ that satisfies

(*) (i) $G$ is connected, has no cutvertex in $V(T)$, and for every simplex $S$
separating $G$ each component of $G \setminus S$ contains a vertex from $V(T)$,
(ii) $\mathcal{N}_G \subset \mathcal{A}(G)$, i.e. all sets in $\mathcal{N}_G$ are admissible,
(iii) $G$ is $\mathcal{N}_G$-maximal and $e_{1,3}$ clashes with $e_{2,4}$,

and

(**) for $\mathcal{G} = \mathcal{G}(T)$: $G \not\cong H_1$ contains no $TX$, $X \in \Gamma$, with a branch-vertex in $H_1 \setminus G$,
for $\mathcal{G} = \mathcal{G}(H_T)$: $G \not\cong H_1$ contains no $HX$, $X \in \Gamma$, with a branch set that is a subset of $V(H_1 \setminus G)$.

Then $\mathcal{G}$ has uncountably many prime and maximal elements.

**Proof.** Let $G \in \mathcal{G}$ be given as stated.

For every $0$-1 sequence $\alpha : \mathbb{N} \to \{0, 1\}$ let $G_\alpha := G \not\cong H_\alpha$. We begin our proof by showing that these $G_\alpha$'s are in $\mathcal{G}$. We then extend each $G_\alpha$ to a graph $G_\alpha^* \supset G_\alpha$, $V(G_\alpha^*) = V(G_\alpha)$, that is maximal in $\mathcal{G}$, and prove that all these $G_\alpha^*$'s are prime. Finally, we show that uncountably many $G_\alpha^*$'s are pairwise non-isomorphic.

Every $G_\alpha$ is in $\mathcal{G}$. We consider the case $\mathcal{G} = \mathcal{G}(TT)$, the proof for $\mathcal{G}(HT)$ is similar. Suppose $T_0 = TX \subset G_\alpha$ for some 0-1 sequence $\alpha$ and $X \in \Gamma$. Since by (**), and $H_1 \supset IH_\alpha$ all branchvertices of $T_0$ are in $G$, $T_0$ differs from $T_0 \setminus G$ at most by some edge-disjoint $G-G$ paths serving as (parts of) subdivided edges of $X$. If one
of these paths has endvertices $v_i, v_j$ with $|i - j| \mod 4 = 1$ then it contains the edges $v_i v_{i+1}$ and $v_j v_{j+1}$, so there can be at most one other such path, which will have the remaining two vertices of $T$ as its endvertices. This, however, implies that either $G \cup \{e_2, e_3, e_4\}$ or $G \cup \{e_2, e_4, e_1\}$ contains a $T$, contradicting (ii) of condition (\(*\)).

Therefore every non-trivial $G$--$G$ path in $T_0$ must be of the form $P_1 = v_1 \ldots v_3$ or $P_2 = v_4 \ldots v_n$. Now the inverse images of such $P_1$ and $P_2$ under the contraction $H \mapsto H_v$ are connected in $H$ and therefore have a non-empty intersection, by Lemma 2.1. But this implies $P_1 \cap P_2 \neq \emptyset$, so $T_0$ contains at most one of $P_1$ or $P_2$, say $P_1$. This, however, indicates that $e_{1,3}$ is not admissible to $G$, again contradicting (ii) of (\(*\)).

For each $\alpha$ extend $G_{\alpha}$ to $G_{\alpha}^*$ as indicated above.

Every $G_{\alpha}^*$ is prime. Suppose $S$ is a separating simplex in $G_{\alpha}^*$. Clearly $S$ does not contain $G_{\alpha}^*[v_1, \ldots, v_4]$ (= $T$). Therefore (i) of (\*) implies that $S$ separates $G_{\alpha}^*[H_v] = H_v^*$, since by the $N_G$-maximality of $G$ every edge of $G_{\alpha}^*[V(G)]$ was already an edge of $G$. By Lemma 2.2 all vertices of $S$ are contained in at most two cycles $C_i$ and $C_{i+1}$ of $H_v$, $0 \leq i \leq 1$. Since every $v \in C_i \setminus S$ is adjacent to a vertex in $C_{i+1}$ and every $v \in C_{i+1} \setminus S$ is joined to a vertex of $C_i$, the second components of the vertices of $S$ must cover all numbers $1, \ldots, 4$ (at least if $i > 1$; the case $i = 0$ needs some easy extra checking making full use of (i) of (\*)). This, however, contradicts the fact that by Lemma 2.2, $|v_i - v_{i'}| \mod 4 = 1$ for edges $v_i v_{i'} \in E(G_{\alpha}^*) \setminus E(G_v)$, whenever $i, i' \in \{1, \ldots, 4\}$.

Uncountably many $G_{\alpha}^*$'s are pairwise non-isomorphic. Suppose not. Then at least one isomorphism class $\mathcal{J}$ of $\{G_{\alpha}^* \mid \alpha \in \{0, 1\}^N\}$ is uncountable: let $G_{\alpha}^* \in \mathcal{J}$ and $\sigma_\alpha : G_{\alpha}^* \to G_{\beta}^*$ be isomorphisms from all $G_{\alpha}^* \in \mathcal{J}$ to $G_{\beta}^*$. As $G_{\beta}^*$ has only countably many $|G|$-tuples of vertices, at least two of the $\sigma_\alpha$'s (say $\sigma_\alpha$ and $\sigma_\alpha'$) agree pointwise on $V(G)$. Then $\sigma := \sigma_\alpha^{-1} \sigma_\alpha'$ is an isomorphism from $G_{\alpha}^*$ to $G_{\beta}^*$ that fixes the vertices of $G$. Now Lemma 2.2 implies that in both $G_{\alpha}^*$ and $G_{\beta}^*$, the cycle $C_i$ contains exactly the vertices of $H_{\alpha}$ (or $H_{\beta}$) that have distance $i$ from $T$ in $H_{\alpha}$ (or $H_{\beta}^*$, respectively), $0 \leq i \leq 4$. But $\sigma$ preserves these distances, contradicting $\alpha \neq \alpha'$. □

Remark. The careful reader will have noticed that in the case of $\mathcal{G} = \mathcal{G}(H)$ the graph $T \subset G$ must be a 4-cycle (and $N_G = \{e_{1,3}, e_{2,4}\}$), whenever $G$ satisfies (\*).

For if $e$ is a non-edge of $T$ other than $e_{1,3}$ or $e_{2,4}$ then, by the definition of $N_G$, for every $N \in \mathcal{N}_G$ the set $N \cup \{e\}$ is again in $\mathcal{N}_G$ and therefore by (ii) admissible, contradicting (iii). In the case of $\mathcal{G} = \mathcal{G}(TT)$, however, this need not be so.

3. Some specific results

In this section we present a selection of applications of Theorem 2.3.

To verify condition (\***) of Theorem 2.3 we shall repeatedly need the following technical lemma. Its proof is straightforward throughout, though in one case ($\mathcal{G} = \mathcal{G}(H)$ and $n = 5$) a little long and tedious.
Lemma 3.1. Let \( \Gamma \) be a set of finite graphs of minimal degree at least 5, and \( \mathcal{G} = \mathcal{G}(TT) \) or \( \mathcal{G} = \mathcal{G}(HT) \). Then every finite graph \( G \in \mathcal{G} \) with specified non-edges \( e_{1,3} \) and \( e_{2,4} \) satisfies (**). \( \Box \)

The following result is an immediate and yet already fairly comprehensive consequence of our main theorem.

Theorem 3.2. \( \mathcal{G}(TT) \) and \( \mathcal{G}(HT) \) have uncountably many prime and maximal elements if \( \Gamma \) is any of the following:

\[
\Gamma = \{ X \mid \delta(X) \geq n \}, \quad \Gamma = \{ X \mid \kappa(X) \geq n \}, \quad \Gamma = \{ X \mid \lambda(X) \geq n \}, \quad \Gamma = \{ X \mid \chi(X) \geq n + 1 \}.
\]

where \( n \geq 5 \), \( \delta, \kappa, \lambda \) and \( \chi \) denote minimal degree, vertex-connectivity, edge-connectivity and chromatic number, and all graphs \( X \) are finite. (For \( n \leq 4 \), characterizations in terms of bases can be obtained for such classes \( \mathcal{G} \), see [4].)

Proof. (**J is clear if \( G \) is obtained from \( K^{n+1} \) by deleting two independent edges \((= : e_{1,3} \) and \( e_{2,4} \)). Lemma 3.1 implies (**) for \( \Gamma = \{ X \mid \delta(X) \geq n \} \), which in turn implies (**) for the other choices of \( \Gamma \) as well (for the chromatic number consider critical graphs). \( \Box \)

Let us now look at a few classes \( \mathcal{G} \) in which subdivisions or contractions of only one graph are forbidden.

Theorem 3.3. Let \( n_1 \geq n_2 \geq \ldots \geq n_r \) and \( n \) be natural numbers, where \( n, n_1, n_2 \geq 5 \). Then \( \mathcal{G}(TX) \) and \( \mathcal{G}(HX) \) have uncountably many prime and maximal elements if

(i) \( X = K^n \), or
(ii) \( X = K(n_1, \ldots, n_r) \).

Proof. (i) Let \( G \) be obtained from \( K^n \) by removing two independent edges \((= : e_{1,3} \) and \( e_{2,4} \)). \( G \) clearly satisfies (*). If \( n = 5 \) then \( G \) satisfies (**) because \( G \not\cong H_1 \) is planar, if \( n \geq 6 \) because of Lemma 3.1.

(ii) By Proposition 1.2 we may assume that \( r = 2 \), i.e. \( X = K^{n-m} \) where \( 5 \leq n \leq m \). Let \( v_1, v_2, \ldots, v_{n+m} \) be the vertices of a \( K^{n-m-2} \). Partition them into three classes \( V_1, V_2, V_3 \) such that \( v_4, v_5 \in V_1 \) and \( |V_1| = n, |V_2| = k - 1, |V_3| = m - k - 1 \), where \( k \) is any number between 2 and \( \min(n, m - 2) \). Pick vertices \( x \in V_2, y \in V_3 \) and delete the edge \( xy \). Finally, obtain \( G \) from this graph by adding two adjacent vertices \( v_1 \) and \( v_2 \), joining \( v_1 \) to all vertices of \( V_1 \cup V_2 \) except \( v_3 \), and \( v_2 \) to all vertices of \( V_1 \cup V_2 \) except \( v_4 \) (Fig. 2). It is easily checked that \( G \) is in \( \mathcal{G} \) and satisfies (*), while (**) again follows from Lemma 3.1. \( \Box \)

Clearly, when we add \( e_{1,3} \) and \( e_{2,4} \) to the graph \( G \) of the proof of Theorem 3.3(ii) then not only a \( K^{n-m} \) arises but some other graphs \( X' \cong K^{n-m} \) as well. In other words, \( e_{1,3} \) and \( e_{2,4} \) clash in \( G \) not only w.r.t. \( \mathcal{G}(TX) \) (or \( \mathcal{G}(HX) \)) but also w.r.t. \( \mathcal{G}(TX') \) (or \( \mathcal{G}(HX') \)).
This observation gives us the following strengthening of the bipartite case of Theorem 3.3.

Denote by $G(k)$ the graph $G[V_2 \cup V_3 \cup \{v_1, v_2\}]$ from the proof of Theorem 3.3(ii).

**Corollary 3.4.** For any $n, m \in \mathbb{N}$, $2 \leq n \leq m$, let $X_1, X_2$ be graphs such that $|X_1| = n$, $|X_2| = m$ and $X_2 \subset G(k)$ for some $k \geq 2 \leq k \leq m - 2$ (Fig. 3). If $\delta(X_1 + X_2) \geq 5$, then $\mathcal{S}(T(X_1 + X_2))$ and $\mathcal{S}(H(X_1 + X_2))$ have uncountably many prime and maximal elements.

**Proof.** If $X_1 = X_2 = K^3$, then $X_1 + X_2 = K^6$, and the assertion follows from Theorem 3.3(i). We may therefore assume that $m \geq 4$. 

...
Define \( G \) as in the proof of Theorem 3.3(ii). Then we have again \( \mathcal{N}_G \subseteq \mathcal{A}(G) \), since \( G \cup e \) does not even contain a \( K_t \) for \( e \in \{e_1, e_2, e_3, e_4\} \). \( G \) is therefore contained in an \( \mathcal{N}' \)-maximal graph \( G' \supset G \) with \( \mathcal{N}' \subseteq \mathcal{N} \). Since \( e_{1,3} \) and \( e_{2,4} \) clash in \( G \), we have in fact \( \mathcal{N}' = \mathcal{N}_G = \{\{e_{1,3}\}, \{e_{2,4}\}\} \). Also, \( G' \) is still prime, so it satisfies (*) \( \Box \).

In our search for a graph \( G \) satisfying (*) for a given class \( \mathcal{G}(TX) \) or \( \mathcal{G}(HX) \), the following approach is perhaps the most natural: specify independent edges \( e_{1,3} \) and \( e_{2,4} \) in \( X \), delete them, and extend the resulting graph \( X^- \) to \( G := \max(X^-) \). Of course we would have to find such \( e_{1,3} \) and \( e_{2,4} \) that \( \mathcal{N}_{X^-} \subseteq \mathcal{A}(X^-) \), i.e. every set of non-edges of \( T \subseteq X^- \) that could simultaneously be realized in \( X^- \neq H_1 \) must be admissible—a condition which is usually easy to check. Because \( e_{1,3} \) and \( e_{2,4} \) clash in \( X^- \), they will still be clashing non-edges in \( G \), so \( G \) will satisfy (ii) and (iii) of (*).

The advantage of this method is that we do not have to specify \( G \) explicitly, which is particularly useful when we are trying to deal with classes \( \mathcal{G}(TX) \) and \( \mathcal{G}(HX) \) for a variety of \( X \)'s simultaneously. For the more specific features of \( X \) we had to allow for, the less accurate could we be when designing \( G \). Its disadvantage is, that \( G \) may in general fail to satisfy (i) of (*): in fact even when \( X \) is prime (which implies that (i) holds for \( X^- \)), \( X^- \) may acquire new separating simplices when it is extended to \( G \).

When using the described method we shall therefore have to choose \( X^- \supset X \setminus \{e_{1,3}, e_{2,4}\} \) such that \( X^- \cup N \) satisfies (i) of (*) for any set \( N \) of non-edges other than \( e_{1,3} \) or \( e_{2,4} \), as is illustrated by the proof of Corollary 3.4. One way of achieving this is to ensure that every vertex of \( X^- \setminus T \) is adjacent either to \( v_1 \) and \( v_3 \) or to \( v_2 \) and \( v_4 \), for this property will carry over to \( G = \max(X^-) \), which will therefore satisfy (i) of (*).

**Lemma 3.5.** Let \( \Gamma \) be a set of finite graphs, \( \mathcal{G} = \mathcal{G}(TT) \) or \( \mathcal{G} = \mathcal{G}(HG) \), and \( X^- \in \mathcal{G} \) finite with non-edges \( e_{1,3} = v_1v_3 \) and \( e_{2,4} = v_2v_4 \) such that

\( (*) \quad (i') \) every vertex of \( X^- \setminus T \) is adjacent either to \( v_1 \) and \( v_3 \) or to \( v_2 \) and \( v_4 \),
\( (i'') \mathcal{N}_{X^-} \subseteq \mathcal{A}(X^-) \),
\( (i''') e_{1,3} \) and \( e_{2,4} \) clash.

Then \( \mathcal{G} \) contains a graph \( G \) that satisfies (*). \( \Box \)

However, (i') seems far from necessary for a graph satisfying (i) of (*) w.r.t. \( \mathcal{G}(TX) \) or \( \mathcal{G}(HX) \), as soon as we consider any one fixed \( X \). It is therefore hardly surprising that the following more general result seems cruder than it should be.

For \( x, y \in V(X) \) let us define

\[ \delta(x, y) := \min(d(x), d(y)), \quad \Gamma_X(x) := \{v \in V(X) \mid xv \in E(X)\}, \]
\[ \gamma(x, y) := \gamma_X(x, y) := |\Gamma_X(x) \cap \Gamma_X(y)| \quad \text{and} \]
\[ \gamma(X) := \min\{\gamma(x, y) \mid x, y \in V(X)\}. \]
So $\gamma(X)$ denotes the minimal number of common neighbours for any two vertices of $X$.

**Theorem 3.6.** Let $X$ be a finite graph of minimal degree $\delta(X) \geq 5$. Then $\mathcal{G}$ has uncountably many prime and maximal elements if

(i) $\mathcal{G} \neq \mathcal{G}(TX)$ and $\gamma(X) = 2$,

(ii) $\mathcal{G} = \mathcal{G}(HX)$ and $\gamma(X) = 2$.

**Proof.** (i) Choose $v_1, v_2 \in V(X)$ such that $\gamma(v_1, v_2) = \gamma(X)$ and $\delta(v_1, v_2) \leq \delta(x, y)$ for all $x, y$ with $\gamma(x, y) = \gamma(X)$. Assume that $d(v_2) = \delta(v_1, v_2)$. Let $v_3$ be a common neighbour of $v_1$ and $v_2$, and let $v_4$ be another neighbour of $v_2$. Obtain $X^-$ from $X$ by omitting $e_{1,2}$ and $e_{2,4}$, and adding all edges of $N(X \setminus \{v_1, v_2\})$, as well as the edges $v_1v$ for every $v \in X \setminus T$ that is neither adjacent to $v_1$ nor to $v_2$ in $X$.

Clearly $X^- \in \mathcal{G}$. In order to show $K_{X^-} \subseteq \mathcal{G}(X^-)$, suppose that $N(T)$ contains a non-edge $e$ such that $X^- \cup e$ has a subgraph isomorphic to $X$. Then $e$ provides $v_1$ and $v_2$ with a new common neighbour (by $\gamma_X(v_1, v_2) < \gamma(X)$) and at the same time increases the degree of $v_2$. But this means that $e = e_{2,4}$ and $e_{1,4} \in E(X)$, so $v_4$ (as well as $v_3$) is a common neighbour of $v_1$ and $v_2$ in $X$. Hence $\gamma(X^- \cup e) \leq \gamma_X(X \setminus \{v_1, v_2\}) = |T_X(v_1) \cap T_X(v_2)| < \gamma(X)$, a contradiction.

The assertion follows by Lemmas 3.1 and 3.5.

(ii) Choose $v_1, v_2 \in V(X)$ such that $\gamma(v_1, v_2) = \gamma(X)$ and let $v_3$, $v_4$ be distinct common neighbours of $v_1$ and $v_2$. Define $X^-$ from $X$ by deleting $e_{1,3}$ and $e_{2,4}$ and adding all edges of $N(X \setminus \{v_1, v_2\})$, the edge $e_{1,2}$, and the edges $v_1v$ where $v \in X \setminus T$ and $v$ is neither adjacent to $v_1$ nor to $v_2$. $\square$

Although Theorem 3.6 is certainly far from being best possible, it is of considerable help in proving the following direct strengthening of Theorem 3.3.

The details of the proof are left to the reader.

**Theorem 3.7.** Let $n, m \in \mathbb{N}$, $2 \leq n \leq m$, and let $X_1, X_2$ be graphs such that $|X_1| = n$, $|X_2| = m$ and $\delta(X) \geq 5$ where $X := X_1 + X_2$. Then $\mathcal{G}(TX)$ and $\mathcal{G}(HX)$ have uncountably many prime and maximal elements. $\square$

**Conjecture.** If $\mathcal{G}(TX) \neq \mathcal{G}(HX)$ has an uncountable base then so does $\mathcal{G}(TX')$ ($\mathcal{G}(HX')$), for every $X' \supset X$ with $V(X') = V(X)$.

**References**

Subdivision and homomorphism bases for graphs


