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On the dynamics of two exponential type systems of difference equations

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ABSTRACT

In this paper we study the asymptotic behavior of the positive solutions of the systems of the two difference equations

$$\text{(i)}\ x_{n+1} = a + by_{n-1}e^{-x_n}, \qquad y_{n+1} = c + dx_{n-1}e^{-y_n},$$
$$\text{(ii)}\ x_{n+1} = a + by_{n-1}e^{-y_n}, \qquad y_{n+1} = c + dx_{n-1}e^{-x_n},$$

where the constants a, b, c, d are positive real numbers, and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers.

1. Introduction

In recent years, systems of nonlinear difference equations have attracted the attention of many researchers for varied reasons. Firstly, the mathematical modeling of a biological problem very often leads to such systems and so difference equations have many applications in Biology, Biomathematics, Bioengineering, Population Dynamics, Genetics and other sciences. Moreover, a biological model, which depicts the competition between two populations, may be represented by a system of two difference equations with solutions (x_n, y_n), $n = 0, 1, \ldots$ where x_n and y_n correspond to the two populations at the time n. So, an extended literature has been developed referring to such systems, as we can see in papers [1–3] and the references cited therein. Furthermore, results concerning difference equations and systems of difference equations of exponential form are included in the papers [2,4–8].

Motivated by all the above reasons, we study in this manuscript a system of nonlinear difference equations which comes from the following difference equation

$$x_{n+1} = a + bx_{n-1}e^{-x_n}$$

that has been studied in [5]. In more detail, in this paper we investigate the boundedness and the persistence of the positive solutions, the existence of a unique positive equilibrium and the global asymptotic stability of the equilibrium of the following systems of difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \qquad y_{n+1} = c + dx_{n-1}e^{-y_n}, \tag{1.1}$$
$$x_{n+1} = a + by_{n-1}e^{-y_n}, \qquad y_{n+1} = c + dx_{n-1}e^{-x_n}, \tag{1.2}$$

where the constants a, b, c, d are positive real numbers and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers. We note that if $x_{-1} = y_{-1}, x_0 = y_0$ then $x_n = y_n$, for all $n = -1, 0, \ldots$ and so both systems reduce the previous difference equation which has been studied in [5]. In addition, in [2] the authors extended results obtained in [5] by studying an analogous system of difference equations of exponential form.

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It is important that the above systems can be considered as models of two directional interactive and invasive species model where species x_n and y_n are affecting each other's population in both directions. So it is obvious that it is very crucial for every positive solution of these systems to be bounded, since the population of species x_n and y_n cannot get infinitely large due to the limited resources. Furthermore, convergence to the equilibrium point $(\bar{x}, \bar{y})$ will apply that the population of both species tends to the natural ideal population. Finally, system (1.2) represents the rule by which two discrete, competing populations reproduce from one generation to the next. Variables x and y denote population sizes during the n-th generation and the sequence or orbit $(x_n, y_n),\ n = 0, 1, 2, \ldots$ describes how the populations evolve over time. Competition between the two populations is reflected by the fact that the transition function for the population of species x_n during the n-th generation is a decreasing function with respect to the population of species y_{n-1} during the $(n-1)$-th generation and the transition function for the population of species y_n during the n-th generation is a decreasing function with respect to the population of species x_{n-1} during the $(n-1)$-th generation.

2. Boundedness and persistence

In the first section we study the boundedness and persistence of the solutions of systems (1.1) and (1.2).

Proposition 2.1. *Let a, b, c, d be positive real numbers such that*

$$p = bde^{-a-c} < 1. \tag{2.1}$$

Then the following statements are true:

(i) *Every solution of (1.1) is positive, bounded and persists.*
(ii) *Every solution of (1.2) is positive, bounded and persists.*

Proof. (i) Since the initial x_{-1}, x_0, y_{-1}, y_0 of (1.1) are positive, every solution of (1.1) is positive.

Let (x_n, y_n) be an arbitrary solution of (1.1). From (1.1) it is obvious that

$$x_n \geq a, \qquad y_n \geq c, \quad n = 1, 2, \ldots. \tag{2.2}$$

Every solution of (1.1) persists.

Moreover from (1.1) and (2.2) it follows that for $n = 2, 3, \ldots$

$$\begin{aligned} x_{n+1} &= a + b(c + dx_{n-3}e^{-y_{n-2}})e^{-x_n} \leq a + bce^{-a} + px_{n-3}, \\ y_{n+1} &= c + d(a + by_{n-3}e^{-x_{n-2}})e^{-y_n} \leq c + dae^{-c} + py_{n-3}. \end{aligned} \tag{2.3}$$

We consider the system of difference equations

$$u_{n+1} = a + bce^{-a} + pu_{n-3}, \qquad v_{n+1} = c + dae^{-c} + pv_{n-3}, \quad n = 2, 3, \ldots. \tag{2.4}$$

Let (u_n, v_n) be a solution of (2.4) such that

$$\begin{aligned} &u_{-1} = x_{-1}, \qquad u_0 = x_0, \qquad u_1 = x_1, \qquad u_2 = x_2, \\ &v_{-1} = y_{-1}, \qquad v_0 = y_0, \qquad v_1 = y_1, \qquad v_2 = y_2. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$u_3 = a + bce^{-a} + px_{-1} > 0, \qquad v_3 = c + dae^{-c} + py_{-1} > 0$$

and working inductively it follows that

$$u_n > 0, \qquad v_n > 0, \quad n = 2, 3, \ldots.$$

Moreover, from (2.4) for $n = 3, 4, \ldots$, we have

$$u_n = \lambda_1 p^{\frac{n}{4}} + \lambda_2(-p)^{\frac{n}{4}} + \lambda_3 p^{\frac{n}{4}} \cos\left(\frac{n\pi}{2}\right) + \lambda_4 p^{\frac{n}{4}} \sin\left(\frac{n\pi}{2}\right) + \frac{a + bce^{-a}}{1-p}, \tag{2.6}$$

$$v_n = \mu_1 p^{\frac{n}{4}} + \mu_2(-p)^{\frac{n}{4}} + \mu_3 p^{\frac{n}{4}} \cos\left(\frac{n\pi}{2}\right) + \mu_4 p^{\frac{n}{4}} \sin\left(\frac{n\pi}{2}\right) + \frac{c + dae^{-c}}{1-p}, \tag{2.7}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ (resp. $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$) are constants defined by $x_{-1},\ x_0,\ x_1,\ x_2$ (resp. $y_{-1},\ y_0,\ y_1,\ y_2$).

Using (2.3)–(2.5) we can prove by induction that

$$x_n \leq u_n, \qquad y_n \leq v_n, \quad n = -1, 0, \ldots. \tag{2.8}$$

Then from (2.2) and (2.6)–(2.8) we obtain that every solution of (1.1) is bounded.

Hence, the proof of Statement (i) is completed. □

(ii) Let (x_n, y_n) be an arbitrary solution of (1.2). Then arguing as in Statement (i), we can show that (x_n, y_n) is positive, bounded and persists. This completes the proof of the proposition.

In the next proposition we study the existence of invariant intervals for Systems (1.1) and (1.2).

Proposition 2.2. *Let a, b, c, d be positive numbers such that (2.1) hold. Then the following statements are true:*

(i) *Consider the intervals*

$$I_1 = \left[a, \frac{a + bce^{-a}}{1-p}\right], \qquad I_2 = \left[c, \frac{c + ade^{-c}}{1-p}\right],$$
$$I_3 = \left[a, \frac{a + bce^{-a} + \epsilon}{1-p}\right], \qquad I_4 = \left[c, \frac{c + ade^{-c} + \epsilon}{1-p}\right],$$

where p is defined in relation (2.1) and ϵ is an arbitrary positive number.
Then, if (x_n, y_n) is a positive solution of (1.1) such that

$$x_{-1}, x_0 \in I_1, \qquad y_{-1}, y_0 \in I_2, \tag{2.9}$$

we have

$$x_n \in I_1, \qquad y_n \in I_2, \quad n = 1, 2, \ldots.$$

Moreover, if (x_n, y_n) is an arbitrary positive solution of (1.1), then there exists an $m \in N$ such that

$$x_n \in I_3, \qquad y_n \in I_4, \quad n \geq m. \tag{2.10}$$

(ii) *Consider the intervals*

$$J_1 = \left[a, \frac{a + bce^{-c}}{1-p}\right], \qquad J_2 = \left[c, \frac{c + ade^{-a}}{1-p}\right],$$
$$J_3 = \left[a, \frac{a + bce^{-c} + \epsilon}{1-p}\right], \qquad J_4 = \left[c, \frac{c + ade^{-a} + \epsilon}{1-p}\right],$$

where p is defined in relation (2.1) and ϵ is an arbitrary positive number.
Then, if (x_n, y_n) is a positive solution of (1.2) such that

$$x_{-1}, x_0 \in J_1, \qquad y_{-1}, y_0 \in J_2,$$

we have

$$x_n \in J_1, \qquad y_n \in J_2, \quad n = 1, 2, \ldots.$$

In addition, if (x_n, y_n) is an arbitrary positive solution of (1.2), then there exists an $m \in \mathbb{N}$ such that

$$x_n \in J_3, \qquad y_n \in J_4, \quad n = m, m+1, \ldots.$$

Proof. (i) Let (x_n, y_n) be a positive solution of (1.1), such that (2.9) hold. Then, from (1.1) we obtain

$$a \leq x_1 = a + by_{-1}e^{-x_0} \leq a + b\frac{c + ade^{-c}}{1-p}e^{-a} = \frac{a + bce^{-a}}{1-p}$$
$$c \leq y_1 = c + dx_{-1}e^{-y_0} \leq c + d\frac{a + bce^{-a}}{1-p}e^{-c} = \frac{c + ade^{-c}}{1-p}$$

and working inductively we can prove that

$$a \leq x_n \leq \frac{a + bce^{-a}}{1-p}, \qquad c \leq y_n \leq \frac{c + ade^{-c}}{1-p}, \quad n = 2, 3, \ldots.$$

This completes the proof of the first part of (i). $\square$

Let (x_n, y_n) be an arbitrary positive solution of (1.1). Then, from Statement (i) of Proposition 2.1, we have

$$\limsup_{n\to\infty} x_n = M < \infty, \qquad \limsup_{n\to\infty} y_n = L < \infty. \tag{2.11}$$

Therefore from (2.3) and (2.11) we get

$$M \leq \frac{a + bce^{-a}}{1-p}, \qquad L \leq \frac{c + ade^{-c}}{1-p},$$

and so there exists an $m \in \mathbb{N}$ such that (2.10) hold. This completes the proof of Statement (i).

(ii) Arguing as in Statement (i), we can prove Statement (ii). This completes the proof of the proposition.

3. Attractivity

In this section we investigate the existence of a unique positive equilibrium for system (1.1) and the attractivity of the unique positive equilibrium. Arguing as in Theorem 1.6.5 of [9], in Theorems 1.11–1.16 of [4] and in Theorems 1.4.5–1.4.8 of [10] we state the following lemma.

Lemma 3.1. *Let $f, g, f : R^+ \times R^+ \to R^+, g : R^+ \times R^+ \to R^+$ be continuous functions, $R^+ = (0, \infty)$ and a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1, a_2 < b_2$.*

(i) *Suppose that*

$$f : [a_1, b_1] \times [a_2, b_2] \to [a_1, b_1], \qquad g : [a_1, b_1] \times [a_2, b_2] \to [a_2, b_2].$$

In addition, assume that $f(x, y)$ (resp. $g(x, y)$) is decreasing with respect to x (resp. y) for every y (resp. x) and increasing with respect to y (resp. x) for every x (resp. y). Finally suppose that, if the real numbers m, M, r, R satisfy the system

$$M = f(m, R), \quad m = f(M, r), \ R = g(M, r), \ r = g(m, R), \ m \le M, \ r \le R$$

then $m = M$ and $r = R$. Then the following system of difference equations

$$x_{n+1} = f(x_n, y_{n-1}), \qquad y_{n+1} = g(x_{n-1}, y_n) \tag{3.1}$$

has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution (x_n, y_n) of the system (3.1) which satisfies

$$x_{n_0} \in [a_1, b_1], \qquad x_{n_0+1} \in [a_1, b_1], \qquad y_{n_0} \in [a_2, b_2], \qquad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N}$$

tends to the unique positive equilibrium of (3.1).

(ii) *Suppose that*

$$f : [a_2, b_2] \times [a_2, b_2] \to [a_1, b_1], \qquad g : [a_1, b_1] \times [a_1, b_1] \to [a_2, b_2].$$

In addition, assume that $f(u, v)$ is a decreasing (resp. increasing) function with respect to u (resp. v) for every v (resp. u) and $g(z, w)$ is a decreasing (resp. increasing) function with respect to z (resp. w) for every w (resp. z). Finally suppose that if the real numbers m, M, r, R satisfy the system

$$M = f(r, R), \quad m = f(R, r), \ R = g(m, M), \ r = g(M, m), \ m \le M, \ r \le R$$

then $m = M$ and $r = R$. Then the system of difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \qquad y_{n+1} = g(x_n, x_{n-1}) \tag{3.2}$$

has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution (x_n, y_n) of the system (3.2) which satisfies

$$x_{n_0} \in [a_1, b_1], \qquad x_{n_0+1} \in [a_1, b_1], \qquad y_{n_0} \in [a_2, b_2], \qquad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N}$$

tends to the unique positive equilibrium of (3.2).

Proposition 3.1. *Let a, b, c, d be positive numbers. Then the following statements are true:*

(i) *Assume that*

$$\theta_1 = be^{-a} < 1, \qquad \theta_2 = de^{-c} < 1. \tag{3.3}$$

Suppose also that

$$(1 + a)p + c\theta_1 < 1, \qquad (1 + c)p + a\theta_2 < 1 \tag{3.4}$$

and

$$\lambda = \frac{p(1 - p)^2}{\Big[1 - (1 + a)p - c\theta_1\Big]\Big[1 - (1 + c)p - a\theta_2\Big]} < 1. \tag{3.5}$$

Then the system (1.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every solution of (1.1) tends to the unique positive equilibrium of (1.1) as $n \to \infty$.

(ii) *Assume that*

$$\zeta_1 = be^{-c} < 1, \qquad \zeta_2 = de^{-a} < 1 \tag{3.6}$$

and

$$\mu = \frac{p(1 - p + c + a\zeta_2)(1 - p + a + c\zeta_1)}{(1 - p)^2} < 1. \tag{3.7}$$

Then the system (1.2) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of (1.2) tends to the unique positive equilibrium of (1.2) as $n \to \infty$.

Proof. (i) Let $f : R^+ \times R^+ \to R^+$, $g : R^+ \times R^+ \to R^+$ be continuous functions, such that

$$f(x, y) = a + bye^{-x}, \qquad g(x, y) = c + dxe^{-y}.$$

Then, if $x \in I_3$, $y \in I_4$ from (3.3) we have

$$a \le f(x, y) \le a + b\frac{c + ade^{-c} + \epsilon}{1 - p}e^{-a} = \frac{a + c\theta_1 + \epsilon\theta_1}{1 - p} < \frac{a + c\theta_1 + \epsilon}{1 - p}$$
$$c \le g(x, y) \le c + d\frac{a + bce^{-a} + \epsilon}{1 - p}e^{-c} = \frac{c + a\theta_2 + \epsilon\theta_2}{1 - p} < \frac{c + a\theta_2 + \epsilon}{1 - p}.$$

Therefore f, g are continuous functions such that $f : I_3 \times I_4 \to I_3$, $g : I_3 \times I_4 \to I_4$.

Let now, $m, M \in I_3$, $r, R \in I_4$ be positive real numbers such that

$$M = a + bRe^{-m}, \quad m = a + bre^{-M}, \ R = c + dMe^{-r}, \ r = c + dme^{-R}, \ m \le M, \ r \le R. \tag{3.8}$$

Then, from (3.8), we have

$$m = a + bce^{-M} + bdme^{-R}e^{-M}, \qquad r = c + dae^{-R} + bdre^{-M}e^{-R}$$

and so

$$m = \frac{a + bce^{-M}}{1 - bde^{-R-M}}, \qquad r = \frac{c + ade^{-R}}{1 - bde^{-R-M}}. \tag{3.9}$$

Then since $M \ge a$, $R \ge c$ it holds

$$m \le \frac{a + bce^{-a}}{1 - p} = \frac{a + c\theta_1}{1 - p}, \qquad r \le \frac{c + ade^{-c}}{1 - p} = \frac{c + a\theta_2}{1 - p}. \tag{3.10}$$

Furthermore, there exists a ξ, $m \le \xi \le M$ such that

$$e^M - e^m = e^{\xi}(M - m). \tag{3.11}$$

From (3.8) and (3.11) and since $M, m \ge a$ we get

$$\begin{aligned} M - m &= b(Re^{-m} - re^{-M}) = be^{-m}(R - r) + bre^{-m-M}(e^M - e^m) \\ &= be^{-m}(R - r) + bre^{-m-M+\xi}(M - m) \le \theta_1(R - r) + r\theta_1(M - m). \end{aligned} \tag{3.12}$$

Hence from (3.10) and (3.12) it follows that

$$M - m \le \theta_1(R - r) + \frac{\theta_1(c + a\theta_2)}{1 - p}(M - m). \tag{3.13}$$

Then since $p = \theta_1\theta_2$, from (3.13) we obtain

$$(M - m)\left(\frac{1 - p - c\theta_1 - ap}{1 - p}\right) \le \theta_1(R - r). \tag{3.14}$$

Therefore from (3.4) and (3.14) we have

$$M - m \le \frac{\theta_1(1 - p)}{1 - c\theta_1 - (a + 1)p}(R - r). \tag{3.15}$$

Similarly, we have

$$R - r \le \frac{\theta_2(1 - p)}{1 - a\theta_2 - (c + 1)p}(M - m). \tag{3.16}$$

Relations (3.15) and (3.16) imply that

$$M - m \le \lambda(M - m). \tag{3.17}$$

Therefore from (3.5) and (3.17) we have $M = m$ and so from (3.8) $r = R$. Consequently, from Lemma 3.1, System (1.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of System (1.1) tends to $(\bar{x}, \bar{y})$. This completes the proof of the Statement (i). □

(ii) We define the functions $f : R^+ \times R^+ \to R^+$, $g : R^+ \times R^+ \to R^+$ as follows

$$f(u,v) = a + bve^{-u}, \qquad g(z,w) = c + dwe^{-z}.$$

Then, if $z, w \in J_3$, $u, v \in J_4$ and arguing as in Statement (i) we have

$$f(u,v) \in J_3, \qquad g(z,w) \in J_4.$$

So f and g are continuous functions such that

$$f : J_4 \times J_4 \to J_3, \qquad g : J_3 \times J_3 \to J_4.$$

Let now, $m, M \in J_3$, $r, R \in J_4$ be real numbers such that

$$M = a + bRe^{-r}, \qquad m = a + bre^{-R}, \quad R = c + dMe^{-m},\ r = c + dme^{-M},\ m \le M,\ r \le R. \tag{3.18}$$

Moreover, there exists a ξ, $r \le \xi \le R$ such that

$$Re^R - re^r = (1+\xi)e^{\xi}(R-r). \tag{3.19}$$

Then from (3.18) and (3.19) and since $r, R \ge c$ we get

$$M - m = b(Re^{-r} - re^{-R}) = be^{-r-R}(Re^R - re^r) = be^{-r-R+\xi}(1+\xi)(R-r) \le be^{-c}(1+\xi)(R-r). \tag{3.20}$$

Moreover, from (3.18), we obtain

$$r = c + dae^{-M} + bdre^{-R}e^{-M}, \qquad R = c + dae^{-m} + bdRe^{-r}e^{-m}$$

which implies that

$$r = \frac{c + ade^{-M}}{1 - bde^{-R-M}} \le \frac{c + a\zeta_2}{1-p}, \qquad R = \frac{c + ade^{-m}}{1 - bde^{-r-m}} \le \frac{c + a\zeta_2}{1-p}. \tag{3.21}$$

Furthermore since $\xi \le R$, from (3.21) it follows that

$$\xi \le \frac{c + a\zeta_2}{1-p}. \tag{3.22}$$

Thus, from (3.20) and (3.22), we get

$$M - m \le \frac{\zeta_1(1 - p + c + a\zeta_2)}{1-p}(R-r). \tag{3.23}$$

Similarly, we obtain

$$R - r \le \frac{\zeta_2(1 - p + a + c\zeta_1)}{1-p}(M-m). \tag{3.24}$$

So, from (3.23) and (3.24) we have

$$M - m \le \mu(M-m). \tag{3.25}$$

Then, from (3.7), (3.18) and (3.25) it is obvious that $M = m$ and $R = r$. Therefore, from Lemma 3.1, System (1.2) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of System (1.2) tends to $(\bar{x}, \bar{y})$. This completes the proof of the proposition.

Proposition 3.2. *Let a, b, c, d be positive numbers. Then the following statements are true:*

(i) *Assume that (3.3)–(3.5) hold. Suppose also that*

$$\kappa = \frac{c\theta_1 + a\theta_2 + (a+c)p}{1-p} + \frac{p(a + c\theta_1)(c + a\theta_2)}{(1-p)^2} + p < 1 \tag{3.26}$$

Then the unique positive equilibrium $(\bar{x}, \bar{y})$ of (1.1) is globally asymptotically stable.

(ii) *Assume that (3.6) and (3.7) hold. Then the unique positive equilibrium $(\bar{x}, \bar{y})$ of (1.2) is globally asymptotically stable.*

Proof. (i) First we will prove that $(\bar{x},\bar{y})$ is locally asymptotically stable. The linearized system of (1.1) about $(\bar{x},\bar{y})$ is the following:

$$x_{n+1} = -b\bar{y}e^{-\bar{x}}x_n + be^{-\bar{x}}y_{n-1}, \qquad y_{n+1} = de^{-\bar{y}}x_{n-1} - d\bar{x}e^{-\bar{y}}y_n, \tag{3.27}$$

which is equivalent to the system

$$w_{n+1} = Aw_n, \quad A = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \gamma & \delta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$
$$\alpha = -b\bar{y}e^{-\bar{x}}, \qquad \beta = be^{-\bar{x}} \qquad \gamma = -d\bar{x}e^{-\bar{y}}, \qquad \delta = de^{-\bar{y}}.$$

Then the characteristic equation of A is

$$\lambda^4 - (\alpha+\gamma)\lambda^3 + \alpha\gamma\lambda^2 - \beta\delta = 0. \tag{3.28}$$

Using Remark of 1.3.1 of [11] all the roots of Eq. (3.28) are of modulus less than 1, if

$$|\alpha| + |\gamma| + |\alpha\gamma| + |\beta\delta| < 1. \tag{3.29}$$

Since $(\bar{x},\bar{y})$ is an equilibrium for (1.1) we have that

$$\bar{x} = a + b(c + d\bar{x}e^{-\bar{y}})e^{-\bar{x}}, \qquad \bar{y} = c + d(a + b\bar{y}e^{-\bar{x}})e^{-\bar{y}}.$$

Hence

$$\bar{x} = \frac{a + bce^{-\bar{x}}}{1 - bde^{-\bar{x}-\bar{y}}} \le \frac{a + c\theta_1}{1-p}, \qquad \bar{y} = \frac{c + ade^{-\bar{y}}}{1 - bde^{-\bar{x}-\bar{y}}} \le \frac{c + a\theta_2}{1-p}. \tag{3.30}$$

Then, since $\bar{x} \ge a$, $\bar{y} \ge c$, from (3.26) and (3.30), we get

$$|\alpha| + |\gamma| + |\alpha\gamma| + |\beta\delta| = b\bar{y}e^{-\bar{x}} + d\bar{x}e^{-\bar{y}} + bd\bar{x}\bar{y}e^{-\bar{x}-\bar{y}} + bde^{-\bar{x}-\bar{y}} \le \kappa < 1$$

and so (3.29) is satisfied. Therefore $(\bar{x},\bar{y})$ is locally asymptotically stable. So, since from Statement (i) of Proposition 3.1, every positive solution of (1.1) tends to the unique positive equilibrium of (1.1), the proof of Statement (i) is completed. □

(ii) First we will prove that $(\bar{x},\bar{y})$ is locally asymptotically stable. The linearized system of (1.2) about the unique positive equilibrium of (1.2) $(\bar{x},\bar{y})$ is the following:

$$x_{n+1} = be^{-\bar{y}}y_{n-1} - b\bar{y}e^{-\bar{y}}y_n, \qquad y_{n+1} = de^{-\bar{x}}x_{n-1} - d\bar{x}e^{-\bar{x}}x_n$$

which is equivalent to the system

$$v_{n+1} = Bv_n, \quad B = \begin{pmatrix} 0 & \alpha & 0 & \beta \\ \gamma & 0 & \delta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad v_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$
$$\alpha = -b\bar{y}e^{-\bar{y}}, \qquad \beta = be^{-\bar{y}} \qquad \gamma = -d\bar{x}e^{-\bar{x}}, \qquad \delta = de^{-\bar{x}}.$$

Then the characteristic equation of B is

$$\lambda^4 - \alpha\gamma\lambda^2 - (\alpha\delta + \beta\gamma)\lambda - \beta\delta = 0. \tag{3.31}$$

Using Remark of 1.3.1 of [11] all the roots of Eq. (3.31) are of modulus less than 1, if

$$|\alpha\gamma| + |\alpha\delta| + |\beta\gamma| + |\beta\delta| < 1. \tag{3.32}$$

Since $(\bar{x},\bar{y})$ satisfies

$$\bar{x} = a + b(c + d\bar{x}e^{-\bar{x}})e^{-\bar{y}}, \qquad \bar{y} = c + d(a + b\bar{y}e^{-\bar{y}})e^{-\bar{x}}$$

we get

$$\bar{x} = \frac{a + bce^{-\bar{y}}}{1 - bde^{-\bar{x}-\bar{y}}} \le \frac{a + c\zeta_1}{1-p}, \qquad \bar{y} = \frac{c + ade^{-\bar{x}}}{1 - bde^{-\bar{x}-\bar{y}}} \le \frac{c + a\zeta_2}{1-p}. \tag{3.33}$$

Then since $\bar{x} \geq a$, $\bar{y} \geq c$, $p = \zeta_1 z_2$, from (3.7) and (3.33) we obtain

$$\begin{aligned}|\alpha\gamma| + |\alpha\delta| + |\beta\gamma| + |\beta\delta| &\leq bde^{-a-c}\left(\frac{(a + c\zeta_1)(c + a\zeta_2)}{(1-p)^2} + \frac{c + a\zeta_2}{1-p} + \frac{a + c\zeta_1}{1-p} + 1\right)\\ &= p\left(1 + \frac{a + c\zeta_1 + c + a\zeta_2}{1-p} + \frac{(a + c\zeta_1)(c + a\zeta_2)}{(1-p)^2}\right)\\ &= \frac{p(1-p)^2 + p(1-p)(a + c\zeta_1 + c + a\zeta_2) + p(a + c\zeta_1)(c + a\zeta_2)}{(1-p)^2} = \mu < 1.\end{aligned}$$

Then inequality (3.32) is satisfied. Therefore $(\bar{x}, \bar{y})$ is locally asymptotically stable. So, from Statement (ii) of Proposition 3.1, the proof of Statement (ii) is completed. This completes the proof of the proposition.

4. Unbounded solutions

In this section we find unbounded solutions for systems (1.1) and (1.2).

Proposition 4.1. *The following statements are true:*

(i) *Suppose that*

$$\theta_1 > 1, \qquad \theta_2 > 1, \tag{4.1}$$

where θ_1, θ_2 are defined in (3.3). Then there exist unbounded solutions (x_n, y_n) of (1.1) such that one of the following relations hold:

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} x_{2n} = a, \qquad \lim_{n\to\infty} y_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = c \tag{4.2}$$

$$\lim_{n\to\infty} x_{2n+1} = a, \qquad \lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = c, \qquad \lim_{n\to\infty} y_{2n} = \infty. \tag{4.3}$$

(ii) *Suppose that*

$$\zeta_1 > 1, \qquad \zeta_2 > 1, \tag{4.4}$$

where ζ_1, ζ_2 are defined in (3.6). Then there exist unbounded solutions (x_n, y_n) of (1.2) such either relations (4.2) or (4.3) hold.

Proof. (i) First we find solutions of (1.1) such that (4.2) are satisfied. Let (x_n, y_n) be a solution of (1.2) with initial values x_{-1}, x_0, y_{-1}, y_0 which satisfy

$$x_0 < m_1, \qquad x_{-1} > M, \qquad y_0 < m_2, \qquad y_{-1} > M \tag{4.5}$$

where

$$m_1 = \ln b, \qquad m_2 = \ln d, \qquad M = \max\left\{\ln\left(\frac{dm_1}{m_2 - c}\right), \ln\left(\frac{bm_2}{m_1 - a}\right)\right\}.$$

Then using (1.1) and (4.5) we have

$$\begin{aligned}&x_1 = a + by_{-1}e^{-x_0} > a + by_{-1}e^{-m_1} = a + y_{-1},\\ &y_1 = c + dx_{-1}e^{-y_0} > c + dx_{-1}e^{-m_2} = c + x_{-1},\\ &x_2 = a + by_0e^{-x_1} < a + bm_2e^{-y_{-1}} < a + bm_2\left(\frac{m_1 - a}{bm_2}\right) = m_1,\\ &y_2 = c + dx_0e^{-y_1} < c + dm_1e^{-x_{-1}} < c + dm_1\left(\frac{m_2 - c}{dm_1}\right) = m_2,\end{aligned}$$

and working inductively we obtain

$$x_{2n+1} > a + y_{2n-1}, \qquad y_{2n+1} > c + x_{2n-1}, \qquad x_{2n} < m_1, \qquad y_{2n} < m_2, \quad n = 1, 2, \ldots. \tag{4.6}$$

Using (1.1) and (4.6) we can prove that (4.2) hold.

Let now (x_n, y_n) be a solution such that

$$x_{-1} < m_1, \qquad x_0 > M, \qquad y_{-1} < m_2, \qquad y_0 > M.$$

Then arguing as above we can prove relations (4.3). This completes the proof of (i). $\square$

(ii) Let (x_n, y_n) be a solution of (1.2) with initial values x_{-1}, x_0, y_{-1}, y_0 satisfying

$$x_0 < p_2, \qquad x_{-1} > L, \qquad y_0 < p_1, \qquad y_{-1} > L, \tag{4.7}$$

where

$$p_2 = \ln d, \qquad p_1 = \ln b, \qquad L = \max\left\{\ln\left(\frac{bp_1}{p_2 - a}\right), \ln\left(\frac{dp_2}{p_1 - c}\right)\right\}.$$

Then using (4.7) and arguing as above we can prove that (4.2) hold.

Finally suppose that

$$x_{-1} < p_2, \qquad x_0 > L, \qquad y_{-1} < p_1, \qquad y_0 > L.$$

Then arguing as above we have that (4.3) are satisfied. This completes the proof of the proposition.

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