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ScienceDirect
Appl. Comput. Harmon. Anal. 22 (2007) 198–216
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Multivariate orthonormal interpolating scaling vectors

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Received 22 August 2005; revised 22 March 2006; accepted 22 June 2006

Available online 24 July 2006

Communicated by Qingtang Jiang

Abstract

In this paper we introduce an algorithm for the construction of interpolating scaling vectors on $\mathbb{R}^d$ with compact support and orthonormal integer translates. Our method is substantiated by constructing several examples of bivariate scaling vectors for quincunx and box–spline dilation matrices. As the main ingredients of our recipe we derive some implementable conditions for accuracy and orthonormality of an interpolating scaling vector in terms of its mask.

Keywords: Interpolating scaling vector; Multiwavelet; Orthogonal bases; Expanding scaling matrices

1. Introduction

In recent years, wavelet analysis has become a very powerful tool in applied mathematics. Wavelet algorithms have been successfully applied in image/signal analysis/compression as well as in numerical analysis, geophysics, meteorology, and in many other fields. Usually, wavelets are constructed by means of a *scaling function*, i.e., a function $\varphi \in L_2(\mathbb{R}^d)$ that satisfies the *refinement equation*

$$\varphi(x) = \sum_{\beta\in\mathbb{Z}^d} a_\beta \varphi(Mx-\beta), \quad (a_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_2\big(\mathbb{Z}^d\big), \tag{1}$$

where M is an expanding integer scaling matrix.

For practical reasons, it is often convenient to use *interpolating* scaling functions, i.e., φ is refinable, at least continuous, and satisfies

$$\varphi(\beta) = \delta_{0,\beta} \quad \text{for all } \beta \in \mathbb{Z}^d. \tag{2}$$

In the past, several examples of interpolating scaling functions with various additional properties have been constructed, see, e.g., [9,10,12,13,18,19,25,29] and references therein. Since the number of wavelets corresponding to a given scaling function is determined by $|\det(M)|$, one is interested in scaling matrices with a small determinant. On

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1 This work has been supported by Deutsche Forschungsgemeinschaft, Grant Da 360/4-2.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2006.06.002

the other hand, scaling functions with orthonormal integer translates give rise to orthonormal wavelet bases. Therefore, orthonormality is also a very desirable property. However, this classical wavelet setting is somewhat restricted. One can show that for the case $|\det(M)| = 2$ a compactly supported interpolating scaling function with orthonormal integer translates does not exist.

One promising approach to bypass this lack of flexibility is to use scaling vectors and multiwavelets. These appear as a natural generalization of the classical wavelet setting. For the univariate case, the concept of interpolating or *cardinal* scaling vectors was introduced by Selesnick [30]. There, some univariate examples of interpolating scaling vectors for dyadic scaling are constructed. In [26], the univariate case is revisited and put on a sound mathematical foundation. Another univariate approach which focuses on biorthogonal systems has been derived in [11]. First results on multivariate scaling vectors which satisfy a Hermite interpolation condition have been obtained in [20] for several scaling matrices. In [6,8] new concepts of interpolatory vector subdivision schemes have been introduced which are closely related to interpolating scaling vectors. For recent developments in the field of multivariate multiwavelets and scaling vectors we refer to the survey articles [3,4] and references therein.

This paper appears as the direct generalization of the work done in [26] to the multivariate setting. Our main result is a universal algorithm for the construction of multivariate interpolating scaling vectors for dilation matrices with $|\det(M)| = 2$. In addition, the scaling vectors obtained by this method possess compact support and orthonormal integer translates simultaneously. Furthermore, we will show that these scaling vectors are very desirable for application purposes, since they are automatically balanced, cf. Section 3.3, and the corresponding multiwavelets are very easy to derive, see Section 5 for details.

This paper is organized as follows. In Section 2, we introduce the general setting and establish an interpolation property for the multivariate case which appears as a straightforward generalization of the univariate condition in [30]. Section 3 is devoted to setting up the main ingredients for our construction method. These consist of implementable conditions for orthonormal integer translates and for the approximation order provided by an interpolating scaling vector in terms of its mask or symbol. Furthermore, we will show in Section 3.3 that interpolating scaling vectors are automatically balanced. Then, in Section 4, we will state our main result and construct two families of bivariate interpolating scaling vectors for the quincunx and a box–spline matrix, respectively. Finally, in Section 5, we show that multiwavelets corresponding to our interpolating scaling vectors can be obtained in a very natural and effortless way.

2. General setting

In this section, we introduce the basic setting and state some auxiliary results. An *r-scaling vector* $\Phi := (\phi_0, \ldots, \phi_{r-1})^\top$, $r > 0$, is a vector of $L_2(\mathbb{R}^d)$-functions which satisfies a *matrix refinement equation*

$$\Phi(x) = \sum_{\beta\in\mathbb{Z}^d} A_\beta \Phi(Mx - \beta) \tag{3}$$

with the *mask* $(A_\beta)_{\beta\in\mathbb{Z}^d}$ of real $r \times r$ matrices and a *scaling matrix* $M \in \mathbb{Z}^{d\times d}$. M has to be *expanding*, i.e., all eigenvalues of M have to be larger than one in modulus. Throughout this paper, we will use the notation $m := |\det(M)|$. Applying the Fourier transform component-wise to (3) yields

$$\hat{\Phi}(\omega) = \frac{1}{m}\mathbf{A}\big(e^{-iM^{-\top}\omega}\big)\hat{\Phi}\big(M^{-\top}\omega\big), \quad \omega \in \mathbb{R}^d, \tag{4}$$

where $e^{-i\omega}$ is a shorthand notation for $(e^{-i\omega_1}, \ldots, e^{-i\omega_d})^\top$. The *symbol* $\mathbf{A}(z)$ is the matrix valued Laurent series

$$\mathbf{A}(z) := \sum_{\beta\in\mathbb{Z}^d} A_\beta z^\beta, \quad z \in \mathbb{T}^d,$$

and $\mathbb{T}^d := \{z \in \mathbb{C}^d\colon\ |z_i| = 1,\ i = 1, \ldots, d\}$ denotes the d-dimensional torus. All elements of $\mathbb{T}^d$ have the form $z = e^{-i\omega}$, $\omega \in \mathbb{R}^d$, thus we have $z^\beta = e^{-i\langle\omega,\beta\rangle}$, and for $\xi \in \mathbb{R}^d$ we use the notation $z_\xi := e^{-i(\omega+2\pi\xi)}$.

The aim of this paper is the construction of families of *interpolating m-scaling vectors* Φ with compact support, i.e., all components of Φ are at least continuous and satisfy

$$\phi_n\big(M^{-1}\beta\big) = \delta_{\rho_n,\beta} \quad \text{for all } \beta \in \mathbb{Z}^d,\ 0 \leqslant n < m, \tag{5}$$

where $R := \{\rho_0, \dots, \rho_{m-1}\}$ denotes a complete set of representatives of $\mathbb{Z}^d/M\mathbb{Z}^d$. Furthermore, we will focus on interpolating scaling vectors with compact support. Note that, in contrast to the scalar case, the interpolation condition (and the length of the scaling vector) is determined by the determinant of the scaling matrix. Nevertheless, it can be considered as a natural generalization of the scalar interpolation condition (2). Let φ be an interpolating scaling function, then the rule

$$\Phi(x) := \big(\varphi(Mx-\rho_0), \dots, \varphi(Mx-\rho_{m-1})\big)^\top \tag{6}$$

defines an interpolating m-scaling vector.

One advantage of interpolating scaling vectors is that they give rise to a Shannon-like sampling theorem as follows. For a compactly supported function vector $\Phi \in L_2(\mathbb{R}^d)^m$ let us define the shift-invariant space

$$S(\Phi) := \left\{ \sum_{\beta\in\mathbb{Z}^d} c_\beta \Phi(\cdot-\beta) \mid (c_\beta) \in \ell\big(\mathbb{Z}^d\big)^{1\times m} \right\},$$

where $\ell(\mathbb{Z}^d)^{n\times k}$, $n, k > 0$, denotes the space of all sequences of real $n \times k$ matrices on $\mathbb{Z}^d$. It is easy to see that, if Φ is a compactly supported interpolating m-scaling vector, then for all $f \in S(\Phi)$ the representation

$$f(x) = \sum_{\beta\in\mathbb{Z}^d} \sum_{i=0}^{m-1} f\big(\beta + M^{-1}\rho_i\big)\phi_i(x-\beta) \tag{7}$$

holds.

A direct consequence of the interpolation property is (algebraically) linearly independent integer translates. Furthermore, it was shown in [23] that for a continuous m-scaling vector with compact support linear independence implies ℓ_2-stability.

As a scaling vector is completely determined by its symbol, cf. [2], our construction will be based on finding suitable conditions for the desired properties in terms of the symbol or the mask, respectively.

Lemma 2.1. *Let $\rho_k \in M\mathbb{Z}^d$, then the mask of an interpolating m-scaling vector has to satisfy*

$$a^{(i,k)}_{M\alpha+\rho_j-M^{-1}\rho_k} = \delta_{0,\alpha}\delta_{i,j} \quad \text{for all } \alpha \in \mathbb{Z}^d,\ 0 \leqslant i, j < m,$$

with $a^{(i,j)}_\beta := (A_\beta)_{i,j}$.

Proof. For each $\gamma \in \mathbb{Z}^d$ there exists an $\alpha \in \mathbb{Z}^d$ and $j \in \{0, \dots, m-1\}$ such that $\gamma = M\alpha + \rho_j$. Thus, we have

$$\phi_i\big(M^{-1}\gamma\big) = \phi_i\big(M^{-1}\rho_j + \alpha\big) = \sum_{\beta\in\mathbb{Z}^d} \big(a^{(i,0)}_\beta, \dots, a^{(i,m-1)}_\beta\big)\Phi(\rho_j + M\alpha - \beta).$$

Since $\rho_k \in M\mathbb{Z}^d$, the interpolation condition yields $\Phi(\beta) = \delta_{\beta,M^{-1}\rho_k} e_k$ for all $\beta \in \mathbb{Z}^d$, where e_k denotes the kth unit vector. Therefore, we obtain

$$\phi_i\big(M^{-1}\gamma\big) = a^{(i,k)}_{M\alpha+\rho_j-M^{-1}\rho_k}.$$

On the other hand, (5) implies $\phi_i(M^{-1}\gamma) = \delta_{0,\alpha}\delta_{i,j}$ which completes the proof. □

Throughout this paper, we shall assume $\rho_0 = 0 \in \mathbb{Z}^d$ without loss of generality. Then the above lemma implies that the symbol of an interpolating m-scaling vector has to have the form

$$\mathbf{A}(z) = \begin{pmatrix} z^{\rho_0} & a^{(0,1)}(z) & \cdots & a^{(0,m-1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ z^{\rho_{m-1}} & a^{(m-1,1)}(z) & \cdots & a^{(m-1,m-1)}(z) \end{pmatrix}. \tag{8}$$

For the case $m = 2$ we can choose $R = \{0, \rho\}$ and obtain

$$\mathbf{A}(z) = \begin{pmatrix} 1 & a^{(0)}(z) \\ z^\rho & a^{(1)}(z) \end{pmatrix}. \tag{9}$$

Lemma 2.1 also connects the notion of interpolating scaling vectors with recent concepts of interpolatory vector subdivision schemes developed in [6,8]. It was shown in [24] that a stable scaling vector with compact support satisfies the sum rules of order 1, cf. Section 3.2. Thus, using Lemma 3.5 in Section 3.2 and Lemma 2.1, Theorem 3.8 in [8] implies that the masks of interpolating scaling vectors define vector subdivision schemes that interpolate on $M^{-1}\mathbb{Z}^d$ in the notion of [8]. These schemes are also interpolatory in the more general sense of [6], cf. Proposition 1 in [6].

The following lemma, stated in [2], provides us with a sufficient condition on the symbol for the existence of a compactly supported solution of the refinement equation (3).

Lemma 2.2. *For $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{r\times r}$ let $\mathbf{A}(\mathbf{1})$ have the eigenvalues $\lambda_1 = m$, $|\lambda_2|, \ldots, |\lambda_r| < m$ with $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{C}^d$, then the refinement equation* (3) *has a compactly supported distributional solution which is unique up to multiplication with a constant.*

Here, $\ell_0(\mathbb{Z}^d)^{n\times k}$ denotes the subspace of all finitely supported sequences in $\ell(\mathbb{Z}^d)^{n\times k}$.

We close this section with some additional notation. A complete set of representatives of $\mathbb{Z}^d/M^\top\mathbb{Z}^d$ shall be denoted by $\tilde{R} := \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{m-1}\}$. Furthermore, the cosets corresponding to $\rho \in R$ (or $\tilde{\rho} \in \tilde{R}$) are denoted by $[\rho]$ (and $[\tilde{\rho}]$, respectively) and for their characteristic functions we use the notation $\mathbb{1}_{[\rho]}(\cdot)$ (and $\mathbb{1}_{[\tilde{\rho}]}(\cdot)$). For $\tilde{\rho} \in \tilde{R}$ we define $\zeta_{\tilde{\rho}}$ by

$$\zeta_{\tilde{\rho}} := e^{-i2\pi M^{-\top}\tilde{\rho}}.$$

In this paper we will excessively use the following lemma on character sums, cf. [7], which connects the sets R and $\tilde{R}$.

Lemma 2.3. *For $\rho_i, \rho_j \in R$ it holds that*

$$\sum_{\tilde{\rho}\in\tilde{R}} \zeta_{\tilde{\rho}}^{\rho_i} \zeta_{\tilde{\rho}}^{-\rho_j} = m \cdot \delta_{i,j}.$$

3. Additional properties

Though our main aim is to construct interpolating scaling vectors with compact support, we also intend to incorporate some additional properties, namely orthonormality, regularity, and approximation order. The first property gives rise to orthonormal multiwavelet bases of $L_2(\mathbb{R}^d)$, the latter two are needed for the characterization of several function spaces and are very desirable for application purposes. Unfortunately, conditions for high regularity of scaling vectors are rather complicated and thus hard to incorporate. Therefore, we will use regularity estimates as a measure of quality for the outcome of our construction. On the other hand, implementable orthonormality and approximation order conditions have been derived in [1,24]. These enter our construction as the main ingredients. Furthermore, we will show in this section that all orthogonal interpolating scaling vectors are balanced.

3.1. Orthonormality

In the following, we will focus on r-scaling vectors $\Phi = (\phi_0, \ldots, \phi_{r-1})^\top$ that satisfy

$$\langle \phi_i, \phi_j(\cdot - \beta)\rangle = c \cdot \delta_{0,\beta}\delta_{i,j}, \quad \beta \in \mathbb{Z}^d,\ 0 \leqslant i, j < r, \tag{10}$$

for a constant $c > 0$, i.e., the integer translates of all component functions are mutually orthogonal, and, in addition, the norms of all components are identical. If we have $c = 1$, then Φ is called *orthonormal*.

Similar to the univariate case, the symbol of a scaling vector satisfying (10) necessarily has to stem from a *conjugate quadrature filter*. The following lemma provides the corresponding condition in terms of the symbol, see, e.g., [24].

Lemma 3.1. *Let Φ be an r-scaling vector with respect to M such that* (10) *holds, then its symbol $\mathbf{A}(z)$ has to satisfy*

$$\sum_{\tilde{\rho}\in\tilde{R}} \mathbf{A}(z_{M^{-\top}\tilde{\rho}}) \overline{\mathbf{A}(z_{M^{-\top}\tilde{\rho}})}^\top = m^2\, \mathbf{I}_r. \tag{11}$$

For the special case of an interpolating 2-scaling vector with compact support we obtain the following simplified conditions.

Theorem 3.2. *Let* $\mathbf{A}(z)$ *be the symbol of an interpolating* 2*-scaling vector with mask* $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{2\times 2}$. $\mathbf{A}(z)$ *satisfies* (11) *if and only if the symbol entries* $a^{(0)}(z)$ *and* $a^{(1)}(z)$ *in* (9) *satisfy*

$$\left|a^{(0)}(z)\right|^2 + \left|a^{(0)}(z_{M^{-\top}\tilde{\rho}})\right|^2 = 2 \tag{12}$$

and

$$a^{(1)}(z) = \pm z^\alpha \overline{a^{(0)}(z_{M^{-\top}\tilde{\rho}})} \tag{13}$$

for some $\alpha \in [\rho]$ *and with* $\tilde{R} = \{0, \tilde{\rho}\}$.

Proof. Combining the interpolation condition (9) with Lemma 3.1, condition (11) is equivalent to the matrix

$$\mathcal{A}(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} a^{(0)}(z) & a^{(0)}(z_{M^{-\top}\tilde{\rho}}) \\ a^{(1)}(z) & a^{(1)}(z_{M^{-\top}\tilde{\rho}}) \end{pmatrix} \tag{14}$$

being unitary for all $z \in \mathbb{T}^d$. Therefore, $a^{(0)}(z)$ has to satisfy (12) and we have

$$\det\bigl(\mathcal{A}(z)\bigr) = \pm z^\alpha$$

for some $\alpha \in \mathbb{Z}^d$. By Cramer's rule we obtain

$$a^{(1)}(z) = \pm z^\alpha \overline{a^{(0)}(z_{M^{-\top}\tilde{\rho}})}$$

and

$$a^{(1)}(z_{M^{-\top}\tilde{\rho}}) = \mp z^\alpha \overline{a^{(0)}(z)}.$$

Lemma 2.3 yields

$$z^\alpha_{M^{-\top}\tilde{\rho}} = \zeta^\alpha_{\tilde{\rho}} z^\alpha = \begin{cases} z^\alpha, & \text{if } \alpha \in [0], \\ -z^\alpha, & \text{if } \alpha \in [\rho], \end{cases}$$

and with $(z_{M^{-\top}\tilde{\rho}})_{M^{-\top}\tilde{\rho}} = z$ we have (13). On the other hand, if (12) and (13) are satisfied, then $\mathcal{A}(z)$ is unitary. □

A simple computation yields the following corollary which provides us with the corresponding conditions in terms of the mask.

Corollary 3.3. *With the notation* $a^{(0)}(z) := \sum_{\beta\in\mathbb{Z}^d} a_\beta z^\beta$ *it holds that*

(a) *condition* (12) *is equivalent to*

$$\sum_{\beta\in\mathbb{Z}^d} a_\beta a_{\beta-M\gamma} = \delta_{0,\gamma} \quad \textit{for all } \gamma \in \mathbb{Z}^d \tag{15}$$

and

(b) *condition* (13) *is equivalent to*

$$a^{(1)}(z) = \pm z^\alpha \sum_{\beta\in\mathbb{Z}^d} (-1)^{\mathbb{1}_{[\rho]}(\beta)} a_\beta z^{-\beta}. \tag{16}$$

Since we require that Φ satisfies the orthogonality condition (10), the problem of computing the constant c has to be addressed. The next theorem shows that, under very mild conditions, for an interpolating scaling vector Φ the constant c is completely determined by the length of Φ.

Theorem 3.4. *Let $\Phi = (\phi_0, \dots, \phi_{m-1})^\top$ be an interpolating m-scaling vector with compact support that satisfies* (10) *and let $S(\Phi)$ contain all constant functions. Then we have*

$$\|\phi_i\|_{L_2}^2 = \int_{\mathbb{R}^d} \phi_i(x)\,\mathrm{d}x = \frac{1}{m} \tag{17}$$

for $i \in \{0, \dots, m-1\}$.

Proof. For an arbitrary index set $\Lambda \subset \mathbb{Z}^d$ let us define the function

$$\Xi_\Lambda(x) := \sum_{\beta\in\Lambda} \sum_{i=0}^{m-1} \phi_i(x-\beta),$$

and we denote $\Xi := \Xi_{\mathbb{Z}^d}$. Since Φ is compactly supported, there exists a finite index set $\tilde{\Lambda}$ such that the identity $\Xi(x) = \Xi_{\tilde{\Lambda}}(x)$ holds for all $x \in \operatorname{supp}(\Phi)$. Due to (10) we have $\|\phi_i\|_{L_2} = \sqrt{c}$ for $i = 0, \dots, m-1$, thus $c^{-1/2}\Phi$ is an orthonormal scaling vector and we obtain the representation

$$\Xi_{\tilde{\Lambda}}(x) = \sum_{\beta\in\tilde{\Lambda}} \sum_{i=0}^{m-1} \bigl\langle \Xi_{\tilde{\Lambda}}, c^{-1/2}\phi_i(x-\beta)\bigr\rangle c^{-1/2}\phi_i(x-\beta). \tag{18}$$

Furthermore, since $S(\Phi)$ contains the constant functions, the sampling property (7) yields $\Xi \equiv 1$. Therefore, combining (7) and (18) yields

$$1 = \frac{1}{\sqrt{c}}\left\langle \Xi_{\tilde{\Lambda}}, \frac{1}{\sqrt{c}}\phi_i \right\rangle = \frac{1}{\|\phi_i\|_{L_2}^2}\langle 1, \phi_i\rangle.$$

Thus, we have proven the first identity in (17).

By definition, Ξ is periodic, and since Φ is compactly supported, we can expand Ξ into its Fourier series. Then the βth Fourier coefficient $\hat{c}_\Xi(\beta)$ of Ξ satisfies

$$(2\pi)^{-d/2}\hat{c}_\Xi(\beta) = \sum_{i=0}^{m-1} \hat{\phi}_i(2\pi\beta).$$

On the other hand, $\Xi \equiv 1$ implies $\hat{c}_\Xi(\beta) = \delta_{0,\beta}$ and we obtain

$$1 = \sum_{i=0}^{m-1} \int_{\mathbb{R}^d} \phi_i(x)\,\mathrm{d}x.$$

With $\|\phi_0\|_{L_2} = \cdots = \|\phi_{m-1}\|_{L_2}$ and the first identity in (17) the proof is complete. □

3.2. Sum rules

A mask $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{r\times r}$ of an r-scaling vector with respect to a scaling matrix M satisfies the *sum rules of order k* if there exists a set of vectors $\{y_\mu \in \mathbb{R}^r \mid \mu \in \mathbb{Z}_+^d,\ |\mu| < k\}$ with $y_0 \neq 0$ such that

$$\sum_{0\leqslant\nu\leqslant\mu} (-1)^{|\nu|} \left(\sum_{\beta\in\mathbb{Z}^d} \frac{(M^{-1}\rho+\beta)^\nu}{\nu!} A_{\rho+M\beta}^\top \right) y_{\mu-\nu} = \sum_{|\nu|=|\mu|} m(\mu,\nu) y_\nu \tag{19}$$

holds for all $\mu \in \mathbb{Z}_+^d$ with $|\mu| < k$ and all $\rho \in R$. The numbers $m(\mu,\nu)$ are uniquely determined by

$$\frac{(M^{-1}x)^\mu}{\mu!} = \sum_{|\nu|=|\mu|} m(\mu,\nu)\frac{x^\nu}{\nu!} \quad \text{for all } x \in \mathbb{R}^d. \tag{20}$$

In [1,24] it was proven that if the mask of a compactly supported scaling vector Φ satisfies the sum rules of order k then Φ provides accuracy of order k, i.e., $\pi_k^d \subset S(\Phi)$, where π_{k-1}^d denotes the space of all polynomials of total degree less than k in $\mathbb{R}^d$. Furthermore, it was shown by Jia [21] that if a compactly supported scaling vector Φ has linear independent integer translates or is at least stable, then the order of accuracy is equivalent to the approximation order provided by Φ.

First of all, to obtain an applicable version of the sum rules, the vectors y_μ, $|\mu| < k$, have to be determined. Fortunately, for interpolating scaling vectors, they can be given explicitly.

Lemma 3.5. *Let $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{m\times m}$ be the mask of an interpolating m-scaling vector Φ satisfying the sum rules of order k. Then the vectors y_μ, $|\mu| < k$, satisfy*

$$y_\mu = \left(\frac{(M^{-1}\rho_0)^\mu}{\mu!}, \dots, \frac{\big(M^{-1}\rho_{m-1}\big)^\mu}{\mu!}\right)^\top.$$

Proof. As stated above, the interpolation condition (5) implies ℓ_p-stability. Therefore, the requirements of Theorem 2.4 in [15] are met and the vectors y_μ satisfy

$$\frac{x^\mu}{\mu!} = \sum_{0\leqslant\nu\leqslant\mu}\sum_{\beta\in\mathbb{Z}^d}\frac{\beta^\nu}{\nu!}y_{\mu-\nu}^\top\Phi(x-\beta)$$

for all $\mu \in \mathbb{Z}_+^d$ with $|\mu| < k$. Since Φ provides accuracy of order k, it holds that $x^\mu \in S(\Phi)$ for $|\mu| < k$, and, due to the sampling property (7), we have

$$\frac{x^\mu}{\mu!} = \sum_{\beta\in\mathbb{Z}^d}\sum_{i=0}^{m-1}\frac{(\beta + M^{-1}\rho_i)^\mu}{\mu!}\phi_i(x-\beta).$$

Since the integer translates of Φ are linearly independent we obtain

$$\sum_{0\leqslant\nu\leqslant\mu}\frac{\beta^\nu}{\nu!}y_{\mu-\nu}^{(i)} = \frac{(\beta + M^{-1}\rho_i)^\mu}{\mu!}$$

for $i = 0, \dots, m-1$ where $y_\mu^{(i)}$ denotes the ith component of y_μ. Induction over $|\mu|$ yields the result. □

This enables us to state some handy conditions for accuracy order in terms of the mask.

Theorem 3.6. *Let $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{m\times m}$ be the mask of an interpolating m-scaling vector Φ as in* (8) *and let $a_\beta^{(i,j)}$ denote the βth coefficient of $a^{(i,j)}(z)$. (A_β) satisfies the sum rules of order k if and only if*

$$\sum_{j=0}^{m-1}\sum_{\beta\in\mathbb{Z}^d}a_{\rho+M\beta}^{(j,i)}\big(M^{-1}(\rho_j-\rho)-\beta\big)^\mu = \big(M^{-2}\rho_i\big)^\mu$$

holds for all $1 \leqslant i < m$, $\rho \in R$ and $\mu \in \mathbb{Z}_+^d$ with $|\mu| < k$.

Proof. Applying Lemma 3.5 to the ith component of the vector valued equation (19) we obtain for $0 \leqslant i < m$

$$\sum_{|\nu|=|\mu|} m(\mu,\nu)\frac{(M^{-1}\rho_i)^\nu}{\nu!} = \sum_{0\leqslant\nu\leqslant\mu}(-1)^{|\nu|}\sum_{\beta\in\mathbb{Z}^d}\frac{(M^{-1}\rho+\beta)^\nu}{\nu!}\sum_{j=0}^{m-1}a_{\rho+M\beta}^{(j,i)}\frac{(M^{-1}\rho_j)^{\mu-\nu}}{(\mu-\nu)!}.$$

Due to Eq. (20) this is equivalent to

$$\frac{(M^{-2}\rho_i)^\mu}{\mu!} = \sum_{0\leqslant\nu\leqslant\mu}(-1)^{|\nu|}\sum_{\beta\in\mathbb{Z}^d}\frac{(M^{-1}\rho+\beta)^\nu}{\nu!}\sum_{j=0}^{m-1}a_{\rho+M\beta}^{(j,i)}\frac{(M^{-1}\rho_j)^{\mu-\nu}}{(\mu-\nu)!}.$$

The multivariate binomial theorem yields

$$\big(M^{-2}\rho_i\big)^\mu = \sum_{j=0}^{m-1}\sum_{\beta\in\mathbb{Z}^d}a_{\rho+M\beta}^{(j,i)}\big(M^{-1}(\rho_j-\rho)-\beta\big)^\mu.$$

This equation holds always true for $i = 0$, since (8) implies $a_\beta^{(j,0)} = \delta_{\beta,\rho_j}$. Thus, the proof is complete. □

For an interpolating scaling vector satisfying (10) and $m = 2$ we obtain the following simplification.

Corollary 3.7. *If we choose $a^{(1)}(z) = z^\rho \sum_{\beta\in\mathbb{Z}^d} (-1)^{\mathbb{1}_{[\rho]}(\beta)} a_\beta z^{-\beta}$ in* (16)*, then for an interpolating* 2*-scaling vector satisfying* (10) *the sum rules are reduced to*

$$\left(M^{-2}\rho\right)^\mu = \sum_{\beta\in\mathbb{Z}^d} a_\beta\left(-M^{-1}\beta\right)^\mu, \qquad \left(M^{-2}\rho\right)^\mu = \sum_{\beta\in\mathbb{Z}^d} a_\beta\left(M^{-1}\beta\right)^\mu(-1)^{\mathbb{1}_{[\rho]}(\beta)}$$

with $R = \{0, \rho\}$.

Proof. For the nontrivial representative $\rho \in R$, Theorem 3.6 yields

$$\begin{aligned}
\left(M^{-2}\rho\right)^\mu &= \sum_{\beta\in\mathbb{Z}^d} a_{M\beta+\rho}\left(-M^{-1}\rho-\beta\right)^\mu + (-1)^{\mathbb{1}_{[\rho]}(-M\beta)} a_{-M\beta}(-\beta)^\mu \\
&= \sum_{\beta\in\mathbb{Z}^d} a_{M\beta+\rho}\left(-M^{-1}(\rho+M\beta)\right)^\mu + a_{M\beta}\left(M^{-1}(M\beta)\right)^\mu \\
&= \sum_{\beta\in\mathbb{Z}^d} a_\beta\left((-1)^{\mathbb{1}_{[\rho]}(\beta)} M^{-1}\beta\right)^\mu = \sum_{\beta\in\mathbb{Z}^d} a_\beta\left(M^{-1}\beta\right)^\mu(-1)^{|\mu|\cdot\mathbb{1}_{[\rho]}(\beta)}.
\end{aligned}$$

And for $0 \in R$ we obtain

$$\begin{aligned}
\left(M^{-2}\rho\right)^\mu &= \sum_{\beta\in\mathbb{Z}^d} a_{M\beta}(-\beta)^\mu + (-1)^{\mathbb{1}_{[\rho]}(\rho-M\beta)} a_{\rho-M\beta}\left(M^{-1}\rho-\beta\right)^\mu \\
&= \sum_{\beta\in\mathbb{Z}^d} a_{M\beta}\left(-M^{-1}(M\beta)\right)^\mu - a_{M\beta+\rho}\left(M^{-1}(\rho+M\beta)\right)^\mu \\
&= \sum_{\beta\in\mathbb{Z}^d} a_\beta\left((-1)^{\mathbb{1}_{[0]}(\beta)} M^{-1}\beta\right)^\mu(-1)^{\mathbb{1}_{[\rho]}(\beta)} = \sum_{\beta\in\mathbb{Z}^d} a_\beta\left(M^{-1}\beta\right)^\mu(-1)^{(1+|\mu|)\mathbb{1}_{[\rho]}(\beta)+|\mu|}.
\end{aligned}$$

With

$$(-1)^{|\mu|\cdot\mathbb{1}_{[\rho]}(\beta)} = \begin{cases} (-1)^{|\mu|}, & \text{if } |\mu| \text{ is even}, \\ (-1)^{\mathbb{1}_{[\rho]}(\beta)}, & \text{else}, \end{cases}$$

and

$$(-1)^{(1+|\mu|)\mathbb{1}_{[\rho]}(\beta)+|\mu|} = \begin{cases} (-1)^{\mathbb{1}_{[\rho]}(\beta)}, & \text{if } |\mu| \text{ is even}, \\ (-1)^{|\mu|}, & \text{else}, \end{cases}$$

the proof is complete. □

Remark 3.8. With the above choice, if $(A_\beta)_{\beta\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)^{2\times2}$ satisfies the sum rules of order 1, we have

$$\mathbf{A}(\mathbf{1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus $\mathbf{A}(\mathbf{1})$ has the eigenvalues 2 and 0. Therefore, Lemma 2.2 ensures the existence and uniqueness of a compactly supported scaling vector corresponding to $(A_\beta)_{\beta\in\mathbb{Z}^d}$.

3.3. Balancing

In many applications, when applying the discrete (multi-)wavelet transform, one has to determine the coefficients $c_f(\beta) \in \mathbb{R}^r$ of a function $f \in S(\Phi)$ given only the sample values $f(\Lambda)$ on a lattice $\Lambda \subset \mathbb{R}^d$. A common practice is to use the sampled values directly as coefficients. Strang and Nguyen [31] refer to this as a *wavelet crime*, though, in the scalar case, it works quite well. For most scaling vectors Φ, however, one observes that the coefficients $c_p(\beta)$ of a polynomial $p \in \pi_k^d$ are not necessarily the sample values of a polynomial in π_k^d and vice versa, even if Φ provides accuracy order $k+1$. Consequently, the corresponding filter banks lack some important approximation properties.

To bypass this problem in the univariate case, Lebrun and Vetterli [28] invented the notion of *balancing*. A generalization of the balancing concept to multivariate (bi-) orthogonal scaling vectors can be found in [5], see also [4]. There, also the following characterization is given.

Definition 3.9. An orthogonal r-scaling vector Φ is k-balanced relative to $\{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^d$ if and only if

$$\sum_{\beta \in \mathbb{Z}^d} \big((\beta+\xi_1)^\alpha, \dots, (\beta+\xi_r)^\alpha\big)\Phi(x-\beta) \in \pi_{|\alpha|}^d \tag{21}$$

for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| < k$.

Finally, we obtain that compactly supported interpolating scaling vectors that satisfy (10) are automatically balanced up to their order of accuracy.

Proposition 3.10. *Let Φ be a compactly supported interpolating m-scaling vector that satisfies* (10) *and provides accuracy order k. Then Φ is m-balanced relative to the set* $\{M^{-1}\rho_0, \dots, M^{-1}\rho_{m-1}\}$.

Proof. Since Φ provides accuracy order k, we have $x^\alpha \in S(\Phi)$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| < k$. For all these α, the sampling property (7) yields

$$x^\alpha = \sum_{\beta \in \mathbb{Z}^d} \sum_{i=0}^{m-1} \big(\beta + M^{-1}\rho_i\big)^\alpha \phi_i(x-\beta)$$

and thus (21) is satisfied. $\square$

4. Explicit construction

In this section, we give an explicit construction method for the symbols of interpolating 2-scaling vectors on $\mathbb{R}^d$ with compact support that satisfy (10). Since the construction involves necessary conditions on the mask only, we also explain how to verify sufficient conditions. To substantiate our approach, several examples for the case $d=2$ are presented.

4.1. General method

Based on the results in the preceding section we suggest the following construction principle:

(1) Choose a scaling matrix M with $|\det(M)| = 2$ and the nontrivial representative ρ of $\mathbb{Z}^d/M\mathbb{Z}^d$ such that $R = \{0, \rho\}$.
(2) Start with the first symbol entry

$$a^{(0)}(z) = \sum_{\beta \in J} a_\beta z^\beta$$

by choosing the support $J \subset \mathbb{Z}^d$ of $(a_\beta)_{\beta \in J}$. We observe that centering the coefficients around a_0 seems to provide the highest regularity, therefore we suggest the choice of $J = [-n, n]^d \cap \mathbb{Z}^d$.
(3) According to Theorem 3.2 and Corollary 3.3 the second symbol entry $a^{(1)}(z)$ has to have the form

$$a^{(1)}(z) = \pm z^\alpha \sum_{\beta \in J} (-1)^{\mathbb{1}_{[\rho]}(\beta)} a_\beta z^{-\beta}$$

with $\alpha \in [\rho]$. Based on our observations we suggest choosing $\alpha = \rho$ and a positive sign, since this seems to provide the highest regularity and the smallest support.

(4) Apply the orthogonality condition (15) to the coefficient sequence $(a_\beta)_{\beta\in J}$. This will consume about one half of the degrees of freedom.
(5) Finally, apply the sum rules of Corollary 3.7 up to the highest possible order to the coefficient sequence $(a_\beta)_{\beta\in J}$.

By this method we obtain a system of linear and quadratic equations in the variables $(a_\beta)_{\beta\in J}$. Due to the large number of equations, it is rather impossible to solve this system analytically. Therefore, a numerical method has to be applied. For the following examples we use an implementation of the Gauß–Newton method. As we have to deal with quadratic equations, the solutions of this system are by no means unique. So, as a screening process, we suggest to measure/estimate the regularity of the scaling vectors corresponding to the obtained solutions. This can be performed by applying an implementation of the method stated in [22], see also [16].

4.1.1. *Sufficient conditions*

So far, the conditions on the mask involved in our approach are necessary only. Therefore, during the construction process, one has to check whether the corresponding scaling vectors actually do possess the desired properties. For this purpose ℓ_2-stability of the scaling vector is crucial.

It was shown in [24] that stability in combination with the orthogonality condition (11) implies that the corresponding scaling vector satisfies (10). In [24] also an implementable version of the stability condition in terms of the symbol is given. The examples constructed in this paper have been validated with an implementation of this method.

On the other hand, to ensure that a scaling vector Φ is interpolating, we have to check that Φ is continuous and satisfies $\Phi(M^{-1}\beta) = (\delta_{0,\beta}, \delta_{\rho,\beta})^\top$ for all $\beta \in \mathbb{Z}^d$. This can be performed as follows: After verifying that Φ is stable, the results in [22] can be used to compute the critical Sobolev exponent s of Φ. If s is strictly larger than one, the Sobolev embedding theorem implies that Φ is continuous. Furthermore, the values of a continuous scaling vector Φ on $\mathbb{Z}^d$ can be obtained by the well known eigenvector–eigenvalue trick, i.e., the sequence $(\Phi(\beta))_{\beta\in\mathbb{Z}^d}$ is an eigenvector of the *transition operator* T_A defined by

$$(T_A v)_\alpha := \sum_{\beta\in\mathbb{Z}^d} A_{M\alpha-\beta} v_\beta, \quad \alpha \in \mathbb{Z}^d,\ v \in \ell_0^{r\times 1}(\mathbb{Z}^d),$$

corresponding to the eigenvalue 1. To obtain the finite matrix representation of T_A, e.g., Lemma 2.3 in [17] can be used. It is easy to verify that $(\delta_{0,\beta}, 0)^\top_{\beta\in\mathbb{Z}^d}$ is an eigenvector of T_A corresponding to the eigenvalue 1, and thus condition (9) implies that Φ is interpolating.

4.2. *The case $d = 2$*

To show the potential of our approach, several examples of interpolating 2-scaling vectors on $\mathbb{R}^2$ are constructed in the sequel. We focus on two of the most popular scaling matrices with determinant ± 2, i.e., the quincunx matrix M_q and a box–spline matrix M_b, defined by

$$M_q := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_b := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Both matrices are idempotent and they generate the same lattice, i.e., $M_q\mathbb{Z}^2 = M_b\mathbb{Z}^2 = \{(i,j)^\top \in \mathbb{Z}^2 \mid i + j \text{ is even}\}$. Therefore, the cosets of $\mathbb{Z}^2/M_q\mathbb{Z}^2$ and $\mathbb{Z}^2/M_b\mathbb{Z}^2$ coincide and we choose $\rho := (0,1)^\top$ in both cases.

The next theorem shows that for both matrices also the solutions of our equation systems are closely related.

Theorem 4.1. *The sequence $(a^q_\beta) \in \ell_0(\mathbb{Z}^2)$ satisfies the orthogonality condition* (15) *and the sum rules of order k in Corollary* 3.7 *with respect to $M = M_q$ if and only if the sequence $(a^b_\beta) \in \ell_0(\mathbb{Z}^2)$, defined by*

$$a^b_\beta := (-1)^{\mathbb{1}_{[\rho]}(\beta)} a^q_{U\beta} \quad \text{with } U := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

satisfies the corresponding conditions with respect to $M = M_b$.

Proof. Let us show the orthogonality condition first. For $\gamma \in \mathbb{Z}^2$ it holds that

$$\sum_{\beta\in\mathbb{Z}^2} a^b_\beta a^b_{\beta-M_b\gamma} = \sum_{\beta\in\mathbb{Z}^2} a^q_{U\beta} a^q_{U(\beta-M_b\gamma)} = \sum_{\beta\in\mathbb{Z}^2} a^q_\beta a^q_{\beta-UM_b\gamma}.$$

Since $UM_b\mathbb{Z}^2 = M_b\mathbb{Z}^2 = M_q\mathbb{Z}^2$, there exists a $\tilde{\gamma} \in \mathbb{Z}^2$ with $UM_b\gamma = M_q\tilde{\gamma}$. Therefore, we have

$$\sum_{\beta\in\mathbb{Z}^2} a^b_\beta a^b_{\beta-M_b\gamma} = \sum_{\beta\in\mathbb{Z}^2} a^q_\beta a^q_{\beta-M_q\tilde{\gamma}}$$

and with $\delta_{0,\tilde{\gamma}} = \delta_{0,\gamma}$ the orthogonality conditions (15) for (a^b_β) and for (a^q_β) are equivalent.

On the other hand, for $\mu \in \mathbb{Z}^2_+$ with $|\mu| < k$ we have

$$\sum_{\beta\in\mathbb{Z}^2} a^b_\beta\bigl(-M_b^{-1}\beta\bigr)^\mu = \sum_{\beta\in\mathbb{Z}^2} a^q_{U\beta}\bigl(-M_b^{-1}\beta\bigr)^\mu(-1)^{\mathbb{1}_{[\rho]}(\beta)} = \sum_{\beta\in\mathbb{Z}^2} a^q_\beta\bigl(-M_b^{-1}U\beta\bigr)^\mu(-1)^{\mathbb{1}_{[\rho]}(U\beta)}.$$

Using the notation $\mu := (\mu_0, \mu_1)^\top$ it holds that

$$\bigl(-M_b^{-1}U\beta\bigr)^\mu = \bigl(M_q^{-\top}\beta\bigr)^\mu = (-1)^{\mu_0}\bigl(M_q^{-1}\beta\bigr)^{E\mu}$$

with $E := \bigl(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\bigr)$. With $\mathbb{1}_{[\rho]}(U\beta) = \mathbb{1}_{[\rho]}(\beta)$, this leads to

$$\sum_{\beta\in\mathbb{Z}^2} a^b_\beta\bigl(-M_b^{-1}\beta\bigr)^\mu = (-1)^{\mu_0}\sum_{\beta\in\mathbb{Z}^2} a^q_\beta\bigl(M_q^{-1}\beta\bigr)^{E\mu}(-1)^{\mathbb{1}_{[\rho]}(\beta)}.$$

Furthermore, it holds that

$$\bigl(M_b^{-2}\rho\bigr)^\mu = (-1)^{\mu_0}\bigl(M_q^{-2}\rho\bigr)^{E\mu}.$$

Therefore, due to $|\mu| = |E\mu|$, the first part of the sum rules of order k in Corollary 3.7 for (a^b_β) is equivalent to the second part of the sum rules of order k for (a^q_β). The opposite direction can be shown analogously. $\square$

4.2.1. Numerical issues

In the numerical treatment of our equation system we are confronted with two major problems. The first problem is related to the local convergence of the Gauß–Newton method. We have to select a sufficiently large set of starting vectors $\tilde{a} \in \mathbb{R}^{|J|}$ in order to find at least some good solutions. This can be performed by either uniformly or randomly sampling $\mathbb{R}^{|J|}$ (or the unit sphere in $\mathbb{R}^{|J|}$, as Eq. (15) implies $\|(a_\beta)\|_2 = 1$). In both cases, for J becoming large, we have to deal with a rapidly increasing amount of starting vectors. For the quincunx matrix M_q, this difficulty can be eased by the observation that many solutions which correspond to scaling vectors with a high regularity share the structure

$$a_\beta = \begin{cases} a_{E\beta}, & \text{if } \beta \in M_q\mathbb{Z}^2,\\ -a_{-E\beta}, & \text{else,}\end{cases} \quad \text{with } E = \bigl(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\bigr). \tag{22}$$

According to Theorem 4.1, a similar structure can be found for M_b. For large index sets J, we focus on these specific structures.

The second problem is to distinguish true solutions from local minima. It turns out that residuals that are even close to machine accuracy in norm do not necessarily correspond to a true solution. To solve this, we test the solutions with a multiple precision implementation of the Gauß–Newton method. If the norm of the residual becomes smaller than $\varepsilon := 10^{-25}$, then we assume the corresponding solution to be true. For application purposes, this should be sufficiently small.

4.3. Examples

Starting with an index set $J = \{-n, \dots, n\}^2$ we obtain a sequence of scaling vectors denoted by Φ_n with increasing accuracy order and regularity. It turns out that for both scaling matrices, those solutions of our equation systems which correspond to the scaling vectors with the highest regularity are linked via Theorem 4.1. The corresponding scaling vectors shall be denoted by Φ_n^q for the quincunx case and Φ_n^b for the box–spline case. In Table 1 the properties of the constructed examples are shown, a selection of the corresponding masks can be found in Appendix A. For a complete list of these masks, see the preprint version of this paper [27].

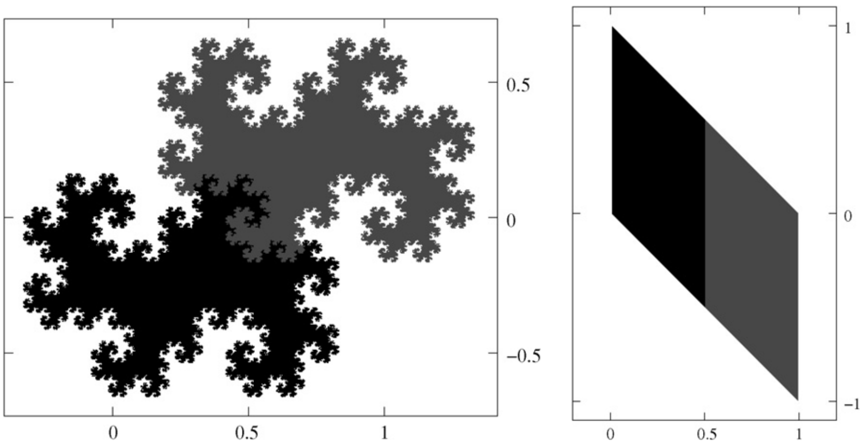
For the case $n = 0$ our solutions are the characteristic functions of the multi-tiles shown in Fig. 1. Via the rule (6), these scaling vectors coincide with the characteristic functions of the classical tiles corresponding to the scaling matrices M_q and M_b, cf. [14]. Both symbols have the form

$$\mathbf{A}(z) = \begin{pmatrix} 1 & 1 \\ z^\rho & z^\rho \end{pmatrix}.$$

For $n \geqslant 2$ all our solutions have critical Sobolev exponents strictly larger than one. Therefore, by the Sobolev embedding theorem, all these scaling vectors are at least continuous. Figures 2 and 3 show the component functions of Φ_2^q and Φ_2^b, respectively. Furthermore, for $n = 5$ we obtain an example that is continuously differentiable. The corresponding functions are graphed in Figs. 4 and 5. The reader should note that all these scaling vectors are very well localized.

Table 1
Properties of the Φ_n

n	Accuracy order	$\sup\{s \mid \Phi_n \in H^s(\mathbb{R}^2)\}$	$\|\text{Residual}\|_\infty$
Quincunx case			
0	1	0.2382	$< \varepsilon$
1	1	0.7425	$< \varepsilon$
2	2	1.3549	$< \varepsilon$
3	3	1.6994	$< \varepsilon$
4	3	1.8186	$< 10^{-18}$
5	4	2.0023	$< 10^{-22}$
Box–spline case			
0	1	0.5	$< \varepsilon$
1	1	0.7361	$< \varepsilon$
2	2	1.3709	$< \varepsilon$
3	3	1.6952	$< \varepsilon$
4	3	1.9337	$< 10^{-18}$
5	4	2.0987	$< 10^{-22}$



(a) Quincunx case (b) Box–spline case

Fig. 1. (Multi-)tiles corresponding to Φ_0.

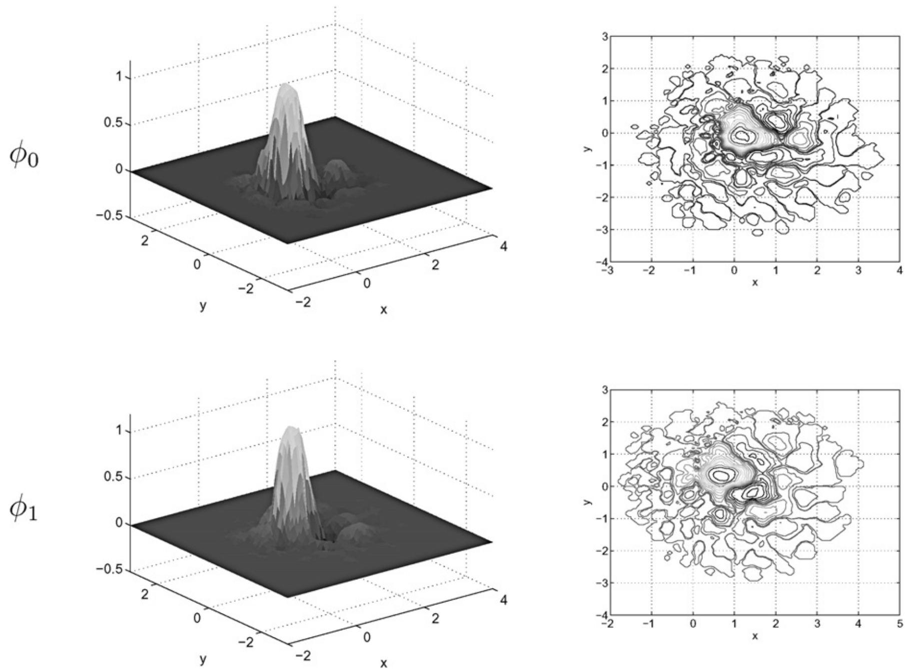


Fig. 2. Component functions of Φ_2^q.

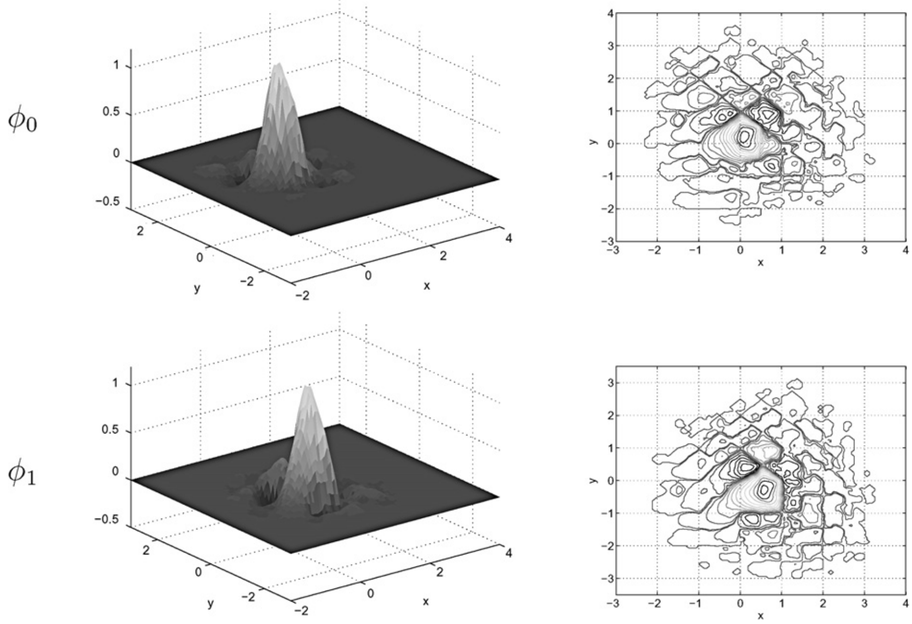


Fig. 3. Component functions of Φ_2^b.

5. Multiwavelets

For almost all application purposes not only the scaling vectors but also some corresponding multiwavelets are needed. Generally, the construction of these multiwavelets is rather complicated. We will show in this section that our interpolating scaling vectors lead to (orthonormal) multiwavelet bases in a very simple and natural way. In the following, we focus on the case $r = m = 2$.

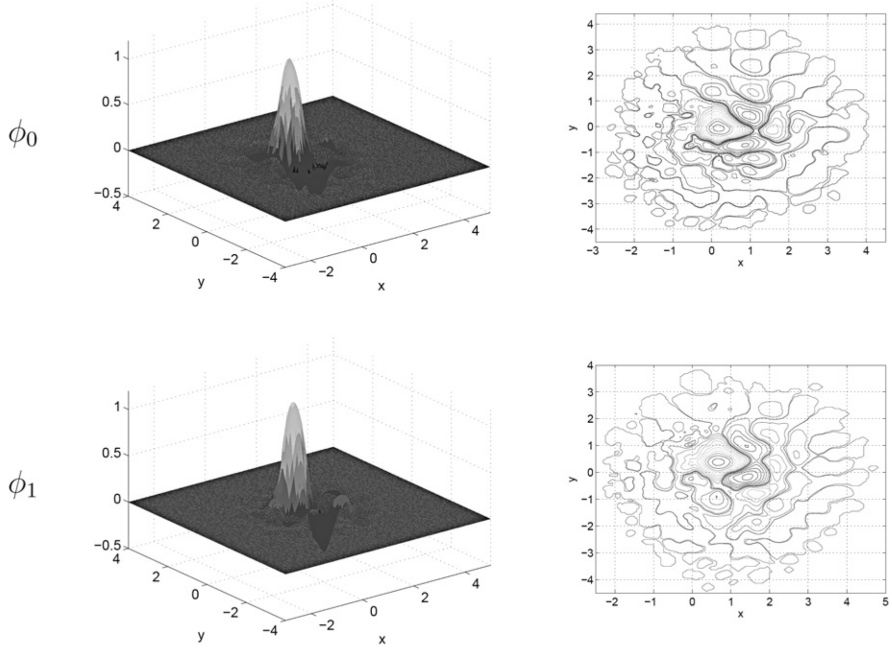


Fig. 4. Component functions of Φ_5^q.

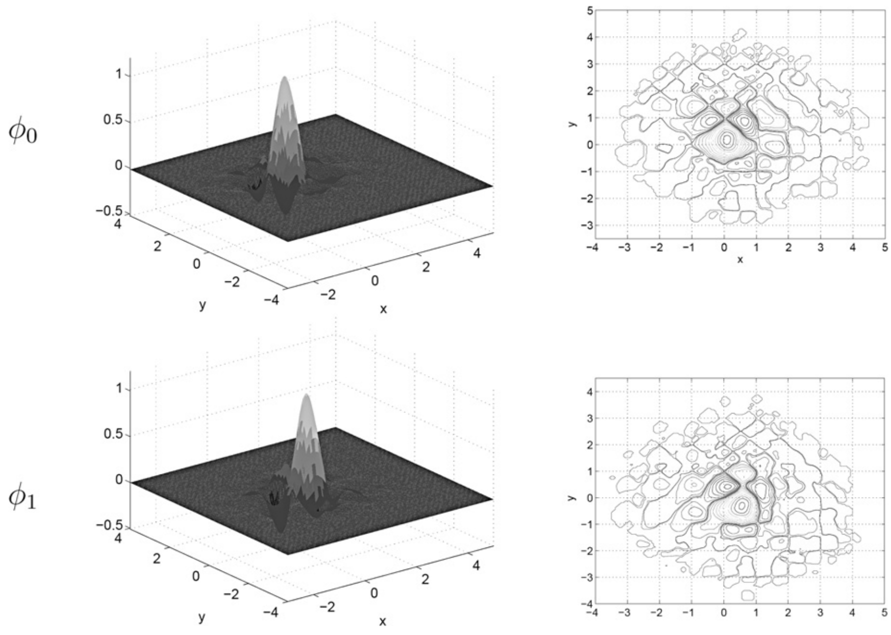


Fig. 5. Component functions of Φ_5^b.

A vector $\Psi := (\psi_0, \psi_1)^\top$ of $L_2(\mathbb{R}^d)$-functions is called a 2-*multiwavelet* if the set $\{2^{-j/2}\psi_i(M^j \cdot -\beta) \mid 0 \leqslant i < 2, j \in \mathbb{Z}, \beta \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L_2(\mathbb{R}^d)$. Let Φ be an orthonormal 2-scaling vector that generates a multiresolution analysis (MRA), then Ψ can be represented as

$$\Psi(x) = \sum_{\beta \in \mathbb{Z}^d} B_\beta \Phi(Mx - \beta) \tag{23}$$

with real 2×2 matrices B_β. Similar to the scalar case and the univariate case, the task of finding a multiwavelet can be converted into a matrix extension problem, cf. [3] and references therein. In our setting this extension problem has the following form. Let $\mathbf{B}(z)$ denote the symbol corresponding to Ψ and let $\mathbf{A}(z)$ denote the symbol of Φ. If we choose $\tilde{R} = \{0, \tilde{\rho}\}$, then Ψ is a 2-multiwavelet if and only if the matrix

$$\mathcal{P}(z) := \frac{1}{2}\begin{pmatrix} \mathbf{A}(z) & \mathbf{A}(z_{M^{-\top}\tilde{\rho}}) \\ \mathbf{B}(z) & \mathbf{B}(z_{M^{-\top}\tilde{\rho}}) \end{pmatrix} \tag{24}$$

is unitary for all $z \in \mathbb{T}^d$. If Φ is interpolating and Ψ is interpolating as well, i.e., we have

$$\Psi\left(M^{-1}\beta\right) = \begin{pmatrix} \delta_{0,\beta} \\ \delta_{\rho,\beta} \end{pmatrix} \quad \text{for all } \beta \in \mathbb{Z}^d$$

and for $R = \{0, \rho\}$, then the extension problem has a unique solution of the following form.

Theorem 5.1. *Let $\mathbf{A}(z)$ be the symbol of a compactly supported interpolating 2-scaling vector Φ that satisfies* (10). *Furthermore, let $\mathbf{B}(z)$ be the symbol of an interpolating function vector Ψ defined by* (23). *The matrix $\mathcal{P}(z)$ in* (24) *is unitary for all $z \in \mathbb{T}^d$ if and only if*

$$\mathbf{B}(z) = \begin{pmatrix} 1 & -a^{(0)}(z) \\ z^\rho & -a^{(1)}(z) \end{pmatrix} \tag{25}$$

holds with $a^{(0)}(z)$ and $a^{(1)}(z)$ as in (9).

Proof. Since Φ and Ψ are interpolating, a direct computation using (23) yields that the symbol $\mathbf{B}(z)$ has to have the form

$$\mathbf{B}(z) = \begin{pmatrix} 1 & b^{(0)}(z) \\ z^\rho & b^{(1)}(z) \end{pmatrix}$$

for some Laurent polynomials $b^{(0)}(z)$ and $b^{(1)}(z)$. Let $\mathcal{P}(z)$ be unitary. If we define $\mathcal{A}(z)$ as in (14) and

$$\mathcal{B}(z) := \frac{1}{\sqrt{2}}\begin{pmatrix} b^{(0)}(z) & b^{(0)}(z_{M^{-\top}\tilde{\rho}}) \\ b^{(1)}(z) & b^{(1)}(z_{M^{-\top}\tilde{\rho}}) \end{pmatrix},$$

then, with $z^\alpha + z^\alpha_{M^{-\top}\tilde{\rho}} = 0$ for $\alpha \in [\rho]$, the unitarity of $\mathcal{P}(z)$ implies

$$\mathcal{A}(z)\overline{\mathcal{B}(z)}^\top = -\mathbf{I}_2. \tag{26}$$

Due to Theorem 3.2, we have $a^{(1)}(z) = \pm z^\alpha \overline{a^{(0)}(z_{M^{-\top}\tilde{\rho}})}$, $\alpha \in [\rho]$, and it holds that $\det(\mathcal{A}(z)) = \mp z^\alpha$. Applying Cramer's rule to (26), we obtain (25).

On the other hand, if a symbol $\mathbf{B}(z)$ corresponding to a mask $(B_\beta) \in \ell_0(\mathbb{Z}^d)^{2\times 2}$ satisfies (25), it is easy to verify that $\mathcal{P}(z)$ is unitary for all $z \in \mathbb{T}$. □

This theorem provides us with a convenient method to construct multiwavelets corresponding to compactly supported interpolating scaling vectors which satisfy (10).

Corollary 5.2. *Under the assumptions of Theorem* 5.1 *let* (25) *be satisfied. Then $\sqrt{2}\Psi$ defines a multiwavelet.*

Proof. Due to Theorem 3.4, $\sqrt{2}\Phi$ is an orthonormal scaling vector that satisfies the refinement equation (4) with the same symbol as Φ. Furthermore, Theorem 4.4 in [3] and the second identity in (17) ensure that $\sqrt{2}\Phi$ generates an MRA. Thus, $\sqrt{2}\Psi$ is a multiwavelet. □

Though the component functions of Ψ in 5.1 are not normalized, Ψ may be called an *interpolating multiwavelet*. Some examples of such interpolating multiwavelets are shown in Figs. 6 and 7. These examples correspond to our scaling vectors Φ_n^q and Φ_n^b for $n = 2$ and $n = 5$ and are denoted by Ψ_n^q and Ψ_n^b, respectively.

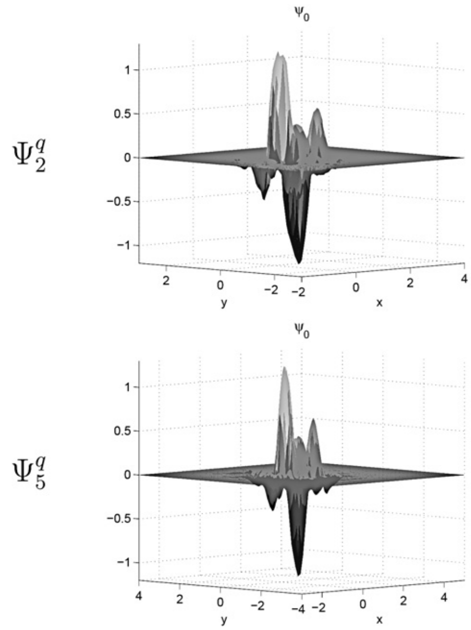


Fig. 6. Multiwavelets corresponding to Φ^q.

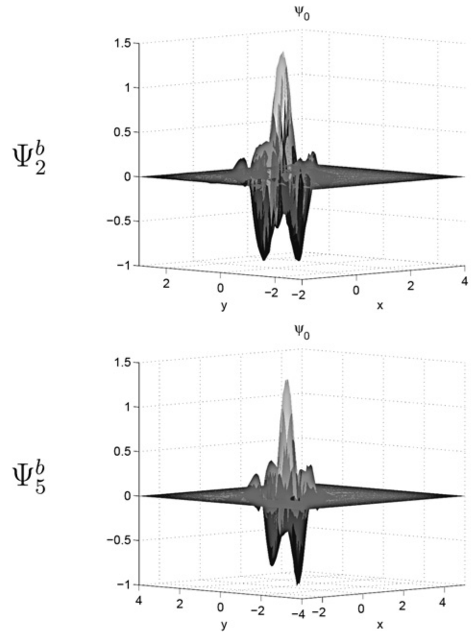


Fig. 7. Multiwavelets corresponding to Φ^b.

Remark 5.3. At first glance, the choice of interpolating multiwavelets seems to be artificial to some extent. But, in addition to their effortless construction, interpolating multiwavelets allow a very efficient implementation of the discrete multiwavelet transform (DMWT) as follows. It is well known that both, the DMWT and the discrete wavelet transform (DWT) act as tree-structured filter banks. Therefore, both algorithms belong to the complexity class $\mathcal{O}(N)$. However, the filter bank within the DWT is scalar valued, the DMWT consists of a multirate filter bank, cf. [31,32]. Thus, for the DMWT input data has to be vectorized, i.e., for scaling vectors of length 2, input data consisting of N elements is split into 2 input streams of size $N/2$. On the other hand, each scalar multiplication within the DWT is substituted by a matrix–vector multiplication within the DMWT. Therefore, for a scalar and a 2×2-matrix valued mask of the same size/support, a naive implementation of the DMWT requires twice as many arithmetic operations (multiplications) as the discrete wavelet transform (DWT). But, exploiting the specific structure of the masks of interpolating scaling vectors and multiwavelets, one half of the multiplications within the DMWT can be omitted, since the first column of all mask elements is trivial. It follows that, in this case, the DMWT can cope with the DWT concerning computational effort.

Remark 5.4. A small software package for computing the critical Sobolev exponent of a scaling vector and for plotting scaling vectors and multiwavelets can be downloaded from: http://www.mathematik.uni-marburg.de/~dahlke/ag-numerik/research/software.

Acknowledgments

The author thanks Stephan Dahlke, Martin Ehler, and Thorsten Raasch for fruitful discussion and helpful comments, and Bastiaan Zapf for his support concerning the MATLAB routines.

Appendix A. Masks

In this appendix the upper right entries $(a_\beta)_{\beta\in J}$ of the masks of Φ_n^q for $n=2$ and $n=5$ are given. A complete list of these mask entries can be found in the preprint version of this paper [27]. The mask entries corresponding to the box–spline case can be derived by applying Theorem 4.1.

Structure of $(a_\beta)_{\beta\in J}$, $n=2$

β	-2	-1	0	1	2
2	$\cdot$	e	$-d_2$	b_1	$\cdot$
1	$-e$	$-d_1$	$-b_0$	a_1	$-c_1$
0	d_0	b_0	a_0	$-c_0$	$-d_2$
-1	$-b_1$	a_2	c_0	$-d_1$	$\cdot$
-2	a_3	c_1	d_0	$\cdot$	$\cdot$

Coefficients of $(a_\beta)_{\beta\in J}$, $n=2$

a_0	8.432940468331447E–1	b_0	3.389216872059052E–1	d_0	4.932029262350007E–2
a_1	1.358816741784712E–1	b_1	9.752076062522183E–4	d_1	4.249567159311051E–2
a_2	1.805671110607621E–2	c_0	1.232795955187407E–1	d_2	6.824621030389556E–3
a_3	2.767567882307886E–3	c_1	1.936777548856011E–2	e	1.737898637156958E–2

Structure of $(a_\beta)_{\beta\in J}$, $n=5$

β	-5	-4	-3	-2	-1	0	1	2	3	4	5
5	$\cdot$	$-n_0$	m_1	$-k_1$	j_3	$-h_2$	g_5	$-e_3$	d_7	$-b_4$	$\cdot$
4	n_0	$-m_0$	k_0	$-j_0$	h_1	$-g_3$	e_2	$-d_5$	$-b_3$	a_7	c_4
3	m_2	$-k_0$	j_1	$-h_0$	g_2	$-e_1$	d_4	b_2	$-a_5$	$-c_3$	d_7
2	k_1	j_2	h_0	g_1	e_0	$-d_3$	b_1	$-a_3$	c_2	$-d_5$	$-f_3$
1	j_4	$-h_1$	$-g_0$	$-e_0$	$-d_1$	$-b_0$	a_1	$-c_1$	d_4	$-f_2$	g_5
0	h_2	g_4	e_1	d_0	b_0	a_0	$-c_0$	$-d_3$	f_0	$-g_3$	$-i_2$
-1	g_6	$-e_2$	$-d_2$	$-b_1$	a_2	c_0	$-d_1$	$-f_1$	g_2	$-i_1$	j_3
-2	e_3	d_6	$-b_2$	$-a_4$	c_1	d_0	f_1	g_1	$-i_0$	$-j_0$	$-l_1$
-3	d_8	b_3	a_6	$-c_2$	$-d_2$	$-f_0$	$-g_0$	i_0	j_1	l_0	m_1
-4	b_4	a_8	c_3	d_6	f_2	g_4	i_1	j_2	$-l_0$	$-m_0$	$\cdot$
-5	a_9	c_4	d_8	f_3	g_6	i_2	j_4	l_1	m_2	$\cdot$	$\cdot$

Coefficients of $(a_\beta)_{\beta \in J}$, $n = 5$

a_0	7.787365342903857E–1	e_1	8.992853890361197E–3	g_5	4.876740403433029E–5
a_1	1.563206966265842E–1	e_2	4.867871825262807E–4	g_6	1.569912663076266E–5
a_2	7.749999830045678E–2	e_3	3.477462756715321E–5	h_0	1.024618153667222E–2
a_3	8.271155512703776E–3	f_0	4.955451636014475E–3	h_1	2.407816279003956E–3
a_4	4.291537021864232E–3	f_1	4.729118599754660E–3	h_2	1.352189335862617E–4
a_5	2.915647698692102E–4	f_2	1.999788847486403E–4	i_0	2.948892987237558E–5
a_6	2.550502699488863E–4	f_3	5.643548015985818E–6	i_1	2.276548845913641E–5
a_7	3.760716383906979E–5	d_0	5.468640976381089E–2	i_2	5.191667165275111E–6
a_8	4.321779481775408E–6	d_1	5.028215879034868E–2	j_0	3.835472950458153E–4
a_9	4.887374074329685E–8	d_2	3.833252785442346E–3	j_1	2.658082630184461E–4
b_0	3.654554805658362E–1	d_3	2.232738245573430E–3	j_2	6.986281217643783E–5
b_1	2.228105086040924E–3	d_4	1.658784423038543E–3	j_3	3.357475727011834E–5
b_2	1.058617395881934E–3	d_5	1.752875013720132E–4	j_4	1.430146258081304E–5
b_3	1.056413112561339E–4	d_6	1.625484670874893E–4	k_0	1.005097425068883E–3
b_4	4.586135448870520E–7	d_7	1.197168631220558E–5	k_1	9.243533118832179E–5
c_0	1.856456585832774E–1	d_8	3.722982487333346E–6	l_0	4.082324106247016E–6
c_1	1.137766276335270E–2	g_0	1.271177367436286E–2	l_1	1.181464415989627E–6
c_2	1.547666424679079E–3	g_1	1.023267728924880E–2	m_0	9.536737680593730E–6
c_3	5.507259305046014E–5	g_2	2.754417575852340E–3	m_1	6.292715888320798E–6
c_4	1.275804213824031E–6	g_3	8.987066789608154E–4	m_2	3.244021792272933E–6
e_0	8.546073131670083E–2	g_4	5.589189575574359E–4	n_0	1.727830918819457E–5

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