Frames of Poisson wavelets on the sphere

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Abstract

In this paper we show that for sufficiently dense grids Poisson wavelets on the sphere constitute a weighted frame. In the proof we will only use the localization properties of the reproducing kernel and its gradient. This indicates how this kind of theorem can be generalized to more general reproducing kernel Hilbert spaces. With the developed technique we prove a sampling theorem for weighted Bergman spaces.

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1. The main theorem

A family of vectors \( \{ f_i, i \in I \} \subset \mathcal{H} \) in a Hilbert space \( \mathcal{H} \) indexed by some measure space \( I \) with positive measure \( \rho \) is called a frame with weight \( \rho \) if the mapping \( i \mapsto f_i \) is weakly measurable, i.e., \( i \mapsto \langle f_i, s \rangle \) is measurable, and if for some \( 0 \leq \varepsilon < 1 \) we have

\[
(1 - \varepsilon) \| s \|^2 \leq \int_I |\langle f_i, s \rangle|^2 \, d\rho(i) \leq (1 + \varepsilon) \| s \|^2
\]

for all \( s \in \mathcal{H} \). Equivalently, the frame condition reads

\[
\left| \int_I |\langle f_i, s \rangle|^2 \, d\rho(i) - \| s \|^2 \right| \leq \varepsilon \| s \|^2.
\]

If \( \varepsilon = 0 \), we call it a tight frame. Most of the time, the measure is the discrete measure on some countable index set.

If \( \{ f_i, i \in I \} \) is a frame, then Neumann theorem applies to the frame operator \( F^* F \), where \( F : \mathcal{H} \to L^2(I, \rho) \), \( F(s) = \langle f_i, s \rangle \), and its inverse can be computed by Neumann series. With a little modification one obtains the following algorithm to invert the operator \( F \).

**Proposition 1.** Let \( \{ f_i, i \in I \} \) be a frame with weight \( \rho \). Given the samples \( v_0(i) = \langle f_i, s \rangle, i \in I \), of a signal \( s \in \mathcal{H} \), construct recurrently

\[
s_k = F^* v_k = \int_{i \in I} v_k(i) f_i d\rho(i),
\]

\[
v_{k+1}(i) = (F s_k)(i) - v_k(i) = \langle f_i, s_k \rangle - v_k(i), \quad i \in I,
\]

for \( k \in \mathbb{N}_0 \). Then the sequence \( s_k \) converges to \( s \) in the topology of \( \mathcal{H} \).

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Some good surveys on the theory of frames can be found in [5] and [6].

1.1. The wavelets

We consider families of Poisson wavelets on the unit sphere $\Omega$. They are defined for $a > 0$ and $x, y \in \Omega$ through

$$\mathbf{g}^d_{y,a}(x) = \sum_{l=1}^{\infty} \gamma_l(a) \mathbf{Q}_l(\cos \theta) \quad \text{for} \quad \gamma_l(t) = t^d e^{-t}, \quad \theta = \angle(x, y).$$

$\mathbf{Q}_l : [-1, 1] \to \mathbb{R}$ is given by $\frac{2l+1}{4\pi} P_l$, where $P_l$ is the Legendre polynomial of order $l$. By the Funcke–Hecke formula,

$$\mathbf{Q}_l \star s = \delta_{k,ls} \quad \text{for} \ s \in \overline{\text{Harm}_l},$$

it is the reproducing kernel of the space $\text{Harm}_l$ of harmonic functions of degree $l$. We refer to [13] for further details on spherical Poisson wavelets. It is shown there that $\{\mathbf{g}_{x,a} \mid (x, a) \in \Omega \times \mathbb{R}^+\}$ is a tight frame with weight $(1/\Gamma(2d)) d\omega(x) da/a$, with $d\omega$ normalized surface measure on $\Omega$. The parameter space of wavelets is $(x, a) \in \Omega \times \mathbb{R}^+ =: \mathbb{H}$ so that $x$ is the position of the wavelet and $a$ is the scale. For a justification of $a$ as scale see also [13]. The mapping $L^2(\Omega, d\omega) \otimes \{C_{\text{te}}\} \ni s \mapsto \mathcal{W}s(x, a) = \{\mathbf{g}_{x,a}, s\}$, where $\{C_{\text{te}}\}$ denotes the subspace of constant functions, is called the Poisson wavelet transform and is an isometry

$$\int_\Omega |s(x)|^2 d\omega(x) = \frac{1}{\Gamma(2d)} \int_\Omega \int_0^\infty |\mathcal{W}s(x, a)|^2 \frac{da}{a} d\omega(x),$$

and the range of $\mathcal{W}$ is a Hilbert space with reproducing kernel

$$\Pi(x, a; y, b) = \frac{1}{\Gamma(2d)} \langle \mathbf{g}_{x,a}, \mathbf{g}_{y,b} \rangle.$$

1.2. The grid

In this paper we want to show that purely discrete frames of Poisson wavelets exist. We introduce the density of a grid of points in the unit ball with respect to the natural measure of Poincaré’s model of a hyperbolic space, compare, e.g., [19,7].

**Definition 1.** We say a grid of points inside the unit ball $\mathbb{B}$ is of density $\rho$ if any hyperbolic ball inside $\mathbb{B}$ with radius $\rho$ with respect to the metrics $d\zeta_0 := \frac{2}{1-r^2} (dr, h d\theta, h \sin \theta d\phi)$

(in spherical coordinates, $h \in \mathbb{R}^+$) contains at least one grid point.

With this notion we may state the main theorem.

**Theorem 1.** For any $d \geq 3$ and any $h > 0$ there exists a number $\rho$ such that for any grid $\mathcal{Y}$ in $\mathbb{B}$ with density $\rho$ with respect to the metrics $\zeta_0$ the corresponding family of Poisson wavelets, $\{\mathbf{g}_{(\theta, \phi), \log \lambda} \mid (\lambda, \phi, \zeta) \in \mathcal{Y}\}$, is a weighted frame for $L^2(\Omega)$.

In the case of wavelet analysis based on group representations, the existence of a density bound for frames has been shown in [9]. For wavelets $g$ over $\mathbb{R}$ with supp $\check{g} \subset [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$ explicit bounds for the density of sampling are given in [10] so that the set of sampled coefficients constitutes a weighted frame. In this case, first scales, and then positions are sampled. Another result concerning frames of wavelets over the real line sampled first over the scales, and then over positions, is presented in [15] and it states that for band-limited wavelets $g$ with a certain decay, the existence of frames can be ensured by a condition on the density of the set of dilations. However, a sampling density for the corresponding translations is not explicitly given. Some more results on sampling density for wavelet frames over $\mathbb{R}$ can be found in [16]. Similar sampling theorems for functions in the unit ball have been proven in [7].

Our proof will be based on localization of the reproducing kernel. The proof is constructive, in such a way that it gives, in principle, an explicit formula for a density bound.

A similar problem of constructing frames of wavelets over the sphere was considered in [1]. The authors prove the existence of discrete frames of stereographic wavelets defined in [2,3], where first the scales and then the positions are discretized. They consider only semi-angelic discretizations of positions.

Weighted frames of Poisson wavelets satisfying conditions of Theorem 1 have been successfully used in numerical applications, compare [14,4]. In both articles, scales $a = \log \lambda$ are chosen to be in a geometric progression. For each scale, one has a discrete set of positions $(\theta, \phi)$ with a proper density. Holschneider et al. [14] adapt the projection of hierarchical
subdivision of a cube onto the sphere, whereas Chambodut et al. [4] employ an icosahedron and subdivision of its facets into congruent triangles.

Our result is a justification for these applications and a generalization to more irregular grids.

The strategy of the proof is as follows: We first establish the existence of semi-discrete frames, where only the scales are discretized. The positions are discretized in a second step. The central estimate for the frame bounds relies on some general principles about frames in reproducing kernel Hilbert spaces. In order to apply this estimate to the wavelet case, we have to prove the localization of the reproducing kernel of Poisson wavelets. Finally, we vary the positions along the scales and formulate the density result.

2. Frames in reproducing kernel Hilbert spaces

The proof of the main theorem will be based on the following general principles which link tight frames with reproducing kernel Hilbert spaces and the characterization of general frames in such Hilbert spaces.

Let $\mathcal{V} = L^2(I, d\rho)$ be a Hilbert space of functions over $I$ with reproducing kernel $K(i, j)$

$$s(i) = \int K(i, j)s(j) d\rho(i).$$

The family of functions $\{f_i = K(i, \cdot)\}$ with $i \in I$ is a tight frame with weight $d\rho$. Indeed, we may write using $(f_i, s) = s(i)$

$$\int |(f_i, s)|^2 d\rho(i) = \int |s(i)|^2 d\rho(i).$$

Vice versa, a tight frame $\{f_i, i \in I\}$ and weight $\rho$ in some Hilbert space $\mathcal{H}$ are naturally associated with a reproducing kernel Hilbert space of functions in $L^2(I, d\rho)$, as shown by the next theorem.

**Theorem 2.** The mapping $\mathcal{F} : \mathcal{H} \rightarrow L^2(I, d\rho)$, $\mathcal{F}s(i) = (f_i, s)$ is a partial isometry and the image $\mathcal{U}$ of this mapping is characterized by the reproducing kernel

$$K(i, j) = (f_i, f_j).$$

That means, $u \in L^2(I, d\rho)$ is in the range of $\mathcal{F}$ if and only if

$$\int K(i, j)u(j) d\rho(j) = u(i).$$

Note that the last integral is absolutely convergent since $K(i, \cdot)$ is in $L^2(I, d\rho)$.

**Proof.** That the mapping is an isometry is simply the definition of what it means to be a tight frame. For the second statement set $u(i) = \mathcal{F}s(i) = (f_i, s)$. We have $|u(i)| \leq \|f_i\|\|s\|$, and thus by isometry $\|s\| = \|u\|$,

$$|u(i)| \leq \|f_i\|\|u\|,$$

and the point evaluation is a continuous functional. That the reproducing kernel is actually given by the expression above, follows by an application of Fubini's theorem. □

We shall make use of these general statements in the following way.

**Proposition 2.** Let $\{f_i, i \in I\}$ be a tight frame with weight $\rho$ on $\mathcal{H}$. A family $\{f_z\}$ with $z \in \Lambda \subset I$ and a measure $d\lambda$ is a frame for $\mathcal{H}$ if and only if $\{K(z, \cdot), x \in \Lambda\}$, $K(x, y) = (f_x, f_y)$, is a frame for $\mathcal{U}$, the image of $\mathcal{F}$.

**Proof.** Use the fact that $\mathcal{F}$ is an isometry. □

Now, frames of the form $\{K(z, \cdot)\}$ can be characterized as follows:

**Theorem 3.** Let $\Lambda \subset I$ and let $d\lambda$ be a measure on $\Lambda$. The family of functions $\{s_z = K(z, \cdot), z \in \Lambda\} \subset L^2(I, d\rho)$ (where $\rho$ is a measure on $I$) is a frame with weight $d\lambda$, for $\mathcal{U}$ if and only if

$$F(x, y) = \int_{\Lambda} K(x, z)K(z, y) d\lambda(z) - K(x, y)$$

is the kernel of a bounded operator $\mathcal{F}$ on $\mathcal{U}$ with $\|F\| < 1$. 


Note that in view of the identity \( K(x, y) = \int K(x, z)K(z, y)\, d\rho(z) \) the theorem shows that the existence of frames is intimately linked to the existence of good quadrature rules for functions in \( \mathcal{U} \).

**Proof.** The proof is adapted from [12]. We may write for arbitrary \( g \in \mathcal{U} \subset L^2(I, d\rho) \)

\[
\int_A |\langle s_z, g \rangle|^2 \, d\lambda(z) = \int_A \left( \int K(z, x)g(x)\, d\rho(x) \right)^2 \, d\lambda(z).
\]

This integral is absolutely convergent since \( K(z, \cdot) \in L^2(I, d\rho) \), and hence it is equal to

\[
\int \int K(x, z)K(z, y)\, dg(x)g(y)\, d\rho(x)\, d\rho(y)\, d\lambda(z).
\]

By hypothesis we have \( \int A K(x, z)K(z, y)\, dg(x)g(y)\, d\rho(x)\, d\rho(y) = F(x, y) + K(x, y) \). Thus,

\[
\int |\langle s_z, g \rangle|^2 \, d\lambda(z) - \|g\|^2 = \int \int F(x, y)\, dg(x)g(y)\, d\rho(x)\, d\rho(y),
\]

and we may conclude since \( \mathbb{F} \) is self-adjoint. \( \square \)

We will use this general principle together with the following perturbation result.

**Corollary 1.** Suppose, for a set \( A \) the family \( \{s_z = K(z, \cdot), \ z \in A\} \) is a weighted frame for \( \mathcal{U} \) with weight \( \lambda \). If now for some other set \( J \) we have for \( \{u_j = K(j, \cdot), \ j \in J\} \subset \mathcal{U} \) and some weight \( \gamma \) that

\[
G(x, y) = \int_A K(x, z)K(z, y)\, d\lambda(z) - \int J K(x, j)K(j, y)\, d\gamma(j)
\]

is the kernel of an operator \( G \) with operator norm \( \|G\| \leq 1 - \|\mathbb{F}\| \), where the kernel of \( \mathbb{F} \) is given by (2), then \( \{u_j, \ j \in J\} \) is a frame with weight \( \gamma \).

**Proof.** Simply apply triangular inequality. \( \square \)

### 3. Semi-continuous frames of Poisson wavelets

In this section, we establish the existence of semi-discrete frames based on discretization of the scales. In Theorem 4 we show that for any decreasing sequence of scales such that the ratio of two successive scales is bounded from below and from above the corresponding family of Poisson wavelets is a frame. Then, in Theorem 5 we prove the existence of parameters for sequences of scales such that the corresponding wavelet families are frames with prescribed \( \epsilon \). These frames are then perturbed through further sampling of the positions.

**Theorem 4.** Let \{\( g^d_j \)\} be a Poisson wavelet family of order \( d \), \( A = \{a_j\}_{j \in \mathbb{N}_0} \) a decreasing sequence of scales, and \( v_j = \log(a_j/a_{j+1}) \) the corresponding weights. Further, for all \( j \in \mathbb{N}_0 \) let \( v_j \) satisfy

\[
c_1 \leq v_j \leq c_2
\]

for some constants \( c_1, c_2 > 0 \). Then, there exists a constant \( C \) such that \{\( g_{x, a_j}: x \in \Omega, \ a_j \in A \)\} is a frame with weight \( C \sum v_j \delta_{a_j}(\alpha)\, d\omega(x) \) for \( L^2(\Omega, d\omega) \).

**Proof.** Let the degree \( d \) be fixed. Consider the function \( \gamma_d: t \mapsto t^d e^{-t} \). By the Funcke–Hecke formula it is enough to show that for all \( t > 0 \) we have

\[
A \leq \sum_{j} |\gamma_d(a_j t)|^2 v_j \leq B,
\]

with constants \( 0 < A < B \). Once it is shown, one sets the constant \( C \) to be equal to \( \frac{2}{\pi e^2} \).

The function \( \gamma_d \), as well as \( \gamma_d^2 \), has its maximum for \( d \). Fix an integer number \( l \). In the case \( \log(a_0) < \log(d) - c_2 \), there exists an index \( k \) such that \( a_0 \leq a_k l \leq d \) and the sum \( \sum_{j=0}^\infty \gamma_d^2(a_j l) v_j \) is larger than the integral \( \frac{\gamma_d^2}{\alpha} \, d\alpha \). Consequently, it is larger than \( \int_{0}^{\alpha} \gamma_d^2(\alpha)\, d\alpha \). Otherwise, one can find an index \( k \) such that \( \log(a_k l) \leq \log(d) - c_2 \leq \log(a_{k-1} l) \) and

\[
\sum_{j=0}^\infty \gamma_d^2(a_j l) v_j \geq \sum_{j=a_{k-1}}^{\infty} \gamma_d^2(a_j l) v_j \geq \int_{0}^{\alpha} \gamma_d^2(\alpha)\, d\alpha \geq \int_{0}^{\alpha} \gamma_d^2(\alpha)\, d\alpha.
\]
Hence, the first inequality in (3) is satisfied with

$$A = \min \left\{ \int_0^{a_0} \gamma_d^2(\alpha) \frac{d\alpha}{\alpha}, \int_{c_2}^{d\alpha} \gamma_d^2(\alpha) \frac{d\alpha}{\alpha} \right\}. $$

For the upper bound, let $k$ be again the smallest index such that $\log(a_k l) \leq \log(d) - c_2$. Then

$$\sum_{j=k}^{\infty} \gamma_d^2(a_j l) \cdot v_j \leq \sum_{j=k}^{\infty} \gamma_d^2(a_j l) \cdot v_{j-1} \frac{c_2}{c_1},$$

and since $a_{k-1} l \leq d$, the sum on the right-hand side is less than

$$\int_0^{d} \gamma_d^2(\alpha) \frac{d\alpha}{\alpha} \cdot \frac{c_2}{c_1}.$$ 

Further, let $\kappa$ be the largest index such that $a_k l \geq d$, supposed to exist. If $\kappa > 0$, the sum $\sum_{j=0}^{\kappa} \gamma_d^2(a_j l) v_j$ is majorized by the integral $\int_{a_{\kappa} l}^{\infty} \gamma_d^2(\alpha) d\alpha/\alpha$. Further, for indices $j$ between $\kappa$ and $k - 1$, the value of $\gamma_d^2(a_j l)$ is not larger than $\gamma_d^2(d)$, and the sum of $v_j$ (equal to the difference between $\log(a_k)$ and $\log(a_k)$) cannot exceed $3c_2$. Altogether, the second inequality in (3) is satisfied with

$$B = \frac{c_2}{c_1} \int_0^{d} \gamma_d^2(\alpha) \frac{d\alpha}{\alpha} + \int_{a_{\kappa} l}^{\infty} \gamma_d^2(\alpha) \frac{d\alpha}{\alpha} + 3c_2 \gamma_d^2(d). \quad \square$$

**Theorem 5.** Let $\{g_d^a \colon a \in \mathbb{R}_+\}$ be a Poisson wavelet family of order $d$. For any $\epsilon > 0$ there exist constants $a_0$ and $X$ such that for any sequence $A = (a_j)_{j \in \mathbb{N}_0}$ with $a_0 \geq a_0$ and $1 < a_j / a_{j+1} < X$ the family $\{g_{x, a_j} \colon x \in \Omega, \ a_j \in A\}$ is a semi-continuous frame for $L^2(\Omega)$, satisfying the frame condition (1) with the prescribed $\epsilon$.

**Proof.** Let $v_j$ be given by $\frac{\log(a_j)}{I} \int_0^I |\gamma(\alpha)|^2 \frac{d\alpha}{\alpha}$ for $I = \int_0^{a_0} |\gamma(\alpha)|^2 \frac{d\alpha}{\alpha}$. For $\gamma_d \colon t \to t^d e^{-t}$, compare the value of the series $I \cdot \sum_{j=0}^{a_0} |\gamma_d(a_j l)|^2 v_j$ with $I$:

$$\left| I \cdot \sum_{j=0}^{a_0} |\gamma_d(a_j l)|^2 v_j - \int_0^{a_0} |\gamma_d(\alpha)|^2 \frac{d\alpha}{\alpha} \right| \leq \sum_{j=0}^{a_0} \|\gamma_d(a_j l)|^2 \int_{a_j l}^{a_{j+1} l} \frac{d\alpha}{\alpha} - \int_{a_j l}^{a_{j+1} l} |\gamma_d(\alpha)|^2 \frac{d\alpha}{\alpha} + \int_{a_j l}^{a_{j+1} l} |\gamma_d(\alpha)|^2 \frac{d\alpha}{\alpha},$$

The inner integral in the series may be estimated by $\int_{a_j l}^{a_{j+1} l} \|\gamma_d^2(\beta)| \frac{d\beta}{\alpha}$ independently of $\alpha$, and the integral $\int_{a_j l}^{a_{j+1} l} \frac{d\alpha}{\alpha}$ is bounded by $\log X$. When summing up over all scales and decreasing the lower bound in the second integral to $a_0$ we obtain

$$I \cdot \sum_{j=0}^{a_0} |\gamma_d(a_j l)|^2 v_j - 1 \leq \log X \cdot \int_0^{a_0} \|\gamma_d^2(\beta)| \frac{d\beta}{\alpha} + \int_{a_0}^{a_\ell} |\gamma_d(\alpha)|^2 \frac{d\alpha}{\alpha}.$$ 

Since the integral $\int_0^{a_0} \|\gamma_d^2(\beta)| \frac{d\beta}{\alpha}$ is finite, there exist $X$ and $a_0$ such that the right-hand side of this inequality is less than $I \cdot \epsilon$. By the Funcke–Hecke formula, the frame condition is satisfied. \quad \square

**4. Discrete frames in wavelet phase space**

In this section we show a general theorem that links the localization of the reproducing kernel and its derivative to the existence of fully discrete frames as perturbation of purely scale discrete frames. We formulate it in a slightly abstract way, to stress the influence of the localization of the reproducing kernel in this theorem.

Consider a Hilbert space of functions over $\Omega \times \mathbb{R}_+$ with reproducing kernel $\Pi$

$$s(x, a) = \int \Pi(x, a; y, b) s(y, b) \, d\omega(y) \frac{db}{b}.$$ 

Examples of such spaces are precisely given by the wavelet coefficients.
Let $\mathcal{B} \subset \mathbb{R}^+$ be a set of scales $b_k$ with $b_{k+1} < b_k$ and $v_k = \log(b_k/b_{k+1})$ bounded $x \leq v_k \leq X$ for some $x, X > 1$.

We suppose that $[\Pi(x, a; \cdot, \cdot), (x, a) \in \Omega \times \mathcal{B}]$ is a frame with respect to the weight $C \sum v_k \delta_k d\omega(x)$ with some constant $C$,

$$(1 - \Delta)\|s\|^2 \leq C \sum_k \int_{\Omega \times \mathbb{R}_+} \Pi(x, b_k; y, b)s(y, b) d\omega(y) \frac{db}{b} |d\omega(x)| \leq (1 + \Delta)\|s\|^2.$$ 

The discretization of the grid is performed as in the definition.

**Definition 2.** We say a grid $\Lambda \in \Omega \times \mathbb{R}_+$ is of type $(X, Y, \delta)$ if the following holds: There is a sequence of scales $\mathcal{B} = (b_j)_{j \in \mathbb{N}_0}$ such that the ratio $b_j/b_{j+1}$ is uniformly bounded from below and from above with the lower bound larger than 1 (i.e., the sequence $(b_j)_{j \in \mathbb{N}_0}$ is decreasing) and upper bound equal to $X$

$$\Xi \leq b_j/b_{j+1} \leq X, \quad \Xi > 1.$$ 

At each scale $b = b_j$, there is a measurable partition $\mathcal{P}_b = \bigcup_{k=1}^{K_b} \Omega(b_k)$ into simply connected sets such that the diameter of each set (measured in central angle) is not larger than $Yb^{1+\delta}$. Each point of the grid is in exactly one of these sets.

The theorem of this section can now be formulated as follows.

**Theorem 6.** If in addition the reproducing kernel $\Pi$ satisfies

$$\begin{align*}
|\Pi(x, a; y, b)| &\leq (a + b)^2 + \varepsilon \cdot \frac{\delta}{\lambda a + (2 - \varepsilon)b}, \\
|a + b| &\Pi(x, a; y, b)| &\leq \lambda \|a + b\|_\infty + \varepsilon \cdot \frac{\delta}{\lambda a + (2 - \varepsilon)b},
\end{align*}$$

for $a, b \leq b_0$ and for some positive constants $\delta, \lambda, \varepsilon$ and $\varepsilon < 1/2$, where $\Pi_\ast$ is the surface gradient with respect to any of the variables $x$ or $y$, then there exists a constant $\rho$, such that for any grid $\Lambda \subset \Omega \times \mathcal{B}$ of type $(\delta, Y)$ with $\rho \leq \rho$ the family $[\Pi(y, b; \cdot, \cdot), (y, b) \in \Lambda]$ is a frame with weight $C \sum \mu(y, b) \delta_k \delta_y$ for $\mu(y, b) = \omega(\Omega(b))$, $y \in \Omega(b)$.

The proof makes use of a convolution estimate for functions over the parameter space $\mathbb{H} = \Omega \times \mathbb{R}_+$. First we need a lemma, which is somehow analogous to Young inequality for $\mathbb{R}^n$.

**Lemma 1.** Denote by $\mathbb{H}$ the space $\mathbb{R}_+ \times \mathbb{R}_+$ with the measure $(\theta d\theta, da/a)$. Let $F$ be such a function $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ that

$$F(x, a; y, b) = \frac{1}{b^2} \int \left( \frac{\Pi(x, y; a)}{b} \right) f, \quad f \in L^1(\mathbb{H}),$$

and $T \in L^p(\mathbb{H})$, $p \geq 1$. Then the following holds

$$\|F \circ T\|_{L^p(\mathbb{H})} \leq 2\pi \|f\|_{L^1(\mathbb{H})} \|T\|_{L^p(\mathbb{H})},$$

where the operation $\circ$ is defined by

$$F \circ T(x, a) = \int_{\mathbb{H}} F(x, a; y, b) T(y, b) d\omega(y) \frac{db}{b}.$$ 

**Proof.** Let $R$ be a non-negative function in $L^q(\mathbb{H})$ with $p^{-1} + q^{-1} = 1$. We may also suppose, that $F$ and $T$ are non-negative. Then

$$\langle F \circ T, R \rangle = \int_{\mathbb{H}} \int_{\mathbb{H}} F \circ T(x, a) R(x, a) d\omega(x) \frac{da}{a}$$

$$= \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{1}{b^2} f \left( \frac{\Pi(x, y; a)}{b} \right) T(y, b) R(x, a) d\omega(y) \frac{db}{b} d\omega(x) \frac{da}{a}.$$ 

By change of variables $a/b \mapsto a$ and exchanging the integrals (since all functions are positive, the integrals may only converge absolutely) we obtain

$$\langle F \circ T, R \rangle = \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{1}{b^2} f \left( \frac{\Pi(x, y; a)}{b} \right) R(x, ab) d\omega(x) \frac{da}{a} T(y, b) d\omega(y) \frac{db}{b}.$$ 


Consider the inner integral with respect to $\omega(x)$, which we write for simplicity as $\int g(x \cdot y) r(x) \, d\omega(x)$. Let $A = A_y$ be the isometry of the sphere which maps $y$ to the North Pole $\hat{e}$ and $\hat{e}$ to $y$.

Then

$$\int g(x \cdot y) r(x) \, d\omega(x) = \int g(Ax \cdot y) r(Ax) \, d\omega(Ax)$$

$$= \int g(x \cdot A^* y) r(Ax) \, d\omega(x) = \int g(x \cdot \hat{e}) r(Ax) \, d\omega(x).$$

Now, $Ax$ describes the position of the point $x$ relative to the point $y$ (depending also on the position of the North Pole). Let $x$ be fixed; by $R_x$ we denote the function $(y, a) \mapsto R(A_y x, a) (= r(Ax))$. Since $A$ was an isometry, we have

$$\int_{\Omega} R_x(y, a) \, d\omega(y) = \int_{\Omega} R(y, a) \, d\omega(y).$$

Then we have (once again exchanging the integrals)

$$\langle F \circ T, R \rangle = \int_{\mathbb{H}} \int_{\mathbb{R}_+} \int_{\Omega} R_x(y, ab) T(y, b) \, d\omega(y) \frac{1}{b^2} f\left(\frac{\theta}{b}, a\right) \frac{d\omega(x)}{a}.$$

where $\theta = \angle(x, \hat{e})$, and further, by Hölder inequality,

$$\langle F \circ T, R \rangle \leq \int_{\mathbb{H}} \int_{\mathbb{R}_+} \|R(\cdot, ab)\|_{L^q(\Omega)} \|T(\cdot, b)\|_{L^p(\Omega)} \frac{1}{b^2} f\left(\frac{\theta}{b}, a\right) \frac{d\omega(x)}{a}.$$

Now, the integral over $\Omega$ may be estimated as follows:

$$\int_{\Omega} \frac{1}{b^2} f\left(\frac{\theta}{b}, a\right) \, d\omega(x) = 2\pi \int_{0}^{\pi/b} \frac{1}{b^2} f\left(\frac{\theta}{b}, a\right) \sin \theta \, d\theta = 2\pi \int_{0}^{\pi/b} f\left(\frac{\theta}{b}, a\right) \frac{\sin(b\theta)}{b} \, d\theta$$

$$\leq 2\pi \int_{0}^{\pi/b} f\left(\frac{\theta}{b}, a\right) \, d\theta \leq 2\pi \int_{0}^{\pi} f\left(\frac{\theta}{a}\right) \, d\theta = 2\pi \|f(\cdot, a)\|_{L^1(\mathbb{R}_+, b \, d\theta)},$$

and therefore, by Hölder inequality with respect to $db/b$,

$$\langle F \circ T, R \rangle \leq 2\pi \int_{\mathbb{R}_+} \|f(\cdot, a)\|_{L^1(\mathbb{R}_+, b \, d\theta)} \|T(\cdot, b)\|_{L^p(\Omega)} \|R(\cdot, ab)\|_{L^q(\Omega)} \frac{da \, db}{b}.$$

$$\leq 2\pi \int_{\mathbb{R}_+} \|f(\cdot, a)\|_{L^1(\mathbb{R}_+, b \, d\theta)} \frac{da}{a} \cdot \|T\|_{L^p(\mathbb{E})} \|R\|_{L^q(\mathbb{E})}$$

$$= 2\pi \|f\|_{L^1(\mathbb{E})} \|T\|_{L^p(\mathbb{E})} \|R\|_{L^q(\mathbb{E})}.$$

Therefore, we have by the Riesz representation theorem

$$\|F \circ T\|_{L^p(\mathbb{E})} \leq 2\pi \|f\|_{L^1(\mathbb{E})} \|T\|_{L^p(\mathbb{E})}.$$

Since by assumption all the norms are finite, the exchanges of integrals were justified. □

We can now proceed with the proof of Theorem 6.

**Proof of Theorem 6.** According to the above convolution estimate and the general perturbation of frame theorem it is enough to show that

$$D = \left| \sum_{(y, b) \in A} \Pi(x, a; y, b) \Pi(y, b; z, c) \mu(y, b) - C \sum_{b \in B} \Pi(x, a; y, b) \Pi(y, b; z, c) \, d\omega(y) \nu(b) \right|$$

is less than
\[
\frac{1}{C} \cdot \delta \cdot \frac{1}{c^2} f \left( \frac{\zeta(x, z)}{c} \right)
\]

for some \( f \in C^1(\mathbb{R}) \) with \( \|f\| = \frac{1}{C^2} \) and \( \delta \in (0, 1 - \Delta) \).

For fixed \((x, a), (z, c)\) and \(b\), set \( F(y) = \Pi(x; a, y, b) \) and \( G(y) = \Pi(z; c, y, b) \). Let \( K^x \) denote the set of points, where \( F \) is ‘large’, i.e., \( K^x = \{ y \in \Omega : \zeta(x, y) \leq \lambda(a + b) \} \). Similarly, denote by \( K^z \) the set ‘large’, i.e., \( K^z = \{ y \in \Omega : \zeta(y, z) \leq \lambda(c + b) \} \).

If the sets \( K^x \) and \( K^z \) are not disjoint, we split the error that one makes by exchanging integration over \( \Omega \) by summation over \( \{ y \in \Omega : (y, b) \in A \} \) into two parts

- \( l_1(b) : F(y) \) ‘large’ or \( G(y) \) ‘large’, i.e., over the set \( \mathcal{D} = K^x \cup K^z \);
- \( l_4(b) : F(y) \) ‘small’ and \( G(y) \) ‘small’, i.e., for \( G = \Omega \setminus (K^x \cup K^z) \).

In the other case, if the sets \( K^x \) and \( K^z \) have an empty intersection, we consider three parts:

- \( l_2(b) : F(y) \) ‘large’, \( G(y) \) ‘small’, i.e., for \( \mathcal{L} = K^x \);
- \( l_3(b) : F(y) \) ‘small’, \( G(y) \) ‘large’, i.e., for \( \mathcal{E} = K^z \);
- \( l_4(b) : F(y) \) ‘small’, \( G(y) \) ‘small’, i.e., for \( G = \Omega \setminus (K^x \cup K^z) \).

Each of the errors may be estimated in the following way: for every set \( \mathcal{O} = \mathcal{O}_k^{(b)} \) the difference between the highest and the lowest value of \( F(\eta) \cdot G(\eta) \), \( \eta \in \mathcal{O} \), is less than or equal to

\[
\sup_{\eta \in \mathcal{O}} \left| \nabla \left( F(\eta) \cdot G(\eta) \right) \right| \cdot \text{diam}(\mathcal{O}),
\]

and hence the difference between

\[
\begin{align*}
\left( \sup_{\eta \in \mathcal{O}} \left| \nabla_s F(\eta) \right| \cdot \sup_{\eta \in \mathcal{O}} |G(\eta)| \right) + \left( \sup_{\eta \in \mathcal{O}} |F(\eta)| \cdot \sup_{\eta \in \mathcal{O}} \left| \nabla_s G(\eta) \right| \right) \cdot \text{diam}(\mathcal{O}) \cdot \mu(y, b).
\end{align*}
\]

When summing up over all the sets \( \mathcal{O} \) that have a non-empty intersection with one of the sets \( \{ D, E, F \text{ or } G \} \), we may calculate suprema over the whole set and choose the largest possible \( \text{diam}(\mathcal{O}) = Yb \), which we denote by \( r \). The sum of \( \mu(y, b) \) is then not larger than the area of \( \mathcal{Y} \) multiplied by \( v(b) \), with \( \mathcal{Y} \) denoting the \( r \)-parallel extension of \( \mathcal{Y} \), i.e.,

\[
\mathcal{Y} = \left\{ \eta \in \Omega \mid \exists y \in \mathcal{Y} : \zeta(y, \eta) \leq r \right\},
\]

where \( \mathcal{Y} \) means one of the sets \( D, E, F \text{ or } G \).

We introduce the notation \( a = a/c, \beta = b/c, \theta = \zeta(x, z), \vartheta = \theta/c \) and \( f_j(\alpha, \vartheta) = \sum_b c^2 I_j(b) \) for \( j = 1, 2, 3, 4 \) and \( b \in B \), but possibly not all the scales. The constant \( c \) may change its value from line to line.

**Part 1.** For \( l_1 \) we have

\[
C^2 I_1(b) \leq C^2 \cdot \left( \frac{1}{a + b} + \frac{1}{c + b} \right) \cdot \frac{(ab)^{2+\varepsilon}}{(a + b)^{6+2\varepsilon}} \cdot \frac{(bc)^{2+\varepsilon}}{(c + b)^{6+2\varepsilon}} \cdot Yb \cdot \omega(Dr) \cdot v(b).
\]

The set \( D \) contained in \( K^x \cup K^z \) and hence, the area of \( D \) is bounded by \( 2 \cdot \text{area}(K_r) \), where \( K \) is the larger of the circles \( K^x \) and \( K^z \). This is given by \( 2\pi \lambda^2 \cdot (c + (1 + Y/\lambda)b)^2 \leq \varepsilon(a + b)^2 \) if \( a < c \), respectively \( 2\pi \lambda^2 \cdot (a + (1 + Y/\lambda)b)^2 \leq \varepsilon(a + b)^2 \) if \( a \geq c \). In the case \( \alpha \leq 1 \), we obtain from (8):

\[
C^2 I_1(b) \leq CY \cdot \frac{\alpha^{2+\varepsilon}}{(a + \beta)^{2+\varepsilon/2}} \cdot \frac{\beta^{5+3\varepsilon/2}}{(a + \beta)^{5+3\varepsilon/2}} \cdot \frac{\beta^{\varepsilon/2}}{(1 + \beta)^{\delta+2\varepsilon}} \cdot v(b).
\]

The second fraction is smaller than 1, and the last one ensures the summability over \( b \), thus, for \( \vartheta \leq \lambda(\alpha + 1) \) we have the estimation

\[
\begin{align*}
f_1 & \leq CY \cdot \alpha^{\varepsilon/2}.
\end{align*}
\]

**A1**

For large \( \vartheta \), \( \vartheta > \lambda(\alpha + 1) \), we use the fact that the sets \( K^x \) and \( K^z \) have a non-empty intersection only for \( b \) such that \( \lambda(\alpha + 2\beta + 1) \geq \vartheta \), i.e., \( 2(1 + \beta) \geq \vartheta/\alpha + 1 - \alpha \), and therefore we may enlarge the last fraction in the estimation (9), and write

\[
\frac{\beta^{\varepsilon/2}}{(1 + \beta)^{\delta+2\varepsilon}} \leq \frac{\beta^{\varepsilon/2}}{(1 + \beta)^{\delta+2\varepsilon}} \cdot \frac{c}{\vartheta + \lambda(1 - \alpha)}.
\]

Consequently, we obtain

\[
\begin{align*}
f_1 & \leq CY \cdot \frac{\alpha^{\varepsilon/2}}{\vartheta + \lambda(1 - \alpha)}.
\end{align*}
\]

**B1**
In the other case, $\alpha > 1$, we get
\[
c^2 l_1(b) \leq c Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha + \beta)^{4+2\epsilon}} \cdot \frac{\beta^{5+2\epsilon}}{(1 + \beta)^{7+2\epsilon}} \cdot v(b).
\]
For $\vartheta \leq \lambda(1 + \alpha)$ we then have
\[
f_1 \leq c Y \cdot \frac{1}{\alpha^{2+\epsilon}}, \tag{C1}
\]
and for $\vartheta > \lambda(1 + \alpha)$ we write
\[
f_1 \leq c Y \cdot \frac{\vartheta/\lambda > \alpha}{(\alpha + \beta)^{2+\epsilon}} \cdot c Y \cdot \frac{1}{[\vartheta + \lambda(\alpha - 1)]^{2+\epsilon}}, \tag{D1}
\]
since for $b$ we take into account the relation $2(\alpha + \beta) \geq \vartheta/\lambda + \alpha - 1$ holds.

**Part 2.** In the second case, $l_2(b)$, we consider only the scales for which $K^x$ and $K^z$ have an empty intersection, i.e., $b$ such that $\vartheta > \lambda(\alpha + 2\beta + 1)$. For the error made in the whole set $E$ we use the formula (7) with $\mu(y, b)$ replaced by the area of $E_c$ (i.e., the area of $(K^z)_c$ multiplied by $v(b)$). The supremum of the modules of $G$ and $\nabla \times G$ is estimated by their values in the point nearest $K^z_c$. Since we have to consider all the sets $C_k(b)$ that have a non-empty intersection with $E_c$, we choose the angular argument in (5) to be equal to $\vartheta - \lambda(a + b) - r$.

We have to assume that the maximal diameter of a partition set is less than $c \cdot \lambda b$, with some $c < 1/2$. For the sake of simplicity, we set $r \leq \lambda b/3$. Altogether we obtain
\[
c^2 l_2(b) \leq c^2 \cdot c \cdot \left(\frac{1}{a + b} + \frac{1}{c + b}\right) \cdot \frac{(ab)^{2+\epsilon}}{(a + b)^{5+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{(c + b)^{6+2\epsilon}} \cdot Y b \cdot (a + b)^2 \cdot v(b). \tag{10}
\]
Further, in the considered range of scales we have $\vartheta/\lambda > \alpha + 2\beta + 1$, and this inequality implies $\vartheta - \lambda(\alpha + 4\beta/3) > [\vartheta + \lambda(2 - \alpha)]/3$ as well as $\vartheta - \lambda(a + 4b/3) > c(a + c)$.

For $\alpha \leq 1$, we write the estimation (10) in the form
\[
c^2 l_2(b) \leq c Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha + \beta)^{4+\epsilon}} \cdot \frac{\beta^{4+\epsilon}}{(1 + \beta)^{1+2\epsilon}} \cdot \frac{1}{[\vartheta + \lambda(2 - \alpha)]^2} \cdot v(b),
\]
that yields
\[
f_2 \leq c Y \cdot \frac{\alpha}{[\vartheta + \lambda(2 - \alpha)]^2}, \tag{A2}
\]
and for $\alpha > 1$ we have
\[
c^2 l_2(b) \leq c Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha + \beta)^{3+\epsilon}} \cdot \frac{\beta^{1+\epsilon}}{(1 + \beta)^{1+2\epsilon}} \cdot \frac{1}{[\vartheta + \lambda(2 - \alpha)]^2} \cdot v(b);
\]
consequently,
\[
f_2 \leq \frac{c Y}{\alpha [\vartheta + \lambda(2 - \alpha)]^2}. \tag{B2}
\]

**Part 3.** Similarly as in the previous case, we obtain from
\[
c^2 l_3(b) \leq c^2 \cdot c \cdot \left(\frac{1}{a + b} + \frac{1}{c + b}\right) \cdot \frac{(ab)^{2+\epsilon}}{[\vartheta - \lambda(c + 4b/3)]^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{(c + b)^{6+2\epsilon}} \cdot Y b \cdot (c + b)^2 \cdot v(b) \tag{11}
\]
the estimations
\[
c^2 l_3(b) \leq c Y \cdot \alpha^{2+\epsilon} \cdot \frac{1}{[\vartheta + \lambda(2\alpha - 1)]^3} \cdot \frac{\beta^{4+2\epsilon}}{(\alpha + \beta)^{4+2\epsilon}} \cdot \frac{\beta}{(1 + \beta)^{4+2\epsilon}} \cdot v(b)
\]
for $\alpha \leq 1$ and
\[
c^2 l_3(b) \leq c Y \cdot \alpha^{2+\epsilon} \cdot \frac{1}{[\vartheta + \lambda(2\alpha - 1)]^3} \cdot \frac{\beta^{2+\epsilon}}{(\alpha + \beta)^{2+\epsilon}} \cdot \frac{\beta^5}{(1 + \beta)^{5+2\epsilon}} \cdot v(b)
\]
for $\alpha > 1$. They yield
\[ f_2 \leq c_Y \cdot \frac{\alpha^{2+\epsilon}}{[\theta + \lambda(2\alpha - 1)]^2} \] (A3)
for $\alpha \leq 1$ and
\[ f_2 \leq c_Y \cdot \frac{\alpha^{2+\epsilon}}{[\theta + \lambda(2\alpha - 1)]^5} \] (B3)
for $\alpha > 1$.

**Part 4.** a) Consider first large $\theta$ and small scales $b$, that is, satisfying the condition $\theta > \lambda(a + 2b + c)$. For the points $y$ on the sphere that lie closer to the spherical circle $K^x$, i.e., elements of the set
\[ \mathcal{R}_x := \{ y \in \Omega \setminus K^x : \zeta(x, y) - \lambda x \leq \zeta(z, y) - \lambda c \}, \] (12)
and for one set $\mathcal{O} = \mathcal{O}_k^{(b)}$, we estimate the error using formula (7); the terms $\sup_{\eta \in \mathcal{O}} |G(\eta)|$ and $\sup_{\eta \in \mathcal{O}} |\nabla G(\eta)|$ may be replaced by the largest possible value in the $r$-parallel extension of $\mathcal{R}_x$, i.e.
\[ \sup_{\eta \in \mathcal{O}} |G(\eta)| \leq \frac{c \cdot (cb)^{2+\epsilon}}{(c + b) \cdot \theta^2_{b+2\epsilon}} \] resp. $\sup_{\eta \in \mathcal{O}} |\nabla G(\eta)| \leq \frac{c \cdot (cb)^{2+\epsilon}}{(c + b) \cdot \theta^2_{b+2\epsilon}} \] (13)
with
\[ \theta = \lambda c + \frac{\theta - \lambda(a + c)}{2}, \quad r \geq \frac{\theta + \lambda(2c - a)}{3} \geq \lambda \left( c + \frac{2b}{3} \right). \]

Further, $\sup_{\eta \in \mathcal{O}} |\nabla G(\eta)| \cdot \mu(y, b)$ resp. $\sup_{\eta \in \mathcal{O}} |F(\eta)| \cdot \mu(y, b)$ may be estimated by
\[ \frac{(ab)^{2+\epsilon}}{a + b} \int_{\mathcal{O}} \frac{d\omega(y)}{(\zeta(x, y) - r)^{6+2\epsilon}} \] resp. $\frac{(ab)^{2+\epsilon}}{a + b} \int_{\mathcal{O}} \frac{d\omega(y)}{(\zeta(x, y) - r)^{6+2\epsilon}}$ multiplied by $\nu(b)$. The bound we obtain for the error is larger if we sum up over all the partition sets having a non-empty intersection with the complement of $K^x$ (with $\sup_{\eta \in \mathcal{O}} |G(\eta)|$ given by (13), a property that does not hold in the whole $(\Omega \setminus K^y)_y$). Since $r \leq \lambda b/3$, we obtain
\[ c^2 l_4^{(y)}(b) \leq c^2 \cdot \left( \frac{1}{a + b} + \frac{1}{c + b} \right) \int_{\mathcal{O}_x} \frac{(ab)^{2+\epsilon}}{(\zeta(x, y) - \lambda b/3)^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{\theta_{b+2\epsilon}} \cdot \nu(b), \]
where $l_4^{(y)}(b)$ means the error made in the set $\mathcal{R}_x$ and $\Omega_x$ is the set $\{ y \in \Omega : \zeta(x, y) \geq \lambda(a + 2b/3) \}$. Denote $\zeta(x, y)$ by $\sigma$, then the integral is given by
\[ c^2 l_4^{(y)}(b) \leq c^2 \cdot \left( \frac{1}{a + b} + \frac{1}{c + b} \right) \cdot \int_{\mathcal{O}_x} \frac{(ab)^{2+\epsilon}}{(\sigma - \lambda b/3)^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{\theta_{b+2\epsilon}} \cdot \nu(b), \]
and upon replacing $\sin \sigma$ by $\sigma = (\sigma - \lambda b/3) + \lambda b/3$ and the upper integration bound $\pi$ by $\infty$, we obtain
\[ c^2 l_4^{(y)}(b) \leq c^2 \cdot \left( \frac{1}{a + b} + \frac{1}{c + b} \right) \cdot \frac{(ab)^{2+\epsilon}}{(a + b/3)^{4+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{(a + b/3)^{5+2\epsilon}} \cdot \nu(b). \]

For $\alpha \leq 1$ we can write:
\[ c^2 l_4^{(y)}(b) \leq c^2 \cdot \frac{\alpha^2}{\alpha + \beta} \cdot \frac{\alpha^2}{(\alpha + \beta/3)^{4+2\epsilon}} \cdot \frac{\beta^{1+\epsilon}}{\theta_{b+2\epsilon}} \cdot \frac{1}{\theta_{2}} \cdot \nu(b). \]

In the second case, $\alpha > 1$, the inequality (15) yields
\[ c^2 l_4^{(y)}(b) \leq c^2 \cdot \frac{\alpha^{2+\epsilon}}{(\alpha + \beta/3)^{3+\epsilon}} \cdot \frac{\beta^{2+\epsilon}}{(1 + \beta)(\alpha + \beta/3)^{1+\epsilon}} \cdot \frac{\beta^{3+\epsilon}}{(a + \beta/3)^{5+2\epsilon}} \cdot \nu(b). \]
Analogously for points closer to the other spherical circle, i.e., elements of
\[ \mathcal{R}_2 := \{ y \in \Omega \setminus \mathcal{K}^2 : \angle(x, y) - \lambda a > \angle(z, y) - \lambda c \}, \tag{18} \]
we obtain
\[ c_2 I_{24}^{(2)}(b) \leq c Y \cdot \left( \frac{1}{\alpha + \beta} + \frac{1}{1 + \beta} \right) \cdot \frac{\alpha^{2+\varepsilon} \beta^{2+2\varepsilon}}{\theta_x^{6+2\varepsilon} (1 + \beta/3)^{4+2\varepsilon}} \cdot v(b), \tag{19} \]
where \( \Omega_2 = \{ y \in \Omega : \angle(z, y) \geq \lambda(c + 2b/3) \} \) and
\[ \theta_x = \frac{\lambda a + \theta - \lambda(a + c)}{2} - r \geq \frac{\theta + \lambda(2a - c)}{3} \geq \frac{\lambda}{3} \left( a + \frac{2}{3}b \right) \]
and \( I_{24}^{(2)} \) is the error made in the set \( \mathcal{R}_2 \). The right-hand side of the inequality (19) may be enlarged so that we get
\[ c_2 I_{24}^{(2)}(b) \leq c Y \cdot \left( \frac{1}{\alpha + \beta} + \frac{1}{1 + \beta} \right) \cdot \frac{\alpha^{2+\varepsilon} \beta^{5+2\varepsilon}}{\theta_x^{6+2\varepsilon} (1 + \beta/3)^{4+2\varepsilon}} \cdot v(b), \tag{20} \]
and we write it for \( \alpha \leq 1 \) as
\[ c_2 I_{24}^{(2)}(b) \leq c Y \cdot \frac{\alpha^{2+\varepsilon}}{\theta_x^{6+2\varepsilon}} \cdot \frac{\beta^{4+2\varepsilon}}{(1 + \beta/3)^{4+2\varepsilon}} \cdot \frac{\beta}{(1 + \beta/3)^{4+2\varepsilon}} \cdot v(b). \tag{21} \]
If \( \alpha > 1 \), we use the factorization
\[ c_2 I_{24}^{(2)}(b) \leq c Y \cdot \frac{\alpha^{2+\varepsilon}}{\theta_x^{6}} \cdot \frac{\beta^{1+2\varepsilon}}{(1 + \beta/3)^{4+2\varepsilon}} \cdot \frac{1}{(1 + \beta/3)^{4+2\varepsilon}} \cdot v(b). \tag{22} \]
b) If \( \theta > \lambda(a + c) \) and \( b \) is such that \( \theta \leq \lambda(a + 2b + c) \), we estimate the error in a similar way, but we set
\[ \theta_x = \lambda(a + b) - r \quad \text{and} \quad \theta_z = \lambda(c + b) - r. \tag{23} \]
We obtain again the estimations (16), (17), (21) and (22). In the first two of them, the denominator of the third fraction is always larger than or equal to powered \( \lambda(1 + 2\beta/3) \), and hence it ensures the summability over \( b \); the second fraction is not larger than a constant. In the inequalities (21) and (22), one can replace the second fraction by a constant, since \( \theta_x \geq \lambda(\alpha + 2\beta/3) \). Further, the estimations
\[ \theta_x \geq \frac{\theta + \lambda(2a - c)}{3} \quad \text{and} \quad \theta_z \geq \frac{\theta + \lambda(2c - a)}{3} \]
are also valid for \( \theta_x \) and \( \theta_z \) defined by (23) if the range of scales is bounded by \( \theta/\lambda \leq a + 2b + c \), which is the case here. Consequently, we obtain from (16) and (21)
\[ f_4 \leq c Y \cdot \frac{\alpha}{(\theta + \lambda(2a - \alpha))^3 + c} \cdot \frac{\alpha^{2+\varepsilon}}{(\theta + \lambda(2a - 1))^3} \tag{A4} \]
for \( \alpha \leq 1 \) and from (17) and (22)
\[ f_4 \leq c Y \cdot \frac{1}{\alpha(\theta + \lambda(2a - \alpha))^3} + c \cdot \frac{\alpha^{2+\varepsilon}}{(\theta + \lambda(2a - 1))^6} \tag{B4} \]
for \( \alpha > 1 \).
c) Now, for \( \theta \leq \lambda(a + c) \), the spherical circles \( \mathcal{K}^x \) and \( \mathcal{K}^z \) have a non-empty intersection for all scales \( b \).
Since \( \Omega \setminus (\mathcal{K}^x \cup \mathcal{K}^z) \subseteq \Omega \setminus \mathcal{K}^2 \) and \( \sup_{\gamma \in \Omega \setminus \mathcal{K}^2}|G(\gamma)| = |G(\lambda(c + b) - r)| \), the inequality (15) with \( \theta_x \geq \lambda(c + 2b/3) \geq c(c + b) \):
\[ c_2 I_{4}(b) \leq c Y \cdot \left( \frac{1}{\alpha + \beta} + \frac{1}{1 + \beta} \right) \cdot \frac{\alpha^{2+\varepsilon} \beta^{3+2\varepsilon}}{(\alpha + \beta/3)^{4+2\varepsilon} (1 + \beta/3)^{6+2\varepsilon}} \cdot v(b) \]
yields an estimation of the error made in the whole set \( I_4 \). For \( \alpha \leq 1 \) we write it as
\[ c_2 I_{4}(b) \leq c Y \cdot \frac{\alpha^{2+\varepsilon}}{\alpha + \beta} \cdot \frac{\beta^{4+2\varepsilon}}{(\alpha + \beta/3)^{4+2\varepsilon}} \cdot \frac{\beta}{(1 + \beta/3)^{6+2\varepsilon}} \cdot v(b) \]
and obtain for the sum over all scales:
\[ f_4 \leq c Y \cdot \alpha^{1+\varepsilon}. \tag{C4} \]
In the opposite case, $\alpha > 1$, one has
\[
c^2 l_4(b) \leq c Y \cdot \frac{\alpha^{2+\epsilon}}{\lambda^2 + 3} \cdot \lambda^{5+2\epsilon} \cdot v(b),
\]
and consequently
\[
f_4 \leq c Y \cdot \frac{1}{\lambda^{2+\epsilon}}.
\]

The following table sorts the obtained estimations:

<table>
<thead>
<tr>
<th>$\theta \leq \lambda (\alpha + 1)$</th>
<th>$\theta &gt; \lambda (\alpha + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &gt; 1$</td>
<td>$[C1] \ [D4]$</td>
</tr>
</tbody>
</table>

Explicitly, we have
\[
f(\alpha, \theta) \leq Y \left( c \cdot \alpha^{\epsilon/2} + \frac{c \cdot \alpha^{1+\epsilon}}{\theta + \lambda (1 - \alpha)} \right) \quad \text{for } \alpha \leq 1 \text{ and } \theta \leq \lambda (\alpha + 1),
\]
\[
f(\alpha, \theta) \leq Y \left( \frac{c \cdot \alpha^{\epsilon}}{[\theta + \lambda (1 - \alpha)]^3} + \frac{c \cdot \alpha}{[\theta + \lambda (2 - \alpha)]^5} + \frac{c \cdot \alpha^{2+\epsilon}}{[\theta + \lambda (2\alpha - 1)]^5} \right) \quad \text{for } \alpha \leq 1 \text{ and } \theta > \lambda (\alpha + 1),
\]
\[
f(\alpha, \theta) \leq \frac{c Y}{\lambda^{2+\epsilon}} \quad \text{for } \alpha > 1 \text{ and } \theta \leq \lambda (\alpha + 1),
\]
\[
f(\alpha, \theta) \leq Y \left( \frac{c}{[\theta + \lambda (\alpha - 1)]^3} + \frac{c}{\alpha [\theta + \lambda (2 - \alpha)]^5} + \frac{c \cdot \alpha^{2+\epsilon}}{[\theta + (2\alpha - 1)]^5} \right) \quad \text{for } \alpha > 1 \text{ and } \theta > \lambda (\alpha + 1),
\]
and hence, $f$ is an $C^1$-integrable function over $\mathbb{R}^d$. Since the value of the integral depends linearly on the constant $Y$, it can be arbitrarily small. □

**Remark.** The choice $\lambda b/3 \leq \delta b$ does not influence the generality of the statements. If one takes as claimed $\lambda b/3 = \delta b$ for some $\delta > 0$, the resulting integrals change, but they are still convergent.

5. Localization of $g^d_l$ and its surface gradient

In order to apply the above general result to Poisson wavelets, we have to prove that they are localized as required in Theorem 6.

5.1. Localization of the wavelet

We now show the following localization:
\[
a^2 s^d_{N, a}(a\theta, \phi) \leq c \cdot \frac{e^{-a}}{\theta^{d+2}}, \quad \theta \in \left[0, \frac{\pi}{a} \right],
\]
holds uniformly in $a$ for some constant $c$. $N$ denotes the North Pole of the sphere. Since the wavelet is rotation invariant, we shall denote it by $g^d_l(a\theta)$.

The proof needs a technical lemma first.

**Lemma 2.** Let $E_d$, $d \in \mathbb{N}$, be a sequence of polynomials in two variables satisfying the recursion
\[
E_{d+1}(\lambda, y) = a_d(\lambda, y) \cdot E_d(\lambda, y) + b(\lambda, y) \cdot \frac{\partial}{\partial \lambda} E_d(\lambda, y)
\]
with
\[
a_d(\lambda, y) = 1 - 2(d + 1)\lambda^2 + (2d + 1)\lambda y \quad \text{and} \quad b(\lambda, y) = (1 + \lambda^2 - 2\lambda y)\lambda,
\]
and such that
\[
E_1(1, 1) = 0, \quad \text{and} \quad \frac{\partial}{\partial \lambda} E_1(1, 1) \bigg|_{\lambda = 1} \neq 0.
\]
Then the polynomial $E_d(1, \cdot)$, $d \geq 2$, has a $(d + 1)/2$-fold root in 1.
Proof. First, we shall prove by induction that
\[ \frac{\partial^k}{\partial \lambda^k} E_d(\lambda, 1) \bigg|_{\lambda=1} = 0 \quad \text{for } k = 0, 1, \ldots, d - 1, \quad \text{and} \quad \frac{\partial^d}{\partial \lambda^d} E_d(\lambda, 1) \bigg|_{\lambda=1} \neq 0. \] (27)

Let \( e_d = E_d(\cdot, 1) \); for \( d = 1 \) we have
\[ e_1(1) = 0 \quad \text{and} \quad e'_1(1) \neq 0 \]
by assumption. Suppose, for some \( d \),
\[ e_d^{(k)}(1) = 0 \quad \text{for } k = 0, 1, \ldots, d - 1, \quad e_d^{(d)}(1) \neq 0. \] (28)

We rewrite the relation (25) in the form
\[ e_{d+1}(\lambda) = a_d(\lambda) \cdot e_d(\lambda) + b(\lambda) \cdot e'_d(\lambda) \]
with
\[ a_d(\lambda) = 1 + (2d + 1) \lambda - 2(d + 1)^2 \quad \text{and} \quad b(\lambda) = (1 - \lambda)^2 \lambda. \]

Then, the \( k \)-th derivative of \( e_{d+1} \) is given by
\[ e_{d+1}^{(k)}(\lambda) = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) [a_d^{[j]}(\lambda) \cdot e_d^{(k-j)}(\lambda) + b^{[j]}(\lambda) \cdot e_d^{(k-j+1)}(\lambda)], \]
and since only the first and the second derivative of \( a_d \) and only the second and the third derivative of \( b \) do not vanish in \( \lambda = 1 \), we obtain
\[ e_{d+1}(1) = 0, \quad e'_{d+1}(1) = a'_d(1) \cdot e_d(1) \]
and
\[ e_{d+1}^{(k)}(1) = \left( \begin{array}{c} k \\ 1 \end{array} \right) a'_d(1) \cdot e_d^{(k-1)}(1) + \left( \begin{array}{c} k \\ 2 \end{array} \right) a''_d(1) \cdot e_d^{(k-2)}(1) + \left( \begin{array}{c} k \\ 3 \end{array} \right) b''(1) \cdot e_d^{(k-1)}(1) + \left( \begin{array}{c} k \\ 4 \end{array} \right) b'''(1) \cdot e_d^{(k-2)}(1) \quad \text{for } k \geq 2. \]

Consequently, \( e_{d+1}^{(d)}(1) = 0 \) for \( k \leq d \) and
\[ e_{d+1}^{(d+1)}(1) = -(d + 1)(d + 3) e_d^{(d)}(1) \neq 0. \]

Now, using the relation (28) we are able to prove that
\[ \frac{\partial^k}{\partial y^k} E_d(1, y) \bigg|_{y=1} = 0 \quad \text{for } k = 0, 1, \ldots, \left[ \frac{d + 1}{2} \right] - 1, \]
\[ \frac{\partial^k}{\partial y^k} E_d(1, y) \bigg|_{y=1} \neq 0 \quad \text{for } k = \left[ \frac{d + 1}{2} \right] \]
for \( d \geq 2 \). The formula (25) yields
\[ \frac{\partial^k}{\partial y^k} E_d(1, y) \bigg|_{y=1} = \frac{\partial^k}{\partial y^k} E_d(\lambda, y) \bigg|_{\lambda=1 \atop y=1} \]
\[ = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \left[ \frac{\partial^{j}}{\partial y^j} a_{d-1}(\lambda, y) \cdot \frac{\partial^{k-j}}{\partial y^{k-j}} E_{d-1}(\lambda, y) + \frac{\partial^{j}}{\partial y^j} b(\lambda, y) \cdot \frac{\partial^{k-j}}{\partial y^{k-j}} \frac{\partial}{\partial \lambda} E_{d-1}(\lambda, y) \right]_{\lambda=1 \atop y=1}. \]

Since \( E_{d-1} \) is a \( C^\infty \)-function, we can exchange the differentiation and the limit. The polynomials \( a_{d-1} \) and \( b \) are linear in \( y \), further, \( a_{d-1}(1, 1) = b(1, 1) = 0 \), and therefore the terms with \( j \neq 1 \) vanish and thus \( E_d(1, 1) = 0 \). Further, for \( k \geq 1 \),
\[ \frac{\partial^k}{\partial y^k} E_d(1, y) \bigg|_{y=1} = k \lambda \left( 2d - 2 \lambda \frac{\partial}{\partial \lambda} \right) \frac{\partial^{k-1}}{\partial y^{k-1}} E_{d-1}(\lambda, y) \bigg|_{\lambda=1 \atop y=1}. \] (29)
The last equation means, we were able to reduce the order of differentiation in \(y\) and the index of the polynomial; however, one more differentiation in \(\lambda\) is needed. A \(k\)-fold application of this procedure yields:

\[
\frac{\partial^k}{\partial y^k} E_d(1, y) \bigg|_{y=1} = \sum_{j=0}^{k} c_j(\lambda) \cdot \frac{\partial^j}{\partial \lambda^j} E_{d-k}(\lambda, 1) \bigg|_{\lambda=1},
\]

where \(c_j\) are some polynomials. If \(k \leq [(d+1)/2] - 1\), then \(2k \leq d+1-2 = d-1\), and further, \(k \leq (d-k) - 1\). Therefore, all the derivatives on the right-hand side of (30) vanish, and consequently \(\frac{\partial^k}{\partial y^k} E_d(1, y) \big|_{y=1} = 0\). For \(k = [(d+1)/2]\) one has \(k \geq d-k\) and hence \(\frac{\partial^k}{\partial y^k} E_{d-k}(\lambda, 1) \big|_{\lambda=1} \neq 0\). The polynomial \(c_k(\lambda)\) is equal to \((-2)^k \lambda^{2k!}\) (compare (29)), i.e., different from zero in \(\lambda = 1\). This yields \(\frac{\partial^{[(d+1)/2]}}{\partial y^{[(d+1)/2]}} E_d(1, y) \big|_{y=1} \neq 0\). \(\Box\)

**Lemma 3.** Let \(\{f_d\}\) be a family of functions over \((0, 1) \times [0, \pi]\) given by

\[
f_1(\lambda, \theta) = \frac{\lambda E_1(\lambda, \cos \theta)}{(1 + \lambda^2 - 2\lambda \cos \theta)^{1/2}},
\]

\[
f_{d+1}(\lambda, \theta) = \lambda \frac{\partial}{\partial \lambda} f_d(\lambda, \theta),
\]

where \(E_1\) is a polynomial satisfying (26). Then, for any \(k \geq 2[d/2] + 1\) there exists a constant \(c\) such that

\[
|f_d(\lambda, \theta)| \leq c \cdot \frac{\lambda}{k^2}, \quad \theta \in (0, \pi],
\]

uniformly in \(\lambda\). For \(d \geq 2\), the number \(2[d/2] + 1\) is the smallest possible exponent \(k\). If \(E_1(1, y)\) has a simple root in \(1\), then \(1\) is the smallest possible exponent \(k\) on the right-hand side of (31) for \(d = 1\).

**Proof.** For any \(d \in \mathbb{N}\), the function \(f_d\) is given by

\[
f_d(\lambda, \theta) = \frac{\lambda E_d(\lambda, \cos \theta)}{(1 + \lambda^2 - 2\lambda \cos \theta)^{d+1/2}},
\]

where \(E_d\) is a polynomial obtained recursively via (25). Consider the function \(\tilde{f} : (\lambda, \theta) \mapsto \frac{\partial^k}{\lambda^k} f_d(\lambda, \theta)\) and define \(F : [0, 1] \times [0, \pi]\) by

\[
F(\lambda, \theta) = \begin{cases}
E_d(0, \cos \theta), & \lambda = 0, \\
\tilde{f}(\lambda, \theta), & 0 < \lambda < 1, \\
\frac{1}{\lambda^k} \cdot E_d(1, \cos \theta), & \lambda = 1, \ \theta > 0, \\
0, & \lambda = 1, \ \theta = 0.
\end{cases}
\]

Since \(\lim_{\theta \to 0^+} \frac{2(1-\cos \theta)}{\theta^2} = 1\), one has

\[
\lim_{\theta \to 0^+} F(1, \theta) = \lim_{\theta \to 0^+} \frac{\theta^k}{\theta^2(1+2d+1/2)} \cdot \theta^{2(d+1)/2} : \frac{E_d(1, \cos \theta)}{(1 - \cos \theta)^{(d+1)/2}}.
\]

The sum of powers of \(\theta\) is equal to \(k - (2[d/2] + 1) \geq 0\); the limit of the last fraction exists according to the previous lemma. Therefore, the function \(F\) is continuous. It is a continuous extension of \(\tilde{f}\) to the compact set \([0, 1] \times [0, \pi]\). Consequently, the function \(\tilde{f}\) is bounded; this yields the desired inequality (31). For the minimality of \(k\) note that \([(d+1)/2]\) (multiplicity of the root of \(E_d(1, y)\) in \(y = 1\)) is the largest possible exponent in the last fraction that ensures that the limit of the fraction exists; further, \(\theta^k/\theta^{2(d+1)/2} \cdot \theta^{2(d+1)/2}\) would be divergent for \(\theta \to 0\) if \(k < 2[d/2] + 1\). \(\Box\)

Note that the critical point is around 0. For arguments \(\theta\) far from 0 the inequality (32) is valid for any \(k\).

Functions that satisfy the conditions of the lemma are, e.g., those describing the field generated by a multipole inside the Earth.

**Lemma 4.** Let

\[
\Psi^d_{\ldots}(\theta, \phi) \mapsto \sum_{l=0}^{\infty} P_l(\cos \theta) \lambda^l, \quad d \in \mathbb{N}_0,
\]

be the field on the sphere generated by the multipole (monopole for \(d = 0\)) \(\mu = (\lambda, \partial_\lambda)^d \delta_{\lambda, \phi, \cdot}\). For any \(k \geq 2[d/2] + 1\) there exists a constant \(c\) such that
\[ |\Psi_{\lambda}^{d}(\theta, \phi)| \leq c \cdot \frac{\lambda}{\theta^{d+2}}, \quad \theta \in (0, \pi], \quad (33) \]

uniformly in \( \lambda \). \( 2[d/2] + 1 \) is the smallest possible exponent on the right-hand side of this inequality.

**Proof.** The first multipole is given by

\[ \Psi_{\lambda}^{1}(\theta, \phi) = \frac{\lambda}{(1 + \lambda^2 - 2\lambda \cos \theta)^{3/2}} \cdot E_{1}(\lambda, \cos \theta), \]

with \( E_{1}(\lambda, y) = y - \lambda \). This polynomial satisfies the conditions (26) and its restriction to \( \lambda = 1 \) has a simple root in \( y = 1 \). Further

\[ \Psi_{\lambda}^{d+1} = \lambda \frac{\partial}{\partial \lambda} \Psi_{\lambda}^{d}. \]

Thus, the previous lemma applies and (33) holds for \( d \geq 1 \). For \( d = 0 \) the estimation may be proven in the same way as in the last lemma using the direct representation of the monopole:

\[ \Psi_{\lambda}^{0}(\theta, \phi) = \frac{1}{\sqrt{1 + \lambda^2 - 2\lambda \cos \theta}}. \quad \square \]

Now we may come to the localization of Poisson wavelets.

**Theorem 7.** Let \( g_{\alpha}^{d} \) be a Poisson wavelet family of order \( d \). Then there exists a constant \( c \) such that

\[ |a^{2}g_{\alpha}^{d}(a\theta)| \leq c \cdot \frac{e^{-a}}{\theta^{d+2}}, \quad \theta \in \left(0, \frac{\pi}{a}\right], \quad (34) \]

uniformly in \( a \). \( d + 2 \) is the largest possible exponent in this inequality.

**Proof.** The function

\[ f_{d} : (a, \theta) \mapsto g_{\alpha}^{d}(\theta)/a^{d} \]

can be expressed as a sum of fields generated by multipoles, \( f_{d}(a, \theta) = 2\Psi_{\alpha}^{d+1}(\theta, \phi) + \Psi_{\alpha}^{d}(\theta, \phi) \), see the article [13]. Since \( d + 2 \geq 2[(d + 1)/2] + 1 \geq 2[d/2] + 1 \) and according to the last lemma, the relation

\[ |f_{d}(a, \theta)| = \left| \frac{a^{2}g_{\alpha}^{d}(\theta)}{a^{d+2}} \right| \leq c \cdot \frac{e^{-a}}{\theta^{d+2}}, \quad \theta \in (0, \pi], \]

holds uniformly in \( a = -\log \lambda \). Upon replacing \( \theta \) by \( a\theta \) and multiplying both sides by \( a^{d+2} \), we obtain the desired inequality. For the second statement note that \( f_{d}(0, \theta) \) is a non-vanishing function of \( \theta \) and therefore \( (\frac{d}{a})^{d} f_{d}(a, \theta) \) diverges for \( a \to 0 \) for any positive exponent \( \epsilon \); thus, \( (a, \theta) \mapsto (\theta/a)^{d+2+\epsilon} a^{d+2}(a\theta) \) is not bounded. \( \square \)

**Remark.** This theorem may be proven directly with use of Lemma 3. One chooses \( E_{1}(\lambda, y) = 1 - \lambda^{2} \), then, \( f_{d+1}(e^{-a}, \theta) = g_{\alpha}^{d}(\theta)/a^{d} \).

**Corollary 2.** The functions \( (a, \theta) \mapsto a^{2}g_{\alpha}^{d}(a\theta) \) are uniformly bounded.

**Proof.** Take exponent 0 in (34). \( \square \)

Note that we investigate the behaviour of \( g_{\alpha}^{d}(a\theta) \), whereas in the Euclidean limit one has the expression \( g_{\alpha}^{d}(\Phi^{-1}(a\rho)) \) (where \( \Phi^{-1} \) is understood as a function of radius in spherical coordinates). The inequality

\[ a^{2}g_{\alpha}^{d}(\Phi^{-1}(a\rho)) \leq \frac{c}{\rho^{d+2}} \]

corresponding to (34) does not hold, we merely have

\[ a^{2}g_{\alpha}^{d}(\Phi^{-1}(a\rho)) = a^{2}g_{\alpha}^{d}\left(2 \arctan \frac{a\rho}{2}\right) \leq \frac{ca^{d+2}}{[2 \arctan(a\rho/2)]^{d+2}}, \]

and for \( \rho \) tending to infinity, the last fraction tends to \( c(a/\pi)^{d+2} \), i.e., does not vanish.
5.2. Localization of the colatitudinal derivative of the wavelet

Analogous statements can be made for the colatitudinal derivative of the wavelet \( g_d^d \). Since the longitudinal one is equal to zero, we immediately have

\[
|\nabla_\phi g_d^d(\theta, \phi)| \leq \frac{\partial}{\partial \theta} g_d^d(\theta, \phi)
\]

with

\[
\nabla_\phi g_d^d(\theta, \phi) = \left( \frac{\partial}{\partial \theta} g_d^d(\theta, \psi), \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} g_d^d(\theta, \phi) \right).
\]

**Lemma 5.** Let \([f_d]\) be a family of functions over \((0, 1) \times [0, \pi]\) given by

\[
f_1(\lambda, \theta) = \frac{\lambda \sin \theta E_1(\lambda, \cos \theta)}{(1 + \lambda^2 - 2\lambda \cos \theta)^{3/2}},
\]

\[
f_d+1(\lambda, \theta) = \frac{\lambda \partial}{\lambda} f_d(\lambda, \theta),
\]

where \(E_1\) is a polynomial satisfying (26). Then, for any \(k \geq 2[d/2] + 2\) there exists a constant \(\epsilon\) such that

\[
|f_d(\lambda, \theta)| \leq \epsilon \cdot \frac{\lambda}{\theta^k}, \quad \theta \in (0, \pi].
\]  

(35)

uniformly in \(\lambda\). For \(d \geq 2\), the number \(2[d/2] + 2\) is the smallest possible exponent \(k\). If \(E_1(1, y)\) has a simple root in 1, then 1 is the smallest possible exponent \(k\) on the right-hand side of (35) for \(d = 1\).

**Proof.** For any \(d \in \mathbb{N}\), the function \(f_d\) is given by

\[
f_d(\lambda, \theta) = \frac{\lambda \sin \theta \cdot E_d(\lambda, \cos \theta)}{(1 + \lambda^2 - 2\lambda \cos \theta)^{d+3/2}},
\]

where \(E_d\) is a polynomial obtained recursively via (25). Consider the function \(\tilde{f} : (\lambda, \theta) \mapsto \frac{\theta^k}{\lambda} \cdot f_d(\lambda, \theta)\) and define \(F : [0, 1] \times [0, \pi]\) by

\[
F(\lambda, \theta) = \begin{cases} 
\frac{\sin \theta \cdot E_d(0, \cos \theta),}{\tilde{f}(\lambda, \theta)}, & \lambda = 0, \\
\frac{\theta^k \sin \theta}{(1 - \cos \theta)^{d+3/2}} E_d(1, \cos \theta), & 0 < \lambda < 1, \\
0, & \lambda = 1, \theta > 0, \\
, & \lambda = 1, \theta = 0.
\end{cases}
\]

Since \(\lim_{\theta \to 0^+} \frac{2(1 - \cos \theta)}{\theta^d} = \lim_{\theta \to 0^+} \frac{2(1 - \cos \theta)^{1/2}}{\sin \theta} = 1\), one has

\[
\lim_{\theta \to 0} \frac{\theta^k}{\lambda \theta^{2(d+1)/2} - \theta^{2([d+1]/2)} \cdot \frac{E_d(1, \cos \theta)}{(1 - \cos \theta)^{d+1/2}}}. 
\]

The sum of powers of \(\theta\) is equal to \(k - (2[d/2] + 2) \geq 0\); the limit of the last fraction exists according to Lemma 2. Thus, the function \(F\) is a continuous extension of \(\tilde{f}\) to the compact set \([0, 1] \times [0, \pi]\), and further, the function \(\tilde{f}\) is bounded. This yields the desired inequality (35). For the minimality of \(k\) note that \([d + 1/2] \cdot \text{multiplicity of the root of } E_d(1, y) \text{ in } y = 1\) is the largest possible exponent in the last fraction that ensures that the limit of the fraction exists; further, \(\theta^k / \theta^{2(d+1)/2} \cdot \theta^{2([d+1]/2)}\) would be divergent for \(\theta \to 0\) if \(k < 2[d/2] + 2\).

Again, this lemma may be applied to (colatitudinal derivative of) the field generated by a multipole.

**Lemma 6.** Let

\[
\psi^{d}_{\lambda} : (\theta, \phi) \mapsto \sum_{l=0}^{\infty} P_l(\cos \theta)^d \lambda^l, \quad d \in \mathbb{N}_0,
\]

be the field on the sphere generated by the multipole (monopole for \(d = 0\)) \(\mu = (\lambda, \lambda)^d \delta_{\lambda,\delta}\). For any \(k \geq 2[d/2] + 2\) there exists a constant \(\epsilon\) such that

\[
\left| \frac{\partial}{\partial \theta} \psi^{d}_{\lambda}(\theta, \phi) \right| \leq \epsilon \cdot \frac{\lambda}{\theta^k}, \quad \theta \in (0, \pi],
\]

(36)

uniformly in \(\lambda\). \(2[d/2] + 2\) is the smallest possible exponent on the right-hand side of this inequality.
Proof. For the first multipole we have
\[
\frac{\partial}{\partial \theta} \Psi_\lambda^1(\theta, \phi) = \frac{\lambda \sin \theta E_1(\lambda, \cos \theta)}{(1 + \lambda^2 - 2\lambda \cos \theta)^{3/2}},
\]
with \( E_1(\lambda, y) = \lambda (-1 + 2\lambda^2 - \lambda \cos y) \). This polynomial satisfies the conditions (26) and its restriction to \( \lambda = 1 \) has a simple root in \( y = 1 \). Further
\[
\frac{\partial}{\partial \theta} \Psi_\lambda^d(\theta, \phi) = \frac{\lambda \sin \theta}{(1 + \lambda^2 - 2\lambda \cos \theta)^{3/2}}.
\]
Thus, the previous lemma applies and (36) holds for \( d \geq 1 \). For \( d = 0 \) the estimation may be proven in the same way as in the previous lemma, with use of the representation:
\[
\frac{\partial}{\partial \theta} \Psi_\lambda^0(\theta, \phi) = \frac{-\lambda \sin \theta}{(1 + \lambda^2 - 2\lambda \cos \theta)^{3/2}}.
\]
And now we may come to the localization of (the derivatives of) Poisson wavelets.

Theorem 8. Let \( \{g_0^d\} \) be a Poisson wavelet family of order \( d \). Then there exists a constant \( c \) such that
\[
\left| a^2 \frac{\partial}{\partial \theta} g_b^d(a\theta) \right|_{\theta=a} \leq \frac{c \cdot e^{-a}}{\theta^{d+2}}, \quad \theta \in \left( 0, \frac{\pi}{a} \right],
\]
uniformly in \( a \). \( d + 3 \) is the largest possible exponent in this inequality.

Proof. The function
\[
f_d : (a, \theta) \mapsto \frac{1}{a^d} \frac{\partial}{\partial \theta} g_a^d(\theta)
\]
can be written as
\[
f_d(a, \theta) = 2 \frac{\partial}{\partial \theta} \Psi_{e-a}^{d+1}(\theta, \phi) + \frac{\partial}{\partial \theta} \Psi_{e-a}^{d}(\theta, \phi).
\]
Since \( n + 3 \geq 2(d + 1)/2 + 2 \geq 2[d/2] + 2 \) and according to the last lemma, the relation
\[
\left| f_d(a, \theta) \right| = \frac{1}{a^d} \left| g_a^n(\theta) \right| \leq \frac{c \cdot e^{-a}}{\theta^{d+2}}, \quad \theta \in [0, \pi],
\]
holds uniformly in \( a = -\log \lambda \). Upon replacing \( \theta \) by \( a\theta \) and multiplying both sides by \( a^{d+3} \), we obtain the desired inequality. For the second statement note that \( f_d(0, \theta) \) is a non-vanishing function of \( \theta \) and therefore \( (\frac{\partial}{\partial \theta} g_b^d(a\theta) \) diverges for \( a \to 0 \) for any positive exponent \( \epsilon \); thus, \( (a, \theta) \mapsto (\theta/a)^{d+3+\epsilon} a^2 g_b^d(a\theta) \) is not bounded.

Corollary 3. The functions
\[
(a, \theta) \mapsto a^2 \frac{\partial}{\partial \theta} g_b^d(a\theta) \bigg|_{\theta=a}\n\]
are uniformly bounded.

Proof. Choose exponent 0 in (37).

6. Discrete frames of Poisson wavelets

Theorem 6 may be applied to Poisson wavelets of order \( d \geq 3 \).

Corollary 4. Let \( \{g_b^d\}, a \in \mathbb{R}_+ \), be a Poisson wavelet family of order \( d \geq 3 \). Then there exists a constant \( \rho \) such that for any grid of type \((b, Y)\) with \( Y \leq \rho \) the family \( \{g_b^d : y, b \in A\} \) is a frame with weight \( C \sum \mu(y, b) \delta_y \delta b \) for \( L^2(\Omega) \) for some \( C > 0 \).

Proof. One has to check that the assumptions of Theorem 6 are satisfied. The set of scales \( B \) is constructed in the same way as in Theorem 4 in Section 3, therefore, \( \{g_b^d : b \in B\} \) is a semi-continuous frame with weight \( \frac{2}{\lambda(b)} \) for \( v(b) = \log(b_1/b_{j+1}) \), and \( A, B \) as in formula (3).
It remains to check if the estimations on the kernel and its gradient hold. As shown in [13], the kernel is given by

$$\Pi(x, a; y, b) = \frac{(ab)^d}{(a + b)^{2d}} g_{a+b}^d(\zeta(x, y)),$$

and from Section 5, Theorem 7 we have

$$|g_{a+b}^d(\zeta(x, y))| \leq c \cdot \frac{(a + b)^{2d}}{\zeta(x, y)^{2d+2}}$$

uniformly in $\zeta(x, y), a, b$, further

$$|g_{a+b}^d(\theta)| \leq c \frac{c}{(a + b)^{2d}}$$

uniformly in $a, b$, see Section 5, Corollary 2. Since

$$\frac{(ab)^{d-1-\epsilon}}{\theta^{d(1-\epsilon)}} \leq c$$

for $\theta \geq \lambda(a + \epsilon b)$ and $\epsilon < 1$ and

$$\frac{(ab)^{d-1-\epsilon}}{(a + b)^{2d(1-\epsilon)}} \leq 1$$

for $\epsilon < 1$, the inequalities (5) are satisfied for the kernel.

For the surface gradient of the kernel we have

$$\nabla_\ast \Pi(x, a; y, b) = \frac{(ab)^d}{(a + b)^{2d}} \nabla_\ast g_{a+b}^d(\zeta(x, y)),$$

and since the longitudinal derivative of the wavelet vanishes, the absolute value of the gradient $\nabla_\ast g_{a+b}^d(\zeta(x, y))$ for any $x, y$ is less than or equal to the absolute value of the derivative with respect to $\theta$ for $\theta = \zeta(x, y)$. Theorem 8, Section 5 yields

$$|\nabla_\ast \Pi(x, a; y, b)| \leq c \cdot \frac{(ab)^d}{(a + b)^{2d}} \cdot \frac{(a+b)^{2d+3}}{\theta^{2d+3}}.$$

uniformly in $\theta$ and consequently

$$|(a + b)\nabla_\ast \Pi(x, a; y, b)| \leq c \cdot \frac{(ab)^d}{(a + b)^{2d}} \cdot \frac{(ab)^{2+\epsilon}}{\theta^{2d+2}}$$

for $\theta \geq \lambda(a + \epsilon b)$. On the other hand, we have

$$|\nabla_\ast g_{a+b}^d(\zeta(x, y))| \leq c \frac{c}{(a + b)^{2d}}$$

compare Corollary 3 in Section 5, and therefore

$$|(a + b)\nabla_\ast \Pi(x, a; y, b)| \leq c \cdot \frac{(ab)^d}{(a + b)^{2d+2}} \cdot \frac{(ab)^{2+\epsilon}}{(a + b)^{6+2\epsilon}}.$$

Thus, the inequalities (5) are satisfied and Theorem 6 applies to Poisson wavelets of order $d \geq 3$. \(\square\)

### 7. Density results for discrete frames

In this section we prove Theorem 1. First we show that a grid of density $\rho$ corresponds to a grid in $\Omega \times \mathbb{R}_+$ having some density properties and then we show that wavelet families corresponding to such grids are frames.

**Definition 3.** We say a grid $A \subset \Omega \times \mathbb{R}_+$ is of type $(b_0, X, Y)$ if the following holds: There is a decreasing sequence of scales $B = (b_j)_{j \in \mathbb{N}_0}$ such that $b_0 > b_0$ and the ratio $b_j/b_{j+1}$ is uniformly bounded from above by $X$. At each scale $b = b_j$, there is a measurable partition $\mathcal{P}_b = \{C_k^{(b)}: k = 1, 2, \ldots, K_b\}$ into simply connected sets such that the diameter of each set (measured in central angle) is not larger than $Yb$. In any of the sets $C_k^{(b)} \times (b_{j+1}, b_j]$ there is at least one point of the grid.

We need a technical lemma first.

**Lemma 7.** For any $d \leq 6\pi$ there exists a measurable partition of $\Omega$ into simply connected sets such that the diameter of each set (measured in central angle) is not larger than $d$ and the radius of the inscribed spherical circle is larger than $d/12$. 

Proof. If \( d \in [\pi, 6\pi] \), choose the partition into two half-spheres. Otherwise, if \( d < \pi \), divide the sphere into \( K = [3\pi/d] \) sets: \( \Omega_k := \{ (\theta, \phi) \in \Omega : \theta \in [k\pi/K, (k+1)\pi/K) \}, k = 0, \ldots, K-2 \), and \( \Omega_{K-1} := \{ (\theta, \phi) \in \Omega : \theta \in [(K-1)\pi/K, \pi) \} \). Since \( K \geq 4 \), the colatitudinal height of each of the sets is not larger than

\[
\frac{\pi}{3\pi/d} \cdot \frac{3\pi/d + 1}{3\pi/d} \leq d \cdot \frac{5}{4} \leq d.
\]

On the other hand, it is larger than

\[
\frac{\pi}{3\pi/d} \geq \frac{d}{3}.
\]

Further, divide each of the slices \( \Omega_k, k = 1, \ldots, K-2 \), into \( L = [3U/d] \) sets

\[
\Omega_{kl} := \{ (\theta, \phi) \in \Omega : \theta \in [j\pi/K, (j+1)\pi/K), \phi \in [2l\pi/L] \}, \quad l = 0, \ldots, L-1,
\]

where \( U \) is the circumference of the larger circle on the boundary of \( \Omega_k \). Then, for each \( \Omega_{kl} \), the length of the sides with constant \( \theta \) is not less than \( d/3 \). Further, since \( U \geq 2\pi \cdot \max\{\sin \frac{2d}{\pi}, 1\} \), we have \( [3U/d] \geq 7 \), and hence, the length of the larger side of \( \Omega_{kl} \) with constant \( \theta \) is not larger than

\[
\frac{U}{3\pi/d} \cdot \frac{3\pi/d + 1}{3\pi/d} \leq d \cdot \frac{8}{7} \leq d.
\]

The diameter of each of the sets \( \Omega_0, \Omega_{K-1} \), and \( \Omega_{kl}, k = 1, \ldots, K-2, l = 0, \ldots, L-1 \), is smaller than \( d \). Further, the inradius is larger than \( d/12 \) (for the sets \( \Omega_{kl} \) compare the inradius of plane equilateral triangle with sides larger than \( d/3 \)). \( \square \)

Now we show the relation between the grids.

Lemma 8. Let \( \Upsilon \) be a grid of density \( \rho \). The grid \( \Lambda := \{ (-\log \lambda, \theta, \phi) : (\lambda, \theta, \phi) \in \Upsilon \} \) is of type \((b_0, X, Y)\) for some \( b_0, X, \) and \( Y \).

Proof. Let \( r_j \) be given by \( \tanh((j+1)\rho) \) and \( b_j = -\log r_j, j \in \mathbb{N}_0 \). The sequence \((b_j)\) is decreasing and such that \( b_0 \) is larger than a given \( b_0 \) if \( \rho \) is small enough.

Now, consider the function \( f : t \mapsto -\log \tanh t \). We have

\[
\frac{b_j - b_{j+1}}{b_{j+1}} = -\rho \cdot g_\tau(j + 2) \rho \quad \text{with} \quad g_\tau(t) = \frac{f'(t + \tau)}{f(t)}
\]

for some \( \tau \in (-\rho, 0) \). The function \( g_\tau \) is continuous on \((0, \infty)\), and such that both limits \( t \to 0 \) and \( \tau \to \infty \) exist. Therefore, it is bounded and there exists a common bound \( \epsilon \) for all \( g_\tau \) with \( \tau \in [0, \rho] \). (Note that \( \epsilon \) increases if \( \rho \) increases.) Hence,

\[
\frac{b_j - b_{j+1}}{b_{j+1}} \leq \rho \cdot \epsilon
\]

independently of \( j \), and consequently the ratio \( b_j/b_{j+1} \) is arbitrarily close to 1 if \( \rho \) is small enough.

On every sphere with radius \( r_j \) choose a partition into simply connected sets \( O_{jk}, k = 1, \ldots, K_j \) for some \( K_j \), having diameter not larger than \( 12\rho \) and inradius larger than \( \rho \) (with respect to the metrics \( \tau_h \)). Its image under the transformation

\[
(\lambda, \theta, \phi) \mapsto (-\log \lambda, \theta, \phi)
\]

is a partition of the sphere with radius \( b_j \) into simply connected sets \( O^{(b_j)}_k \) having diameter not larger than

\[
b_j \cdot \frac{1 - r_j^2}{2h} \cdot 12\rho \leq \frac{6\rho}{h} \cdot b_j.
\]

It is smaller than or equal to \( Yb_j \) for \( \rho \leq Yh/6 \). Since the distance between \( r_j \) and \( r_{j+1} \) is equal to \( 2\rho \), every set \( O_{jk} \times [r_j, r_{j+1}) \) contains a ball of radius \( \rho \), consequently it contains at least one point of the grid \( \Upsilon \). Therefore, each set \( O^{(b_j)}_k \times (b_{j+1}, b_j) \) contains at least one point of the grid \( \Lambda \).

Hence, \( \Lambda \) is a grid of type \((b_0, X, Y, 0)\). Moreover, we have shown that any grid \( \Lambda \) corresponding to a grid \( \Upsilon \) with density \( \rho \) is of type \((b_0, X, Y)\) for a given set of parameters \( b_0, X, \) and \( Y \) if \( \rho \) is small enough. \( \square \)

In the end of the section, we prove a theorem on the existence of discrete wavelet frames.
Theorem 9. Let \( \{g_a : a \in \mathbb{R}^+\} \) be a wavelet family with the generating function \( \gamma \) such that \( \int_0^\infty ||\gamma_n^2(t)|| \, dt < \infty \) and the kernel \( \Pi \) satisfying
\[
|\Pi(x, a; y, b)| \leq (ab)^{2+\epsilon} \cdot \frac{c}{(a+b)^{3+\epsilon}} \cdot \frac{\epsilon}{(x,y)^{2\epsilon}} \cdot \begin{cases} \lambda |a + (2-\epsilon)b|, & \text{if } \gamma(x, y) \leq \lambda, \\ \lambda (a + \epsilon b), & \text{if } \gamma(x, y) > \lambda, \end{cases}
\]
for any \( a, b \leq b_0 \) and for some positive constants \( \alpha, \lambda, \epsilon, \) and \( \epsilon < 1/2 \). Then there exist constants \( b_0, X, \) and \( Y \) such that for any grid of type \( B_0, X, Y \) the family \( \{g_{x,b} : (x, b) \in A \} \) is a weighted frame for \( L^2(\Omega) \).

Proof. According to Theorem 5 there exist \( b_0 \) and \( X \) such that for any sequence of scales constructed as in Definition 3 the family \( \{g_{x,b} : x \in \Omega, b_j \in B \} \) is a semi-continuous frame for \( L^2(\Omega) \) satisfying \( \frac{8}{3} < A \leq 1 \leq B < \frac{10}{3} \). Then, by Corollary 1, we have to show that
\[
D = \left| \sum_{(y,b) \in A} \Pi(x, a; y, b)\Pi(y, b; z, c) \mu(y, b) - \sum_{b \in B \Omega} \int \Pi(x, a; y, b)\Pi(y, b; z, c) \, d\omega(y) \, v(b) \right|
\]
is less than
\[
\delta \cdot \frac{1}{c^2} f \left( \frac{\gamma(x, z)}{c} ; \frac{a}{c} \right)
\]
for some \( f \in L^1(\mathbb{K}) \) with \( \|f\| = \frac{1}{2\pi} \) and \( \delta \in (0, \frac{2}{3}) \). For any \( b \in (b_{j+1}, b_j] \) we set \( J(b) = b \) and split \( D \) as follows
\[
D \leq \left| \sum_{(y,b) \in A} \Pi(x, a; y, b)\Pi(y, b; z, c) \mu(y, b) - \sum_{(y,b) \in A} \Pi(x, a; y, J(b))\Pi(y, J(b); z, c) \mu(y, b) \right|
+ \left| \sum_{(y,b) \in A} \Pi(x, a; y, J(b))\Pi(y, J(b); z, c) \mu(y, b) - \sum_{b \in B_\Omega} \int \Pi(x, a; y, b)\Pi(y, b; z, c) \, d\omega(y) \, v(b) \right|
\]
The second summand is bounded by a multiple of \( Y \), compare the proof of Theorem 6. For the first summand we make similar estimations as in that theorem. If there is more than one grid point in a set \( \mathcal{O}_k \times (b_{j+1}, b_j] \), we choose the corresponding measure for each point to be the measure of the set divided by the number of points in it. This yields the same upper bound for the error made in that set as if there was only one point therein.

For fixed \( (x, a), (z, c) \) and \( y \), set \( F(b) = \Pi(x, a; y, b) \) and \( G(b) = \Pi(z, c; y, b) \). For each \( b \in \mathcal{O} \times (b_{j+1}, b_j] \) the difference between \( F(b)G(b) \) and \( F(b)G(b) \) is less than or equal to
\[
\int_{b_{j+1}}^{b_j} \left| d\left[ \frac{F(\tilde{b}) \cdot G(\tilde{b})}{db} \right] \right| \, d\tilde{b} \leq \sup_{b \in (b_{j+1}, b_j]} \left| \frac{d}{db} \left[ F(\tilde{b}) \cdot G(\tilde{b}) \right] \right| \cdot \log X
\]
and the supremum may be estimated as follows:
\[
\sup_{b \in (b_{j+1}, b_j]} \left| \frac{d}{db} \left[ F(\tilde{b}) \cdot G(\tilde{b}) \right] \right| \leq \sup_{b \in (b_{j+1}, b_j]} \left| \frac{d}{db} F(\tilde{b}) \right| \cdot \sup_{b \in (b_{j+1}, b_j]} |G(\tilde{b})| + \sup_{b \in (b_{j+1}, b_j]} |F(\tilde{b})| \cdot \sup_{b \in (b_{j+1}, b_j]} \left| \frac{d}{db} G(\tilde{b}) \right|.
\]
Using (38) and the fact that \( \Pi \) is symmetric with respect to the first and second pairs of variables we obtain
\[
\left| \frac{d}{db} F(\tilde{b}) \right| \leq (ab)^{2+\epsilon} \cdot \frac{c}{(a+b)^{3+\epsilon}} \cdot \begin{cases} |a + (2-\epsilon)b|, & \text{if } \gamma(x, y) \leq \lambda, \\ (a + \epsilon b), & \text{if } \gamma(x, y) > \lambda, \end{cases}
\]
for any \( \tilde{b} \in (b_{j+1}, b_j] \), consequently for the supremum over this interval. Analogous estimations (with \( a \) replaced by \( c \) and \( \gamma(x, y) \) replaced by \( \gamma(x, y) \)) hold for \( G \). When summing up over all \( \mathcal{O} \)'s on all scales, we obtain for the error analogous expressions as in the proof of Theorem 6. The differences are: the factor \( \frac{1}{b_{j+1}} + \frac{1}{b_j} \) vanishes and the upper bound for \( \text{diam}(\mathcal{O}) \), \( Yb_j \), is replaced by \( \log X \). Further, \( b_j \)'s in the numerator are replaced by \( b_j \leq Xb_{j+1} \) and in the denominator by \( b_{j+1} \). Therefore, we can use estimations from the proof of the last theorem multiplied by \( \frac{a+b}{\rho} \) in the case \( \alpha \leq 1 \), resp. \( \frac{1+\beta}{\rho} \) in the case \( \alpha > 1 \), with \( Y \) replaced by \( \log X \) and with a different constant \( c \). In all the cases, expressions for \( f_j \)'s are
similar as in the previous case, only with another constants ζ and Υ replaced by log X. Thus, the first summand of D is arbitrarily small for X small enough. Since by decreasing X the bounds of the semi-continuous frame become closer to 1, this proves the theorem. □

This theorem applies to Poisson wavelets of order \( d \geq 3 \).

8. Sampling sequences for Bergman spaces

Using Theorem 1, we prove that some weighted Bergman functions are uniquely defined by a countable set of values. Similarly as in [13], \( \mathcal{H}_{\text{int},d}(\Omega) \) denotes the homogeneous weighted interior Bergman space of harmonic functions in \( \text{Int} \Omega \), which are square integrable with respect to some weight, so that they satisfy

\[
\|s\|_{\mathcal{H}_{\text{int},d}(\Omega)}^2 = \frac{1}{4\pi I'(2d)} \int_{\text{Int} \Omega} |s(x)|^2 \log^{2d-1} (1/|x|^2) \frac{dx}{|x|^4} < \infty.
\]

Equivalently, the norm of \( \mathcal{H}_{\text{int},d}(\Omega) \) can be expressed in terms of Fourier coefficients as follows

\[
\|s\|_{\mathcal{H}_{\text{int},d}(\Omega)}^2 = \sum_{l>0} I^{-2d} \sum_{m=-l}^l |\hat{s}_{l,m}|^2,
\]

as can be shown by integration term by term of the Fourier series. Therefore, this space is actually a Hilbert space. Note that the classical Bergman space \( B(\Omega) \), which consists of all harmonic functions which are square summable over the ball, has the norm

\[
\|s\|_{B(\Omega)}^2 = \frac{1}{4\pi} \int_{\text{Int} \Omega} |s(x)|^2 dx = \sum_{l=0}^{\infty} \frac{1}{2l+3} \sum_{m=-l}^l |\hat{s}_{l,m}|^2.
\]

Thus, for functions of zero mean, \( \int s = 0 \), the Bergman norm and the norm in \( \mathcal{H}_{\text{int},1/2}(\Omega) \) are equivalent:

\[
\int s = 0 \quad \Rightarrow \quad \frac{1}{5} \|s\|_{\mathcal{H}_{\text{int},1/2}(\Omega)}^2 \leq \|s\|_{B(\Omega)}^2 \leq \frac{1}{2} \|s\|_{\mathcal{H}_{\text{int},1/2}(\Omega)}^2.
\]

As proven in Theorem 5 in [13], the image of the wavelet transform with respect to Poisson wavelet of order \( d \) is exactly the interior Bergman space of order \( d \). (Note the differences in terminology: Poisson wavelets and Poisson wavelet transform as defined in Section 1, in [13] are called external Poisson wavelets and internal Poisson wavelet transform, respectively.)

**Theorem 10.** For any \( d \geq 3 \) and any \( h > 0 \) there exists a number \( \rho \) such that for any grid \( \mathcal{Y} \) in \( B \) with density \( \rho \) with respect to the metrics \( \zeta_h \), any function \( f \in \mathcal{H}_{\text{int},d}(\Omega) \) is uniquely defined through its samples on the grid points. Moreover the inversion is stable in the sense that there is a weight \( \mu : \mathcal{Y} \rightarrow \mathbb{R}_+ \), such that the weighted sampling operator

\[
S : \mathcal{H}_{\text{int},d}(\Omega) \rightarrow L^2(\mathcal{Y}), \quad s \mapsto \mu s|_{\mathcal{Y}}
\]

is bounded and has a bounded left inverse \( K \), \( KS = I \).

In other words, the sequence of grid points is sampling for \( \mathcal{H}_{\text{int},d}(\Omega) \).

The operator \( K \) can be found using the algorithm from Proposition 1.

This result is analogous to one obtained by Seip [18] for classical Bergman spaces \( A^p \) over the unit disk in \( \mathbb{R}^2 \), i.e., harmonic functions with finite \( \mathcal{L}^p \)-norm, compare also [8, Theorem 10, p. 188]. It is shown there that every set in the unit disk in \( \mathbb{C} \) with density less than or equal to \( (1 + \sqrt{2}/\rho)^{-1} \) with respect to the pseudohyperbolic metrics \( \zeta_1 \) is a set of uniqueness for \( A^p \).

Similar results about weighted Bergman spaces over the unit disk can be found, e.g., in [18,17,11].

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**References**
