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Extreme points and isometries on vector-valued Lipschitz spaces

Fernanda Botelho[a], Richard J. Fleming[b,*], James E. Jamison[a]

[a] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, United States

[b] Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, United States

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ABSTRACT

For a Banach space E and a compact metric space (X,d), a function $F: X \to E$ is a Lipschitz function if there exists $k > 0$ such that

$$\|F(x) - F(y)\| \leqslant kd(x,y) \quad \text{for all } x, y \in X.$$

The smallest such k is called the Lipschitz constant $L(F)$ for F. The space $\mathrm{Lip}(X,E)$ of all Lipschitz functions from X to E is a Banach space under the norm defined by

$$\|F\| = \max\{L(F), \|F\|_\infty\},$$

where $\|F\|_\infty = \sup\{\|F(x)\|: x \in X\}$. Recent results characterizing isometries on these vector-valued Lipschitz spaces require the Banach space E to be strictly convex. We investigate the nature of the extreme points of the dual ball for $\mathrm{Lip}(X,E)$ and use the information to describe the surjective isometries on $\mathrm{Lip}(X,E)$ under certain conditions on E, where E is not assumed to be strictly convex. We make use of an embedding of $\mathrm{Lip}(X,E)$ into a space of continuous vector-valued functions on a certain compact set.

1. Introduction

For a Banach space E (either real or complex) and a compact metric space (X,d), a function $F: X \to E$ is a Lipschitz function if there exists $k > 0$ such that

$$\|F(x) - F(y)\| \leqslant kd(x,y) \quad \text{for all } x, y \in X.$$

The smallest such k is called the Lipschitz constant $L(F)$ for F, and we also have

$$L(F) = \sup \frac{\|F(x) - F(y)\|}{d(x,y)},$$

where the supremum is taken over all $x, y \in X$ with $x \neq y$. The space $\mathrm{Lip}(X,E)$ of all Lipschitz functions from X to E is a Banach space under the norm defined by

$$\|F\| = \max\{L(F), \|F\|_\infty\},$$

where $\|F\|_\infty = \sup\{\|F(x)\|: x \in X\}$. When it may help to clarify things, we may occasionally write the Lipschitz space norm by $\|\cdot\|_L$. When E is the scalar field, we will write $\mathrm{Lip}(X)$ instead of $\mathrm{Lip}(X,E)$.

Finding characterizations of isometries on specific Banach spaces has a long history; see, for example, [4,5]. Investigations of surjective isometries on spaces of Lipschitz functions in the scalar case go back to de Leeuw in 1961 [10] and include,

* Corresponding author.
E-mail address: flemi1rj@cmich.edu (R.J. Fleming).

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among others, [6,9,11,13,15,16]. Recently, Jiménez-Vargas and Villegas-Vallecillos [8] turned their attention to the vector-valued case, showing that any surjective linear isometry from $\mathrm{Lip}(X,E)$ onto $\mathrm{Lip}(Y,E)$ is a weighted composition operator of the form

$$TF(y) = V(y)F\big(\varphi(y)\big),$$

where φ is a Lipschitz homeomorphism from Y to X, and $y \to V(y)$ is a Lipschitz map from Y into the space of isometries on E. To obtain this result it was assumed that E is strictly convex, and T maps a certain constant function $1_X \otimes e$ to $1_Y \otimes e$.

Even more recently, Araujo and Dubarbie [1], interested in conditions in the vector-valued case that will insure that all isometries are weighted compositions, have shown that results may be obtained by relaxing the compactness requirements on X, Y, and using a weaker condition on the operator T, which they call *property* **P**. This requires that for each $y \in Y$, there is a constant function F such that $TF(y) \neq 0$. In [1], the requirement that the Banach space E be strictly convex is retained.

Our focus in the current paper is to remove the requirement of strict convexity and to use extreme point methods to describe the isometries.

A key notion is that we may regard $\mathrm{Lip}(X,E)$ as a subspace of a continuous function space $C(W, E \oplus_\infty \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, depending on whether E is a real or complex space, respectively. If $\tilde{X} = \{(x,y)\colon\ x,y \in X,\ x \neq y\}$, then $\tilde{X} \times S(E^*)$, where the second component space, $S(E^*)$, the surface of the unit ball of E^*, is given the weak*-topology, is locally compact and completely regular, so has a Stone–Čech compactification $\beta(\tilde{X} \times S(E^*))$. Now $W = X \times \beta(\tilde{X} \times S(E^*))$ is a compact space and we define the map $\Phi_{X,E}$ from $\mathrm{Lip}(X,E)$ into $C(W, E \oplus_\infty \mathbb{K})$ by

$$\Phi_{X,E}F(x,\xi) = \big(F(x), F^\beta(\xi)\big),$$

where F^β denotes the continuous extension to $\beta(\tilde{X} \times S(E^*))$ of the function

$$\tilde{F}\big((a,b), v^*\big) = v^*\left[\frac{F(a) - F(b)}{d(a,b)}\right], \quad (a,b) \in \tilde{X},\ v^* \in S\big(E^*\big).$$

Let us denote by $\mathcal{M}(X,E)$ the subspace of $C(W, E \oplus_\infty \mathbb{K})$ which is the image of $\mathrm{Lip}(X,E)$ under the map $\Phi_{X,E}$. Here we have written the elements of direct sum spaces as pairs. We should point out that the idea of embedding a Lipschitz space into a continuous function space goes back to de Leeuw [10], and was used in the investigations of isometries in the scalar case by Roy [13] as well as by Jiménez-Vargas and Villegas-Vallecillos [7].

It is straightforward to show that $\Phi_{X,E}$ is a linear isometry. By ψ_w we will mean the evaluation function on $C(W, E \oplus_\infty \mathbb{K})$ and by $\tilde{\psi}_x$ we will mean the evaluation function on $\mathrm{Lip}(X,E)$, so that $\tilde{\psi}_x(F) = F(x)$. For any Banach space E, $\mathrm{ext}(E^*)$ will denote the extreme points of the unit ball of E^*. By $B(E)$ we will mean the unit ball of a Banach space E, and $S(E)$ will denote the surface of the unit ball.

In the next section we will obtain a characterization of surjective isometries on $\mathrm{Lip}(X,E)$ to $\mathrm{Lip}(Y,D)$ using extreme point techniques and mild elements of the M-structure theory of Behrends [2], to which we refer the reader for necessary definitions and terminology. Unlike the other investigations of isometries on vector-valued Lipschitz spaces as mentioned above, we do not require that the Banach spaces E, D be strictly convex. Of course some assumptions must be made.

At this point, we wish to introduce a new term.

Definition 1. A Banach space E is *quasi sub-reflexive*, QSR, if every extreme point of the unit ball of E^* is norm attaining. That is, if $e^* \in \mathrm{ext}(E^*)$, then there exists $e \in S(E)$ such that $e^*(e) = 1$.

We note that by a theorem of James, [12, p. 262], a space is reflexive if every bounded linear functional is norm attaining. The term sub-reflexive, due to Phelps, means that the set of norm-attaining members is dense in the dual, but, in fact, every Banach space is known to be sub-reflexive, a theorem of Bishop and Phelps [12, p. 278]. The space ℓ^1 is an example of a non-reflexive, quasi sub-reflexive space, as are the $C(K)$ spaces. A non-reflexive space whose dual is strictly convex would fail to be QSR. Such an example can be found in [12, p. 483].

We also introduce a stronger form of the Araujo and Dubarbie property **P**, which we call *property* **Q**.

Definition 2. An operator $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,D)$ is said to have *property* **Q** if for each $y \in Y$, $u \in D$, there is a constant function $F \in \mathrm{Lip}(X,E)$ such that $TF(y) = u$.

In the third section we replace the assumption of property **Q** with connectedness assumptions on X, Y, as well as slightly stronger restrictions on E, D, while in the last section we include some special results.

2. The non-strictly convex case using property Q

We begin by identifying important extreme points in the dual of the Banach space $\mathrm{Lip}(X,E)$. We make use here of ideas that go back to de Leeuw [10] and Roy [13]. We also note that Johnson [9] gives a characterization of extreme points of the unit ball of a subspace of $C(X)$ that relates to our next theorem.

Theorem 3. *Suppose (X,d) is a compact metric space and E is a quasi sub-reflexive Banach space. If $x \in X$ and $e^* \in \operatorname{ext}(E^*)$, then $e^* \circ \tilde{\psi}_x$ is an extreme point for the unit ball of $\operatorname{Lip}(X,E)^*$.*

Proof. Let $x \in X$ and $e^* \in \operatorname{ext}(E^*)$ be given and suppose

$$e^* \circ \tilde{\psi}_x = \frac{1}{2}\gamma^* + \frac{1}{2}\delta^*, \tag{1}$$

where γ^* and δ^* are elements of the unit sphere of $\operatorname{Lip}(X,E)^*$. By the identification of $\operatorname{Lip}(X,E)$ with a subspace of $C(W, E \oplus_\infty \mathbb{K})$, we know that there are regular Borel measures μ^* and ν^* on W with values in $(E \oplus_\infty \mathbb{K})^* = E^* \oplus_1 \mathbb{K}$ and total variation equal to 1, such that

$$\gamma^*(F) = \int_W \Phi F \, d\mu^* \quad \text{and} \quad \delta^*(F) = \int_W \Phi F \, d\nu^* \quad \text{for all } F \in \operatorname{Lip}(X,E).$$

For convenience we have dropped the subscript X, E on the map Φ. We will let μ^* and ν^* denote both the measures and the functionals they define. We note that the characterization of elements of the dual of a vector-valued continuous function space $C(K,E)$ was first given by Singer [14]. Another, even more general statement of this characterization, is given in [3, p. 182].

Let $e \in S(E)$, $x_0 \in X$, and $\lambda \in \mathbb{R}$ be defined such that

$$e^*(e) = 1, \qquad d(x_0,x) = \sup\{d(y,x)\colon\ y \in X\}, \quad \text{and} \quad \lambda = 2 + d(x_0,x).$$

Note that the existence of such an e is guaranteed by the facts that $e^* \in \operatorname{ext}(E^*)$ and E is quasi sub-reflexive. Define H on X by

$$H(z) = \big(\lambda - d(z,x)\big)e.$$

Then

$$2 \leqslant \big|2 + d(x_0,x) - d(z,x)\big|\|e\| = \big\|H(z)\big\| = \big|\lambda - d(z,x)\big| \leqslant \lambda,$$

and since $\|H(x)\| = \lambda$, we have $\|H\|_\infty = \lambda$. Furthermore, if $(s,t) \in \tilde{X}$, then

$$\big\|H(s) - H(t)\big\| = \big|d(s,x) - d(t,x)\big| \leqslant d(s,t),$$

so that $L(H) \leqslant 1$. Thus $\|H\| = \max\{\|H\|_\infty, L(H)\} = \lambda$. Observe also, that $\|H(z)\| < \lambda$ whenever $z \neq x$, and that $(e^* \circ \tilde{\psi}_x)(H) = e^*(H(x)) = \lambda$. Hence, from (1) we have

$$\lambda = \left|\frac{1}{2}\gamma^*(H) + \frac{1}{2}\delta^*(H)\right| \leqslant \frac{1}{2}\big|\gamma^*(H)\big| + \frac{1}{2}\big|\delta^*(H)\big| \leqslant \frac{1}{2}\|H\| + \frac{1}{2}\|H\| = \lambda.$$

We conclude that

$$\lambda = \big|\gamma^*(H)\big| = \big|\delta^*(H)\big|.$$

Let ξ be a fixed element of $\beta(\tilde{X} \times S(E^*))$. If $|\mu^*|(W \setminus \{(x,\xi)\}) > 0$, there is, by the regularity of μ^*, a compact subset Q of $W \setminus \{(x,\xi)\}$ such that $|\mu^*|(Q) > 0$, where by $|\mu^*|$ we mean the total variation of μ^*. Since Q is compact, there exists λ_0 such that

$$\sup_{w \in Q} \big\|\Phi_{X,E} H(w)\big\| = \lambda_0 < \lambda.$$

Thus,

$$\begin{aligned}
\lambda = \big|\gamma^*(H)\big| &= \left|\int_W \Phi H \, d\mu^*\right| = \left|\int_{\{(x,\xi)\}} \Phi(H)\, d\mu^* + \int_Q \Phi(H)\, d\mu^* + \int_{W \setminus \{(x,\xi)\} \setminus Q} \Phi H \, d\mu^*\right| \\
&\leqslant \int_{\{(x,\xi)\}} \|\Phi H\| \, d\big|\mu^*\big| + \int_Q \|\Phi H\| \, d\big|\mu^*\big| + \int_{W \setminus \{(x,\xi)\} \setminus Q} \|\Phi H\| \, d\big|\mu^*\big| \\
&\leqslant \lambda\big|\mu^*\big|\big(\big\{(x,\xi)\big\}\big) + \lambda_0\big|\mu^*\big|(Q) + \lambda\big|\mu^*\big|\big(W \setminus \big\{(x,\xi)\big\} \setminus Q\big) \\
&< \lambda\big|\mu^*\big|(W) = \lambda.
\end{aligned}$$

This contradiction means that we must have $|\mu^*|(W \setminus \{(x,\xi)\}) = 0$, so $\mu^*(W \setminus \{(x,\xi)\}) = 0$ as well. The same argument implies the same result holds also for ν^*. Let G be any element of $\operatorname{Lip}(X,E)$. From the display above we have

$$
\begin{aligned}
e^*\big(G(x)\big) &= \big(e^* \circ \tilde{\psi}_x\big)(G) = \frac{1}{2}\gamma^*(G) + \frac{1}{2}\delta^*(G) \\
&= \frac{1}{2}\Big[\int\limits_{\{(x,\xi)\}} \Phi G \, d\mu^* + \int\limits_{\{(x,\xi)\}} \Phi G \, d\nu^*\Big] \\
&= \frac{1}{2}\mu^*\big(\{(x,\xi)\}\big)\big(G(x), G^\beta(\xi)\big) + \frac{1}{2}\nu^*\big(\{(x,\xi)\}\big)\big(G(x), G^\beta(\xi)\big) \\
&= \frac{1}{2}\big[\varphi^*\big(G(x)\big) + \mu_1 G^\beta(\xi)\big] + \frac{1}{2}\big[v^*\big(G(x)\big) + \nu_1 G^\beta(\xi)\big].
\end{aligned}
$$

Here we have used the fact that, as a functional, $\mu^*(\{(x,\xi)\}) = \varphi^* \oplus \mu_1 = (\varphi^*, \mu_1)$ and $\nu^*(\{(x,\xi)\}) = v^* \oplus \nu_1 = (v^*, \nu_1)$, where φ^*, v^* are elements of $B(E^*)$ and μ_1, ν_1 are scalars. If in the display above, we let $u \in E$ and $G = 1_X \otimes u$, then $G^\beta(\xi) = 0$ and we get

$$
e^*(u) = \frac{1}{2}\varphi^*(u) + \frac{1}{2}v^*(u),
$$

and since e^* is extreme, we must have $e^* = \varphi^* = v^*$. Hence, $\mu_1 = \nu_1 = 0$. Therefore, using the display above again, we have for any $G \in \mathrm{Lip}(X,E)$ that $\gamma^*(G) = \delta^*(G) = e^* \circ \tilde{\psi}_x(G)$, and the proof is complete. □

Recall that a bounded operator T on a Banach space E is called a *multiplier* of E if every element of $\mathrm{ext}(E^*)$ is an eigenvector for T^*. Thus, for each $e^* \in \mathrm{ext}(E^*)$, there is a scalar $a_T(e^*)$ such that

$$
T^* e^* = a_T\big(e^*\big)e^*.
$$

For such a multiplier T, a multiplier S is said to be an *adjoint* of T if $a_S = \overline{a_T}$. The *centralizer* of E, written $Z(E)$, is the set of all multipliers for which an adjoint exists. The reader should consult [2] for information concerning these ideas.

Let the set $\tilde{Z}(\mathrm{Lip}(X,E))$ consist of all bounded linear operators W on $\mathrm{Lip}(X,E)$ such that $e^* \circ \tilde{\psi}_x$ is an eigenvector for W^* for each $x \in X$, $e^* \in \mathrm{ext}(E^*)$ and where we suppose the corresponding adjoint exists. This set will be called the *pseudo-centralizer* of $\mathrm{Lip}(X,E)$. We observe that $Z(\mathrm{Lip}(X,E)) \subseteq \tilde{Z}(\mathrm{Lip}(X,E))$ if E is quasi sub-reflexive. Later we give examples of Lipschitz spaces with centralizer equal to its pseudo-centralizer and also with centralizer strictly contained in its pseudo centralizer.

Lemma 4. *Suppose (X,d) is a compact metric space and E is a Banach space with trivial centralizer. Then the pseudo-centralizer of $\mathrm{Lip}(X,E)$ is the set $\{M_h\colon h \in \mathrm{Lip}(X)\}$, where by M_h we mean the operator defined by $M_h F(x) = h(x)F(x)$ for all $F \in \mathrm{Lip}(X,E)$ and $x \in X$.*

Proof. First suppose that $h \in \mathrm{Lip}(X)$ and consider the operator M_h. Let $F \in \mathrm{Lip}(X,E)$ be given, and $x, y \in X$. Then,

$$
\begin{aligned}
\big\|h(x)F(x) - h(y)F(y)\big\| &\leqslant \big\|h(x)F(x) - h(y)F(x)\big\| + \big\|h(y)F(x) - h(y)F(y)\big\| \\
&\leqslant \big|h(x) - h(y)\big|\big\|F(x)\big\| + \big|h(y)\big|\big\|F(x) - F(y)\big\| \\
&\leqslant \big[L(h)\|F\|_\infty + \|h\|_\infty L(F)\big]d(x,y),
\end{aligned}
$$

which shows that $hF \in \mathrm{Lip}(X,E)$. Furthermore,

$$
\begin{aligned}
(M_h)^*\big(e^* \circ \tilde{\psi}_x\big)(F) &= e^*\big(M_h F(x)\big) = e^*\big(h(x)F(x)\big) \\
&= h(x)e^*\big(F(x)\big) = h(x)\big(e^* \circ \tilde{\psi}_x\big)(F).
\end{aligned}
$$

This shows that M_h has the desired multiplier property, and $M_{\bar{h}}$ would satisfy the adjoint property. Hence, $M_h \in \tilde{Z}(\mathrm{Lip}(X,E))$.

On the other hand, let $W \in \tilde{Z}(\mathrm{Lip}(X,E))$. For $x \in X$, define $P(x)$ on E by

$$
P(x)u = WG(x),
$$

where $G \in \mathrm{Lip}(X,E)$ and $G(x) = u$. To see that this function is well defined, suppose $H \in \mathrm{Lip}(X,E)$ and $H(x) = G(x) = u$. Let $e^* \in \mathrm{ext}(E^*)$. Since $W \in \tilde{Z}(\mathrm{Lip}(X,E))$, there exists a scalar $a_W(e^*,x)$ such that

$$
W^*\big(e^* \circ \tilde{\psi}_x\big) = a_W\big(e^*,x\big)\big(e^* \circ \tilde{\psi}_x\big).
$$

Therefore, we have

$$
\begin{aligned}
e^*\big(WG(x)\big) &= a_W\big(e^*,x\big)e^*\big(G(x)\big) = a_W\big(e^*,x\big)e^*\big(H(x)\big) = a_W\big(e^*,x\big)\big(e^* \circ \tilde{\psi}_x\big)(H) \\
&= W^*\big(e^* \circ \tilde{\psi}_x\big)(H) = e^*\big(WH(x)\big).
\end{aligned}
$$

Since this holds for every element of $\operatorname{ext}(E^*)$, we conclude that $WG(x) = WH(x)$. Next we observe that

$$\left|a_W\left(e^*, x\right)\right| = \left\|a_W\left(e^*, x\right)\left(e^* \circ \tilde{\psi}_x\right)\right\| = \left\|W^*\left(e^* \circ \tilde{\psi}_x\right)\right\| \leqslant \left\|W^*\right\| = \|W\|.$$

From this it is straightforward to show that $\|P(x)u\| \leqslant \|W\|\|u\|$ for all $u \in E$ so that $P(x)$ is a bounded linear operator on E for each $x \in X$. For $u \in E$ and $F \in \operatorname{Lip}(X, E)$ for which $F(x) = u$, we see that

$$\begin{aligned}\left(P(x)\right)^*\left(e^*\right)(u) = e^*\left(P(x)\right)(u) = e^*\left(WF(x)\right) &= W^*\left(e^* \circ \tilde{\psi}_x\right)(F)\\ &= a_W\left(e^*, x\right)\left(e^* \circ \tilde{\psi}_x\right)(F)\\ &= a_W\left(e^*, x\right)e^*(u).\end{aligned}$$

Thus $P(x)$ is a multiplier and will have an adjoint obtained from the adjoint of W, so that $P(x) \in Z(E)$. Since $Z(E)$ is trivial, there will be a scalar $h(x)$ such that

$$P(x) = h(x)I$$

and we have

$$WF(x) = P(x)\left(F(x)\right) = h(x)F(x).$$

This can be done for every $x \in X$, and so we have shown that $W = M_h$. Because $WF = hF$ must be an element of $\operatorname{Lip}(X, E)$ for each $F \in \operatorname{Lip}(X, E)$, given e in the unit sphere of E and $x, y \in X$, it follows that

$$\begin{aligned}\left|h(x) - h(y)\right| &= \left\|h(x)e - h(y)e\right\| = \left\|h(x)(1_X \otimes e)(x) - h(y)(1_X \otimes e)(y)\right\|\\ &= \left\|W(1_X \otimes e)(x) - W(1_X \otimes e)(y)\right\|\\ &\leqslant L\left(W(1_X \otimes e)\right)d(x, y).\end{aligned}$$

This shows that $h \in \operatorname{Lip}(X)$, and the proof is complete. □

Lemma 5. *Suppose E, D are quasi sub-reflexive Banach spaces, X, Y are compact metric spaces, and T is a surjective linear isometry from $\operatorname{Lip}(X, E)$ to $\operatorname{Lip}(Y, D)$ which has property* **Q***. If $v^* \in \operatorname{ext}(D^*)$ and $y \in Y$, then $T^*(v^* \circ \tilde{\psi}_y) = e^* \circ \tilde{\psi}_x$ for some $e^* \in \operatorname{ext}(E^*)$ and $x \in X$.*

Proof. Given v^* and y as in the statement of the lemma, we know from Theorem 3 that $v^* \circ \tilde{\psi}_y$ is an extreme point for $\operatorname{Lip}(Y, D)^*$ and so is the image of an extreme point of the unit ball of $\mathcal{M}(Y, D)^*$ which, by a theorem of Brosowski and Deutsch (among others; see [4, pp. 33, 44]), must be of the form $\gamma^* \circ \psi_w$, where γ^* is an extreme point of $(D \oplus_\infty \mathbb{K})^* = D^* \oplus_1 \mathbb{K}$, and $w \in W$. Since $\gamma^* = (u^*, 0)$ or $(0, \tau)$, where $u^* \in \operatorname{ext}(D^*)$ and τ is a scalar with $|\tau| = 1$, we see from the definition of $\Phi_{Y,D}$ that

$$v^* \circ \tilde{\psi}_y = \Phi^*_{Y,D}\left(\left(v^*, 0\right) \circ \psi_{(y,\xi)}\right), \tag{2}$$

where $\xi \in \beta(\tilde{Y} \times S(D^*))$. (Note that the particular ξ has no importance here.) To justify this equation, we note that if $v^* \circ \tilde{\psi}_y = \Phi^*_{Y,D}((0, \tau) \circ \psi_{(z,\xi)})$, then we would have, for any $G \in \operatorname{Lip}(Y, D)$, that

$$v^*\left(G(y)\right) = \tau G^\beta(\xi),$$

which would yield a contradiction for $G = 1_Y \otimes u$ where $v^*(u) \neq 0$. On the other hand, if $v^* \circ \tilde{\psi}_y = \Phi^*_{Y,D}((d^*, 0) \circ \psi_{(z,\xi)})$, then we would obtain

$$v^*\left(G(y)\right) = d^*\left(G(z)\right), \quad \text{for all } G \in \operatorname{Lip}(Y, D), \tag{3}$$

where $d^* \in \operatorname{ext}(D^*)$ and $(z, \xi) \in Y \times \beta(\tilde{Y} \times S(D)^*)$. If $y \neq z$, we can choose G which has a value at y such that $v^*(G(y)) \neq 0$, while $G(z) = 0$, which gives a contradiction. Now, since $y = z$, we have from (3) that

$$v^*\left(G(y)\right) = d^*\left(G(y)\right),$$

which means that $v^* = d^*$. This justifies (2). Hence,

$$\begin{aligned}T^*\left(v^* \circ \tilde{\psi}_y\right)(F) &= \left(v^* \circ \tilde{\psi}_y\right)(TF)\\ &= \Phi^*_{Y,D}\left(\left(v^*, 0\right) \circ \psi_{(y,\xi)}\right)(TF)\\ &= \left(\left(v^*, 0\right) \circ \psi_{(y,\xi)}\right)\left(\Phi_{Y,D}TF\right)\\ &= \left(\left(v^*, 0\right) \circ \psi_{(y,\xi)}\right)\left(R(\Phi_{X,E}F)\right)\\ &= R^*\left(\left(v^*, 0\right) \circ \psi_{(y,\xi)}\right)(\Phi_{X,E}F),\end{aligned}$$

where R is the isometry from $\mathcal{M}(X,E)$ onto $\mathcal{M}(Y,D)$ defined by $R = \Phi_{Y,D}T(\Phi_{X,E})^{-1}$. Now $R^*((v^*,0)\circ\psi_{(y,\xi)})$ must be an extreme point of $\mathcal{M}(X,E)^*$, and so is of the form $(e^*,0)\circ\psi_{(x,\zeta)}$ or $(0,\tau)\circ\psi_{(x,\zeta)}$. If it is the latter, than we would have from the calculation above that

$$v^*\big(TF(y)\big) = \big[(0,\tau)\circ\psi_{(x,\zeta)}\big](\Phi_{X,E}F) = \tau F^\beta(\zeta),$$

for all $F \in \mathrm{Lip}(X,E)$. Choose $u \in D$ such that $v^*(u) \neq 0$, and, using the fact that T has property **Q**, let F be a constant function such that $TF(y) = u$. Then by the equality above,

$$0 \neq v^*\big(TF(y)\big) = \tau F^\beta(\zeta) = 0,$$

since F is constant. Therefore, we must conclude that $R^*((v^*,0)\circ\psi_{(y,\xi)}) = (e^*,0)\circ\psi_{(x,\zeta)}$ and using the calculation above once again, for any $G \in \mathrm{Lip}(X,E)$, we have

$$\begin{aligned} T^*\big(v^*\circ\tilde\psi_y\big)(G) &= \big((e^*,0)\circ\psi_{(x,\zeta)}\big)(\Phi_{X,E}G) \\ &= \big(e^*,0\big)\big(G(x),G^\beta(\zeta)\big) \\ &= e^*\big(G(x)\big) \\ &= \big(e^*\circ\tilde\psi_x\big)(G). \end{aligned}$$

This shows that $T^*(v^*\circ\tilde\psi_y) = e^*\circ\tilde\psi_x$ and completes the proof. □

Theorem 6. *Let X, Y be compact metric spaces, and let E, D be quasi sub-reflexive Banach spaces with trivial centralizers. Suppose that $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,D)$ is a surjective linear isometry so that both T and T^{-1} have property* **Q**. *Then there exists a Lipschitz homeomorphism φ from Y onto X with $L(\varphi) \leqslant \max\{1,\mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leqslant \max\{1,\mathrm{diam}(Y)\}$ and a Lipschitz map $y \to V(y)$ from Y to the space of surjective linear isometries on E to D such that*

$$TF(y) = V(y)F\big(\varphi(y)\big) \quad \textit{for all } F \in \mathrm{Lip}(X,E),\ y \in Y.$$

Proof. Let $h \in \mathrm{Lip}(X)$ and consider the operator TM_hT^{-1} defined on $\mathrm{Lip}(Y,D)$ to itself. Let $v^* \in \mathrm{ext}(D^*)$, $y \in Y$, and $G \in \mathrm{Lip}(Y,D)$ be given. Then $G = TF$ for some $F \in \mathrm{Lip}(X,E)$. By Lemma 5 there are $e^* \in \mathrm{ext}(E^*)$ and $z \in X$ such that $T^*(v^*\circ\tilde\psi_y) = e^*\circ\tilde\psi_z$. We have

$$\begin{aligned} \big(TM_hT^{-1}\big)^*\big(v^*\circ\tilde\psi_y\big)(G) &= T^*\big(v^*\circ\tilde\psi_y\big)(M_hF) \\ &= \big(e^*\circ\tilde\psi_z\big)(M_hF) \\ &= e^*\big(h(z)F(z)\big) = h(z)e^*\big(F(z)\big) \\ &= h(z)T^*\big(v^*\circ\tilde\psi_y\big)(F) \\ &= h(z)\big(v^*\circ\tilde\psi_y\big)(G). \end{aligned}$$

It follows that TM_hT^{-1} is in the pseudo centralizer of $\mathrm{Lip}(Y,D)$ ($TM_{\bar h}T^{-1}$ serves as an adjoint). Hence, by Lemma 4, there is an element $\tilde h \in \mathrm{Lip}(Y)$ such that

$$TM_hT^{-1} = M_{\tilde h}.$$

Therefore,

$$TM_h = M_{\tilde h}T,$$

from which we obtain

$$\begin{aligned} h(z)e^*\big(F(z)\big) &= \big(e^*\circ\tilde\psi_z\big)(M_hF) \\ &= T^*\big(v^*\circ\tilde\psi_y\big)(M_hF) \\ &= \big(M_h^*T^*\big)\big(v^*\circ\tilde\psi_y\big)(F) \\ &= \big(T^*M_{\tilde h}^*\big)\big(v^*\circ\tilde\psi_y\big)(F) \\ &= v^*\big(M_{\tilde h}TF(y)\big) \\ &= \tilde h(y)v^*\big(TF(y)\big) \\ &= \tilde h(y)T^*\big(v^*\circ\tilde\psi_y\big)(F) \\ &= \tilde h(y)e^*\big(F(z)\big). \end{aligned}$$

Since we may choose F so that $e^*(F(z)) \neq 0$, we must conclude that $h(z) = \tilde{h}(y)$. Suppose for $v^*, w^* \in \operatorname{ext}(D^*)$ it is true that

$$T^*\big(v^* \circ \tilde{\psi}_y\big) = e^* \circ \tilde{\psi}_x \quad \text{and} \quad T^*\big(w^* \circ \tilde{\psi}_y\big) = u^* \circ \tilde{\psi}_z. \tag{4}$$

From our earlier calculations, we must have

$$h(x) = \tilde{h}(y) = h(z).$$

Since the Lipschitz functions on X are numerous enough to separate points, we conclude that $x = z$. Hence, we may define the function φ on Y by $\varphi(y) = x$ if $T^*(v^* \circ \tilde{\psi}_y) = e^* \circ \tilde{\psi}_x$. This function is well defined since (4) implies $x = z$. Next, we define, for each $y \in Y$, an operator $V(y)$ on E to D by

$$V(y)u = TF(y) \quad \text{where } F \in \operatorname{Lip}(X, E) \text{ has the property that } F\big(\varphi(y)\big) = u.$$

To see that $V(y)$ is well defined, suppose that $x = \varphi(y)$ and $F(x) = H(x)$ for $F, H \in \operatorname{Lip}(X, E)$. For any extreme point $v^* \in \operatorname{ext}(D^*)$, and $T^*(v^* \circ \tilde{\psi}_y) = e^* \circ \tilde{\psi}_x$, it follows that

$$v^*\big(TF(y)\big) = e^*\big(F(x)\big) = e^*\big(H(x)\big) = v^*\big(TH(y)\big).$$

Therefore, since this holds for any extreme point $v^* \in \operatorname{ext}(D^*)$, we must have $TF(y) = TH(y)$.

To show that $V(y)$ is an isometry, let $e \in E$ be given and suppose $F \in \operatorname{Lip}(X, E)$ is such that $F(\varphi(y)) = e$, $\|F\| = \|F(\varphi(y))\| = \|e\|$. Let $e^* \in \operatorname{ext}(E^*)$ be such that $e^*(e) = \|e\|$. Since T^{-1} is an isometry with the same properties as T, there exists $v^* \in S(D^*)$ such that $T^*(v^* \circ \tilde{\psi}_y) = e^* \circ \tilde{\psi}_{\varphi(y)}$. Hence,

$$\begin{aligned}
\|e\| &= e^*\big(F\big(\varphi(y)\big)\big) = \big(e^* \circ \tilde{\psi}_{\varphi(y)}\big)(F) \\
&= T^*\big(v^* \circ \tilde{\psi}_y\big)(F) \\
&= \big|v^*\big(TF(y)\big)\big| \\
&\leqslant \big\|TF(y)\big\| \leqslant \|TF\| = \|F\| \\
&= \|e\|.
\end{aligned}$$

Therefore,

$$\|e\| = \big\|TF(y)\big\| = \big\|V(y)F\big(\varphi(y)\big)\big\| = \big\|V(y)e\big\|,$$

which shows that $V(y)$ is an isometry. It must also be surjective, for if $u \in D$ is given, by the surjectivity of T, there is some $F \in \operatorname{Lip}(X, E)$ such that $TF = 1_Y \otimes u$. Hence

$$u = TF(y) = V(y)F\big(\varphi(y)\big),$$

so that u is in the range of $V(y)$.

To see that the map $y \to V(y)$ is a Lipschitz map, let $u \in S(E)$ be given, and let $F = 1_X \otimes u$. Since $TF \in \operatorname{Lip}(Y, D)$, we have, for $y_1, y_2 \in Y$,

$$\big\|V(y_1)(u) - V(y_2)(u)\big\| = \big\|TF(y_1) - TF(y_2)\big\| \leqslant L(TF)d(y_1, y_2) \leqslant d(y_1, y_2).$$

Since this holds for all $u \in S(E)$, we conclude that $\|V(y_1) - V(y_2)\| \leqslant d(y_1, y_2)$.

Finally, let us show that φ is a Lipschitz map. We follow an argument given in [7]. Let $u \in S(E)$ and consider the function $F_x = f_x \otimes u$, where $f_x(z) = d(z, x)$. It is easy to see that $\|F_x\|_\infty \leqslant \operatorname{diam}(X)$, and $L(F_x) \leqslant 1$, so that $\|F_x\| \leqslant k = \max\{1, \operatorname{diam}(X)\}$. Since T is an isometry, $\|TF_{\varphi(y)}\| \leqslant k$ for all $y \in Y$, so that $L(TF_{\varphi(y)}) \leqslant k$ for each $y \in Y$. Let $y, z \in Y$ be given, so that we have

$$\big\|TF_{\varphi(y)}(y) - TF_{\varphi(y)}(z)\big\| \leqslant kd(y, z). \tag{5}$$

Now,

$$TF_{\varphi(y)}(y) = V(y)(F_{\varphi(y)}\big(\varphi(y)\big) = V(y)\big[d\big(\varphi(y), \varphi(y)\big)u\big] = 0,$$

while

$$TF_{\varphi(y)}(z) = V(z)F_{\varphi(y)}\big(\varphi(z)\big) = V(z)\big[d\big(\varphi(y), \varphi(z)\big)u\big] = d\big(\varphi(y), \varphi(z)\big)V(z)u.$$

From (5) and the above two lines, we have

$$d\big(\varphi(y), \varphi(z)\big)\big\|V(z)u\big\| \leqslant kd(y, z),$$

and since $V(z)$ is an isometry, $d(\varphi(y), \varphi(z)) \leqslant kd(y, z)$. Hence, φ is Lipschitz with $L(\varphi) \leqslant \max\{1, \operatorname{diam}(X)\}$. Applying the same argument to the inverse, we get φ^{-1} exists and is Lipschitz with $L(\varphi^{-1}) \leqslant \max\{1, \operatorname{diam}(Y)\}$. $\square$

It is easy to see that if T is a weighted composition operator in the form $TF(y) = V(y)F(\varphi(y))$, where $V(y)$ is a surjective operator from E to D for each y, and φ is a function from Y to X, then T has property **Q**. For, given $y \in Y$, $u \in D$, let $e \in E$ be such that $V(y)e = u$. Then $T(1_X \otimes e)(y) = V(y)e = u$. Let us say that a bounded linear operator T from $\mathrm{Lip}(X, E)$ to $\mathrm{Lip}(Y, D)$ has the *constant function property*, CFP, if for each $u \in D$, there exists $e(u) \in E$ such that $T(1_X \otimes e(u)) = 1_Y \otimes u$. We observe that if T has CFP, then it has property **Q**, and that if T is a weighted composition operator as above with the CFP, where $V(y)$ is assumed to be injective for each y, then the map $y \to V(y)$ is constant. In this last case, if T is an isometry and if $\mathrm{diam}(X) \leqslant 2$, we can show that φ is an isometry. We record these observations in the following corollary.

Corollary 7. *Suppose that T is a surjective linear isometry from* $\mathrm{Lip}(X, E)$ *onto* $\mathrm{Lip}(Y, D)$, *where X, Y are compact metric spaces and E, D are quasi sub-reflexive and have trivial centralizers.*

(i) *The isometry T can be written as a weighted composition operator as in Theorem* 6 *if and only if both T and T^{-1} have property* **Q**.
(ii) *The isometry T can be written as a weighted composition operator in the form $TF(y) = V(y)F(\varphi(y))$ where $V(y) = V$ is constant if and only if both T and T^{-1} have CFP.*
(iii) *If T is as in item* (ii) *above, and* $\mathrm{diam}(X) \leqslant 2$, *then φ is an isometry.*

Proof. The proof of both (i) and (ii) is immediate. Let us prove (iii). Suppose $y_0, y_1 \in Y$ are such that

$$d\big(\varphi(y_0), \varphi(y_1)\big) > d(y_0, y_1).$$

Define F on X by $F(x) = (1 - d(x, \varphi(y_0))e$, where e is a fixed element of $S(E)$. It is straightforward to show that $F \in \mathrm{Lip}(X, E)$ with $L(F) \leqslant 1$. Since $\mathrm{diam}(X) \leqslant 2$, we see that

$$\big\|F(x)\big\| = \big|1 - d\big(x, \varphi(y_0)\big)\big| \leqslant 1,$$

and since $\|F(\varphi(y_0))\| = 1$, we have $\|F\|_\infty = 1$. Thus, $\|F\|_L = 1$, and so $\|TF\|_L = 1$ as well. However,

$$\begin{aligned}\frac{\|TF(y_0) - TF(y_1)\|}{d(y_0, y_1)} &= \frac{\|VF(\varphi(y_0)) - VF(\varphi(y_1))\|}{d(y_0, y_1)} \\ &= \frac{\|F(\varphi(y_0)) - F(\varphi(y_1))\|}{d(y_0, y_1)} = \frac{d(\varphi(y_0), \varphi(y_1))}{d(y_0, y_1)} \\ &> 1.\end{aligned}$$

This contradicts the fact that $L(TF) \leqslant 1$, and we conclude that $d(\varphi(y_0), \varphi(y_1)) \leqslant d(y_0, y_1)$. The opposite inequality is obtained by considering φ^{-1}, and we conclude that φ is an isometry. □

3. The non-strictly convex case using connectedness

The need to assume the property **Q** in the proof of Lemma 5 and hence also for Theorem 6 is less than satisfying since it is an assumption made on the isometry itself. This condition can be removed by adding some conditions to the metric spaces X, Y and the Banach spaces E, D.

Let us introduce some notation here that will be useful. Extreme points of the unit ball of the dual of $\mathcal{M}(X, E)$ of the form $(e^*, 0) \circ \psi_w$, where $e^* \in \mathrm{ext}(E^*)$, $w \in W$, will be said to be extreme points of type I, and the set of all such will be denoted by $\mathrm{Ext}_I(\mathcal{M}(X, E)^*)$. Similarly, the set $\mathrm{Ext}_{II}(\mathcal{M}(X, E)^*)$ will be the set of all extreme points of type II on $\mathcal{M}(X, E)^*$, that is, those of the form $(0, \tau) \circ \psi_w$ where $|\tau| = 1$, and $w \in W$. Thus the action of an extreme point of type I on $\Phi_{X,E}F$ is given by

$$\big[\big(e^*, 0\big) \circ \psi_{(x,\xi)}\big](\Phi_{X,E}F) = e^*\big(F(x)\big) = \big(e^* \circ \tilde{\psi}_x\big)(F),$$

while for an extreme point of type II the action is given by

$$\big[(0, \tau) \circ \psi_{(x,\xi)}\big](\Phi_{X,E}F) = \tau F^\beta(\xi).$$

For convenience later, we will let $\mathrm{Ext}_I(\mathrm{Lip}(X, E)^*)$ denote the extreme points of the dual unit ball of $\mathrm{Lip}(X, E)$ of the form $e^* \circ \tilde{\psi}_x$, for $x \in X$, $e^* \in \mathrm{ext}(E^*)$ and $\mathrm{Ext}_{II}(\mathrm{Lip}(X, E)^*)$ will be the extreme points of the form $\tau\tilde{\psi}_\xi$, for $\xi \in \beta(\tilde{X} \times S(E^*))$, where by $\tilde{\psi}_\xi$ we will mean the function defined on $\mathrm{Lip}(X, E)$ by

$$\tilde{\psi}_\xi(F) = F^\beta(\xi). \tag{6}$$

Note that we have, in general,

$$\mathrm{Ext}_I\big(\mathcal{M}(X, E)^*\big) \subset \Delta_I\big(\mathcal{M}(X, E)^*\big) \subset \Gamma_I\big(\mathcal{M}(X, E)^*\big),$$

where

$$\Delta_I = \{(e^*, 0) \circ \psi_w \colon e^* \in \operatorname{ext}(E^*),\ w \in W\},$$
$$\Gamma_I = \{(e^*, 0) \circ \psi_w \colon e^* \in S(E^*),\ w \in W\}.$$

Recall that in the proof of Lemma 5 we actually showed that $\Phi^*_{X,E}$ maps the set $\operatorname{Ext}_I(\mathcal{M}(X,E)^*)$ onto the set of extreme points of the unit ball of $\operatorname{Lip}(X,E)^*$ of the form $e^* \circ \tilde{\psi}_x$, where $e^* \in \operatorname{ext}(E^*)$ and $x \in X$, that is $\operatorname{Ext}_I(Lip(X,E)^*)$. Furthermore, we showed that R^* maps $\operatorname{Ext}_I(\mathcal{M}(Y,D)^*)$ onto $\operatorname{Ext}_I(\mathcal{M}(X,E)^*)$, where R is the isometry from $\mathcal{M}(X,E)$ to $\mathcal{M}(Y,D)$ given by $R = \Phi_{Y,D} T (\Phi_{X,E})^{-1}$, corresponding to the isometry $T : \operatorname{Lip}(X,E) \to \operatorname{Lip}(Y,D)$, and where T had property **Q**. Of course, the extreme points of type II will be mapped to each other in the same way. We would like to obtain this result about R without assuming property **Q** for T. Observe also that by Theorem 3, we have, under the hypotheses of that theorem, that

$$\operatorname{Ext}_I(\mathcal{M}(X,E)^*) = \Delta_I(\mathcal{M}(X,E)^*).$$

The same equality occurs also for Y, D, of course. If we assume that E^* and D^* are strictly convex, then

$$\operatorname{Ext}_I(\mathcal{M}(X,E)^*) = \Gamma_I(\mathcal{M}(X,E)^*),$$

and this holds for Y, D as well.

The following simple lemma will be useful.

Lemma 8. *If E is a Banach space and $E \neq \mathbb{R}$, then $S(E^*)$ is pathwise connected.*

Proof. Let e^* and v^* be two distinct functionals in $S(E^*)$. Let $w^* \in S(E^*)$ be such that 0 is not on the line segment from e^* to w^*, nor on the segment from w^* to v^*. Let $\varphi(t)$ denote the function describing the path from e^* to v^* consisting of the two line segments from e^* to w^* followed by w^* to v^*. Since $\|\varphi(t)\| \neq 0$ for all t, then $\frac{\varphi(t)}{\|\varphi(t)\|}$ defines a path in $S(E^*)$ connecting e^* with v^*. Note that this holds for any topology on $S(E^*)$ that is equal to or weaker than the norm topology. $\square$

We are going to assume now that our metric spaces X, Y are compact and connected, and that the Banach spaces E, D are both quasi sub-reflexive and have strictly convex dual spaces. By James' theorem, mentioned earlier, these conditions on E, D imply they must be reflexive and smooth. Hence we will now assume that.

Lemma 9. *Let X, Y be compact and pathwise connected metric spaces, and assume that E, D are smooth, reflexive Banach spaces and let T be a surjective linear isometry from $\operatorname{Lip}(X,E)$ to $\operatorname{Lip}(Y,D)$. If $v^* \in \operatorname{ext}(D^*)$ and $y \in Y$, then $T^*(v^* \circ \tilde{\psi}_y) = e^* \circ \tilde{\psi}_x$ for some $e^* \in \operatorname{ext}(E^*)$ and $x \in X$.*

Proof. We are going to show that for the isometry R introduced in the proof of Lemma 5, R^* maps extreme points of type I of $\mathcal{M}(Y,D)^*$ (type II) onto extreme points of type I of $\mathcal{M}(X,E)^*$ (type II, respectively). First we note that since $B(E^*)$ is compact in the w^*-topology and X is compact, then $\operatorname{Ext}_I(\mathcal{M}(X,E)) = \Gamma_I(\mathcal{M}(X,E)^*)$ is compact and easily shown to be disjoint from the w^*-closure of $\operatorname{Ext}_{II}(\mathcal{M}(X,E)^*)$. Therefore, they form a separation of the extreme points of the unit ball of $\mathcal{M}(X,E)^*$. Let us assume that E and D are not equal to $\mathbb{R}$. Since $S(D^*)$ and Y are both pathwise connected, it follows that $\operatorname{Ext}_I(\mathcal{M}(Y,D)^*) = \Gamma_I(\mathcal{M}(Y,D)^*)$ is itself pathwise connected (in the w^*-topology). Its image under the isometry R^* must also be pathwise connected, and is therefore mapped entirely inside $\Gamma_1(\mathcal{M}(X,E)^*)$ or $\operatorname{Ext}_{II}(\mathcal{M}(X,E)^*)$. If the latter were the case, then for every $v^* \in S(D^*)$, $y \in Y$, there would be a scalar τ with $|\tau| = 1$, and $\xi \in \beta(\tilde{X} \times S(E^*))$ such that for every $F \in \operatorname{Lip}(X,E)$, we have

$$v^*(TF(y)) = \tau F^\beta(\xi).$$

It is clearly impossible for this to happen, for we could choose a constant function F and $y \in Y$, $v^* \in S(D^*)$ such that

$$0 \neq v^*(TF(y)) = \tau F^\beta(\xi) = 0.$$

We conclude that $R^*(\Gamma_I) \subset \Gamma_I$, and the desired result follows as in the last part of the proof of Lemma 5. Finally, we observe that if $E = D = \mathbb{R}$, then the above argument still works. Although $\operatorname{Ext}_I(\mathcal{M}(Y,D^*)$ is not connected, it is the union of two disjoint connected sets, and each must map entirely into $\operatorname{Ext}_I(\mathcal{M}(X,E)^*)$ or $\operatorname{Ext}_{II}(\mathcal{M}(X,E)^*)$. $\square$

Theorem 10. *Let X, Y be compact and pathwise connected metric spaces, and assume that E, D are smooth, reflexive Banach spaces and let T be a surjective linear isometry from $\operatorname{Lip}(X,E)$ to $\operatorname{Lip}(Y,D)$. Then there exists a Lipschitz homeomorphism φ from Y onto X with $L(\varphi) \leqslant \max\{1, \operatorname{diam}(X)\}$ and $L(\varphi^{-1}) \leqslant \max\{1, \operatorname{diam}(Y)\}$ and a Lipschitz map $y \to V(y)$ from Y to the space of surjective linear isometries on E to D such that*

$$TF(y) = V(y)F(\varphi(y)) \quad \text{for all } F \in \operatorname{Lip}(X,E),\ y \in Y. \tag{7}$$

Proof. The proof that T has the form given by (7) is, by using Lemma 9 instead of Lemma 5, exactly the same as the proof of Theorem 6. $\square$

4. Pointed metric spaces

Let us recall here that the notation introduced earlier, in (6), implies that, for $(a, b, v^*) \in \tilde{X} \times S(E^*)$,

$$\tilde{\psi}_{(a,b,v^*)}(F) = F^\beta\big(a, b, v^*\big) = \tilde{F}\big(a, b, v^*\big) = v^*\left[\frac{F(b) - F(a)}{d(a, b)}\right],$$

where $F \in \operatorname{Lip}(X, E)$ and we have written $((a, b), v^*) = (a, b, v^*)$ for convenience.

Definition 11. A metric space X is said to be a *pointed* metric space if there exists $x_0 \in X$ such that $d(x_0, x) = 1$ for all $x \neq x_0$. The point x_0 will be called the *special point*.

Pointed metric spaces have been discussed in [11], where they were called *centered* spaces, and in [16]. We observe that a pointed metric space must have diameter no more than 2, and any compact metric space with diameter less than or equal to 2 can be made into a pointed compact metric space by adding one point.

We need a lemma due to de Leeuw [10] which has been used by a number of authors through the years. We state it without proof.

Lemma 12 *(de Leeuw). Let Z be a compact Hausdorff space, and let A be a closed subspace of $C(Z)$. If $z \in Z$, then a sufficient condition for $\psi_z \in \operatorname{ext}(A^*)$ is that there exists a function $f \in B(A)$ such that*

(i) $f(z) = 1$,
(ii) $|f(y)| = 1$ *if and only if there exists a* $\lambda = \pm 1$ *such that*

$$g(y) = \lambda g(z) \quad \textit{for all } g \in A.$$

In this case it is said that f peaks at z relative to A.

Let $Z = \beta(\tilde{X} \times S(E^*))$, and let $A(X, E)$ denote the subspace of $C(Z)$ consisting of all those functions F^β defined on Z, where $F \in \operatorname{Lip}(X, E)$. As before, ψ_ξ will denote the evaluation functional on $C(Z)$. In this section, we are going to consider real Banach spaces and real-valued functions.

Lemma 13. *Assume that X is a compact, pointed metric space, with x_0 as the special point. If $x \in X$, with $x \neq x_0$, and $v^* \in S(E^*)$, with v^* norm attaining, where E is a real Banach space, then $\psi_{(x_0,x,v^*)} \in \operatorname{ext}(A(X, E)^*)$.*

Proof. The idea here is to show that (x_0, x, v^*) is a peak point for a function in $A(X, E)$. Choose $v \in S(E)$ such that $v^*(v) = 1$. We now set

$$F(z) = \begin{cases} \frac{1}{2}v & \text{if } z = x_0, \\ -\frac{1}{2}v + \frac{d(z,x)}{2(1+\operatorname{diam}(X))}v & \text{if } z \neq x_0. \end{cases}$$

The function F is Lipschitz. First, we observe that

$$F^\beta\big(x_0, x, v^*\big) = 1.$$

If (x_1, x_2, w^*) is a point in $\tilde{X} \times S(E^*)$, different from (x_0, x, v^*), then $F^\beta(x_1, x_2, w^*) = w^*(\frac{F(x_1)-F(x_2)}{d(x_1,x_2)})$ satisfies one of the following possibilities:

$$\left|w^*\left(\frac{F(x_1) - F(x_2)}{d(x_1, x_2)}\right)\right| = \begin{cases} \left|1 - \frac{d(x_2,x)}{2(1+\operatorname{diam}(X))}\right||w^*(v)| < 1, & x_2 \neq x \text{ or } w^* \neq -v^*, \\ \text{or} & \\ \left|1 - \frac{d(x_1,x)}{2(1+\operatorname{diam}(X))}\right||w^*(v)| < 1, & x_1 \neq x \text{ or } w^* \neq -v^*, \\ \text{or} & \\ \frac{|d(x_1,x)-d(x_2,x)|}{2(1+\operatorname{diam}(X))}\frac{|w^*(v)|}{d(x_1,x_2)} \leqslant \frac{1}{2} & \text{otherwise.} \end{cases}$$

This is sufficient to conclude that $|F^\beta(\xi)| < 1$, for every $\xi \in Z$ except when $\xi = (x_0, x, \pm v^*)$ or $\xi = (x, x_0, \pm v^*)$. Since

$$G^\beta\big(x_0, x, -v^*\big) = -G^\beta\big(x_0, x, v^*\big),$$

and

$$G^\beta\big(x, x_0, \pm v^*\big) = \mp G^\beta\big(x_0, x_1, v^*\big),$$

for all $G \in \operatorname{Lip}(X, E)$, Lemma 12 implies that $\psi_{(x_0,x,v^*)}$ is an extreme point of the unit ball of $A(X, E)^*$. $\square$

Note that if E is reflexive, then the above holds for any $v^* \in S(E^*)$, or for any $v^* \in \mathrm{ext}(E^*)$ if E is QSR.

We recall that candidates for extreme points of type II for $\mathcal{M}(X,E)^*$ are of the form $(0,\tau)\circ\psi_{(x,\xi)}$, where $\tau = \pm 1$, $x \in X$, and $\xi \in Z = \beta(\tilde{X} \times S(E^*))$. Observe that

$$\big[(0,\tau)\circ\psi_{(x,\xi)}\big](\Phi_{X,E}F) = \tau F^\beta(\xi) = \tau\psi_\xi\big(F^\beta\big) = \tau\tilde{\psi}_\xi(F)$$

for all $F \in \mathrm{Lip}(X,E)$.

Theorem 14. *Suppose X is a compact, pointed metric space with special point x_0 and E is a real Banach space. Then $\tilde{\psi}_\xi \in \mathrm{Ext}_{II}(\mathrm{Lip}(X,E)^*)$ for $\xi = (x_0, x, v^*)$, where $x \neq x_0$, and v^* is norm attaining.*

Proof. Let x, v^* be as given in the statement of the theorem. Suppose

$$\tilde{\psi}_{(x_0,x,v^*)} = \frac{1}{2}\delta_0^* + \frac{1}{2}\delta_1^*, \tag{8}$$

where δ_0^*, δ_1^* are elements in the unit ball of $\mathrm{Lip}(X,E)^*$. As we have seen before, the action of δ_0^* on $\mathrm{Lip}(X,E)$ is given by an integral with respect to a regular Borel measure μ^* on W with values in $E^* \oplus_1 \mathbb{R}$, so that

$$\delta_0^*(F) = \int_W (\Phi_{X,E}F)\,d\mu^* = \int_X F\,du_0^* + \int_Z F^\beta\,d\sigma_0^*,$$

where μ_0^* is a measure on X with values in E^* and σ_0^* is a real-valued measure on $Z = \beta(\tilde{X} \times S(E^*))$. Furthermore, $|\mu_0^*|(X) + |\sigma_0^*|(Z) = 1$, where we are using the total variation of the measures. The breakup of the integral into two pieces as above is straightforward to show, and is especially clear if one uses the alternate form of W as a disjoint union of X and Z. Of course, δ_1^* has a similar makeup.

If we consider the Lipschitz function F defined in the previous lemma, we recall that $\|F\|_\infty < L(F) = 1$. We have

$$\begin{aligned}
1 = L(F) &= \tilde{\psi}_{(x_0,x,v^*)}F \\
&= \frac{1}{2}\delta_0^*(F) + \frac{1}{2}\delta_1^*(F) \\
&= \frac{1}{2}\bigg[\int_X F\,d\mu_0^* + \int_X F\,d\mu_1^*\bigg] + \frac{1}{2}\bigg[\int_Z F^\beta d\sigma_0^* + \int_Z F^\beta\,d\sigma_1^*\bigg].
\end{aligned}$$

If $|\mu_0^*|(X) > 0$ or $|\mu_1^*|(X) > 0$, then

$$\begin{aligned}
1 = L(F) &\leqslant \frac{1}{2}\big[\|F\|_\infty\big|\mu_0^*\big|(X) + \|F\|_\infty\big|\mu_1^*\big|(X) + L(F)\big|\sigma_0^*\big|(Z) + L(F)\big|\sigma_1^*\big|(Z)\big] \\
&< \frac{1}{2}\big[\big|\mu_0^*\big|(X) + \big|\mu_1^*\big|(X) + \big|\sigma_0^*\big|(Z) + \big|\sigma_1^*\big|(Z)\big] = 1.
\end{aligned}$$

This contradiction means that $\mu_0^*(X) = \mu_1^*(X) = 0$, so that the actions of δ_0^* and δ_1^* reduce to functionals on the space we called $A(X,E)$. Hence, Eq. (8) is really a statement to which Lemma 13 applies. It follows that $\delta_0^* = \delta_1^* = \tilde{\psi}_{(x_0,x,v^*)}$, and the theorem is proved. $\square$

We note in passing, that a result of the above proof is that for all $y \in X$, $x \neq x_0$, and v^* norm attaining, we have $(0,\tau)\circ\psi_{(y,(x_0,x,v^*))} \in \mathrm{Ext}_{II}(\mathcal{M}(X,E)^*)$.

Theorem 15. *Let X, Y be pointed, compact metric spaces and suppose that E, D are quasi sub-reflexive Banach spaces with trivial centralizers. If T is a surjective linear isometry from $\mathrm{Lip}(X,E)$ to $\mathrm{Lip}(Y,D)$ such that*

$$T^*\big[\mathrm{Ext}_I\big(\mathrm{Lip}(Y,D)^*\big)\big] = \mathrm{Ext}_I\big(\mathrm{Lip}(X,E)^*\big),$$

then T and T^{-1} have the constant function property and there exists a surjective isometry $V : E \to D$ and an isometry $\varphi : Y \to X$ such that

$$TF(y) = VF\big(\varphi(y)\big),$$

for every $F \in \mathrm{Lip}(X,E)$ and $y \in Y$.

Proof. Since T^* maps extreme points of type I to extreme points of type I, it must do the same for extreme points of type II. Suppose y_0 is the special point in Y, $y \in Y$, and $v^* \in \mathrm{ext}(D^*)$. By Theorem 14, $\tilde{\psi}_{(y_0,y,v^*)}$ is an extreme point of type II for $\mathrm{Lip}(Y,D)^*$, so that there exist $\xi \in \beta(\tilde{X} \times S(E^*))$ and $\tau = \pm 1$ such that

$$T^*[\tilde{\psi}_{(y_0,y,v^*)}](F) = \tau\tilde{\psi}_\xi(F).$$

If F is a constant function, then we obtain

$$v^*\big[TF(y) - TF(y_0)\big] = \tau F^\beta(\xi) = 0.$$

Since this holds for all $v^* \in \mathrm{ext}(D^*)$, we conclude that $TF(y) - TF(y_0) = 0$, which implies that $TF(y) = TF(y_0)$ for all $y \in Y$, and TF is constant. The same argument applies to T^{-1}, and so both T and T^{-1} have CFP. It follows from Corollary 7, part (ii) that T is a composition operator given by $TF(y) = VF(\varphi(y))$, with φ a Lipschitz homeomorphism, and $V(y) = V$ for all $y \in Y$. The fact that φ is an isometry follows from Corollary 7, part (iii), since a pointed metric space has $\mathrm{diam}(X) \leqslant 2$. □

Remark 16.

(i) We notice that if X is a compact, pointed metric space with special point x_0, and E has trivial centralizer, the pseudo-centralizer $\tilde{Z}(\mathrm{Lip}(X,E))$ contains strictly the centralizer of $\mathrm{Lip}(X,E)$. Lemma 4 states that $\tilde{Z}(\mathrm{Lip}(X,E)) = \{M_h\colon h \in \mathrm{Lip}(X)\}$. Therefore, an extreme point of the type $\tilde{\psi}_{(x_0,x,v^*)}$ must be an eigenvector of the adjoint of a multiplier operator M_h, for M_h to be in the centralizer. This implies that

$$M_h^*(\tilde{\psi}_{(x_0,x,v^*)}) = a_{(x_0,x,v^*)}\tilde{\psi}_{(x_0,x,v^*)}.$$

For every $F \in \mathrm{Lip}(X,E)$ we have

$$v^*\big(h(x)F(x) - h(x_0)F(x_0)\big) = a_{(x_0,x,v^*)}v^*\big(F(x) - F(x_0)\big).$$

For F the constant function equal to v, where $v^*(v) \neq 0$, we have $h(x) = h(x_0)$ for all $x \in X$ and h is a constant function. Therefore the centralizer of the real space $\mathrm{Lip}(X,E)$ is trivial. In particular, it follows that the centralizer of real $\mathrm{Lip}(X)$ is trivial for any compact pointed metric space X.

(ii) On the other hand, if we take X to be a two point space $\{x, y\}$ with $d(x, y) > 2$, and E a Banach space with trivial centralizer, then $\mathrm{Lip}(X,E)^*$ has no extreme points of type II, so the centralizer and pseudo-centralizer are the same.

(iii) Following in the spirit of an example given in [11, p. 68], let X be a compact, pointed metric space with special point x_0. Let E be a Banach space, and define $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(X,E)$ by $TF(x_0) = F(x_0)$, and $TF(x) = F(x_0) - F(x)$ if $x \neq x_0$. It can be shown that T is an isometry, although it is not a weighted composition operator. Of course, T does not satisfy CFP, nor property **Q**, nor even property **P**.

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