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Lattice Boltzmann model for two-dimensional unsteady Burgers' equation☆

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Abstract

In this paper, a special lattice Boltzmann model is proposed to simulate two-dimensional unsteady Burgers' equation. The maximum principle and the stability are proved. The model has been verified by several test examples. Excellent agreement is obtained between numerical predictions and exact solutions. The cases of steep oblique shock waves are solved and compared with the two-point compact scheme results. The study indicates that lattice Boltzmann model is highly stable and efficient even for the problems with severe gradients.

MSC: 74S30; 65M12; 34E05

Keywords: Lattice Boltzmann method; Two-dimensional Burgers' equation; 4-bit model; Multi-scale analysis; Maximum principle

1. Introduction

In recent years, the lattice Boltzmann method (LBM) has been developed into an alternative and promising numerical scheme for simulating fluid flows [1,8,3] and solving various mathematical–physical problems [7,4,10–12]. This method can be either regarded as an extension of the lattice gas automaton [6] or as a special discrete form of the Boltzmann equation for kinetic theory [5]. Unlike conventional numerical schemes based on discretizations of partial differential equations describing macroscopic conservation laws, the LBM is based on solving the discrete-velocity Boltzmann equation from statistical physics. It describes the microscopic picture of particles movement in an extremely simplified way, while on the macroscopic level it gives a correct average description.

The Burgers' equation, which is also called the nonlinear advection–diffusion equation, is a simplified model of Navier–Stokes equations. It retains the nonlinear aspects of the governing equation in many practical transport problems such as aggregation interface growth, the formation of large-scale structures in the adhesion model for cosmology, turbulence transport, shock wave theory, wave processes in thermoelastic medium, transport and dispersion of pollutants

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in rivers and sediment transport. Therefore, it is usually used to test different numerical methods. The unsteady two-dimensional Burgers' equation in one unknown variable take the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial y^2}, \quad x_0 \leqslant x \leqslant x_N, \quad y_0 \leqslant y \leqslant y_N, \quad t > 0, \tag{1}$$

with the initial condition $u(x, y, 0) = u_0(x, y)$. Here the viscous coefficient $v = 1/Re > 0$, Re is the Reynolds number. For a small value of v, Burgers' equation behaves merely as hyperbolic partial differential equation and the problem becomes very difficult to solve as a steep shock-like wave fronts developed.

Elton [4] and Shen et al. [11] have proposed lattice Boltzmann models for 2D Burgers' equation in which there is only one convective term. In the paper, we developed a 4-bit model for Eq. (1). By using Taylor expansion and multi-scale analysis, the time-dependent two-dimensional Burgers' equation is recovered from the lattice Boltzmann equation, and the local equilibrium distribution functions are obtained. It is generally recognized that the LBM is a Lagrangian discretization of a discrete-velocity Boltzmann equation. In this view, we find such lattice Boltzmann scheme satisfies maximum principle, therefore, we complete the proof of stability.

The rest of the paper is organized as follows. Section 2 proposes a lattice Boltzmann model and derives the 2D Burgers' equation from the model. A stability analysis of the LBM is also given in Section 3. In Section 4 some numerical experiments are made using our model. And the conclusions are given in the end.

2. Lattice Boltzmann method

According to the theory of the LBM, it consists of two steps: (1) streaming, where each particle moves to the nearest node in the direction of its velocity; and (2) colliding, which occurs when particles arriving at a node interact and possibly charge their velocity directions according to scattering rules. Usually, with the single-relaxation-time BGK approximation [2], these two steps can be combined into the following LBE:

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{\mathrm{eq}}), \tag{2}$$

where f_α is the distribution function of particles; f_α^{eq} is the local equilibrium distribution function of particles; Δx and Δt are space and time increments, respectively; $c = \Delta x/\Delta t$ is "the speed of light" in the system; $\mathbf{e}_\alpha$ is the velocity vector of a particle in the α link and τ is the dimensionless single-relaxation-time which controls the rate of approach to equilibrium. Virtually it is a full discretization of time, space, and velocity. The macroscopic velocity, u is defined in terms of the distribution functions as

$$u = \sum_\alpha f_\alpha = \sum_\alpha f_\alpha^{\mathrm{eq}} = \sum_\alpha f_\alpha^{(0)}. \tag{3}$$

The lattice Boltzmann schemes are established on the square grids with four perpendicular directions:

$$e_1 = (c, 0), \quad e_2 = (0, c), \quad e_3 = (-c, 0), \quad e_4 = (0, -c).$$

This is a 4-bit model shown in Fig. 1. To derive the macroscopic equation from the lattice BGK model, we employ the Taylor expansion and multi-scale analysis. The distribution functions are expanded up to linear terms in the small expansion parameter ε

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \mathrm{O}(\varepsilon^2).$$

From the kinetic equation (2), we expand the distribution function $f_\alpha(x + \Delta t e_\alpha, t + \Delta t)$ in its Taylor expansion and calculate an approximation of $f_\alpha^{(1)}$,

$$\begin{aligned} f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) &= f_\alpha(\mathbf{x}, t) + \Delta t e_{\alpha i} \partial_{x_i} f_\alpha + \Delta t \partial_t f_\alpha + \mathrm{O}(\varepsilon^2) \\ &= \left(1 - \frac{1}{\tau}\right) f_\alpha(\mathbf{x}, t) + \frac{1}{\tau} f_\alpha^{\mathrm{eq}}(\mathbf{x}, t) \end{aligned}$$

$$\varepsilon f_\alpha^{(1)} = -\tau \Delta t (e_{\alpha i} \partial_{x_i} f_\alpha + \partial_t f_\alpha) + \mathrm{O}(\varepsilon^2), \tag{4}$$

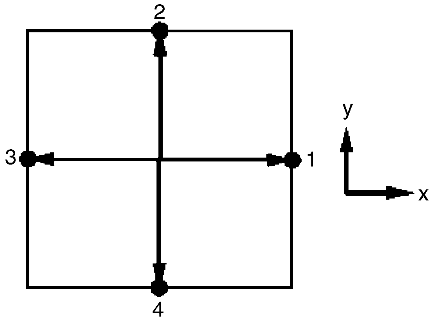

Fig. 1. The 4-bit lattice configuration.

where the Greek subscript α is used to label discrete velocities and the Roman subscripts i, j denote the Cartesian coordinates x, y.

Introduce time scale $t_2 = \varepsilon^2 t$, and space scale $x_1 = \varepsilon x$, then the time derivation and the space derivation can be expanded formally

$$\partial_t = \varepsilon^2 \partial_{t_2}, \quad \partial_x = \varepsilon \partial_{x_1}.$$

Inserting the expansion and the scalings into the conservation relation, we obtains up to second order in ε

$$0 = \sum_\alpha [f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t)] = \sum_\alpha [f_\alpha(\mathbf{x}, t) + \Delta t \varepsilon^2 \partial_{t_2} f_\alpha + \Delta t \varepsilon e_{\alpha i} \partial_{x_{1i}} f_\alpha + \frac{1}{2} \Delta t^2 \varepsilon^2 \partial_{x_{1i}} \partial_{x_{1j}} e_{\alpha i} e_{\alpha j} f_\alpha^{(0)} - f_\alpha(\mathbf{x}, t) + \mathrm{O}(\varepsilon^3)]$$

and

$$\sum_\alpha \Delta t \varepsilon e_{\alpha i} \partial_{x_{1i}} f_\alpha = \Delta t \varepsilon \partial_{x_{1i}} \sum_\alpha e_{\alpha i} f_\alpha^{(0)} + \Delta t \varepsilon^2 \sum_\alpha \partial_{x_{1i}} e_{\alpha i} f_\alpha^{(1)} + \mathrm{O}(\varepsilon^3).$$

Substituting Eq. (4) into the above equation and neglecting the terms of order $\mathrm{O}(\varepsilon^3)$, we get multi-scale equation

$$\Delta t \varepsilon^2 \partial_{t_2} \sum_\alpha f_\alpha^{(0)} + \Delta t \varepsilon \partial_{x_{1i}} \sum_\alpha e_{\alpha i} f_\alpha^{(0)} + \Delta t^2 \varepsilon^2 \left(\frac{1}{2} - \tau\right) \partial_{x_{1i}} \partial_{x_{1j}} \sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(0)} = 0.$$

Corresponding to the macroscopic equation (1), we let

$$\sum_\alpha e_{\alpha i} f_\alpha^{(0)} = \frac{u^2}{2}, \quad \sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(0)} = \lambda u \delta_{ij}. \tag{5}$$

From (3) and (5) we get the equilibrium distributions

$$f_\alpha^{\mathrm{eq}} = \begin{cases} \dfrac{u}{4} + \dfrac{u^2}{4c}, & \alpha = 1, 2, \\ \dfrac{u}{4} - \dfrac{u^2}{4c}, & \alpha = 3, 4 \end{cases} \tag{6}$$

and $\lambda = c^2/2$. So the viscosity v is defined by

$$v = \lambda(\tau - \tfrac{1}{2})\Delta t = \tfrac{1}{2}(\tau - \tfrac{1}{2})\Delta t c^2.$$

then the Burgers equation (1) can be obtained.

3. Stability analysis

In this section, we will prove the lattice Boltzmann model is stable. First LBE (2) can be rewritten by classical finite difference notation,

$$\begin{cases} f_{1,i,j}^{n+1} = \left(1 - \frac{1}{\tau}\right) f_{1,i,j}^{n} + \frac{u_{i,j}^{n}}{4\tau} + \frac{(u_{i,j}^{n})^2}{4\tau c}, \\ f_{2,i,j+1}^{n+1} = \left(1 - \frac{1}{\tau}\right) f_{2,i,j}^{n} + \frac{u_{i,j}^{n}}{4\tau} + \frac{(u_{i,j}^{n})^2}{4\tau c}, \\ f_{3,i-1,j}^{n+1} = \left(1 - \frac{1}{\tau}\right) f_{3,i,j}^{n} + \frac{u_{i,j}^{n}}{4\tau} - \frac{(u_{i,j}^{n})^2}{4\tau c}, \\ f_{4,i,j-1}^{n+1} = \left(1 - \frac{1}{\tau}\right) f_{4,i,j}^{n} + \frac{u_{i,j}^{n}}{4\tau} - \frac{(u_{i,j}^{n})^2}{4\tau c}, \end{cases} \tag{7}$$

where n denotes the nth layer time, i, j are spatial grids,

$$\tau = \frac{2v}{\Delta t c^2} + \frac{1}{2}, \tag{8}$$

subject to initial distribution

$$f_{1,i,j}^{0} = f_{2,i,j}^{0} = \frac{u_{i,j}^{0}}{4} + \frac{(u_{i,j}^{0})^2}{4c}, \quad f_{3,i,j}^{0} = f_{4,i,j}^{0} = \frac{u_{i,j}^{0}}{4} - \frac{(u_{i,j}^{0})^2}{4c}. \tag{9}$$

At time $(n+1)\Delta t$, u is

$$u_{i,j}^{n+1} = f_{1,i,j}^{n+1} + f_{2,i,j}^{n+1} + f_{3,i,j}^{n+1} + f_{4,i,j}^{n+1}. \tag{10}$$

We assume initial value $u_0(\mathbf{x})$ is bounded and smooth enough. We will prove lattice Boltzmann schemes (7) are stable in $L^1(R^2) \cap L^\infty(R^2)$. Suppose

$$u_0(\mathbf{x}) \in L^1, \quad |u_0(\mathbf{x})| \leqslant 1. \tag{11}$$

By combining (10) with (7), it is not difficult to see that if $u_{i,j}^{n} \leqslant 1$, and

$$\tau \geqslant 1, \quad \Delta t/\Delta x \leqslant \tfrac{1}{2}, \tag{12}$$

then the scheme (7) is monotonic. $\tau \geqslant 1$ means

$$\frac{v\Delta t}{\Delta x^2} \geqslant \frac{1}{4}. \tag{13}$$

Now we will point out that the solution of the scheme (7) satisfies the maximum principle.

Lemma 1 (*maximum principle*). *If initial value* $|u_0(\mathbf{x})| \leqslant 1$ *and the restrictions* (12) *holds*, *then for* $\forall i, j \in Z$, *there are*

$$\min_{l,m} u_{l,m}^{0} \leqslant u_{i,j}^{n+1} \leqslant \max_{l,m} u_{l,m}^{0}, \quad n \geqslant 0. \tag{14}$$

Using (7), (9), (10) and induction we can prove (14). Assume that $\widetilde{u}(\mathbf{x})$ is another solution of Eq. (1), with subject to initial condition

$$\widetilde{u}(\mathbf{x}, 0) = \widetilde{u}_0(\mathbf{x}), \tag{15}$$

where $\widetilde{u}_0(\mathbf{x}) \leqslant 1$. The numerical schemes and restrictions are still (7) and (12), respectively.

So we can prove that

Lemma 2. *If the conditions of Lemma* 1 *are fulfilled*, *there are inequalities*

$$\sum_{i,j} \max(u_{i,j}^{n+1}, \widetilde{u}_{i,j}^{n+1}) \leqslant \sum_{i,j} \max(u_{i,j}^{0}, \widetilde{u}_{i,j}^{0}), \tag{16}$$

$$\sum_{i,j} \min(u_{i,j}^{n+1}, \widetilde{u}_{i,j}^{n+1}) \geqslant \sum_{i,j} \min(u_{i,j}^{0}, \widetilde{u}_{i,j}^{0}). \tag{17}$$

Denote that $u_{\Delta x}^n = \{u_{i,j}^n, i, j \in Z\}$ is the discrete solution of LBE (7)–(10) at the time $n\Delta t$, and $\|u_{\Delta x}^n\|_{L^1(R^2)} = \sum_{i,j} |u_{i,j}^n| \Delta x$ is the L^1 norm of discrete function $u_{\Delta x}^n$. Then the solution is stable in the meaning of L^1.

Theorem. *If* $u_{\Delta x}^n, \widetilde{u}_{\Delta x}^n$ *are the solutions of Eq.* (7), $u_{\Delta x}^0, \widetilde{u}_{\Delta x}^0 \in L^1(R^2)$, *with subject to the corresponding initial conditions* (11), (15) *and restrictions* (12), *then there are*

$$\|u_{\Delta x}^n - \widetilde{u}_{\Delta x}^n\|_{L^1(R^2)} \leqslant \|u_{\Delta x}^0 - \widetilde{u}_{\Delta x}^0\|_{L^1(R^2)}, \tag{18}$$

$$\|u_{\Delta x}^n\|_{L^1(R^2)} \leqslant \|u_{\Delta x}^0\|_{L^1(R^2)}. \tag{19}$$

Proof.

$$|u_{i,j}^{n+1} - \widetilde{u}_{i,j}^{n+1}| = \max(u_{i,j}^{n+1}, \widetilde{u}_{i,j}^{n+1}) - \min(u_{i,j}^{n+1}, \widetilde{u}_{i,j}^{n+1}),$$

Summing the absolute value to all *i*, *j*, by Lemma 2, we have

$$\sum_{i,j} |u_{i,j}^{n+1} - \widetilde{u}_{i,j}^{n+1}| \leqslant \sum_{i,j} |u_{i,j}^{0} - \widetilde{u}_{i,j}^{0}|.$$

If we let $\widetilde{u}_{\Delta x}(.,.,t) = 0$ in (18), we can get (19). □

Remark. The restrictions (12), (13) are sufficient but not necessary. In the following experience, we give out on purpose the spatial and time steps violating one of conditions, but we can still obtain the correct result.

4. Numerical experiments

To show the effectiveness of the 4-bit lattice Boltzmann model, we present three numerical examples in this section. In the first two examples, the numerical solutions are compared with the analytical solutions or available finite difference results. In the third example, we take numerical solutions computed on a very fine mesh as the "exact" solutions, and then compare the numerical solutions with them.

Example 1. The first test is the solution of two-dimensional unsteady Burgers' equation that is dominated by moderate gradients [9]. Eq. (1) with the following Dirichlet boundary conditions is solved by the LBM with 4-bit model

$$u(0, y, t) = \frac{1}{1 + e^{(y-t)/2v}}, \quad u(1, y, t) = \frac{1}{1 + e^{(1+y-t)/2v}},$$

$$u(x, 0, t) = \frac{1}{1 + e^{(x-t)/2v}}, \quad u(x, 1, t) = \frac{1}{1 + e^{(1+x-t)/2v}},$$

and the initial condition

$$u(x, y, 0) = \frac{1}{1 + e^{(x+y)/2v}}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1.$$

Table 1
L_∞ and L_2 errors at different viscosities and grids at time = 0.25

Viscosities	Grids	L_∞ error	L_2 error
$v = 1$	4×4	4.423E − 005	1.914E − 005
	10×10	7.882E − 005	2.016E − 005
	20×20	1.483E − 004	2.652E − 005
	40×40	2.921E − 004	3.681E − 005
	80×80	5.820E − 004	5.181E − 005
$v = 0.1$	10×10	1.061E − 002	2.565E − 003
	20×20	3.077E − 003	4.536E − 004
	40×40	1.626E − 002	1.875E − 003
	80×80	3.820E − 002	3.125E − 003
$v = 0.01$	80×80	5.893E − 002	7.610E − 003
	200×200	5.713E − 002	4.372E − 003

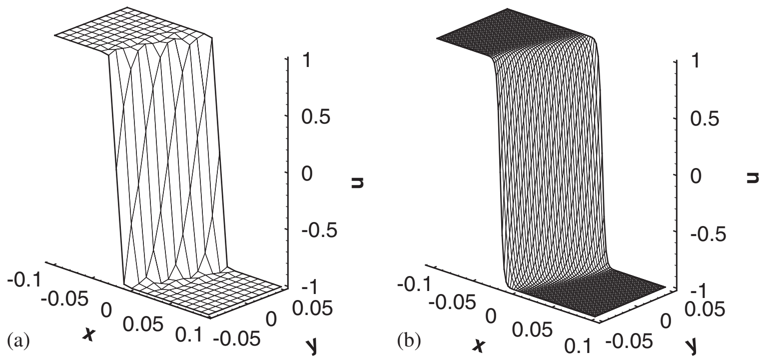

Fig. 2. LBM solutions ($Re = 500$): (a) 20×10 grid; (b) 80×40 grid.

The numerical steady state solutions of Eq. (1), with the above initial and boundary conditions, have been obtained using the LBM at time = 0.25 for different grid sizes and viscosities, $v = 1.0$, $v = 0.1$ and $v = 0.01$. To test the computational efficiency and accuracy of this model, error has been measured using both L_2-norm and L_∞-norm. For every grid we choose $c = \Delta x/\Delta t = 10$. The LBM produces more accurate and hence more efficient solutions, even on a course grid, as shown in Table 1. It is sufficient to obtain convergent steady solutions with only four grid points in each spatial direction with $v = 1.0$. From Table 1, it can be seen that the error increases with an increasing Reynolds number. Hence we need to choose finer grids for high Reynolds number.

Example 2. The second problem is the solution of the 2D unsteady Burgers equation with a steep oblique shock in the domain: $-0.1 \leqslant x \leqslant 0.1$ and $-0.05 \leqslant y \leqslant 0.05$. The following Dirichlet boundary conditions are set to form an "oblique" shock in the domain [9]:

$$u(-0.1, y, t) = -\tanh\left(\frac{-0.1 - 0.4y}{2v}\right), \quad u(0.1, y, t) = -\tanh\left(\frac{0.1 + 0.4y}{2v}\right),$$

$$u(x, -0.05, t) = -\tanh\left(\frac{x + 0.02}{2v}\right), \quad u(x, 0.05, t) = -\tanh\left(\frac{x - 0.02}{2v}\right).$$

The computed steady state solutions using the 4-bit lattice Boltzmann model, $v = 0.002$ and on two different grids (20×10), (80×40), at time = 0.1 are shown in Fig. 2. Here we choose $c = 2$. Again, the LBM is capable of producing convergent and stable solution with steep oblique shock on relatively coarse grid (20×10), the same as the finite

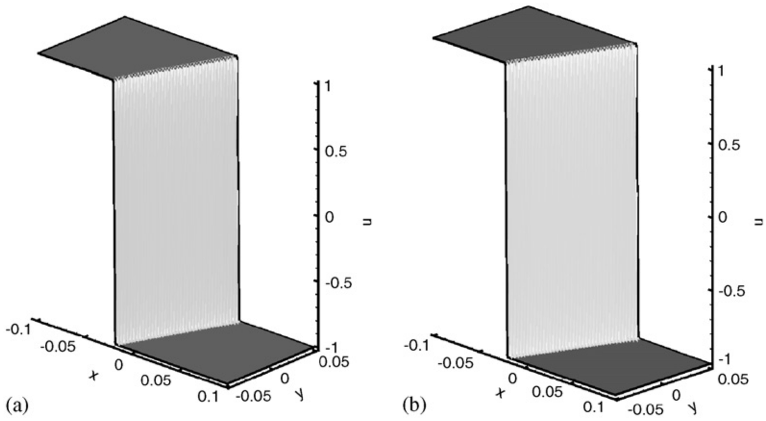

Fig. 3. LBM solutions: (a) $Re = 5000$, 200×100 grid; (b) $Re = 10\,000$, 400×200 grid.

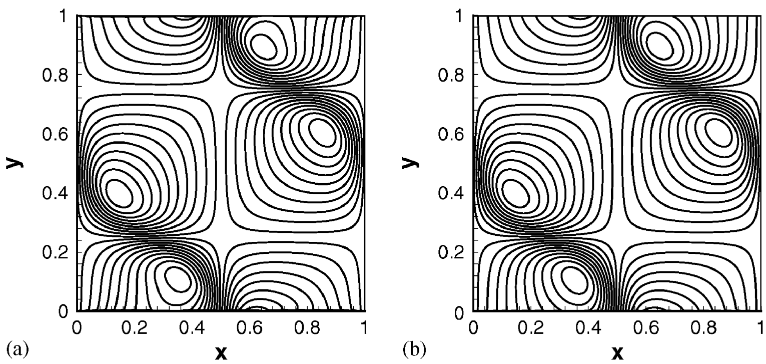

Fig. 4. x–y solution contours: (a) $N = 128$ LBM solution; (b) "exact" solution.

difference solution [9]. Moreover, the 4-bit lattice Boltzmann model efficiently solves the same problem with steeper oblique shock cases of $v = 0.00005 - 0.0002$, even smaller viscosity, without any oscillations. It just needs finer grids with an increasing Reynolds number. The LBM solutions with $v = 0.0002$ and $v = 0.0001$, namely $Re = 5000$ and 10 000, on different grids (200×100), (400×200), respectively, are shown in Fig. 3. This concludes that the 4-bit lattice Boltzmann model is stable and efficient for solving the 2D unsteady Burgers equation, especially with severe gradients.

Example 3. In this example, we consider the Burgers' equation (1) in $(x, y) \in [0, 1] \times [0, 1]$, which satisfies periodic boundary condition, and corresponding initial value is

$$u(x, y, 0) = \sin(2\pi x)\ \cos(2\pi y).$$

We compute the solutions using 4-bit lattice Boltzmann model at $t = \frac{1}{8}$. Where the viscosity $v = 0.01$, and $c = 1$. The exact solution of this problem is unknown. We take the numerical solution computed on a very fine mesh (2048 grid points) as the "exact" solution for the purpose of comparison. The x–y solution contours are plotted in Fig. 4 for grid points $N = 128$, and "exact" solution. We can see that the numerical solution agrees well with the "exact" solution. Fig. 5(a) shows the surface plot of the numerical solution for $N = 64$. Fig. 5(b) are the solutions for $y = 0.5$ at $t = \frac{1}{8}$ on different grids. When the mesh is coarser, the error appears. When $N = 64$ the error is very small, but it converges to the "exact" solution rapidly when the mesh is refined.

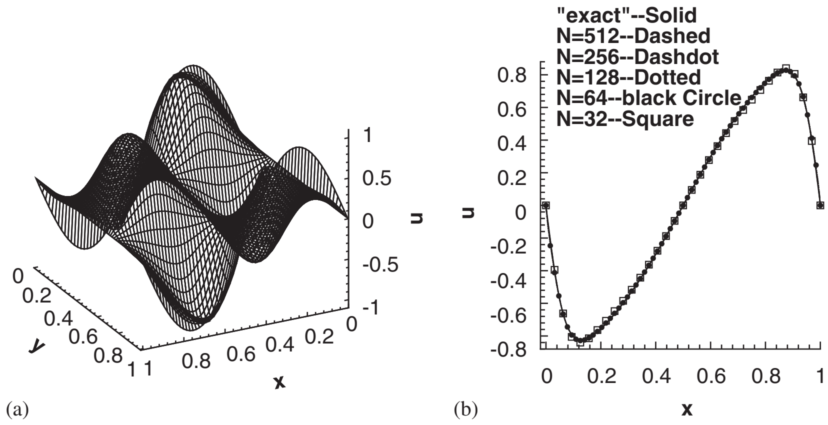

Fig. 5. (a) LBM solutions for $N = 64$; (b) solutions for $y = 0.5$ at $t = \frac{1}{8}$.

5. Conclusion

In conclusion, the 4-bit lattice Boltzmann model is used to solve the 2D unsteady Burgers' equation, for problems that are dominated by moderate to severe internal and boundary gradients. We have proved the maximum principle and the stability which has also been discussed. The computational efficiency and the stability of the model are tested by comparing the numerical results with the exact solutions, especially with the two-point compact scheme solutions in solving steep gradient problems.

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