A Feedback Near-Optimum Control for Nonlinear Systems

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A near-optimum feedback controller based on the sensitivity of state and control functions with respect to a coupling parameter is introduced for nonlinear systems. The method makes use of Pontryagin’s maximum principle and the Riccati formulation of the linear regulator problem. A numerical example with previously known results is solved using the method.

INTRODUCTION

The general purpose of an optimal control problem is to find a control function $u(t)$ which minimizes a given cost functional while satisfying system equations and conditions. Due to the difficulties encountered in optimization of nonlinear systems by the classical methods, i.e., gradient, Newton’s second variation, etc., near-optimum control laws have received special attention in recent years (Kelly, 1964; White and Cook, 1973; Garrard et al., 1967; Garrard, 1969; 1972; Werner and Cruz, 1968; Nishikawa et al., 1971; Kokotovic et al., 1969). Some of the methods suggested in literature have been “equivalent linearization” (White and Cook, 1973), and “approximate solution to the Hamilton–Jacobi–Bellman’s equation” (Garrard et al., 1967; Garrard, 1969, 1972). Another approach in near-optimum control of nonlinear systems is the so called “optimally adaptive,” (Werner and Cruz, 1968), which minimizes the system cost functional regardless of plant parameters or initial condition variations. The control function is expanded in a MacLaurin’s series of plant parameters and initial conditions. It has been asserted (Werner and Cruz, 1968) that for an $r$th-order truncation of the control function series, the optimum cost functional has been approximated up to $(2r + 1)$ terms. Another near-optimum control design developed (Nishikawa et al., 1971) takes on the general class of nonlinear systems given in (Garrard, et al., 1967),

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while making use of a similar parameter expansion as in optimally adaptive control (Werner and Cruz, 1968). The "optimally sensitive controller" (Kokotovic et al., 1969) is a first-order approximation of the optimally adaptive which presents a convenient derivation of sensitivity equations from the maximum principle.

This paper extends the optimally sensitive control to a wider class of nonlinear systems than those considered by (Garrard et al., 1967) and (Nishikawa et al., 1971) while preserving the same order of approximation of the cost functional as in the optimally adaptive.

Consider the state model of any nonlinear controllable system

\[ \dot{x} = Ax + Bu + \epsilon f(x, u) \]  

(1)

where \( x \) is \( n \)-th order state and \( u \) is \( m \)-th order control vectors, \( f \) is a continuously differentiable function of its arguments, \( A \) and \( B \) are constant matrices of appropriate dimensions and \( \epsilon \in [0, 1] \) is a scalar parameter. It is noted that the parameter \( \epsilon \) introduces nonlinearities in an otherwise linear system and furthermore any nonlinear control system can be easily represented by (1). The optimum control problem is to find a control \( u \) which would approximately minimize a cost functional

\[ J = L(x(t_f)) + \int_{t_0}^{t_f} V(x, u, t) \, dt. \]  

(2)

Where \( L \) and \( V \) are scalar functions, \( t_0 \) and \( t_f \) are fixed and the final state \( x(t_f) \) is assumed to be free. The proposed near-optimum control, as in (Garrard et al., 1967) and (Nishikawa et al., 1971), is of feedback form. A numerical example is given and the results are compared with some of the previously proposed methods in the literature.

**Near-Optimum Control**

Consider the system equation (1), rewritten

\[ \dot{x}(t, \epsilon) = h(x, u, t, \epsilon) = Ax(t, \epsilon) + Bu(t, \epsilon) + \epsilon f(x(t, \epsilon), u(t, \epsilon), \epsilon), \]  

(3a)

\[ x(t_0, \epsilon) = x_0. \]  

(3b)

Without loss of generality, the cost functional (2) is assumed to be quadratic,

\[ J = \frac{1}{2} x'(t_f, \epsilon) M x(t_f, \epsilon) + \frac{1}{2} \int_{t_0}^{t_f} (x' Q x + u' R u) \, dt, \]  

(4)
where $M, Q,$ and $R$ satisfy the usual linear state regulator conditions (Kalman, 1960) and prime denotes transposition. The necessary optimality conditions based on the maximum principle are

\[ \dot{x} = H \dot{p} = h = Ax + Bu + ef(x, u, \epsilon), \quad x(t_0, \epsilon) = x_0, \]  
\[ \dot{p} = -H_x = -Qx - A'p - (\partial f / \partial x)'p, \quad p(t_f, \epsilon) = -Mx(t_f, \epsilon), \]  
\[ 0 = H_u = -Ru + (B + \epsilon (\partial f / \partial u))'p, \]

where $H = -\frac{1}{2}(x'Qx + u'Ru) + p'(Ax + Bu + ef(x, u, \epsilon))$ is the Hamiltonian function and the subscripts denote vector gradients. Note that for nonzero values of $\epsilon$, (5)-(7) represent a nonlinear two-point boundary-value problem whose solution is difficult to obtain. In order to overcome this difficulty let us differentiate (5)-(7) with respect to $\epsilon$ and let $\epsilon \to 0$, assuming that all functions are continuously differentiable.

\[ \dot{x}^1 = h_x x^1 + h_u u^1 + f, \quad x^1(t_0) = 0, \]  
\[ \dot{p}^1 = -H_x x^1 - h_x' p^1 - H_{ux} u^1 - f_x' p, \quad p^1(t_f) = 0, \]  
\[ 0 = H_{ux} x^1 + h_u' p^1 + H_{uu} u^1 + f_u' p, \]

where $x^1 \triangleq \lim_{\epsilon \to 0} \partial x(t, \epsilon) / \partial \epsilon$, similarly for $u^1$ and $p^1$, are sometimes referred to as first-order sensitivity functions (Kokotovic et al., 1969). Assuming that $H_{uu}$ is negative definite, eliminating $u^1$ in (8)-(9) by using (10), the following linear two-point boundary value problem results

\[ \dot{x}^1 = F x^1 + E p^1 + \omega_0, \]  
\[ \dot{p}^1 = G x^1 - F' p^1 + \delta_0, \]

where,

\[ F = h_x - h_u H_{uu}^{-1} H_{ux} |_{\epsilon = 0} = A, \]  
\[ G = H_{ux} H_{uu}^{-1} H_{ux} - H_{xx} |_{\epsilon = 0} = Q, \]  
\[ E = -h_u H_{uu}^{-1} h_u' |_{\epsilon = 0} = BR^{-1}B', \]  
\[ \omega_0 = f - h u H_{uu}^{-1} f_u |_{\epsilon = 0}, \]  
\[ \delta_0 = H_{ux} H_{uu}^{-1} f_u' p - f_x p |_{\epsilon = 0}. \]
It is well known (Kalman, 1960) that $x^1$ and $p^1$ are related by

$$p^1 = -Kx^1 + g_1,$$

where $K$ is the symmetric positive-definite solution of matrix Riccati equation

$$\dot{K} = -KA - A'K + KSK - Q,$$

where $K(t_f) = M$, $S = BR^{-1}B'$ and $g_1$ is the solution of an adjoint vector equation

$$\dot{g}_1 = -(A - SK)'g_1 + K\omega_0 + \delta_0$$

whose boundary condition in view of (8), (9), and (13) is

$$g_1(t_f) = K(t_f)x^1(t_f) = Mx^1(t_f).$$

Note that since only the final conditions of $K$ and $g_1$ are known, (14) and (15) must be solved backward in time.

Substituting $p^1$ of (13) in (11) results in

$$\dot{x} = (A - SK)x^1 + Sg_1 + \omega_0,$$

which can be solved for $x^1$. Considering (10) and (13), the first-order control sensitivity $u^1$ can be obtained,

$$u^1 = -R^{-1}B'Kx^1 - R^{-1}(B'g_1 + f_u p^0) = -R^{-1}B'Kx^1 - \theta_1,$$

where $p^0$ is the costate zeroth-order sensitivity or the costate vector which acts as the forcing function for the first-order terms evaluated along the "nominal trajectory." It is noted that (17) constitutes the first-order coefficients of MacLaurin’s series expansions of $x$ and $u$ in the parameter $\epsilon$, i.e.,

$$x = x^0 + \epsilon x^1 + \epsilon^2 x^2/2! + \ldots,$$

$$u = u^0 + \epsilon u^1 + \epsilon^2 u^2/2! + \ldots.$$  

The second-order terms $x^2$ and $u^2$ can be similarly obtained by differentiating (8)-(10) with respect to $\epsilon$ and letting $\epsilon \to 0$. This would require that the state, control and costate vectors be continuously differentiable with respect to $\epsilon$ as many times as the designer wishes, i.e.,

$$\dot{x}^2 = (A - SK)x^2 + Sg_1 + \omega_{i-1},$$

$$u^i = -R^{-1}B'Kx^1 - R^{-1}(B'g_i + f_u p^{i-1}) = -R^{-1}B'Kx^1 - \theta_i.$$
for $i = 1, 2...$, where all coefficients are evaluated at $\epsilon = 0$, i.e., along the “nominal trajectory.” This is obtained by letting $\epsilon \to 0$ in (5)-(6), which reduces the originally nonlinear two-point boundary-value problem to a linear regulator whose solution is known (Kalman, 1960)

\[ u^0 = -R^{-1}B'Kx^0, \quad (20a) \]

\[ \dot{x}^0 = (A - SK)x^0, \quad x^0(t_0) = x_0, \quad (20b) \]

where $K$ is the solution of matrix Riccati equation (14).

The proposed near-optimum control can now be shown to have an exact feedback and an approximate forward term. Substituting (20a) and (19b) in (18b) results in the following:

\[ u = -R^{-1}B'K \sum_{i=0}^{\infty} \epsilon^i x_i/i! \]

\[ = -R^{-1} \sum_{i=1}^{\infty} \epsilon^i \left(B'_ig_i + f_a'p^{i-1}\right)/i! \]

\[ = -R^{-1}B'Kx - \sum_{i=1}^{\infty} \epsilon^i \theta_i/i! \quad (21) \]

Note that the second part of (21) was obtained in lieu of (18a). The proposed $r$th-order near-optimum control can be obtained by truncating the second series in (21) after $r$ terms, i.e.,

\[ u^r = -R^{-1}B'Kx - R^{-1} \sum_{i=1}^{r} \epsilon^i \theta_i/i! \quad (22) \]

which clearly shows that only the forward term contributes to the suboptimality of the control. Substituting (22) in (3) results

\[ \dot{x}(t, \epsilon) = (A - SK)x(t, \epsilon) - BR^{-1} \sum_{i=1}^{r} \epsilon^i \theta_i/i! + ef(x(t, \epsilon), u(t, \epsilon), \epsilon). \quad (23) \]

It must be emphasized that by virtue of (20), (17), and (19) the homogeneous portions of all orders of the sensitivity functions remain the same. Furthermore, the solution of the $(i - 1)$th terms should be used as forcing functions for evaluating the $i$th term. Figures 1 and 2 respectively show the structure of the proposed control and the flow chart of the computational procedure.
Fig. 1. Block diagram for the $r$th-order near-optimum controller

Fig. 2. A flow-chart for the computational procedure.
A Numerical Example

In this section the proposed near-optimum control is applied to the synthesis of a “control logic for a regulation system” (Garrard et al., 1967) whose torque source is a field control dc motor. The dynamic torque, field circuit equations are

\[ J_L \frac{d^2(\delta \theta)}{dt^2} + B_L \frac{d(\delta \theta)}{dt} = (K_t E/R_a) i_f - (K_b K_b/R_a) \frac{d(\delta \theta)}{dt} i_f^2, \]

\[ L_f \frac{di_f}{dt} + R_f i_f = e_f. \]  

(24)

In (24), \(J_L\) = moment of inertia of the load, \(\delta \theta\) = angular position error of the shaft, \(B_L\) = friction loss coefficient, \(\delta \dot{\theta}\) = angular velocity error of the shaft, \(K_t\) = torque constant, \(L_f\) = field inductance, \(R_f\) = field resistance, \(e_f\) = field voltage, \(E\) = constant armature voltage, \(R_a\) = armature resistance, and \(K_b\) = back emf constant. Armature inductance has been neglected. The nonlinear term in (24) arises from the magnetization properties of the motor’s ferromagnetic material. Equation (24) can be rewritten as,

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = a_{22} x_2 + a_{23} x_3 - \epsilon x_2 x_3^2, \]

\[ \dot{x}_3 = a_{33} x_3 + b_3 u, \]

(25)

where \(x_1 = \delta \theta, x_2 = \delta \dot{\theta}, x_3 = i_f, u = e_f, a_{22} = -B_L/J, a_{23} = K_t E/J L_R_a, a_{33} = -R_f/L_f, b_3 = 1/L_f\), and \(\epsilon = K_t K_b/J L_R_a\). Since \(R_a\) has relatively large value, the magnitude of \(\epsilon\) is small compared to \(a_{22}\) and \(a_{23}\), and it is normally set equal to zero and the nonlinear term is neglected (Garrard et al., 1967). The performance index is \(J = \frac{1}{2} \int_0^T (x'Qx + r_1 u^2) \, dt\). The control problem is to find a suitable near-optimum control which approximately minimizes \(J\). For computational purposes the same numerical values of previous work are used so that the result can be compared with those developed by Garrard et al., (1967) and the linearization scheme of White and Cook (1973).

\[ \dot{x} = Ax + Bu + ef(x), \]  

(26a)

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2 & 2 \\ 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f = \begin{bmatrix} 0 \\ -x_2 x_3^2 \\ 0 \end{bmatrix}, Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, r_1 = 1, \]

(26b)
and $\epsilon = 0.20$ The nominal control and state trajectories can be found from

$$u^0 = -R^{-1}B'Kx^0 = x_1^0 - 2.12x_2^0 - 0.872x_3^0,$$

(27a)

$$\dot{x}_1^0 = x_2^0,$$

$$\dot{x}_2^0 = -0.2x_2^0 + 2x_3^0,$$

$$\dot{x}_3^0 = -x_1^0 - 2.12x_2^0 - 5.872x_3^0.$$

(27b)

Table 1 summarizes the cost functional $J$ for different initial conditions, using the linearization scheme (White and Cook, 1973), suboptimal control, using approximate solution of Hamilton–Jacobi–Bellman’s equation (Garrard, 1967), and the present method. As seen from the table, the first-order near-optimum control proposed here results in the best overall performance. It is also noted that in view of Nishikawa (1971), for the case $r = 0$ the optimum cost functional is approximated up to one term while for $r = 1$ it is approximated up to $2.1 + 1 = 3$ terms. Thus it is not surprising that $r = 1$ control law is closer to the exact optimum solution. Figure 3 shows the angular position error vs. time using the three methods.

<table>
<thead>
<tr>
<th>Initial state $x_0$</th>
<th>Control law</th>
<th>Cost functional $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linearization (White and Cook, 1973)</td>
<td>0.2293 1.1297 4.1941</td>
</tr>
<tr>
<td></td>
<td>Approximate Hamilton–Jacobi–Bellman’s (Garrard et al., 1967)</td>
<td>0.2288 1.1259 4.0627</td>
</tr>
<tr>
<td></td>
<td>Present method $r = 0$</td>
<td>0.2165 1.0722 3.9520</td>
</tr>
<tr>
<td></td>
<td>Present method $r = 1$</td>
<td>0.2162 1.0712 3.9349</td>
</tr>
</tbody>
</table>
An essentially feedback control structure is proposed for nonlinear systems. The control is of the near-optimum nature since its forward term, which is a MacLaurian series in the coupling parameter $\epsilon$ is truncated, i.e., the feedback portion of the control remains exact. It must be noted that to obtain the $i$th-order near optimum control, only the $(i-1)$th terms are needed which will act as forcing functions. The results of Table 1 clearly show that the proposed method is certainly competitive with the existing methods.

The basic difficulty in implementing the proposed method is that with each new order of approximation, a number of partial differentiations of the two point boundary value problem representing the previous order is needed. This difficulty may be eliminated by comparing the cost functional and responses resulting from each new order of approximation with the previous one. If the new approximation does not improve the performance by a great deal, the search for a more optimum design may be terminated.

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REFERENCES


