Interpolation estimates for entropy numbers with applications to non-convex bodies

Mieczysław Mastyło

Faculty of Mathematics & Comp. Sci., A. Mickiewicz University, and Institute of Mathematics, Polish Academy of Science (Poznań branch), Umultowska 87, 61-614 Poznań, Poland

Received 30 October 2008; accepted 7 February 2009
Available online 20 February 2009

Communicated by Paul Nevai

Abstract

We complement classical results on the interpolation of entropy numbers as well as certain \(s\)-numbers and present an application to a class of non-convex bodies which are generalizations of \(p\)-convex bodies. In particular we apply the estimates of entropy numbers of operators on Calderón–Lozanovskii spaces to approximation of the volume of \(\varphi\)-absolute convex hull of \(n\) points in \(\mathbb{R}^k\) generated by a class of concave functions.

© 2009 Elsevier Inc. All rights reserved.

Keywords: Entropy numbers; \(s\)-numbers; Interpolation spaces; Non-convex bodies

1. Introduction

In recent years a lot of attention has been paid to the study of entropy numbers and \(s\)-numbers of bounded linear operators between quasi-Banach spaces (see, e.g., [4,13,19–22]). In particular a large number of research articles as well as several monographs are devoted to entropy and approximation numbers of compact embeddings between function spaces (see [13,24,25]). We also refer the reader to [10], where some applications to spectral theory of (pseudo-)differential operators are presented. It is worth recalling that if \(T: X \to X\) is a compact operator on a (complex) quasi-Banach space and \((\lambda_n(T))\) denotes the sequence of eigenvalues of \(T\), ordered
according to non-increasing modulus and counting multiplicities, then the famous Carl–Triebel inequality states that for all positive integers \( n \) the estimate

\[
|\lambda_n(T)| \leq \sqrt{2} e_n(T)
\]

holds, where \( e_n(T) \) is the \( n \)-th entropy number of \( T \) (see, e.g., [13, Theorem 2.d.1], [10, Theorem 1.3.4]). We also recall that König [13] proved that if \( X \) is a Banach space, then

\[
|\lambda_n(T)| = \lim_{k \to \infty} \sqrt[k]{a_n(T^k)},
\]

where \( a_n(T) \) is the \( n \)-th approximation number of \( T \). The results mentioned above found applications in the theory of eigenvalue distributions of compact operators as well as in the local theory of Banach spaces [21].

Complementing and generalizing a classical result concerning one-sided interpolation of entropy numbers and \( s \)-numbers of operators between quasi-Banach spaces, we present some estimates of entropy numbers of operators on Calderón–Lozanovskii spaces. The main motivation is to approximate the volume of non-convex bodies that are generalizations of \( p \)-convex bodies with \( 0 < p < 1 \). Our approach to this problem is based on the interpolation theory, taking into account an idea from Gudéon and Litvak [11]. We will make use of several notions from the theory of Banach spaces and their generalizations, which are quasi-Banach spaces. For the reader’s convenience, we present some definitions and results that will be used in the sequel. A nice general reference to quasi-Banach spaces is [12]. Let \( (X, \| \cdot \|) \) be a quasi-normed space. We call \( \| \cdot \| \) a \( p \)-norm \((0 < p \leq 1)\) whenever

\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad \text{for all } x, y \in X.
\]

The Aoki–Rolewicz theorem (see [12]) states that for any quasi-normed space \( X \) there exists an equivalent \( p \)-norm for some \( 0 < p \leq 1 \). Let \( X \) and \( Y \) be quasi-Banach spaces. The space of all linear bounded operators from \( X \) to \( Y \) equipped with the quasi-norm \( \|T\|_{X \to Y} := \sup\{\|Tx\|_Y : x \in U_X\} \) is denoted by \( L(X, Y) \). Here, as usual, \( U_X := \{x \in X; \|x\|_X \leq 1\} \) stands for the closed unit ball of \( X \). Recall that for an operator \( T : X \to Y \) between quasi-Banach spaces \( X, Y \), the \( n \)-th (dyadic) entropy number \( e_n(T) \) of \( T \) is defined by

\[
e_n(T) := \inf\{\varepsilon > 0; \exists y_1, \ldots, y_{2^n-1} \in Y, \ T(U_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon U_Y)\}.
\]

Throughout this paper we will use the following properties of entropy numbers which are true for any operators \( S, T : A \to B \) and \( R : B \to C \) between quasi-Banach spaces and all \( k, m \in \mathbb{N} \).

(i) (Monotonicity): \( \|T\| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0 \).
(ii) (Multiplicity): \( e_{k+m-1}(S \circ R) \leq e_k(S)e_m(R) \).
(iii) \((p\text{-additivity})\): If \( B \) is a \( p \)-Banach space, then \( e_{k+m-1}^p(S + T) \leq e_k^p(S) + e_m^p(T) \).

Note that if \( T : A \to B \) is a bounded operator between quasi-Banach spaces \((A, \| \cdot \|_A), (B, \| \cdot \|_B)\) and \((A_1, \| \cdot \|_{A_1}), (B_1, \| \cdot \|_{B_1})\) with \( A = A_1, B = B_1 \) are equipped with the quasi-norms \( \| \cdot \|_A \asymp \| \cdot \|_A \) and \( \| \cdot \|_{B_1} \asymp \| \cdot \|_B \), then it is easy to see that

\[
e_n(T : A_1 \to B_1) \asymp e_n(T : A \to B),
\]

where the constants of equivalence depend on the constants of equivalence of the quasi-norms. Combining this fact with the \( p \)-additivity property, we conclude by the Aoki–Rolewicz theorem that...
that there exists a constant $C > 0$ such that for any bounded operators $S, T: A \to B$ between quasi-Banach spaces and for all $k, m \in \mathbb{N}$, we have

$$e_{k+m-1}(S + T) \leq C(e_k(S) + e_m(T)).$$

This paper is organized as follows. Section 2 contains results on one-sided interpolation of entropy numbers of operators between quasi-Banach spaces. We show estimates of entropy numbers in terms of appropriate interpolation functions. In Section 3 we show that for $K$-linearizable Banach couples, one-sided interpolation of $s$-numbers is possible. Sections 4 and 5 are devoted to applications. First, we deal with Calderón–Lozanovskii spaces between quasi-Banach spaces, then the notation operators in Banach spaces in $[\mathbb{R}^k, \|\cdot\|]$. Further, fundamental functions were used in the interpolation theory of Banach spaces as it appears in $[2]$. Interpolation of entropy numbers and Litvak in $[11]$ points in the finite-dimensional Orlicz space $\ell^n_p$-space which generalizes the notion of the $\ell^n_p$-absolute convex hull of a subset in a linear space which generalizes the notion of the $p$-absolute hull for $p \in (0, 1)$. Further, we give some applications of the previously obtained results to the study of the $\varphi$-absolute convex hulls of $n$ points in $\mathbb{R}^k$ generated by a special class of normalized positive concave functions $\varphi$. The key result in Section 5 is Theorem 5.1, which gives an estimate of entropy numbers of operators from the finite-dimensional Orlicz space $\ell^n_p$ to a quasi-Banach space $(\mathbb{R}^n, \|\cdot\|)$, where $\varphi$ is a normalized positive concave function. Next, we apply this theorem to approximate the volume of the $\varphi$-absolute convex hull of $n$ points in $\mathbb{R}^k$. We also extend certain results proved by Guédon and Litvak in [11] for the case of $p$-absolute hulls and $\ell^n_p$-spaces for any $p \in (0, 1)$.

2. Interpolation of entropy numbers

We shall use a minor modification of the notation and terminology commonly used in interpolation theory of Banach spaces as it appears in [1,2]. Let $\overline{A} = (A_0, A_1)$ be a quasi-Banach couple, that is, two quasi-Banach spaces $A_j, j = 0, 1$, which are continuously included in some Hausdorff topological vector space. For each $s, t > 0$ and $a \in A_0 \cap A_1$ (resp., $a \in A_0 + A_1$) we define

$$J(s, t, a) := J(s, t, a; \overline{A}) := \max\{s\|a\|_{A_0}, t\|a\|_{A_1}\}$$

and respectively

$$K(s, t, a) := K(s, t, a; \overline{A}) := \inf\{s\|a\|_{A_0} + t\|a\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}.$$ 

Let $A$ be a quasi-Banach space intermediate with respect to the quasi-Banach couple $\overline{A} = (A_0, A_1)$ (i.e., $A_0 \cap A_1 \leftrightarrow A \leftrightarrow A_0 + A_1$ where, as usual, $\leftrightarrow$ means the continuous inclusion). To characterize the position of $A$ more precisely in terms of some additional parameters we define the fundamental functions. For each $s, t > 0$ set

$$\varphi(s, t) := \varphi_{\overline{A}}(s, t; \overline{A}) := \sup\{K(s, t, a); \|a\|_A = 1\}$$

and

$$\psi(s, t) := \psi_{\overline{A}}(s, t; \overline{A}) := \sup\{\|a\|_A; J(s^{-1}, t^{-1}, a) \leq 1\}.$$ 

The function $\varphi$ (as well as $\psi$) is monotone in each coordinate and homogeneous of degree 1, that is, $\varphi(\lambda s, \lambda t) = \lambda \varphi(s, t)$ for $\lambda > 0$ and $s, t > 0$. Note that in the case of Banach spaces the function $\varphi_A := \varphi(1, \cdot)$ has been first introduced in [8] in the study of interpolation of one-dimensional operators. Further, fundamental functions were used in the interpolation theory of operators in Banach spaces in [9]. In what follows if $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ are quasi-Banach couples, then the notation $T: \overline{A} \to \overline{B}$ means that $T$ is a linear operator from $A_0 + A_1$...
Let $A$ be a quasi-Banach space and $B$ be quasi-Banach spaces with $A$ intermediate with respect to $A$ and let $\varphi = \varphi_A$. Then for each operator $T : A \to Y$ between quasi-Banach spaces $X$ and $Y$, the $n$-th (dyadic) inner entropy number $\varphi_n(T)$ of $T$ is defined by

$$\varphi_n(T) := \sup\{\rho > 0; \exists\{x_1, \ldots, x_p\} \subset U_X \text{ with } p > 2^n - 1, \|Tx_i - Tx_j\|_Y > 2\rho \text{ for } i \neq j\}.$$ 

Note that the proof of the inequality $e_n(T) \leq 2\varphi_n(T)$ for all $n \in \mathbb{N}$, presented in [19, Theorem 12.1.10] for Banach spaces is also valid for quasi-Banach spaces.

Proposition 2.2. Let $A$ be a quasi-Banach space and $B$ be quasi-Banach space intermediate with respect to a couple $B = (B_0, B_1)$ of quasi-Banach spaces and let $\psi = \psi_B$. Then for each operator $T : A \to B$ and all positive integers $k$ and $m$ we have

$$e_{k+m-1}(T : A \to B) \leq C_B \varphi(e_k(T : A_0 \to B), e_m(T : A_1 \to B)).$$

We start with the following two interpolation results that are known in the case of Banach spaces and the power functions. Since the second one will be used later, we present its proof for the sake of completeness. The proof of the first one can be obtained in a similar way to the classical case (see, e.g., [19, Proposition 12.1.12]).
Therefore, setting $C = \max\{C_{B_0}, C_{B_1}\}$, we obtain
\[
\|Ta_i - Ta_j\|_B \leq 2C\psi(s_0, s_1).
\]
Combining this estimate with the fact that the set $\{a_1, \ldots, a_p\}$ with $p > 2^{(k+m-1)−1}$ is included in $U_A$, we conclude that
\[
\varphi_{k+m−1}(T: A → B) ≤ C\psi(e_k(T: A → B_0), e_m(T: A → B_1)).
\]
Since $e_n(T) ≤ 2\varphi_n(T)$ for all $n ∈ \mathbb{N}$, the desired estimate follows.

We conclude this section with the remark that the behaviour of compact operators under interpolation has been studied since the 1960s. In 2001, Cobos, Cwikel and Matos in their paper [6] studied estimates for the measure of non-compactness of operators in Banach spaces and other ideal measures in terms of related interpolation functions. We recall that, if $T: X → Y$ is an operator between quasi-Banach spaces, then the ball measure of non-compactness $\beta(T)$ of $T$ is given by the infimum of all $\varepsilon > 0$ such that there exists a finite number of elements $x_1, \ldots, x_n \in X$ so that
\[
T(U_X) ⊂ \bigcup_{j=1}^n (x_j + \varepsilon U_Y).
\]
Clearly $\beta(T) = 0$ if and only if $T$ is compact. Since $\beta(T) = \lim_{n→∞} e_n(T)$, the results presented in this section imply immediately the corresponding estimates of the measures of non-compactness of operators in quasi-Banach spaces.

### 3. Interpolation of $s$-numbers

The notion of the $s$-number sequence is due to Pietsch (for details see [20]). It turns out that $s$-numbers are very powerful tools for estimating eigenvalues of operators in Banach spaces. As entropy numbers, $s$-numbers have been also extensively studied, especially approximation ($a_n$), Gelfand ($c_n$) and Kolmogorov numbers ($d_n$). If $X, Y$ are Banach spaces and $T ∈ L(X, Y)$, the numbers are defined by
\[
a_n(T) := \inf\{\|T - R\|; \ R ∈ L(X, Y), \ \text{rank}(R) < n\},
\]
\[
c_n(T) := \inf\{\|T\|_{G → Y}; \ G ⊂ X, \ \text{codim}(G) < n\},
\]
and
\[
d_n(T) := \inf\{\|q_ST\|_{X/Y → S}; \ S ⊂ Y, \ \text{dim}(S) < n\},
\]
respectively, where $q_S: Y → Y/S$ denotes the quotient mapping. We note that any of these sequences $s_n ∈ \{a_n, c_n, d_n\}$ satisfies the relations
\[
\|T\| = s_1(T) ≥ s_2(T) ≥ \cdots ≥ 0,
\]
\[
s_{k+m−1}(S + T) ≤ s_k(S) + s_m(T); \ S, T ∈ L(X, Y),
\]
\[
s_{k+m−1}(ST) ≤ s_k(S) s_m(T); \ S ∈ L(Y, Z), \ T ∈ L(X, Y).
\]
Moreover,
\[
a_n(T) ≥ c_n(T), \quad a_n(T) ≥ d_n(T),
\]
with equality for Hilbert space operators $T$. For the basic properties of the $s$-numbers and
applications to eigenvalue and compactness problems, we refer the reader to the monographs [5,20,21] and the references given therein.

Inspired by [7], we show that for a quite rare range of couples the one-sided interpolation of \(s\)-numbers is possible. Following Peetre a Banach couple \(\overline{X} = (X_0, X_1)\) is said to be \(K\)-linearizable if there exist a constant \(C > 0\) and a family of operators \(\{V_j(t)\}_{t>0}, j = 0, 1\), such that

\[
V_0(t) + V_1(t) = I,
\]

where \(I\) is the identity mapping in \(X_0 + X_1\) and we have for \(j = 0, 1\)

\[
\|V_0(t)x\|_{X_0} \leq Ct^j\|x\|_{X_j} \quad \text{for } x \in X_j,
\]

\[
\|V_1(t)x\|_{X_1} \leq Ct^{j-1}\|x\|_{X_j} \quad \text{for } x \in X_j.
\]

It is easy to see then that

\[
K(1, t, x; \overline{X}) \leq \|V_0(t)x\|_{X_0} + t\|V_1(t)x\|_{X_1} \leq CK(1, t, x; \overline{X}).
\]

There are many important examples of \(K\)-linearizable couples. For instance: any regular couple \((H_0, H_1)\) of Hilbert spaces (see [7]), the couple \((L_p, \dot{W}_p^k)\), where \(\dot{W}_p^k\) is the “homogeneous” Sobolev space and the couple of Besov spaces \((B_{p,q}^{s_0}([\mathbb{R}^n]), B_{p,q}^{s_1}([\mathbb{R}^n]))\). On the other hand, if \(p_0 \neq p_1\), the couple \((L_{p_0}, L_{p_1})\) is not of this type (see [26] for more examples and details). We note that Nilsson [16] proved that for couples of separable Banach lattices the definition of the \(K\)-linearizable Banach couple is equivalent to a more explicit quasi-linearization. It is shown in [16, Lemma 4.5] that couples \((E_0, E_1)\) of Banach sequence lattices satisfying certain conditions are quasi-linearizable. In particular, \((\ell_{q_0}(2^{−n\theta_0}), \ell_{q_1}(2^{−n\theta_1}))\) forms a quasi-linearizable Banach couple for any \(0 \leq \theta_0 < \theta_1 \leq 1\) and \(1 \leq q_0, q_1 \leq \infty\).

**Proposition 3.1.** Let \(\overline{B} = (B_0, B_1)\) be a \(K\)-linearizable Banach couple, \(B\) be an intermediate Banach space with respect to \(\overline{B}\), \(\psi = \psi_{\overline{B}}\) and \(A\) be a Banach space. Then, for any additive \(s\)-function and for each operator \(T: A \to \overline{B}\) the following statements are true for all positive integers \(k\) and \(m\):

(i) If \(s_k(T: A \to B_0) = 0\), then

\[
s_{k+m-1}(T: A \to B) \leq (1 + C)\psi(0+, 1)s_n(T: A \to B_1).
\]

(ii) If \(s_m(T: A \to B_1) = 0\), then

\[
s_{k+m-1}(T: A \to B) \leq (1 + C)\psi(1, 0+)s_k(T: A \to B_0).
\]

(iii) If \(s_k(T: A \to B_1) > 0\) and \(s_m(T: A \to B_1) > 0\), then

\[
s_{k+m-1}(T: A \to B) \leq 2(1 + C)\psi(s_k(T: A \to B_0), s_m(T: A \to B_1)).
\]

**Proof.** Since \(V_j(t): B_0 + B_1 \to (B_0, B_1)\) and \(B \hookrightarrow B_0 + B_1\), we conclude that \(V_j(t): B \to (B_0, B_1)\). For any \(b \in B_1\), we have

\[
\|V_0(t)b\|_B \leq \psi(t, 1)J(t^{-1}, 1, V_0(t)b; \overline{B}) = \psi(t, 1)\max\{t^{-1}\|V_0(t)b\|_{B_0}, \|b - V_1(t)b\|_{B_1}\} \leq (1 + C)\psi(t, 1)\|b\|_{B_1}.
\]
Similarly, for any $b \in B_0$, we have
\[
\|V_1(t)b\|_B \leq \psi(1, t^{-1}) J(1, t, V_1(t)b; \overline{B}) = \psi(1, t^{-1}) \max \{ \|b - V_0(t)b\|_{B_0}, t \|V_1(t)b\|_{B_1} \} 
\leq (1 + C) \psi(1, t^{-1}) \|b\|_{B_0}.
\]
This implies that $\|V_0(t)\|_{B_1 \rightarrow B} \leq (1 + C) \psi(t, 1)$ and $\|V_1(t)\|_{B_0 \rightarrow B} \leq (1 + C) \psi(1, t^{-1})$ for any $t > 0$. Let $T: A \rightarrow \overline{B}$. Then by $V_0(t) + V_1(t) = I$, we have $T = V_1(t)T + V_0(t)T$ for any $t > 0$. Thus combining the above estimates with properties of the $s$-numbers, we obtain
\[
s_{k+m-1}(T: A \rightarrow B) \leq s_k(V_1(t): A \rightarrow B) + s_n(V_0(t): A \rightarrow B) 
\leq \|V_1(t)\|_{B_1 \rightarrow B} s_k(T: A \rightarrow B_0) + \|V_0(t)\|_{B_0 \rightarrow B} s_m(T: A \rightarrow B_1) 
\leq (1 + C)(\psi(1, t^{-1})s_k(T: A \rightarrow B_0) + \psi(t, 1)s_m(T: A \rightarrow B_1)).
\]
To conclude the proof, we simply take the limits with $t \rightarrow 0^+$ in case (i) and with $t \rightarrow \infty$ in case (ii). We obtain case (iii) by substituting $t = s_k(T: A \rightarrow B_0)/s_m(T: A \rightarrow B_1)$. □

The proof of the following result is similar (we only need to use the representation $T = TV_0(t) + TV_1(t)$ for any $t > 0$).

**Proposition 3.2.** Let $\overline{A} = (A_0, A_1)$ be a $K$-linearizable regular Banach couple, $A$ be an intermediate Banach space with respect to $\overline{A}$, $\varphi = \varphi_{\overline{A}}$ and $B$ be a Banach space. Then, for any additive $s$-function and for each operator $T: \overline{A} \rightarrow B$ the following statements are true for all positive integers $k$ and $m$:

(i) If $s_k(T: A_0 \rightarrow B) = 0$, then
\[
s_{k+m-1}(T: A \rightarrow B) \leq (1 + C)\varphi(0^+, 1)s_n(T: A_0 \rightarrow B).
\]

(ii) If $s_m(T: A_1 \rightarrow B) = 0$, then
\[
s_{k+m-1}(T: A \rightarrow B) \leq (1 + C)\varphi(1, 0^+)s_k(T: A \rightarrow B_0).
\]

(iii) If $s_k(T: A_0 \rightarrow B) > 0$ and $s_m(T: A_1 \rightarrow B) > 0$, then
\[
s_{k+m-1}(T: A \rightarrow B) \leq 2(1 + C)\varphi(s_k(T: A_0 \rightarrow B), s_m(T: A_1 \rightarrow B)).
\]

We conclude this section with an example of a quasi-linearizable couple useful from the point of view of applications of the above results. Given any sequence $\{X_n\}_{n \in \mathbb{Z}}$ of Banach spaces and a positive sequence $\{w_n\}_{n \in \mathbb{Z}}$, we denote by $E(w_nX_n)$ the space of all sequences $\{x_n\} \in \prod_{n \in \mathbb{Z}} X_n$ such that $\{w_n\|x_n\|X_n\}_{n \in \mathbb{Z}} \in E$. It is a Banach space equipped with the norm $\|\{x_n\}\| = \|\{w_n\|x_n\|X_n\}\|_E$. In the sequel, for a given subset $A \subset \mathbb{Z}$ and $x = \{x_n\} \in E(w_nX_n)$, we denote by $x1_A$ the sequence with coordinates $x_n$ for $n \in A$ and 0 for $n \in \mathbb{Z} \setminus A$.

**Lemma 3.1.** Assume that $E$ is a Banach sequence lattice on $\mathbb{Z}$ and $u = \{u_n\}_{n \in \mathbb{Z}}$, $v = \{v_n\}_{n \in \mathbb{Z}}$ are positive sequences. Then for any sequence $\{X_n\}_{n \in \mathbb{Z}}$ of Banach spaces the couple $(E(u_nX_n), E(v_nX_n))$ is $K$-linearizable.

**Proof.** For any fixed $t > 0$ we put
\[
A_t := \{n \in \mathbb{Z}; u_n \leq tv_n\}, \quad B_t := \mathbb{Z} \setminus A_t.
\]
Let $x = \{x_n\}_{n \in \mathbb{Z}} \in E(u_nX_n) + E(v_nX_n)$. Then for
\[
y(t) = x1_{A_t} \quad \text{and} \quad z(t) = x - y(t),
\]
we have
\[
\|y(t)\|_E \leq J_{1t}(x) = \psi(1, t^{-1}) J(1, t, x; \overline{E}) \leq \psi(1, t^{-1}) \max \{ \|x - y(t)\|_E, t \|y(t)\|_E \} 
\leq (1 + C) \psi(1, t^{-1}) \|x\|_E.
\]
This holds for any $t > 0$, so we may take $t \rightarrow 0^+$ to obtain
\[
\|x\|_E \leq \psi(0^+, 1) \|y(0^+)\|_E,
\]
which implies that $x \in E(u_nX_n)$. Therefore, $E(u_nX_n)$ is $K$-linearizable. □
we have
\[ K(t, x; E(u_n X_n), E(v_n X_n)) \leq \|y(t)\|_{E(u_n X_n)} + t \|z(t)\|_{E(v_n X_n)} \]
\[ = \|x A_t\|_{E(u_n X_n)} + t \|x B_t\|_{E(v_n X_n)}. \]
Since for any \( x_0 \in E(u X_n) \) and \( x_1 \in E(v X_n) \) such that \( x = x_0 + x_1 \),
\[ \min\{u, tv\}x := \min\{u_n x_n, tv_n x_n\} = \min\{u, \nu t\}x_0 + \min\{u, \nu t\}x_1, \]
we conclude that
\[ \|x A_t\|_{E(u_n X_n)} + t \|x B_t\|_{E(v_n X_n)} \leq 2 \min\{u, \nu t\}x \|E(x_n)\| \]
\[ \leq 2(\|x_0\|_{E(u_n X_n)} + t \|x_1\|_{E(v_n X_n)}). \]
This shows that
\[ \|x A_t\|_{E(u_n X_n)} + t \|x B_t\|_{E(v_n X_n)} \leq 2K(t, x; E(u_n X_n), E(v_n X_n)) \]
The estimates obtained give the following equivalence:
\[ K(t, x; E(u_n X_n), E(v_n X_n)) \asymp \|x A_t\|_{E(u_n X_n)} + t \|x B_t\|_{E(v_n X_n)}. \]
To complete the proof it is enough to define
\[ V_0(t)x = x A_t \quad \text{and} \quad V_1(t)x = x B_t \]
for any \( x \in E(u_n X_n) + E(v_n X_n). \quad \square \]
As a direct consequence of the above lemma, we have the following conclusion: since any regular couple \((H_0, H_1)\) of Hilbert spaces is isomorphic to a couple \((\ell_2(G_n), \ell_2(2^{-n}G_n))\) for some sequence \(\{G_n\}_{n \in \mathbb{Z}}\) of Hilbert spaces, so \((H_0, H_1)\) is \(K\)-linearizable (see [7]).

4. Applications to Calderón–Lozanovskii spaces

We present some applications of the results obtained to special constructions including the Calderón–Lozanovskii spaces. Recall that if \( \overline{X} = (X_0, X_1) \) is a couple of quasi-Banach lattices on a \( \sigma \)-finite and complete measure space \((\Omega, \mu) := (\Omega, \Sigma, \mu) \) and \( \varphi \in \mathcal{U} \) (i.e., \( \varphi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) is concave and positively homogeneous of degree 1), then the Calderón–Lozanovskii space \( \varphi(\overline{X}) = \varphi(X_0, X_1) \) consists of all \( x \in L^0(\mu) \) such that
\[ |x| \leq \lambda \varphi(|x_0|, |x_1|) \mu\text{-a.e. on } \Omega \text{ for some } x_j \in E_j \text{ with } \|x_j\| x_j \leq 1, j = 0, 1. \]
The space \( \varphi(\overline{X}) \) is a quasi-Banach lattice equipped with the quasi-norm (cf. [14,15])
\[ \|x\|_{\varphi(\overline{X})} := \inf\{\lambda > 0; |x| \leq \lambda \varphi(|x_0|, |x_1|), \|x_0\|_{\varphi(\overline{X})} \leq 1, \|x_1\|_{\varphi(\overline{X})} \leq 1\}. \]
In the case of the power function \( \varphi(s, t) = s^{1-\theta} t^\theta \), \( \varphi(\overline{X}) \) is the well-known Calderón space (see [3]). Note that if \((X_0, X_1)\) is a couple of Banach lattices, then the following Kőthe duality formula holds (see [14,15]):
\[ \varphi(X_0, X_1)' = \tilde{\varphi}(X_0', X_1'), \]
where the constants of equivalence of norms do not depend on \( \varphi \). Here, for \( \varphi \in \mathcal{U} \), the conjugate function \( \tilde{\varphi} \) is defined by
\[ \tilde{\varphi}(s, t) := \inf\{\alpha s + \beta t; \varphi(\alpha, \beta) \}; \alpha, \beta > 0 \]
for all $s, t \geq 0$. We have $\varphi \in \mathcal{U}$ and $\hat{\varphi} = \varphi$ (see [15]). It is easy to see that $\varphi^* \leq \hat{\varphi} \leq 2\varphi^*$, where for $\varphi \in \mathcal{U}$ we let $\varphi^*(s, t) = 1/\varphi(s^{-1}, t^{-1})$ for $s, t > 0$. Let us note that in many cases the Calderón–Lozanovskii spaces may be easily identified. In particular, it is easy to check that in the case of locally bounded Orlicz spaces $(L_{\varphi_0}, L_{\varphi_1})$ on a measure space for any $\varphi \in \mathcal{U}$, we have (see [17,18])

$$\varphi(L_{\varphi_0}, L_{\varphi_1}) = L_{\varphi}$$

with equivalence of quasi-norms, where $\varphi^{-1}(t) = \varphi(\varphi_0^{-1}(t), \varphi_1^{-1}(t))$ for $t \geq 0$. In the following proposition we present, among other things, the estimates of interpolation functions for Calderón–Lozanovskii spaces. We recall that if $E$ is a Banach lattice on $(\Omega, \mu)$ and $w \in L^0(\mu)$ is a positive weight function, then the weighted Banach lattice $E(w)$ is defined by setting $\|x\|_{E(w)} = \|xw\|_E$.

**Proposition 4.1.** Let $X = (X_0, X_1)$ be a couple of quasi-Banach lattices on $(\Omega, \mu)$ and $X = \varrho(X_0, X_1)$ be a Calderón–Lozanovskii space generated by $\varrho \in \mathcal{U}$. Then we have:

(i) $\varphi_X(s, t; X) \leq \varrho(s, t)$ for all $s, t > 0$.

(ii) $\psi_X(s, t) \leq \hat{\varrho}(s, t)$ for all $s, t > 0$.

(iii) If $\varrho$ is sub-multiplicative (i.e., there exists $C > 0$ such that $\varrho(1, st) \leq C \varrho(1, s)\varrho(1, t)$ for any $s, t > 0$) then the continuous inclusion:

$$\varrho(X_0(w_0), X_1(w_1)) \hookrightarrow \varrho(X_0, X_1)(\varrho^*(w_0, w_1))$$

holds for arbitrary weights $w_0$ and $w_1$.

(iv) If $\varphi$ is super-multiplicative (i.e., there exists $C > 0$ such that $\varrho(1, st) \geq C \varrho(1, s)\varrho(1, t)$ for any $s, t > 0$), then the following continuous inclusion:

$$\varrho(X_0, X_1)(\varrho^*(w_0, w_1)) \hookrightarrow \varrho(X_0(w_0), X_1(w_1))$$

holds for arbitrary weights $w_0$ and $w_1$.

**Proof.** (i) Let $x \in \varrho(X_0, X_1)$. Then $|x| \leq \lambda \varrho(|x_0|, |x_1|)$ for some $\lambda > 0$ and $x_j$ with $\|x_j\|_{X_j} \leq 1, j = 0, 1$. Since

$$\varrho(|x_0|, |x_1|) \leq \frac{\alpha |x_0| + \beta |x_1|}{\hat{\varrho}(\alpha, \beta)} \quad (\alpha, \beta > 0),$$

it follows that

$$K(s, t; x; X) \leq \lambda K(s, t, \varrho(|x_0|, |x_1|)) \leq \frac{\lambda(\alpha s\|x_0\|_{X_0} + \beta t\|x_1\|_{X_1})}{\hat{\varrho}(\alpha, \beta)} \leq \frac{\lambda(\alpha s + \beta t)}{\hat{\varrho}(\alpha, \beta)}$$

for arbitrary $\alpha, \beta > 0$. This implies that

$$K(s, t; x; X) \leq \varrho(s, t)\|x\|_{\varrho},$$

i.e., $\varphi_X(s, t) \leq \varrho(s, t)$ for all $s, t > 0$. (ii) Fix $s, t > 0$ and let $x \in X_0 \cap X_1$ with $\|x\|_{X_0} \leq s$, $\|x\|_{X_1} \leq t$. Then for $y_0 = x/s, y_1 = x/t$, we have $\|y_0\|_{X_0} \leq 1$ and $\|y_1\|_{X_1} \leq 1$. Since

$$|x| = \frac{\rho(\alpha y_0, \beta t y_1)}{\rho(\alpha, \beta)} \leq \frac{\alpha s + \beta t}{\rho(\alpha, \beta)} \rho(y_0, y_1)$$
holds for all positive $\alpha$ and $\beta$, we get
$$
\|x\|_p \leq \inf \left\{ \frac{\alpha s + \beta t}{\rho(\alpha, \beta)} ; \alpha, \beta > 0 \right\} = \hat{\rho}(s, t).
$$
This yields $\psi_X(s, t) \leq \hat{\rho}(s, t)$ for all $s, t > 0$. (iii) If $x \in \rho(X_0(w_0), X_1(w_1))$, then we can find $\lambda > 0$ and $x_j \in X_j(w_j)$ with $\|x_j\|_{X_j(w_j)} \leq 1$ for $j = 0, 1$ such that
$$
|x| \leq \lambda \rho(|x_0|, |x_1|).
$$
By our hypothesis, it follows that $\rho(su, vt) \leq C\rho(u, v)\rho(s, t)$ for all $s, t, u, v > 0$. Thus for $w := \rho^*(w_0, w_1)$, $y_j = x_j w_j$ ($j = 0, 1$), we have
$$
|x|w \leq \lambda \rho(|y_0|/w_0, |y_1|/w_1)\rho^*(w_0, w_1) \leq \lambda C\rho(|y_0|, |y_1|).
$$
Since $\|y_j\|_{X_j} \leq 1$, $j = 0, 1$, we conclude that $x \in X(w)$ and $\|x\|_{X(w)} \leq C\|x\|_{\rho(X_0(w_0), X_1(w_1))}$. This proves the required continuous inclusion. In a similar way we prove (iv). □
Combining the above proposition with the results of Section 1, and the inequalities $\psi^* \leq \hat{\psi} \leq 2\psi^*$, we obtain the following.

**Corollary 4.1.** Let $(X_0, X_1)$ be a couple of quasi-Banach lattices, $\psi \in \mathcal{U}$ and $X$ be a quasi-Banach space. Then there exists a constant $C$ independent of $\psi$ such that for all positive integers $k$ and $n$ we have:

(i) For each operator $T : (X_0, X_1) \rightarrow X$,
$$
e_{k+m-1}(T : \psi(X_0, X_1) \rightarrow X) \leq C \psi(e_k(T : X \rightarrow X_0), e_m(T : X \rightarrow X_1)).
$$

(ii) For each operator $T : X \rightarrow (X_0, X_1)$,
$$
e_{k+m-1}(T : X \rightarrow \psi(X_0, X_1)) \leq C \psi^*(e_k(T : X_0 \rightarrow X), e_m(T : X_1 \rightarrow X)).
$$

5. **Applications to non-convex bodies**

We present applications of the one-sided interpolation of entropy numbers to study bodies in $\mathbb{R}^n$. We recall that by a body we mean a compact set in $\mathbb{R}^n$ containing the origin as an interior point and star shaped with respect to the origin. By the ellipsoid in $\mathbb{R}^n$ we always mean a linear image of the canonical Euclidean ball. Following [11], given bodies $K$ and $B$ in $\mathbb{R}^n$, we define the Banach–Mazur distance by
$$
d(K, B) = \inf\{\lambda > 0 ; K - z \subset T(B - x) \subset \lambda(K - z)\},
$$
where the infimum is taken over all linear operators $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and all $x, z \in \mathbb{R}^n$. We also define the following distance:
$$
d_0(K, B) = \inf\{\lambda > 0 ; K \subset T(B) \subset \lambda K\},
$$
where the infimum is taken over all linear operators $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Clearly, if $K$ and $B$ are centrally symmetric bodies, then $d(K, B) = d_0(K, B)$ and it is the standard Banach–Mazur distance. A quasi-Orlicz function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is defined as a continuous non-decreasing function such that $\varphi(0) = 0$ and $p(\varphi) > 0$. Here $p(\varphi)$ denotes the *lower Matuszewska–Orlicz index* of $\varphi$ defined by $p(\varphi) = \lim_{t \rightarrow 0} \ln \overline{\varphi}(t)/\ln t$, where $\overline{\varphi}(t) := \sup_{s > 0} \varphi(st)/\varphi(s)$. It is easy to see that the condition $p(\varphi) > 0$ is equivalent to $\lim_{t \rightarrow 0} \overline{\varphi}(t) = 0$. With any quasi-Orlicz function
\( \varphi \), we associate the space \( \ell_\varphi \) of all sequences of scalars \( x = (x_k) \) such that \( \sum_{k=1}^{\infty} \varphi(|x_k|/\lambda) < \infty \) for some \( \lambda > 0 \). The space \( \ell_\varphi \) equipped with the quasi-norm
\[
\|x\|_{\ell_\varphi} = \inf\left\{ \lambda > 0; \sum_{k=1}^{\infty} \varphi(|x_k|/\lambda) \leq 1 \right\}
\]
is a symmetric quasi-Banach lattice called an Orlicz sequence space. For an Orlicz sequence space \( \ell_\varphi \), we denote by \( \ell_\varphi^n \) the linear span of the first \( n \) standard unit vectors \( e_1, \ldots, e_n \) equipped with the quasi-norm induced from \( \ell_\varphi \). Given a strictly increasing positive concave function \( \varphi \) on \([0, \infty)\) with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) (called a normalized concave function for short) and a subset \( A \) of a linear space \( X \), its \( \varphi \)-absconv \( A \) is defined as
\[
\varphi\text{-absconv}A = \left\{ \sum_{k=1}^{m} \lambda_k x_k; \ m \in \mathbb{N}, \ x_k \in A, \ \lambda_k \in \mathbb{R}, \ \sum_{k=1}^{m} \varphi(|\lambda_k|) \leq 1 \right\}.
\]
If \( \varphi(t) = t^p \) with \( p \in (0, 1] \), we write \( p\text{-absconv}A \) (resp., absconv if \( p = 1 \)) instead of \( \varphi\text{-absconv}A \). For any concave function \( \varphi: [0, \infty) \to [0, \infty) \), we define a lower index \( a_\varphi \) of Simonenko by the formula (see [23]):
\[
a_\varphi = \inf_{t > 0} \frac{t \varphi'(t)}{\varphi(t)}.
\]
It is easy to see that any concave function \( \varphi: [0, \infty) \to [0, \infty) \) is a quasi-Orlicz function with \( p(\varphi) \geq a_\varphi \). Given any set \( A \subset \mathbb{R}^n \), by vol \( (A) \) we denote the volume of \( A \). We have the following result.

**Theorem 5.1.** Let \( X = (\mathbb{R}^n, \| \cdot \|) \) be a quasi-Banach space and \( T \) be a linear operator from \( \mathbb{R}^n \) to \( X \). Assume that \( \varphi: [0, \infty) \to [0, \infty) \) is a concave function such that \( 0 < a_\varphi < 1 \). Then for all integers \( k \leq n \) and \( m \geq 1 \) one has
\[
e_{k+m-1}(T; \ell_\varphi^n \to X) \leq C_\varphi \min\left\{ 1, \frac{\varphi^{-1}(f(n, k))}{f(n, k)} \right\} e_m(T; \ell_1^n \to X),
\]
where \( f(n, k) = \frac{1}{k} \ln(1 + \frac{k}{n}) \) and \( C_\varphi \) depends only on \( \varphi \).

**Proof.** It is easy to check that our hypothesis \( 0 < p := a_\varphi < 1 \) implies that the function \( t \mapsto \varphi^{-1}(t)/t^{1/p} \) is non-increasing. Since \( \varphi^{-1} \) is a convex function, so \( t \mapsto \varphi^{-1}(t)/t \) is non-decreasing. This implies that the function \( \rho: [0, \infty) \to [0, \infty) \) defined by \( \rho(0) = 0 \) and
\[
\varphi^{-1}(t) = t \rho(t^{1/p}/t)
\]
for all \( t > 0 \)
is a quasi-concave function on \([0, \infty)\). Now take a concave function \( \phi \) such that \( \phi \asymp \rho \) and \( \psi \in \mathcal{U} \) defined by \( \psi(u, v) = v\phi(u/v) \) for \( v > 0 \) and \( 0 \) if \( v = 0 \). Then, we have
\[
\varphi^{-1}(t) \asymp \psi(t^{1/p}, t),
\]
whence some simple calculations show that (see [17] or [18])
\[
\ell_\varphi^n = \psi(\ell_p^n, \ell_1^n)
\]
and
\[
\|x\|_{\ell_\varphi^n} \leq \|x\|_{\psi(\ell_p^n, \ell_1^n)} \leq 2\|x\|_{\ell_p^n}.
\]
By the multiplicity property of entropy numbers, we have
\[ e_{k+m-1}(T: \psi((\ell^n_p, \ell^n_1) \to X) \leq e_k\left(\text{id}: \psi((\ell^n_p, \ell^n_1) \to \ell^n_1)\right) e_m(T: \ell^n_1 \to X) \]
for all integers \(k\) and \(m\). Therefore, in order to complete the proof, it is enough to show that there exists a constant \(C_\varphi\) which depends on \(\varphi\), such that
\[ e_k\left(\text{id}: \psi((\ell^n_p, \ell^n_1) \to \ell^n_1)\right) \leq C_\varphi \min\left\{1, \frac{\varphi^{-1}(f(n,k))}{f(n,k)}\right\}, \]
where \(f(n,k) = \frac{1}{k} \ln(1 + \frac{n}{k})\). In order to see this, note that for all integers \(k \leq n\) one has (see [11, Theorem 6])
\[ e_k(\text{id}: \ell^n_p \to \ell^n_1) \leq C_p \min\left\{1, \left(\frac{\ln(1 + \frac{n}{k})}{k}\right)^{\frac{1}{p}-1}\right\}, \]
where \(C_p \leq \left(\frac{C}{p} \ln(\frac{2}{p})\right)^{1/p}\) for an absolute constant \(c > 0\). Combining this estimate with Corollary 4.1, we obtain (by \(\varphi^{-1}(t) \asymp \psi(t^{1/p}, t)\)) the required estimate:
\[ e_k\left(\text{id}: \psi((\ell^n_p, \ell^n_1) \to \ell^n_1)\right) \leq C_1 \min\left\{1, \psi\left(\left(\frac{\ln(1 + \frac{n}{k})}{k}\right)^{\frac{1}{p}-1}, 1\right)\right\} \]
\[ \leq C_2 \min\left\{1, \frac{\varphi^{-1}(f(n,k))}{f(n,k)}\right\}, \]
where the constants \(C_1, C_2\) depend on \(\varphi\). □

The proof of the following lemma which is an extension of the corresponding result proved by Guédon and Litvak [11, Lemma 3] is standard.

**Lemma 5.1.** Let \(\varphi: [0, \infty) \to [0, \infty)\) be a normalized concave function such that \(0 < a_\varphi < 1\). There exists a positive constant \(C_\varphi\) such that for every set of points \(x_1, \ldots, x_n\) in the \(k\)-dimensional space \(\mathbb{R}^k\), the following estimate holds:
\[ \left(\frac{\text{vol} \left(\varphi\text{-absconv}\{x_1, \ldots, x_n\}\right)}{\text{vol} \left(\text{absconv}\{x_1, \ldots, x_n\}\right)}\right)^{1/k} \leq C_\varphi \min\left\{1, \frac{\varphi^{-1}(f(n,k))}{f(n,k)}\right\}, \]
where \(f(n,k) = \frac{1}{k} \ln(1 + \frac{n}{k})\).

**Proof.** Let \(T: \mathbb{R}^n \to \mathbb{R}^k\) be a linear operator defined by \(T(e_j) = x_j\) for \(j = 1, \ldots, n\). Let \(X := (\mathbb{R}^k, \| \cdot \|)\), where the unit ball of \(X\) is \(U_X := \text{absconv} \{x_1, \ldots, x_n\}\). Applying Theorem 5.1 with \(m = 1\), we have
\[ e_k(T: \ell^n_p \to X) \leq C_\varphi \min\left\{1, \frac{\varphi^{-1}(f(n,k))}{f(n,k)}\right\} \|T: \ell^n_1 \to X\| \]
\[ = C_\varphi \min\left\{1, \frac{\varphi^{-1}(f(n,k))}{f(n,k)}\right\}. \]
A volume argument easily shows that
\[ 2e_k(T: \ell^n_p \to X) \geq \left(\frac{\text{vol} \left(T(U_{\ell^n_p})\right)}{\text{vol} U_X}\right)^{1/k}. \]
Since \(\varphi\) is normalized \(T(U_{\ell^n_p}) = \varphi\text{-absconv}\{x_1, \ldots, x_n\}\) and the proof is complete. □
As an immediate consequence of Theorem 5.1 and the well-known estimate of the volume absconv \{x_1, \ldots, x_n\} (see, e.g., [4]), we obtain the following independently interesting corollary.

**Corollary 5.1.** For any normalized concave function \( \varphi : [0, \infty) \to [0, \infty) \) with \( 0 < a_\varphi < 1 \) there exists a positive constant \( C_\varphi \) such that for every set of points \( x_1, \ldots, x_n \) in the \( k \)-dimensional Euclidean ball \( \mathbb{R}^k \), we have

\[
\left( \frac{\text{vol} (\varphi\text{-absconv}\{x_1, \ldots, x_n\})}{\text{vol} U_{\ell^2_k}} \right)^{1/k} \leq C_\varphi \min \left\{ 1, \frac{\varphi^{-1}(f(n, k))}{\sqrt{f(n, k)}} \right\},
\]

where \( f(n, k) = \frac{1}{k} \ln(1 + \frac{n}{k}) \).

Applying Lemma 5.1, Corollary 5.1 and repeating the argument from the proof of Proposition 4 in [11], we obtain the following result. For the sake of completeness, we present its proof.

**Proposition 5.1.** For every normalized concave function \( \varphi \) with \( 0 < a_\varphi < 1 \), integers \( k = 1, \ldots, n \), and all projections \( P \) of rank \( k \), we have

\[
d(\mathcal{P} U_{\ell^\varphi^2}, U_{\ell^2_k}) \geq \frac{1}{C_\varphi} \max \left\{ 1, \frac{\sqrt{f(n, k)}}{\varphi^{-1}(f(n, k))} \right\},
\]

where \( C_\varphi \) is the same constant as in Lemma 5.1.

**Proof.** Let \( \mathcal{E} \) be an ellipsoid satisfying \( \mathcal{P} U_{\ell^\varphi^2} \subset \mathcal{E} \) and \( d \) be the best constant such that

\[
d^{-1} \mathcal{E} \subset \mathcal{P} U_{\ell^\varphi^2} \subset \mathcal{E}.
\]

Denote by \( v \) the isomorphism on \( \mathbb{R}^k \) such that \( v(\mathcal{E}) = U_{\ell^2_k} \) and define for all \( j = 1, \ldots, n \), \( x_j = v P e_j \). It is clear that \( x_j \in U_{\ell^2_k} \) for all \( j = 1, \ldots, n \) and

\[
\frac{1}{d} \leq \left( \frac{\text{vol} (v(\mathcal{P} U_{\ell^\varphi^2}))}{\text{vol} U_{\ell^2_k}} \right)^{1/k} = \left( \frac{\text{vol} (\mathcal{P} U_{\ell^\varphi^2})}{\text{vol} \mathcal{E}} \right)^{1/k}.
\]

Since \( v(\mathcal{P} U_{\ell^\varphi^2}) = \varphi\text{-absconv}\{x_1, \ldots, x_n\} \), it follows from Lemma 5.1 and Corollary 5.1 that

\[
d \geq \max \left\{ 1, \frac{\sqrt{f(n, k)}}{\varphi^{-1}(f(n, k))} \right\}.
\]

To complete the proof it is enough to choose an ellipsoid \( \mathcal{E} \) which realizes the distance from \( \mathcal{P} U_{\ell^\varphi^2} \) to the Euclidean ball. \( \square \)

**Acknowledgments**

The author would like to thank the referee for valuable suggestions, which helped to improve the presentation of the results.

The work was supported by Committee of Scientific Research, Poland, grant P03A 013 26.

**References**