Linear Algebra and its Applications 431 (2009) 1105–1115

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Linearly independent vertices and minimum semidefinite rank

Philip Hackney [a], Benjamin Harris [b,1], Margaret Lay [c,1], Lon H. Mitchell [d], Sivaram K. Narayan [e,*,1], Amanda Pascoe [f,1]

[a] *Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067, United States*
[b] *Department of Mathematics, Brown University, Providence, RI 02912, United States*
[c] *Department of Mathematics and Computer Science, Grinnell College, Grinnell, IA 50112-1690, United States*
[d] *Department of Mathematics, Virginia Commonwealth University, Richmond, VA 23284-2014, United States*
[e] *Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, United States*
[f] *Department of Mathematics, Furman University, Greenville, SC 29613-1148, United States*

A R T I C L E I N F O

Article history:
Received 13 November 2007
Accepted 15 March 2009
Available online 19 May 2009

Submitted by S. Fallat

AMS classification:
15A18
15A57
05C50

Keywords:
Minimum semidefinite rank
Join
Linearly independent vertices

A B S T R A C T

We study the minimum semidefinite rank of a graph using vector representations of the graph and of certain subgraphs. We present a sufficient condition for when the vectors corresponding to a set of vertices of a graph must be linearly independent in any vector representation of that graph, and conjecture that the resulting graph invariant is equal to minimum semidefinite rank. Rotation of vector representations by a unitary matrix allows us to find the minimum semidefinite rank of the join of two graphs. We also improve upon previous results concerning the effect on minimum semidefinite rank of the removal of a vertex.

* Corresponding author. Tel.: +1 989 774 3566; fax: +1 989 774 2414.
E-mail address: sivaram.narayan@cmich.edu (S.K. Narayan).
1 Research supported in part by NSF Grant DMS 02-43674.

0024-3795/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2009.03.030

1. Introduction

A *graph* G consists of a set of vertices V and a set of edges E, where the elements of E are unordered pairs of vertices. The *order* of G, denoted $|G|$, is the cardinality of V. A graph is *simple* if it has no multiple edges or loops. For Sections 1–3, we assume all graphs to be simple. In Section 4, graphs may have multiple edges but no loops.

The entries of an n-by-n Hermitian matrix $A = (a_{ij})$ over the complex numbers $\mathbb{C}$ naturally determine a graph $G(A)$ with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{\{v_i, v_j\}: a_{ij} \neq 0, i > j\}$. Calculating possible multiplicities of eigenvalues for Hermitian matrices based upon properties of their related graph has been of recent interest [10]. Also of recent interest is the minimum rank problem, which seeks to determine the smallest possible rank of any real symmetric matrix with given graph. For more information on minimum rank problems, see the survey by Fallat and Hogben [7]. In this paper, we consider the related problem of determining the minimum rank among positive semidefinite (henceforth *psd*) matrices with a given graph [2,4,5,9,18].

Given a graph G, let $\mathcal{P}(G)$ represent all psd matrices with graph G. Define the *minimum semidefinite rank* of G as

$$\operatorname{msr}(G) = \min\{\operatorname{rank} A: A \in \mathcal{P}(G)\}.$$

The study of msr was initiated by Barrett et al. [2] using the notation hmr_+.

In what follows, since the direct sum of matrices for connected components of a graph gives a matrix for the entire graph and this process is additive in rank, we assume all graphs are connected unless otherwise specified.

Remark 1.1. Since the Laplacian matrix of a connected graph G on n vertices, $L(G)$, is positive semidefinite and has rank $n - 1$ [15], we have that $\operatorname{msr}(G) \leqslant n - 1$ for all graphs G. We can also provide a positive definite matrix with graph G by taking $L(G) + I$, where I is the identity of $M_n(\mathbb{C})$.

Given a field $\mathbb{F}$, subsets S, A, B, and C of $\mathbb{F}$, a positive integer d, and a nondegenerate bilinear form $b(x, y)$ on $\mathbb{F}^d$, a *vector representation* [16] of a simple graph G with vertices $v_1, \dots, v_n$ is a list of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{F}^d$ whose components are in S such that for all i and j, $b(\vec{v}_i, \vec{v}_i) \in A$, if v_i is adjacent to v_j in G then $b(\vec{v}_i, \vec{v}_j) \in B$, and if v_i is not adjacent to v_j in G then $b(\vec{v}_i, \vec{v}_j) \in C$. For example, Lovász defines an *orthonormal representation* with $\mathbb{F} = \mathbb{R} = S = B, A = \{1\}$ and $C = \{0\}$ in his solution of the Shannon capacity of C_5 [14] and his characterization (with Saks and Schrijver) of k-connected graphs [11,12]. See the survey by Lovász and Vesztergombi [13] for further information.

Given a set of n column vectors in $\mathbb{C}^d, \vec{X} = \{\vec{x}_1, \dots, \vec{x}_n\}$, let X be the matrix $[\vec{x}_1 \cdots \vec{x}_n]$. Then X^*X is a psd matrix called the *Gram matrix* of $\vec{X}$ with regard to the Euclidean inner product. Its associated graph G has n vertices $v_1, \dots, v_n$ corresponding to the vectors $\vec{x}_1, \dots, \vec{x}_n$, and edges corresponding to nonzero inner products among those vectors. Since $X^*X \in \mathcal{P}(G)$ for the graph G, we say $\vec{X}$ is a *vector representation* of G (with $\mathbb{F} = \mathbb{C} = S = A, B = \mathbb{C} \setminus \{0\}$, and $C = \{0\}$). By rank $\vec{X}$, we mean the dimension of the span of the vectors in $\vec{X}$, which is equal to rank X^*X [8, Theorem 7.2.10].

In what follows, when a graph G and vertex v are specified, we will often use $\vec{v}$ to mean a vector representing the vertex v in a vector representation of G. However, we will also use $\vec{v}$ to stand for an arbitrary vector.

Since any psd matrix A may be factored as Y^*Y for some $Y \in M_n(\mathbb{C})$ with $\operatorname{rank} A = \operatorname{rank} Y$, each psd matrix is the Gram matrix of a suitable set of vectors. Therefore, finding a psd matrix with a given graph and finding a vector representation of the graph are equivalent problems.

Recall that the *neighborhood* of a vertex v of a graph G, denoted $N(v)$, is the set of vertices of G adjacent to v. The *closed neighborhood* of a vertex v, $N[v]$ is $\{v\} \cup N(v)$. We say a vertex v is a *duplicate* of a vertex w if $N[v] = N[w]$. Since duplicate vertices may be represented by the same vector, removing a duplicate vertex does not affect the minimum semidefinite rank [5]. In particular, the minimum semidefinite rank of a complete graph on two or more vertices is one.

Remark 1.2. If u and v are not duplicate vertices in a graph G, then $\vec{u} \notin \operatorname{span} \vec{v}$ for all vector representations of G.

2. Joins of graphs

The minimum semidefinite rank of an induced subgraph H of a graph G provides a lower bound for $\mathrm{msr}(G)$. As an example, let $\mathrm{ts}(G)$ be the *tree size* of G, the maximum number of vertices in an induced tree [6]. Since the msr of a tree on $n \geqslant 2$ vertices is $n-1$ [18], if G is a connected graph of order at least two then $\mathrm{msr}(G) \geqslant \mathrm{ts}(G) - 1$.

The *independence number* of G, $\mathrm{i}(G)$, which is the cardinality of the largest *independent* (pairwise disjoint) set of vertices of G, is also a lower bound for $\mathrm{msr}(G)$. It is known that the msr of a cycle on $n \geqslant 3$ vertices is $n-2$ [18], and thus $\mathrm{msr}(G) - \mathrm{i}(G)$ may be arbitrarily large. However, in other cases, such as complete bipartite graphs, the size of the largest independent set does give the minimum semidefinite rank [4].

Given an induced subgraph H of a graph G, one might ask whether, given a vector representation of H of rank $d = \mathrm{msr}(H)$ contained in $\mathbb{C}^d$, it is possible to complete that vector representation to a vector representation of all of G with vectors in $\mathbb{C}^d$. In the case of a complete bipartite graph, this is implied by the above mentioned msr result. We now give two other instances where such a construction can be accomplished, preceded by a lemma giving the actual construction.

Lemma 2.1. *Let $\vec{u}_1, \ldots, \vec{u}_n$ and $\vec{v}_1, \ldots, \vec{v}_m$ be nonzero vectors in $\mathbb{C}^d$ and let S be a (possibly trivial) subspace of $\mathbb{C}^d$ such that no u_i or v_j is contained in S. Then there exists a unitary operator U on $\mathbb{C}^d$ such that U fixes S and $\langle \vec{u}_i, U\vec{v}_j \rangle$ is nonzero for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.*

Proof. Let $S^\perp$ denote the orthogonal complement of S in $\mathbb{C}^d$ and set $\vec{x}_j = \mathrm{proj}_{S^\perp} \vec{u}_j$ and $\vec{y}_k = \mathrm{proj}_{S^\perp} \vec{v}_k$. By the assumptions, the $\vec{x}_j$ and $\vec{y}_k$ are all nonzero. Let $\vec{x}_j^{\perp}$ ($\vec{y}_k^{\perp}$) denote the subspace of $S^\perp$ orthogonal to $\vec{x}_j$ ($\vec{y}_k$), and define

$$R = \left(\bigcup_{j=1}^{n} \vec{x}_j^{\perp} \right) \cup \left(\bigcup_{k=1}^{m} \vec{y}_k^{\perp} \right).$$

Then R is the union of a finite number of hyperplanes (of $S^\perp$), and cannot cover all of $S^\perp$. Thus there exists a nonzero unit vector $\vec{w}$ in $S^\perp$ such that $\langle \vec{w}, \vec{x}_j \rangle$ and $\langle \vec{w}, \vec{y}_k \rangle$ are nonzero for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$. Write $\vec{x}_j = a_j \vec{w} + \vec{x}_j'$ and $\vec{y}_k = b_k \vec{w} + \vec{y}_k'$ where the vectors $\vec{x}_j'$ and $\vec{y}_k'$ are each orthogonal to $\vec{w}$. Extend $\vec{w}$ to a basis of $S^\perp$, and let U_θ be the unitary transformation of $S^\perp$ that has matrix

$$\begin{pmatrix} e^{i\theta} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

with respect to that basis. Let I_S be the identity operator on S. Then

$$\begin{aligned} \langle \vec{u}_j, (U_\theta \oplus I_S)\vec{v}_k \rangle &= \langle a_j \vec{w} + \vec{x}_j' + (\vec{u}_j - \vec{x}_j), e^{i\theta} b_k \vec{w} + \vec{y}_k' + (\vec{v}_k - \vec{y}_k) \rangle \\ &= e^{-i\theta} a_j \overline{b_k} + \langle \vec{u}_j', \vec{v}_k' \rangle + \langle \vec{u}_j - \vec{x}_j, \vec{v}_k - \vec{y}_k \rangle. \end{aligned}$$

By the choice of $\vec{w}$, both a_j and b_k are nonzero, and so if

$$\langle \vec{u}_j, (U_\theta \oplus I_S)\vec{v}_k \rangle = \langle \vec{u}_j, (U_{\theta'} \oplus I_S)\vec{v}_k \rangle$$

then $\theta \equiv \theta' \pmod{2\pi}$. Specifically, for fixed j and k, there is at most one value of θ in the interval $[0, 2\pi)$ for which $\langle \vec{u}_j, (U_\theta \oplus I_S)\vec{v}_k \rangle$ is zero. Since there are finitely many pairs (j, k), there exists a value of θ for which $\langle \vec{u}_j, (U_\theta \oplus I_S)\vec{v}_k \rangle$ is nonzero for every $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$. $\square$

Proposition 2.2. *Let G be a bipartite graph with independent sets X, Y such that $X \cup Y = V(G)$. Let $|X| = m \geqslant |Y| = n$, and suppose $|\bigcap_{v \in Y} N(v)| \geqslant n$. Then $\mathrm{msr}(G) = m$.*

Proof. Because X is an independent set of vertices, $\operatorname{msr}(G) \geqslant m$. To show that $\operatorname{msr}(G) \leqslant m$, we will exhibit a vector representation of G in $\mathbb{C}^m$. Let $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, and $\{x_1, \ldots, x_n\} \subseteq \bigcap_{v \in Y} N(v)$.

Using Remark 1.1, choose a vector representation

$$\vec{V} = \{\vec{y}_1, \ldots, \vec{y}_n, \vec{x}_{n+1}, \ldots, \vec{x}_m\}$$

in $\mathbb{C}^m$ of the subgraph of G induced by $\{y_1, \ldots, y_n, x_{n+1}, \ldots, x_m\}$ that has rank m. Let $S = \operatorname{span}\{\vec{x}_{n+1}, \ldots, \vec{x}_m\}$, and note that by our assumption on the rank of $\vec{V}$, none of the vectors $\vec{y}_1, \ldots, \vec{y}_n$ lie in S. Let $S^\perp$ denote the orthogonal complement of S in $\mathbb{C}^m$, and choose an orthonormal basis $\{\vec{z}_1, \ldots, \vec{z}_n\}$ of $S^\perp$. Let U be the unitary operator resulting from the application of Lemma 2.1 to the vectors $\vec{y}_i$, the vectors $\vec{z}_i$, and the subspace S. Because S is invariant under U, so is $S^\perp$, so that $\langle \vec{x}_i, U\vec{z}_j \rangle = 0$ for all $n+1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, and $\langle \vec{y}_i, U\vec{z}_j \rangle \neq 0$ for all $1 \leqslant i, j \leqslant n$ by Lemma 2.1. Thus

$$\{\vec{y}_1, \ldots, \vec{y}_n, \vec{x}_{n+1}, \ldots, \vec{x}_m, U\vec{z}_1, \ldots, U\vec{z}_n\}$$

represents G in $\mathbb{C}^m$ as desired. $\square$

Definition 2.3 [19]. We say that a graph G is the join of graphs G_1 and G_2, written $G = G_1 \vee G_2$, if

(1) $V(G)$ is the disjoint union of $V(G_1)$ and $V(G_2)$,
(2) if $v, w \in V(G_i)$ then $\{v, w\} \in E(G)$ if and only if $\{v, w\} \in E(G_i)$ for $i = 1, 2$, and
(3) if $v \in V(G_1)$ and $w \in V(G_2)$, then $\{v, w\} \in E(G)$.

Proposition 2.4. *Let G_1, G_2 be connected graphs on two or more vertices. Then* $\operatorname{msr}(G_1 \vee G_2) = \max\{\operatorname{msr}(G_1), \operatorname{msr}(G_2)\}$.

Proof. Without loss of generality, let $\operatorname{msr}(G_1) = n \geqslant \operatorname{msr}(G_2)$. Moreover, let $V(G_1) = \{v_1, v_2, \ldots, v_k\}$ and $V(G_2) = \{w_1, w_2, \ldots, w_l\}$. By assumption, there exist vector representations $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ of G_1 and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l$ of G_2 in $\mathbb{C}^n$. Since G_1 and G_2 have no isolated vertices, these vector representations contain no zero vectors. Let U be the unitary operator on $\mathbb{C}^n$ resulting from the application of Lemma 2.1 to the vectors v_i, the vectors w_j, and the trivial subspace. Then the vectors

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, U\vec{w}_1, U\vec{w}_2, \ldots, U\vec{w}_l\}$$

represent $G_1 \vee G_2$ in $\mathbb{C}^n$. Thus, $\operatorname{msr}(G_1 \vee G_2) \leqslant n$. Since G_1 is an induced subgraph of $G_1 \vee G_2$, $\operatorname{msr}(G_1 \vee G_2) \geqslant \operatorname{msr}(G_1)$. Taken together, we have the desired result. $\square$

A result similar to Proposition 2.4 for the real-symmetric minimum rank problem has been found by Barioli and Fallat [1].

From the definition of the associated graph $G(A)$ for a psd matrix A, vector representations of a graph may include a zero vector. Thus, isolated vertices do not increase the minimum semidefinite rank of a graph. The *minimum vector rank* (mvr) of a graph G is defined to be the minimum rank among vector representations of G that have no zero vectors. Notice that $\operatorname{mvr}(G)$ differs from $\operatorname{msr}(G)$ by exactly the number of isolated vertices of G, and that mvr (like msr) is additive on connected components of a graph.

Lemma 2.5 [9]. *If G is a connected graph and H is an induced subgraph of G, then* $\operatorname{msr}(G) \geqslant \operatorname{mvr}(H)$.

Proposition 2.6. *Let G_1, G_2 be graphs (possibly not connected). Then* $\operatorname{msr}(G_1 \vee G_2) = \max\{\operatorname{mvr}(G_1), \operatorname{mvr}(G_2)\}$.

Proof. Apply Lemma 2.1 as in the proof of Proposition 2.4 to vector representations of G_1 and G_2 that contain no zero vectors to get one direction, and Lemma 2.5 gives the reverse inequality. $\square$

Corollary 2.7. *For any graph G,* $\operatorname{mvr}(G) = \operatorname{msr}(G \vee K_1)$.

Corollary 2.8. *Let G be a complete multipartite graph with at least two nonempty partite sets. Then* $\operatorname{msr}(G) = \mathrm{i}(G)$.

Definition 2.9 [19]. Suppose G is decomposable into two graphs, G_1 and G_2, sharing only one vertex v such that if $u \in V(G_1)$ and $w \in V(G_2)$, then $\{u, w\} \in E(G)$ only if $u = v$ or $w = v$. Then G_1 and G_2 are joined at a *cut vertex*, and we write $G = G_1 \,.\, G_2$.

If $G = G_1 \,.\, G_2$, then $\mathrm{msr}(G) = \mathrm{msr}(G_1) + \mathrm{msr}(G_2)$ [4], which is also true when G is the disjoint union of subgraphs G_1 and G_2. Rephrased using complements, Proposition 2.6 states that if $\overline{G}$ is the disjoint union of $\overline{G_1}$ and $\overline{G_2}$, then $\mathrm{msr}(G)$ is equal to $\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\}$. By analogy, this suggests that if $\overline{G} = \overline{G_1} \,.\, \overline{G_2}$, then perhaps $\mathrm{msr}(G)$ is equal to $\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\}$. This is essentially correct, as the next proposition demonstrates.

Proposition 2.10. *Let $\overline{G} = \overline{G_1} \,.\, \overline{G_2}$ with v the cut vertex for $\overline{G_1}$ and $\overline{G_2}$. If v is an isolated vertex in G, then* $\mathrm{msr}(G)$ *is equal to* $\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} - 1$. *If v is not an isolated vertex in G, and not a duplicate vertex in G_1 and G_2,* $\mathrm{msr}(G)$ *is given by* $\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\}$. *Otherwise*,

$$\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} \leqslant \mathrm{msr}(G) \leqslant \max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} + 1.$$

Proof. Let $V(G_1) = \{v, u_1, \ldots, u_k\}$ and $V(G_2) = \{v, w_1, \ldots, w_l\}$. Note that because $\overline{G} = \overline{G_1} \,.\, \overline{G_2}$, each u_i is adjacent to each w_i in G. Thus, unless v is isolated in G, G is connected.

If v is an isolated vertex in G, then $G - v$ is the join of $G_1 - v$ and $G_2 - v$. Further, v must be an isolated vertex in both G_1 and G_2, so that $\mathrm{mvr}(G_i - v) = \mathrm{mvr}(G_i) - 1$ for both subgraphs. Using this and Proposition 2.6,

$$\begin{aligned}\mathrm{msr}(G) = \mathrm{msr}(G - v) &= \max\{\mathrm{mvr}(G_1 - v), \mathrm{mvr}(G_2 - v)\}\\ &= \max\{\mathrm{mvr}(G_1) - 1, \mathrm{mvr}(G_2) - 1\} = \max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} - 1.\end{aligned}$$

If v is not an isolated vertex in G, then G is connected, and Lemma 2.5 gives that

$$\max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} \leqslant \mathrm{msr}(G).$$

Let $n = \max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\}$. If v is not a duplicate vertex in G_1 and G_2, choose, without loss of generality, vector representations $\{\vec{v}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ of G_1 and $\{\vec{v}', \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ of G_2 in $\mathbb{C}^n$ with no zero vectors such that $\vec{v} = \vec{v}'$. By Remark 1.2, no $\vec{u}_i$ or $\vec{w}_j$ lies in the span of $\vec{v}$. Let U be the unitary operator resulting from the application of Lemma 2.1 to the vectors $\vec{u}_i$, the vectors $\vec{w}_j$, and the subspace $S = \mathrm{span}\{\vec{v}\}$. Then $\{\vec{v}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a vector representation of G_1, $\{U\vec{v}', U\vec{w}_1, U\vec{w}_2, \ldots, U\vec{w}_l\}$ is a vector representation of G_2, and

$$\{U\vec{v}' = \vec{v}' = \vec{v}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, U\vec{w}_1, U\vec{w}_2, \ldots, U\vec{w}_l\}$$

is a vector representation of G of rank n, showing $\mathrm{msr}(G) \leqslant \max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\}$.

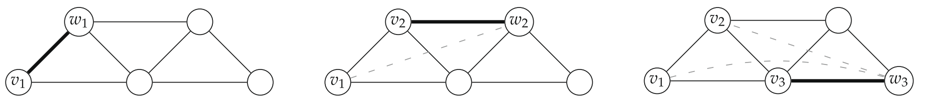
If v is a duplicate vertex in G_1 or G_2, choose, without loss of generality, vector representations $\{\vec{v}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ of G_1 and $\{\vec{v}', \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ of G_2 in $\mathbb{C}^n$ with no zero vectors such that $\vec{v} = \vec{v}'$. For each vector $\vec{z}$ in one of the vector representations, define new vector representations of G_1 and G_2 in $\mathbb{C}^{n+1}$ by setting $\vec{z}' = \vec{z} \oplus c$ if vertex z is a duplicate of v and $\vec{z}' = \vec{z} \oplus 0$ otherwise, where c is defined to be one more than the maximum absolute value taken over inner products of pairs of vectors in each representation (this ensures the result will still be representations of G_1 and G_2). Now, applying Lemma 2.1 as above will result in a vector representation of G of rank $n + 1$, showing $\mathrm{msr}(G) \leqslant \max\{\mathrm{mvr}(G_1), \mathrm{mvr}(G_2)\} + 1$. $\square$

3. Linearly independent vertices

We say that vertices $v_1, \ldots, v_n$ of a graph G are *linearly independent* if in any vector representation $\vec{V}$ of G, $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent vectors. In this section, we present a method for identifying a set of vertices of a graph G whose representing vectors must be linearly independent in any vector representation of G.

Definition 3.1. Let G be a connected graph and let $S = \{v_1, \ldots, v_m\}$ be an ordered set of vertices of G. Denote by G_k the subgraph induced by $v_1, v_2, \ldots, v_k$ for each k, $1 \leqslant k \leqslant m$. Let H_k be the connected component of G_k such that $v_k \in V(H_k)$. If for each k, there exists $w_k \in V(G)$ such that $w_k \neq v_l$ for $l \leqslant k$, $\{w_k, v_k\} \in E(G)$, and $\{w_k, v_l\} \notin E(G)$ for all $v_l \in V(H_k)$ with $l \neq k$, then S is called a *vertex set of ordered subgraphs* (or *OS-vertex set*). The *OS-number of a graph* G, denoted $\mathrm{OS}(G)$, is the maximum cardinality among all OS-vertex sets of G.

Example 3.2. Given below is an example of the construction of an OS-set, with each v_i and w_i shown, and dashed lines indicating non-adjacency showing that each w_i satisfies the definition. Inspection will show the constructed OS-set is maximal.



Proposition 3.3. *Let* $G = (V, E)$ *be a connected graph and let* S *be an OS-vertex set in* G. *Then* $|S| \leqslant \mathrm{msr}(G)$. *In particular*, $\mathrm{msr}(G) \geqslant \mathrm{OS}(G)$.

Proof. Let $S = \{v_1, \ldots, v_m\}$ be an OS-vertex set in G. We prove, by induction on $|S|$, that $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a linearly independent set in any vector representation $\vec{V}$ of G.

If $|S| = 1$, then $\{\vec{v}_1\}$ is linearly independent since $\vec{v}_1$ is nonzero. Assume that the result is true for $|S| < k \leqslant m$. Suppose $|S| = k$, G_k is the subgraph induced by $v_1, v_2, \ldots, v_k$ and H_k is the connected component of G_k containing v_k. By the induction hypothesis, $\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ form a linearly independent set. Suppose $\vec{v}_k = \sum_{i=1}^{k-1} c_i \vec{v}_i$. Let $\{\vec{v}_{n_1}, \ldots, \vec{v}_{n_r}\} \subseteq \{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ be the vectors corresponding to the vertices of $H_k - v_k$. If $\{\vec{v}_{l_1}, \ldots, \vec{v}_{l_s}\} \subseteq \{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ are the vectors corresponding to the vertices in any other component of G_k, then

$$0 = \left\langle \vec{v}_k, \sum_{i=1}^{s} c_{l_i} \vec{v}_{l_i} \right\rangle = \left\| \sum c_{l_i} \vec{v}_{l_i} \right\|^2 .$$

By the induction hypothesis this implies $c_{l_i} = \cdots = c_{l_s} = 0$. Therefore $\vec{v}_k = \sum_{i=1}^{r} c_{n_i} \vec{v}_{n_i}$. Then $\langle \vec{v}_k, \vec{w}_k \rangle = \sum_{i=1}^{r} c_{n_i} \langle \vec{v}_{n_i}, \vec{w}_k \rangle = 0$. This contradicts the assumption that $\{v_k, w_k\} \in E(G)$. Hence $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent and $\mathrm{msr}(G) \geqslant \mathrm{OS}(G)$. $\square$

The sum of two positive semi-definite matrices is positive semi-definite and the rank of a sum is never more than the sum of the ranks [8, p. 13]. If we cover all edges of a graph G with (not necessarily induced) subgraphs of known msr, this can lead to useful upper bounds for $\mathrm{msr}(G)$. First, suppose that G is labeled and that $G_1, \ldots, G_k$ are (labeled) subgraphs of G, that is, each G_i, $i = 1, \ldots, k$ is the result of deleting some edges and/or vertices from G. We say that $G_1, \ldots, G_k$ *cover* G if each edge (vertex) of G is an edge (vertex) of at least one G_i, $1 \leqslant i \leqslant k$. The cover $G_1, \ldots, G_k$ of G is called a *clique cover* of G if each of $G_1, \ldots, G_k$ is a clique of G. The *clique cover number* of G, $\mathrm{cc}(G)$, is the minimum value of k for which there is a clique cover $G_1, \ldots, G_k$ of G [17].

Remark 3.4. If $C_1, C_2, \ldots, C_k$ is a clique cover of a graph G, then $\mathrm{msr}(G) \leqslant k$. In particular, $\mathrm{msr}(G) \leqslant \mathrm{cc}(G)$.

Recall that a graph is *chordal* if it does not have an induced subgraph that is a cycle on four or more vertices, and a vertex is *simplicial* if its neighborhood is a clique. It is well known that every chordal graph has a simplicial vertex [3, p. 175].

Algorithm 3.5. Let G be a simple chordal graph. Define $E_0 = F_0 = G_0 = G$. For the kth step, $k \geqslant 1$,

(1) Select v_k to be a simplicial vertex in G_{k-1}, w_k to be any neighbor of v_k in G_{k-1}, and let C_k be the maximal clique in G containing the closed neighborhood of v_k in G_{k-1}.
(2) Define $E_k = G_{k-1} - v_k$.

(3) Define F_k to be the subgraph of E_k obtained by deleting all of the edges covered by any of the cliques $C_1, \ldots, C_k$.
(4) Define G_k to be the induced subgraph of E_k obtained by removing any vertices of E_k that are isolated in F_k.

The algorithm terminates at a value l when G_l is empty.

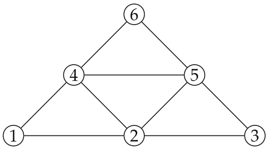
Proposition 3.6. *Let G be a connected, chordal graph. Algorithm 3.5 constructs an OS-vertex set $S \subseteq V(G)$ such that $|S| = \mathrm{OS}(G) = \mathrm{cc}(G)$ and a corresponding minimal clique cover of G.*

Proof. We first note that at each step in Algorithm 3.5, v_k may be chosen to be a simplicial vertex since each G_k is an induced subgraph of the original chordal graph G and hence is also chordal. We now show that the resulting $v_1, \ldots, v_l$ comprise an OS-set in G, and will do so by showing that $v_1, \ldots, v_k$ comprise an OS-set in G at each step k.

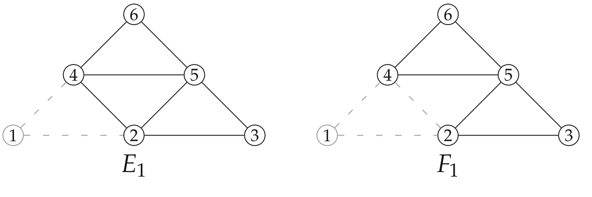
Assume that the vertices $v_1, \ldots, v_{k-1}$ form an OS-set in G. By construction, v_k is not isolated in F_{k-1}. Hence, there exists a vertex w_k in F_{k-1} such that the edge (w_k, v_k) is also in F_{k-1}. Let H_k be the connected component of the subgraph of G induced by $\{v_1, \ldots, v_k\}$ that contains v_k, and label the vertices of H_k as $\{v_{n_1}, \ldots, v_{n_r}, v_k\}$. We must then show that $\{w_k, v_{n_i}\} \notin E(G)$ for $1 \leqslant i \leqslant r$. Let $n_i < k, 1 \leqslant i \leqslant r$ be an index such that $\{w_k, v_{n_i}\} \in E(G)$. If v_{n_i} and v_k are adjacent in G, then the edge $\{w_k, v_k\}$ would be part of the maximal clique C_{n_i}. However, by construction, this would force w_k and v_k to not be adjacent in F_{k-1}, a contradiction. Therefore, if $\{w_k, v_{n_i}\} \in E(G)$ then $\{v_{n_i}, v_k\} \notin E(G)$, and, for any n_i, since v_{n_i} and v_k belong to H_k, there is a shortest path of length at least two in H_k joining v_{n_i} to v_k. Let $n_j < k, 1 \leqslant j \leqslant r$ be an index such v_{n_j} is adjacent to w_k and the distance between v_{n_j} and v_k in H_k is minimal among all vertices v_{n_i} adjacent to w_k. Then no vertex on the shortest path in H_k from v_{n_j} to v_k is adjacent to w_k, and appending the edges $\{v_k, w_k\}$ and $\{w_k, v_{n_j}\}$ to this path produces an induced cycle $C_n, n \geqslant 4$. By construction, this cycle has no chords. This contradicts that G is a chordal graph. Thus $(w_k, v_{n_i}) \notin E(G)$ for all $1 \leqslant i \leqslant r$ and $S = \{v_1, \ldots, v_k\}$ is an OS-vertex set in G.

The only way that a vertex v belongs to G but not to G_k for some fixed k is if all of the edges incident to v are covered by the cliques $C_1, \ldots, C_k$. Thus this construction of the OS-vertex set S produces a clique cover $\mathcal{C}$ of G. Using Remark 3.4 and Proposition 3.3, we see that $|S| \leqslant \mathrm{msr}(G) \leqslant |\mathcal{C}|$. Since $\mathcal{C}$ has the same cardinality as $S, \mathcal{C}$ is a minimal clique cover, S is an OS-set of maximum cardinality, and $\mathrm{OS}(G) = |S| = |\mathcal{C}| = \mathrm{cc}(G)$. $\square$

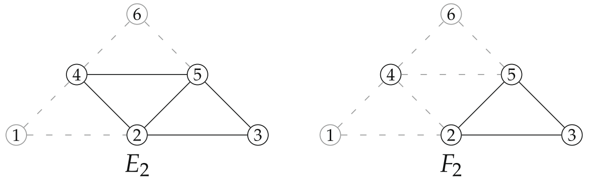
Example 3.7. Consider the chordal graph below.



Using Algorithm 3.5, v_1 may be chosen from among vertices 1, 3, and 6, and w_1 may be any neighbor. Choose vertex 1 to be v_1 and vertex 4 (or 2) as w_1. It follows that vertices 1, 2, and 4 comprise C_1. Removing vertex 1 yields the following graphs:



Since no vertices are isolated in F_1, $G_1 = E_1$. Only vertices 6 and 3 are simplicial in G_1, so choose vertex 6 as v_2 and vertex 4 (or 5) as w_2. Then vertices 4, 5, and 6 comprise C_2. This gives the following graphs:



Because vertex 4 is isolated in F_2, G_2 will be the subgraph of E_2 induced by vertices 2, 3, and 5. Finally, v_3 can be any of the vertices 2, 3, or 5, and w_3 can be either of the two left from the choice of v_3. Then $C_3 = G_2$, G_3 is empty, and the algorithm ends.

Using Proposition 3.6 along with Remark 3.4, we get that

Corollary 3.8 [5]. *If G is a connected, chordal graph, then* $\mathrm{msr}(G) = \mathrm{OS}(G) = \mathrm{cc}(G)$.

Suppose G' is an induced subgraph of G and S is an OS-vertex set in G'. Since each of the G_k in Definition 3.1 are induced subgraphs of G', they are also induced subgraphs of G. Hence S is also an OS-vertex set of G. This, combined with the following result, gives that there is an OS-vertex set of size $\mathrm{ts}(G) - 1$ in any graph G.

Corollary 3.9. *Given a tree T, for each $v \in V(T)$, $V(T) \setminus \{v\}$ is an OS-vertex set.*

For an induced forest of G with components $T_1, T_2, \ldots, T_k$, take the sum of the tree size of each T_i and subtract off the number of components that are not isolated vertices. Among all induced forests of G maximize this count and call this number $\mathrm{fm}(G)$, the *forest measure* of G [5]. Any isolated vertices occurring in an induced subgraph of a connected graph G contribute 1, rather than 0, to $\mathrm{msr}(G)$, as an irreducible positive semidefinite matrix has positive diagonal entries.

Proposition 3.10. *For a connected graph G, $\mathrm{OS}(G) \geqslant \mathrm{fm}(G)$.*

Proof. Let F be an induced forest of G. Each tree T_i that is not a single vertex of F has a nonempty OS-vertex set S_i of cardinality $|V(T_i)| - 1$ by Corollary 3.9. Because no vertex of S_i is adjacent to a vertex of S_j for $i \neq j$, $S = \bigcup S_i$ is an OS-vertex set of F. Let $v_1, \ldots, v_j$ be the set of isolated vertices of the induced forest F. Since G is connected, each v_i is adjacent to some vertex in G which is not in S because $S \subseteq V(F)$. Therefore $S' = S \cup \{v_1, \ldots, v_j\}$ is an OS-vertex set of G. If F is an induced forest realizing $\mathrm{fm}(G)$, then $\mathrm{OS}(G) \geqslant \mathrm{fm}(G)$. $\square$

We end this section with the following conjecture:

Conjecture 3.11. *For any graph G, $\mathrm{OS}(G) = \mathrm{msr}(G)$.*

4. Orthogonal vertex removal

In this section, we consider a generalization of the minimum semidefinite rank problem by van der Holst [18]. Given an undirected graph $G = (V, E)$ on n vertices that has no loops, but may have multiple edges, denote by $H(G)$ the set of all n by n Hermitian matrices $A = [a_{ij}]$ such that

- $a_{i,j} \neq 0$ if i and j are joined by exactly one edge, and
- $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent.

We say $\vec{V} = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{C}^m$ is a *vector representation* of G when $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$ if i and j are joined by a single edge, and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if i and j are not adjacent and $i \neq j$. By the complement $\overline{G}$ of a multigraph G, we will mean the simple graph on the same vertex set where two vertices are adjacent if and only if they are not adjacent in G.

Given a vector representation $\vec{V}$ of a graph G, for fixed i, we may "orthogonally remove" the vector $\vec{v}$ by orthogonally projecting onto the complement of the span of $\vec{v}$. That is, replace each $\vec{v}_j$ with

$$\vec{v}_j - \frac{\langle \vec{v}, \vec{v}_j \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

to get a vector representation $\vec{V} \ominus \vec{v}$ of a graph G' with rank and order decreased by one (this process corresponds to taking a Schur complement in the Gram matrix). The graph G' may be obtained from G by altering edges of the subgraph of G induced by $N(v)$ in the following manner: for $u, w \in N(v)$, if v is connected to u and w by a single edge, and u is not connected to w in G, then u is connected to w by a single edge in G'. In any other case, u and w may or may not be connected in G'.

To reflect this situation, define $G \ominus v$ as follows: in the induced subgraph $G - v$ of G, between any $u, w \in N(v)$ add $e - 1$ edges, where e is the sum of the number of edges between u and v and the number of edges between w and v. This process ensures that if $\vec{V}$ is a vector representation of a graph G, $\vec{V} \ominus \vec{v}$ is a vector representation of $G \ominus v$, and proves $\operatorname{msr}(G) \geqslant \operatorname{msr}(G \ominus v) + 1$. Unfortunately, $\operatorname{msr}(G) - \operatorname{msr}(G \ominus v)$ may be arbitrarily large [5], and seems difficult to calculate.

We are particularly interested, then, in determining conditions on the vertex v that allow us to calculate $\operatorname{msr}(G) - \operatorname{msr}(G \ominus v)$. Some success has already been recorded in this direction. The case where the vertex v to be removed has degree two with connected neighbors has been used to characterize trees in terms of their minimum semidefinite rank [18, Lemma 3.7]. This was expanded to

Theorem 4.1 [5]. *Suppose v is a simplicial vertex of a graph G that is adjacent to at least one neighbor by exactly one edge. Then* $\operatorname{msr}(G) = \operatorname{msr}(G \ominus v) + 1$.

Also, it was shown that $\operatorname{msr}(G) = \operatorname{msr}(G \ominus v) + 1$ for v a vertex of a simple graph G where the subgraph induced by $N(v)$ is either complete or lacks one or two edges [4]. In what follows, we expand upon this result.

A *star* is a tree that has one vertex adjacent to all of the other vertices [19]. A *star forest* is a forest of stars.

Proposition 4.2. *Let G be a connected graph, let v be a vertex of G not adjacent to any of its neighbors by multiple edges, and let H be the graph induced by the vertices of $N(v)$. If $\overline{H}$ is a star forest then* $\operatorname{msr}(G) = \operatorname{msr}(G \ominus v) + 1$.

Proof. First, $\operatorname{msr}(G) \geqslant \operatorname{msr}(G \ominus v) + 1$. To show the reverse inequality, assume that $\vec{V}$ is a vector representation of $G \ominus v$ of rank m in $\mathbb{C}^m$. We will construct from $\vec{V}$ a vector representation of G with rank $m + 1$. First, view the vectors of $\vec{V}$ as vectors in $\mathbb{C}^{m+1}$ orthogonal to some unit vector $\vec{e}$. Now, consider one of the s stars that comprise $\overline{H}$. Let w be the vertex at the center represented by vector $\vec{w}$ in $\vec{V}$, and let $w_1, \ldots, w_k$ be the pendant vertices of the star represented by vectors $\vec{w}_1, \ldots, \vec{w}_k$. Observe that for each $c \in \mathbb{C} \setminus \{0\}$ and each vertex w_i, because v is not adjacent to any of its neighbors by multiple edges, $\langle \vec{w}, \vec{w}_i \rangle$ is nonzero in $\vec{V}$, there exists a unique nonzero complex number a_i such that

$$\langle \vec{w} + c\vec{e}, \vec{w}_i + a_i\vec{e} \rangle = 0.$$

Further, although $\langle \vec{w}_i, \vec{w}_j \rangle$ may be nonzero in $\vec{V}$, for all but finitely many such c, we have

$$\langle \vec{w}_i + a_i\vec{e}, \vec{w}_j + a_j\vec{e} \rangle \neq 0$$

for all i and j. Suppose that $v_1, \ldots, v_r$ are those vertices of G adjacent to v that are isolated in $\overline{H}$. Consider also replacing each vector $\vec{v}_i$ of $\vec{V}$ that represents vertex v_i by $\vec{v}_i + b_i\vec{e}$. To achieve our aim, we must select complex numbers $c_1, \ldots, c_s$ and $b_1, \ldots, b_r$ so that in the vector representation resulting from the replacements described above we avoid

- some a_i or b_i is zero, and
- there exist two orthogonal vectors that represent neighbors of v not belonging to the same star in $\overline{H}$.

However, each of these conditions is satisfied by at most finitely many sets of c_i and b_i. Choosing, then $c_1, \ldots, c_s$ and $b_1, \ldots, b_r$ so that neither condition holds, and letting $\vec{e}$ represent v, we get a new vector representation $\vec{U}$ of some graph G' on the same vertices as G.

Recall that the vectors in $\vec{U}$ representing vertices not adjacent to v in G are orthogonal to $\vec{e}$. Thus, since the a_i and b_i are nonzero, v is adjacent in G' only to its neighbors in G. Also, since $G \ominus v$ has the same edges of G except between neighbors of v, so does G'. Because there do not exist two orthogonal vectors in $\vec{U}$ that represent neighbors of v not belonging to the same star, and because we forced the a_i to depend upon the choice of c in such a way as any two neighbors of the same star are represented by orthogonal vectors in $\vec{U}$, G' and G either both have a single edge or both have multiple edges between any two neighbors of v. Thus $\mathcal{P}(G') = \mathcal{P}(G)$, and $\vec{U}$ is a vector representation of G with rank $m + 1$. $\square$

Proposition 4.3. *Let G be a connected graph with a vertex v that is not adjacent to any of its neighbors by multiple edges, and let H be the graph induced by the vertices of $N(v)$. Suppose there exist m star forests $F_1, \ldots, F_m$ that are subgraphs of $\overline{H}$ and together cover all of the edges of $\overline{H}$. If, for each i, F_i is an induced subgraph of the graph with vertex set $V(\overline{H})$ and edge set $E(\overline{H}) \setminus (\cup_{j<i} E(F_j))$, then* $\operatorname{msr}(G) \leqslant \operatorname{msr}(G \ominus v) + m$.

Proof. Denote v by v_m, and consider the graph G_0 obtained by adding vertices $v_1, \ldots, v_{m-1}$ to G so that v_i is joined by single edges to the vertices of F_i. Consider the graphs

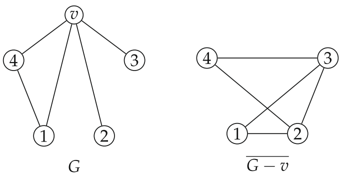
$$G_i = ((\ldots((G_0 \ominus v_1) \ominus v_2)\ldots) \ominus v_i).$$

From the conditions on the F_i, the complement of the subgraph of H induced by the vertices of $N(v_i)$ in G_{i-1} is a single star forest. Therefore, we may apply Proposition 4.2 repeatedly to the vertices $v_1, \ldots, v_m$, to see that $\operatorname{msr}(G_m) = \operatorname{msr}(G_0) - m$. Further, by construction, G_m is a supergraph of $G \ominus v$ on the same vertices which may be obtained from $G \ominus v$ by the possible addition of edges where edges already exist in $G \ominus v$. Therefore, $\operatorname{msr}(G_m) \leqslant \operatorname{msr}(G \ominus v)$. Finally, $\operatorname{msr}(G_0) \geqslant \operatorname{msr}(G)$, so that

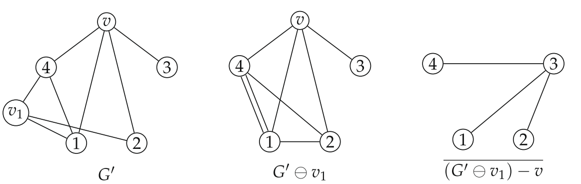
$$\operatorname{msr}(G \ominus v) \geqslant \operatorname{msr}(G_m) = \operatorname{msr}(G_0) - m \geqslant \operatorname{msr}(G) - m,$$

establishing the desired result. $\square$

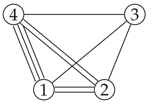
Example 4.4. Consider the graph G with vertex v and the complement of $N(v)$ below.



The subgraph of $\overline{G - v}$ induced by vertices 1, 2, and 4 is a star, so we add a vertex v_1 to G adjacent to those three vertices to get a new graph G', and by Proposition 4.2, $\operatorname{msr}(G' \ominus v_1) = \operatorname{msr}(G') - 1$.



Since $\overline{(G' \ominus v_1) - v}$ is a star, let $G'' = G' \ominus v_1$, and apply Proposition 4.2 to see that $\text{msr}(G'' \ominus v) = \text{msr}(G'') - 1$.



$G'' \ominus v$

By inspection, $\text{msr}(G'' \ominus v) = 1$, so that $\text{msr}(G) \leqslant 3$ by Proposition 4.3.

Acknowledgments

The authors wish to thank the referees for their careful reading and helpful comments and suggestions, including finding an additional case to consider in Proposition 2.10.

References

[1] Francesco Barioli, Shaun Fallat, On the minimum rank of the join of graphs and decomposable graphs, Linear Algebra Appl. 421 (2–3) (2007) 252–263.
[2] Wayne Barrett, Hein van der Holst, Raphael Loewy, Graphs whose minimal rank is two, Electron. J. Linear Algebra 11 (2004) 258–280. (electronic)
[3] Béla Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, vol. 184 , Springer-Verlag, New York, 1998.
[4] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Terry Lenker, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, Brian D. Sutton, On the minimum semidefinite rank of a simple graph, Linear and Multilinear Algebra, submitted for publication.
[5] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, Kelly Steinmetz, Brian D. Sutton, Wendy Wang, On the minimum rank among positive semidefinite matrices with a given graph, SIAM J. Matrix Anal. Appl. 30 (2) (2008) 731–740.
[6] Paul Erdös, Michael Saks, Vera T. Sós, Maximum induced trees in graphs, J. Combin. Theory Ser. B 41 (1) (1986) 61–79.
[7] Shaun M. Fallat, Leslie Hogben, The minimum rank of symmetric matrices described by a graph: a survey, Linear Algebra Appl. 426 (2–3) (2007) 558–582.
[8] Roger A. Horn, Charles R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1990, Corrected reprint of the 1985 original.
[9] Yunjiang Jiang, Lon H. Mitchell, Sivaram K. Narayan, Unitary matrix digraphs and minimum semidefinite rank, Linear Algebra Appl. 428 (7) (2008) 1685–1695.
[10] Charles R. Johnson, António Leal Duarte, On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree, Linear Algebra Appl. 348 (2002) 7–21.
[11] L. Lovász, M. Saks, A. Schrijver, Orthogonal representations and connectivity of graphs, Linear Algebra Appl. 114/115 (1989) 439–454.
[12] L. Lovász, M. Saks, A. Schrijver, A correction: "Orthogonal representations and connectivity of graphs" [Linear Algebra Appl. 114/115 (1989) 439–454; MR0986889 (90k:05095)], Linear Algebra Appl. 313 (1–3) (2000) 101–105.
[13] L. Lovász, K. Vesztergombi, Geometric representations of graphs, in: Paul Erdös and his mathematics, II (Budapest, 1999), Bolyai Soc. Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 471–498.
[14] László Lovász, On the Shannon capacity of a graph, IEEE Trans. Inform. Theory 25 (1) (1979) 1–7.
[15] Russell Merris, A survey of graph Laplacians, Linear and Multilinear Algebra 39 (1–2) (1995) 19–31.
[16] T.D. Parsons, Tomaž Pisanski, Vector representations of graphs, Discrete Math. 78 (1–2) (1989) 143–154.
[17] Norman J. Pullman, Clique coverings of graphs—a survey, in: Combinatorial Mathematics, X (Adelaide, 1982), Lecture Notes in Math., vol. 1036, Springer, Berlin, 1983, pp. 72–85.
[18] Hein van der Holst, Graphs whose positive semi-definite matrices have nullity at most two, Linear Algebra Appl. 375 (2003) 1–11.
[19] Douglas B. West, Introduction to Graph Theory, Prentice-Hall, Inc., Upper Saddle River, NJ, 1996.