Kernel estimates and $L^p$-spectral independence of generators of $C_0$-semigroups

Hisakazu Shindoh

Department of Mathematics, Faculty of Science, Tokyo University of Science, 26 Wakamiya-cho, Shinjuku-ku, Tokyo 162-0827, Japan

Received 19 January 2008; accepted 13 June 2008

Communicated by L. Gross

Dedicated to Professor Shizuo Miyajima on the occasion of his 60th birthday

Abstract

After the appearance of W. Arendt’s result that “Gaussian estimate of a semigroup implies the $L^p$-spectral independence of the generator,” various generalizations have been obtained. This paper shows that a certain kernel estimate of a semigroup implies the $L^p$-spectral independence of the generator, generalizing the case of upper Gaussian estimate and “Gaussian estimate of order $\alpha \in (0, 1]$ [S. Miyajima, H. Shindoh, Gaussian estimates of order $\alpha$ and $L^p$-spectral independence of generators of $C_0$-semigroups, Positivity 11 (1) (2007) 15–39], Definition 3.1.” The proof uses S. Karrmann’s result about the $L^p$-spectral independence and B.A. Barnes’ theorem about the spectrum of integral operators. As an application, the $L^p$-spectral independence of $-\Delta^{\alpha} + V$ ($\alpha \in (0, 1]$) for a suitable $V$ is proved with the help of a recent result by V. Liskevich, H. Vogt and J. Voigt [V. Liskevich, H. Vogt, J. Voigt, Gaussian bounds for propagators perturbed by potentials, J. Funct. Anal. 238 (2006) 245–277].

© 2008 Elsevier Inc. All rights reserved.

Keywords: $L^p$-spectrum; Integral kernel; Banach algebra; Gaussian estimate; Fractional powers of an operator; Spectral mapping theorem; Positive semigroup; Perturbation

1. Introduction

In [14], B. Simon conjectured that the Schrödinger operator $-\Delta/2 + V$ acting in $L^p(\mathbb{R}^N)$ has the spectrum independent of $p \in [1, \infty)$ (cf. [14, (1.10)]) and he gave an affirmative answer...
for a class of potentials $V$ (cf. [14, Theorem 5.1]). Later, R. Hempel and J. Voigt confirmed the conjecture for $V$ in a larger class than that treated by Simon. On the other hand, W. Arendt [1] succeeded in generalizing their results in an abstract direction. In more detail, he introduced a notion of upper Gaussian estimate of a $C_0$-semigroup, and showed that if a $C_0$-semigroup $T$ on $L^2$ satisfies an upper Gaussian estimate and the generator of $T$ is self-adjoint, then the generator of the $C_0$-semigroup on $L^p$ naturally induced by $T$ has the spectrum independent of $p$.

The main objective of this paper is to prove that the spectrum of $(-\Delta)^\alpha + V$ ($\alpha \in (0, 1]$) is independent of $p \in [1, \infty)$. Precisely speaking, the perturbed operator $(-\Delta)^\alpha + V$ is defined via Voigt’s theory of absorption semigroups [15] (for more details, see Section 4). In this paper, the $L^p$-independence of the spectrum of $(-\Delta)^\alpha + V$ in $L^p(\mathbb{R}^N)$ is proved without any assumptions on the space dimension $N$ and $\alpha \in (0, 1]$ (cf. Theorem 4.2). This is a much-improved result compared with that of [13].

In the proof of Theorem 4.2, an estimate of the integral kernel $K_{\alpha, V}(t; x, y)$ of the $C_0$-semigroup $\exp(-t((-\Delta)^\alpha + V))$ is important. To obtain the estimate of $K_{\alpha, V}(t; x, y)$, we use a result by V. Liskevich, H. Voigt and J. Voigt [8, Theorem 3.10]. Roughly speaking, the estimate guarantees that the integral kernel $K_{\alpha, V}(t; x, y)$ decreases polynomially on the off-diagonal part, and the decay is sufficiently fast to prove the $L^p$-independence of the spectrum of $(-\Delta)^\alpha + V$.

Theorem 4.2 is indeed a consequence of an abstract theorem (Theorem 3.9) on $L^p$-spectral independence of generators of $C_0$-semigroups. This result can be applied to $C_0$-semigroups with integral kernels satisfying an estimate (see Assumption 3.1). More precisely, Theorem 3.9 implies that a $C_0$-semigroup on $L^2$ with an integral kernel satisfying Assumption 3.1 naturally induces a $C_0$-semigroup on $L^p$ and the $L^p$-spectral independence of the generators of the $C_0$-semigroups holds. Note that the estimate in Assumption 3.1 is a generalization of upper Gaussian estimates and Gaussian estimates of order $\alpha$ defined in [11, Definition 3.1] (cf. Example 3.3). In the proof of Theorem 3.9, we use S. Karrmann’s result [6, Lemma 6.3] and B.A. Barnes’ theorem [3, Theorem 4.8]. The former states that the $L^p$-spectral independence of the generator of a $C_0$-semigroup is implied by the $L^p$-spectral independence of a power of the resolvents of the generator. The latter gives a sufficient condition for $L^p$-spectral independence of integral operators by using the theory of Banach algebras. On Assumption 3.1, it is proved that a power of a resolvent $(\lambda - A)^{-1}$ has an integral kernel satisfying the assumption of Barnes’ theorem, where $A$ is the generator of the $C_0$-semigroup in question (cf. Proposition 3.8).

In addition, we give another proof of Theorem 3.9 that, the author believes, is of independent interest. In the proof, we use a lemma instead of Karrmann’s result, which states that the $L^p$-spectral independence of the generator $A_p$ of a $C_0$-semigroup $(T_p(t))_{t \geq 0}$ is implied by the $L^p$-independence of $\sigma(T_p^\beta(t))$ for every $t > 0$ and $0 < \beta < 1$, where $(T_p^\beta(t))_{t \geq 0}$ denotes the $C_0$-semigroup generated by $(-A_p)^\beta$. In the course of the proof, an explicit asymptotic expansion of the function $f_{t, \beta}(s)$ is given, where $f_{t, \beta}$ is the function appearing the following expression of the semigroup generated by a fractional power of a generator $A$:

$$e^{-t(-A)^\beta} = \int_0^\infty f_{t, \beta}(s)e^{sA} \, ds$$

(cf. formula (2) in [17, Chapter IX, Section 11]). More details are described in Section 5.
2. Barnes’ theorem

Since we need an abstract result by Barnes [3], to state his result, we define some function spaces and weight functions. In what follows, \( \Omega \) denotes an open subset of \( \mathbb{R}^N \).

**Definition 2.1.** (Cf. [3, pp. 122, 123].) (i) \( A_1 \) denotes the linear space consisting of all measurable functions \( K: \Omega \times \Omega \to \mathbb{C} \) such that

\[
\|K\|_1 := \max \left\{ \text{ess sup}_{x \in \Omega} \int_{\Omega} |K(x, y)| \, dy, \text{ess sup}_{y \in \Omega} \int_{\Omega} |K(x, y)| \, dx \right\} < \infty.
\]

Similarly, \( A_2 \) denotes the linear space of all measurable functions \( K: \Omega \times \Omega \to \mathbb{C} \) such that the following \( \| \cdot \|_2 \)-norm of \( K \) is finite:

\[
\|K\|_2 := \max \left\{ \text{ess sup}_{x \in \Omega} \left( \int_{\Omega} |K(x, y)|^2 \, dy \right)^{1/2}, \text{ess sup}_{y \in \Omega} \left( \int_{\Omega} |K(x, y)|^2 \, dx \right)^{1/2} \right\}.
\]

The space \((A_1, \| \cdot \|_1)\) and \((A_2, \| \cdot \|_2)\) are Banach spaces. Moreover, \( A_1 \) is a Banach \(*\)-algebra with the following involution \( K \mapsto K^\ast \) and multiplication:

\[
K^*(x, y) := \overline{K(y, x)} \quad ((x, y) \in \Omega \times \Omega),
\]

\[
(K \ast L)(x, y) := \int_{\Omega} K(x, z)L(z, y) \, dz \quad (K, L \in A_1).
\]

(ii) The weight function \( w_\delta \) is defined by

\[
w_\delta(x, y) := \left(1 + |x - y|\right)^\delta \quad ((x, y) \in \mathbb{R}^N \times \mathbb{R}^N)
\]

for each \( \delta \in (0, 1) \). Let \( A_{w_\delta} \) be the linear space of all measurable functions \( K: \Omega \times \Omega \to \mathbb{C} \) such that \( K w_\delta \in A_1 \) and \( \| \cdot \|_{w_\delta} \) be defined by \( \|K\|_{w_\delta} := \|K w_\delta\|_1 \) for each \( \delta \in (0, 1) \), where \( K w_\delta \) denotes the pointwise product of \( K \) and \( w_\delta \). Then, \( A_{w_\delta} \) is a \(*\)-subalgebra of \( A_1 \) and \((A_{w_\delta}, \| \cdot \|_{w_\delta})\) is a Banach \(*\)-algebra (cf. [3, Note 4.3]).

(iii) Let \( \Gamma[m] \) be the set

\[
\Gamma[m] := \{(x, y) \in \Omega \times \Omega \mid |x - y| \leq m\}
\]

for each \( m \in \mathbb{N} \) and let \( \chi(\Gamma) \) be the characteristic function of \( \Gamma \subset \mathbb{R}^N \times \mathbb{R}^N \). Then, set

\[
A_1^0 := \left\{K \in A_1 \left| \lim_{m \to \infty} \|\chi(\Gamma[m]^c) K\|_1 = 0 \right.\right\}.
\]

\( A_1^0 \) is a closed \(*\)-subalgebra of \( A_1 \). In addition, \( A_2^0 \) and \( A_{w_\delta}^0 \) are defined as subspaces of \( A_2 \) and \( A_{w_\delta} \) by replacing \( \| \cdot \|_1 \) with \( \| \cdot \|_2 \) and \( \| \cdot \|_{w_\delta} \), respectively in the definition of \( A_1^0 \).

(iv) Let \( A_{w_\delta, 2} := A_{w_\delta} \cap A_2 \), \( A_{w_\delta, 2}^0 := A_{w_\delta}^0 \cap A_2^0 \) for each \( \delta \in (0, 1) \) and \( \|K\|_{w_\delta, 2} := \max\{\|K\|_{w_\delta}, \|K\|_2\} \). Then, \((A_{w_\delta, 2}, \| \cdot \|_{w_\delta, 2})\) is a Banach \(*\)-algebra (cf. [3, Lemma 4.4]) and \( A_{w_\delta, 2}^0 \) is a closed \(*\)-subalgebra of \( A_{w_\delta, 2} \).
Theorem 2.3. (See Barnes [3, Theorem 4.8].) Assume that $K$ is in $A_{w^2,2}^0$ for some $\delta \in (0, 1]$. Then the following assertions hold:

(i) $\sigma_{w^2,2}(K) = \sigma(K_p)$ for all $p \in [1, \infty]$ when $K$ is normal (i.e., $K^* K = K K^*$).

(ii) In general, $\sigma_{w^2,2}(K) = \sigma(K_p) \cup \sigma((K^*)_p)$ for all $p \in [1, \infty]$.

In these assertions, $\sigma_{w^2,2}(K)$ denotes the spectrum of $K$ as an element in the Banach algebra $A_{w^2,2}^0$, and $\sigma(K_p)$ denotes the spectrum of the bounded operator $K_p$ on $L^p(\Omega)$ (cf. Remark 2.2).

Remark 2.4. Let $A_{w^2,2}^0 := A_{w^2}^0 \cap A_2$. [3, Theorem 4.8] states that the same conclusions (i), (ii) in Theorem 2.3 hold for all $K \in A_{w^2,2}^0$. Moreover, in the proof of [3, Theorem 4.8], it is claimed that if $K = K^* \in A_{w^2,2}^0$, then we have

$$\|\chi(\Gamma[m])K - K\|_{w^2,2} \to 0$$

as $m \to \infty$, in other words, $K \in A_{w^2,2}^{0,0}$. However, let $K$ be defined by

$$K(x, y) := \begin{cases} \sqrt{y} & (y \geq 2, \ 2y \leq x \leq 2y + 1/y), \\
0 & \text{(otherwise).} \end{cases}$$

Then, $K + K^*$ is hermitian and belongs to $A_{w^2,2}^{0,0}$ for each $\delta \in (0, 1/2)$ but does not belong to $A_{w^2,2}^{0,0}$ for any $\delta \in (0, 1/2)$. For a proof, see [10, Proposition 3.1]. For this reason, we have replaced $A_{w^2,2}^{0,0}$ in [3, Theorem 4.8] with $A_{w^2,2}^{0,0}$. Once this replacement is made, Theorem 2.3 can be proved in exactly the same way as in [3] except for the part concerning the assertion $K \in A_{w^2,2}^{0,0}$.

Remark 2.5. It is easy to see that for all $K \in A_1$,

$$(K^*_p f)' = K_p f' \quad (f \in L^p(\Omega))$$

for each $p \in [1, \infty)$, where $(K^*)_p'$ is the conjugate operator of $(K^*)_p$ and $p'$ is the conjugate exponent of $p$. Hence, it follows from assertion (ii) that

$$\sigma_{w^2,2}(K) = \sigma(K_p) \cup \sigma(K_{p'})$$

holds for each $p \in [1, \infty)$. 

Remark 2.2. Any $K \in A_1$ defines a bounded linear operator $K_p$ on $L^p(\Omega)$ by

$$(K_p f)(x) := \int_{\Omega} K(x, y) f(y) \, dy \quad (f \in L^p(\Omega), \ x \in \Omega)$$

for each $p \in [1, \infty]$ (cf. [3, p. 122]).

Now, we introduce a result by Barnes [3]. For the reason described in Remark 2.4 below, we state it in a form where its “assumption part” is a little strengthened.
3. The main result

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) and let \( T = (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on \( L^2(\Omega) \) with generator \( A \). Most of the results in this section depends on this assumption.

Assumption 3.1. \( T(t) \) is an integral operator for each \( t > 0 \) and the family of the integral kernels \( (K_t(x,y))_{t > 0} \) satisfies the following condition: There exist measurable functions \( \phi : (0, \infty) \to [0, \infty) \), \( F : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty) \) and a constant \( \kappa > 0 \) such that for all \( t > 0 \),

\[
|K_t(x,y)| \leq \phi(t) F(t^{-\kappa} x, t^{-\kappa} y)
\]

(3.1)

for a.e. \( (x,y) \in \Omega \times \Omega \), where \( \phi, F \) and \( \kappa \) satisfy the following conditions:

(C-i) \( F \in A_{\omega,0,2}^{0,0} \) for some \( \delta_0 \in (0, 1) \).

(C-ii) The function \( t \mapsto t^{\kappa N} \phi(t) \) is bounded on \( (0, \infty) \).

Remark 3.2. On Assumption 3.1, the following assertions hold.

(i) By estimate (3.1) and condition (C-i), the estimate

\[
\|T(t)\| \leq t^{\kappa N} \phi(t) \|F\|
\]

(3.2)

holds for all \( t > 0 \). Hence, by condition (C-ii), the \( C_0 \)-semigroup \( T \) is bounded.

(ii) By estimate (3.2) and the fact \( \lim \inf_{t \downarrow 0} \|T(t)\| \geq 1 \), we have for each \( \varepsilon \in (0, 1) \), the inequality \( 1 - \varepsilon \leq t^{\kappa N} \phi(t) \|F\| \) for sufficiently small \( t > 0 \). Combined with condition (C-ii), this inequality implies that there exist constants \( C_1, C_2 > 0 \) such that \( C_1 t^{-\kappa N} \leq \phi(t) \leq C_2 t^{-\kappa N} \) holds for sufficiently small \( t > 0 \).

Example 3.3. Suppose that a \( C_0 \)-semigroup \( T = (T(t))_{t \geq 0} \) on \( L^2(\Omega) \) with generator \( A \) satisfies a Gaussian estimate of order \( \alpha \) for an \( \alpha \in (0, 1] \) in the sense defined in [11, Definition 3.1]: There exist constants \( M > 0 \), \( \omega \in \mathbb{R} \) and \( b > 0 \) such that

\[
|T(t)f| \leq Me^{\omega t} e^{-bt(-\Delta)^\alpha} |f|
\]

(3.3)

for all \( t \geq 0 \) and \( f \in L^2(\Omega) \). Here, we identify \( L^2(\Omega) \) with a subspace of \( L^2(\mathbb{R}^N) \). Without loss of generality, we may assume \( \omega = 0 \) (if necessary, consider \( A - \omega \) instead of \( A \)). Then, this domination implies, as is proved in [11, Corollary 3.4 and Proposition 3.5], that \( T(t) \) \( (t > 0) \) is an integral operator and the corresponding kernel \( K_t(x,y) \) is estimated as

\[
|K_t(x,y)| \leq C \frac{bt}{((bt)^{1/\alpha} + |x-y|^2)^{N/2+\alpha}}
\]

(3.4)

for each \( t > 0 \) and a.e. \( (x,y) \in \Omega \times \Omega \), where \( C \) is a constant independent of \( t > 0 \) and \( (x,y) \in \Omega \times \Omega \). Hence, the kernel \( K_t(x,y) \) satisfies estimate (3.1) with \( \phi(t) = t^{-N/(2\alpha)} \), \( F(x,y) = (bt^{1/\alpha} + |x-y|^2)^{-N/2-\alpha} \) and \( \kappa = 1/(2\alpha) \). Then, we can easily show that \( \phi, F \) and \( \kappa \) satisfy conditions (C-i) and (C-ii).
The next proposition shows that a $C_0$-semigroup $T$ on $L^2(\Omega)$ satisfying Assumption 3.1 naturally induces a $C_0$-semigroup on $L^p(\Omega)$ for each $p \in [1, \infty)$:

**Proposition 3.4.** Suppose that a $C_0$-semigroup $T$ on $L^2(\Omega)$ satisfies Assumption 3.1. Then, for each $p \in [1, \infty)$, there exists a unique bounded $C_0$-semigroup $T_p = (T_p(t))_{t \geq 0}$ on $L^p(\Omega)$ such that for all $t > 0$ and $f \in L^p(\Omega)$,

$$(T_p(t)f)(x) = \int_\Omega K_t(x, y) f(y) \, dy$$

for a.e. $x \in \Omega$. (Note that $K_t$ is independent of $p \in [1, \infty)$.)

**Proof.** By assumption, for all $t > 0$,

$$|K_t(x, y)| \leq \phi(t) F(t^{-k}x, t^{-k}y)$$

holds for a.e. $(x, y) \in \Omega \times \Omega$ (estimate (3.1)). Since $F$ in the right-hand side of this estimate belongs to $A_{\mu_0, 2}^{0, 0}$ for a $\mu_0 \in (0, 1]$, in particular to $A_1$, there exists a bounded operator $T_p(t)$ on $L^p(\Omega)$ for each $t \geq 0$ and $p \in [1, \infty]$, and the integral kernel of $T_p(t)$ $(t > 0)$ is $K_t(x, y)$ (cf. Remark 2.2). It is clear that the family of the operators $T_p := (T_p(t))_{t \geq 0}$ satisfies the semigroup property for each $p \in [1, \infty)$. Further, $T_p$ is a bounded semigroup since

$$\sup_{t > 0} \|T_p(t)\| \leq \sup_{t > 0} t^{kN} \phi(t) \cdot \|F\|_1 < \infty. \quad (3.5)$$

So, in order to finish the proof of Proposition 3.4, it remains to show that $T_p$ is strongly continuous on $L^p(\Omega)$ for each $p \in [1, \infty)$. For this purpose, note that inequality (3.5) holds also in the case of $p = \infty$. Since $T_p(t)$ coincides with $T_1(t)$ on $L^p(\Omega) \cap L^1(\Omega)$ and $T_\infty(t)$ on $L^p(\Omega) \cap L^\infty(\Omega)$, the strong continuity of $T_1$ implies that of $T_p$ for each $p \in (1, \infty)$. So we have only to prove that $T_1(t)f \to f$ as $t \downarrow 0$ in $L^1(\Omega)$ for all $f \in L^1(\Omega)$ in what follows. Since $\sup_{0 < t \leq 1} \|T_1(t)\| < \infty$, we may assume that $f \in L^1(\Omega) \cap L^2(\Omega)$ with compact support.

Let $f \in L^1(\Omega) \cap L^2(\Omega)$ be such that $f(x) = 0$ ($|x| \geq r$) for some $r > 0$. Then, for each $R > r$, the inequality

$$\|T_1(t)f - f\|_{L^1(\Omega)} = \|T_1(t)f - f\|_{L^1(\Omega \cap B(0, R))} + \|T_1(t)f\|_{L^1(\Omega \cap B(0, R)^c)} \leq \|B(0, R)^{1/2} T_2(t)f - f\|_{L^2(\Omega)} + \|T_1(t)f\|_{L^1(\Omega \cap B(0, R)^c)}$$

holds, where $B(0, R)$ denotes the ball in $\mathbb{R}^N$ with center 0 and radius $R$, and $|B(0, R)|$ denotes the volume of $B(0, R)$. The second term of the right-hand side of this inequality converges to 0 as $R \to \infty$ uniformly in $t \in (0, 1)$:

$$\sup_{0 < t \leq 1} \|T_1(t)f\|_{L^1(\Omega \cap B(0, R)^c)} \to 0 \quad (R \to \infty).$$

Indeed, by estimate (3.1), we obtain that for each $t \in (0, 1]$,
\[ \| T_1(t)f \|_{L^1(\Omega \cap B(0, R)^c)} \leq \phi(t) \int_{|x| \geq R} \left( \int_{|y| \leq r} F(t^{-k}x, t^{-k}y) |\tilde{f}(y)| \, dy \right) \, dx \]

\[ = \phi(t) \int_{|y| \leq r} \left( \int_{|x| \geq R} F(t^{-k}x, t^{-k}y) \, dx \right) |\tilde{f}(y)| \, dy \]

\[ = t^{kN} \phi(t) \int_{|y| \leq r} \left( \int_{|x| \geq t^{-k}R} F(x, t^{-k}y) \, dx \right) |\tilde{f}(y)| \, dy \]

\[ \leq t^{kN} \phi(t) \int_{|y| \leq r} \left( \int_{|x| - t^{-k}y \geq t^{-k}(R-r)} F(x, t^{-k}y) \, dx \right) |\tilde{f}(y)| \, dy \]

\[ \leq t^{kN} \phi(t) \left\| \chi \left( \Gamma^{[t^{-k}(R-r)]} \right) F \right\|_{L^1(\Omega)} \]

\[ \leq \sup_{0 < t \leq 1} t^{kN} \phi(t) \cdot \left\| \chi \left( \Gamma^{[R-r]} \right) F \right\|_{L^1(\Omega)}, \]

where \( \tilde{f} \in L^1(\mathbb{R}^N) \) is the extension of \( f \) defined as zero on \( \Omega^c \). The right-hand side of the last inequality is independent of \( t \in (0, 1] \) and converges to 0 as \( R \to \infty \) by condition (C-i) and (C-ii). Hence, we conclude \( \lim_{t \downarrow 0} \| T_1(t)f - f \|_{L^1(\Omega)} = 0 \).

It should be noted that we can replace the integral kernel \( K_t(x, y) \) in Proposition 3.4 by \( \tilde{K}(t, x, y) \) that is measurable in \( (t, x, y) \).

**Lemma 3.5.** Let \( T_p \) and \( K \) be as in Proposition 3.4. Then, there exists a measurable function \( \tilde{K} : (0, \infty) \times \Omega \times \Omega \to \mathbb{C} \) such that for a.e. \( t \in (0, \infty) \)

\[ K_t(x, y) = \tilde{K}(t, x, y) \quad (a.e. \ (x, y) \in \Omega \times \Omega) \]

and that for each \( p \in [1, \infty) \), a.e. \( t \in (0, \infty) \) and all \( f \in L^p(\Omega) \)

\[ (T_p(t)f)(x) = \int_{\Omega} \tilde{K}(t, x, y) f(y) \, dy \quad (a.e. \ x \in \Omega). \]

**Remark 3.6.** We may assume that \( \tilde{K} \) satisfies the estimate

\[ |\tilde{K}(t, x, y)| \leq \phi(t) F(t^{-k}x, t^{-k}y) \quad (3.6) \]

for all \( (t, x, y) \in (0, \infty) \times \Omega \times \Omega \).

**Proof.** This assertion would be proved by an argument similar to that in the proof of Lemma 3.6. in [10] (we need to replace \( L^2 \) with \( L^1 \) in the proof).

To prove the \( L^p \)-spectral independence of the generator of a consistent family of \( C_0 \)-semigroups, it is sufficient to prove the \( L^p \)-spectral independence of a power of the resolvents of the generator. More precisely, the next lemma holds by Karrmann’s result (cf. [6, Lemma 6.3] and the proof of [6, Theorem 1.7]). The author noticed Karrmann’s result via reference [7].
Lemma 3.7. Let \( T_p = (T_p(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on \( L^p(\Omega) \) with generator \( A_p \) for each \( p \in [1, \infty) \). Then the following assertions hold.

(i) Assume that there exists an \( n_0 \in \mathbb{N} \) such that for sufficiently large \( \lambda \), the set \( \sigma((\lambda - A_p)^{-n_0}) \) is independent of \( p \in [1, \infty) \). Then the spectrum of \( A_p \) is independent of \( p \in [1, \infty) \).

(ii) Assume that there exists an \( n_0 \in \mathbb{N} \) such that for sufficiently large \( \lambda \), the set \( \sigma((\lambda - A_p)^{-n_0}) \cup \sigma((\lambda - A_{p'})^{-n_0}) \) is independent of \( p \in (1, \infty) \), where \( p' \) is the conjugate exponent of \( p \). Then \( \sigma(A_p) \cup \sigma(A_{p'}) \) is independent of \( p \in (1, \infty) \).

Proof. (i) See [6, Lemma 6.3] and the proof of [6, Theorem 1.7].

(ii) This assertion is proved in a way similar to that in the proof of (i). \( \square \)

The next proposition, together with Lemma 3.7, is used to prove the main result of this paper.

Proposition 3.8. Let \( T = (T(t))_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on \( L^2(\Omega) \) with generator \( A \). Assume that \( T \) satisfies Assumption 3.1. Then the following assertions hold.

(i) For each \( \lambda > 0 \) and \( n \in \mathbb{N} \) with \( n > \kappa N \), where \( \kappa \) is as in Assumption 3.1, the operator \( (\lambda - A)^{-n} \) is an integral operator and its kernel \( G_n(\lambda; \cdot, \cdot) \) is given by

\[
G_n(\lambda; x, y) = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} \tilde{K}(t, x, y) \, dt \tag{3.7}
\]

(a.e. \( (x, y) \in \Omega \times \Omega \)), where \( \tilde{K} \) is as in Lemma 3.5. Moreover, \( G_n(\lambda; \cdot, \cdot) \in A^{0,0}_{\delta_0,2} \) for each \( \lambda > 0 \) and \( n \in \mathbb{N} \) with \( n > \kappa N \), where \( \delta_0 \) is as in condition (C-i).

(ii) Let \( R_{n,p}(\lambda) \) be the bounded operator defined by \( G_n(\lambda; \cdot, \cdot) \) on \( L^p(\Omega) \) for each \( \lambda > 0 \), \( n \in \mathbb{N} \) with \( n > \kappa N \) and \( p \in [1, \infty) \) (cf. Remark 2.2). Then for each \( \lambda > 0 \), \( n \in \mathbb{N} \) with \( n > \kappa N \) and \( p \in [1, \infty) \), the operator \( R_{n,p}(\lambda) \) coincides with \( (\lambda - A_p)^{-n} \), where \( A_p \) is the generator of \( T_p \) in Proposition 3.4.

Proof. (i) To prove equality (3.7), we first note that for each \( \lambda > 0 \), \( n \in \mathbb{N} \) with \( n > \kappa N \) and \( u \in L^2(\Omega) \), the function

\[
(t, x, y) \mapsto t^{n-1} e^{-\lambda t} \tilde{K}(t, x, y)u(y)
\]

is integrable on \( (0, \infty) \times E \times \Omega \) \( \tag{3.8} \)

for each bounded measurable subset \( E \) of \( \Omega \), where \( \tilde{K}(t, x, y) \) is the integral kernel of \( e^{tA} \) in Lemma 3.5. For the time being, we prove (3.7) admitting this fact, and then we verify (3.8), which is independent of (3.7). It follows from (3.8) that the function \( (t, x) \mapsto t^{n-1} e^{-\lambda t} \int_\Omega \tilde{K}(t, x, y)u(y) \, dy = t^{n-1} e^{-\lambda t} (\exp(tA)u)(x) \) is integrable on \( (0, \infty) \times E \). Hence it is verified that the \( L^1(E) \)-valued function \( t \mapsto t^{n-1} e^{-\lambda t} \exp(tA)u \mid E \) is integrable on \( (0, \infty) \) and the equality

\[
\left( \int_0^\infty t^{n-1} e^{-\lambda t} e^{tA}u \, dt \right)(x) = \int_0^\infty t^{n-1} e^{-\lambda t} (e^{tA}u)(x) \, dt
\]
holds for a.e. \( x \in E \), hence for a.e. \( x \in \Omega \). The left-hand side of this equality is equal to \((n - 1)![(\lambda - A)^{-n}u](x)\). Thus, we obtain the equality

\[
((\lambda - A)^{-n}u)(x) = \frac{1}{(n - 1)!} \int_0^\infty t^{n-1}e^{-\lambda t} \left( \int_{\Omega} \tilde{K}(t, x, y)u(y) \, dy \right) \, dt
\]

\[
= \frac{1}{(n - 1)!} \int_{\Omega} \left( \int_0^\infty t^{n-1}e^{-\lambda t} \tilde{K}(t, x, y) \, dt \right)u(y) \, dy
\]

for each \( u \in L^2(\Omega) \) and a.e. \( x \in \Omega \), and accordingly we obtain equality (3.7).

Now, we prove (3.8). Set \( \tilde{u}(x) := u(x) \) (\( x \in \Omega \)), 0 otherwise. Then, for each bounded measurable subset \( E \) of \( \Omega \), by estimate (3.6), we can show that

\[
\int_{(0, \infty) \times E \times \Omega} \left| t^{n-1}e^{-\lambda t} \tilde{K}(t, x, y)u(y) \right| \, dt \, dx \, dy
\]

\[
\leq \int_0^\infty t^{n-1}e^{-\lambda t} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)| \, dy \right) \, dx \right) \, dt
\]

\[
\leq \int_0^\infty t^{n-1}e^{-\lambda t} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) \, dy \right)^{1/p'} \, dx \right) \, dt
\]

\[
\times \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)|^p \, dy \right)^{1/p} \, dx \, dt
\]

\[
= \int_0^\infty t^{n-1}e^{-\lambda t} \cdot t^{\kappa N/p'} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) \, dy \right)^{1/p'} \, dx \right) \, dt
\]

\[
\times \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)|^p \, dy \right)^{1/p} \, dx \, dt
\]

\[
\leq \|F\|_{1/p'}^{1/p'} \int_0^\infty t^{n-1}e^{-\lambda t} \cdot t^{\kappa N/p'} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)|^p \, dy \right)^{1/p} \, dx \right) \, dt
\]

\[
\leq \|F\|_{1/p'} |E|^{1/p'} \int_0^\infty t^{n-1}e^{-\lambda t} \cdot t^{\kappa N/p'} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)|^p \, dy \right) \, dx \right)^{1/p} \, dt
\]

\[
= \|F\|_{1/p'} |E|^{1/p'} \int_0^\infty t^{n-1}e^{-\lambda t} \cdot t^{\kappa N/p'} \phi(t) \left( \int_{\Omega} \left( \int_{\mathbb{R}^N} F(\tau^{-x}, \tau^{-y}) |\tilde{u}(y)|^p \, dy \right) \, dx \right)^{1/p} \, dt
\]
\[ \begin{align*}
\leq & \|F\|_{1/p'} |E|^{1/p'} \int_{0}^{\infty} t^{n-1} e^{-\lambda \cdot t} \cdot t^{\kappa N} \phi(t) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, t^{-\kappa} y) \, dx \right) |\tilde{u}(y)|^p \, dy \right)^{1/p} \, dt \\
\leq & \|F\|_1 |E|^{1/p'} \|u\|_{L^p(\Omega)} \int_{0}^{\infty} t^{n-1} e^{-\lambda \cdot t} \cdot t^{\kappa N} \phi(t) \, dt < \infty.
\end{align*} \]

In the last inequality, we used conditions (C-i) and (C-ii). In the inequalities above, \( p' \) is the conjugate exponent of \( p \) and \( |E| \) denotes the measure of \( E \). Needless to say, in the case of \( p = 1 \), \( |E|^{1/p'} \) and so on can be replaced by 1.

Next, we successively prove that for each \( \lambda > 0 \) and \( n \in \mathbb{N} \) with \( n > \kappa N \), the assertions

(a) \( G_n(\lambda; \cdot, \cdot) \in A_{w_{\delta_0}} \),
(b) \( G_n(\lambda; \cdot, \cdot) \in A_2 \),
(c) \( G_n(\lambda; \cdot, \cdot) \in A_{w_{\delta_0}}^0 \), and
(d) \( G_n(\lambda; \cdot, \cdot) \in A_2^0 \) hold,

where \( \delta_0 \) is as in condition (C-i).

(a) By equality (3.7) and estimate (3.6), for a.e. \( x \in \Omega \), we have

\[ 
(n - 1)! \int_{\Omega} w_{\delta_0}(x, y) |G_n(\lambda; x, y)| \, dy 
\leq \int_{\mathbb{R}^N} w_{\delta_0}(x, y) \left( \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \phi(t) F(t^{-\kappa} x, t^{-\kappa} y) \, dt \right) \, dy 
\]

\[ 
= \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \phi(t) \left( \int_{\mathbb{R}^N} w_{\delta_0}(x, y) F(t^{-\kappa} x, t^{-\kappa} y) \, dy \right) \, dt 
\]

\[ 
= \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{\mathbb{R}^N} w_{\delta_0}(x, t^\kappa y) F(t^{-\kappa} x, y) \, dy \right) \, dt 
\]

\[ 
= \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{\mathbb{R}^N} (1 + t^\kappa |t^{-\kappa} x - y|)^{\delta_0} F(t^{-\kappa} x, y) \, dy \right) \, dt 
\]

\[ 
\leq \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{\mathbb{R}^N} (1 + |t^{-\kappa} x - y|)^{\delta_0} F(t^{-\kappa} x, y) \, dy \right) \, dt 
\]

\[ 
+ \int_{0}^{\infty} t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{\mathbb{R}^N} (1 + |t^{-\kappa} x - y|)^{\delta_0} F(t^{-\kappa} x, y) \, dy \right) \, dt 
\]
\[
\leq \left( \int_0^1 t^{n-1} e^{-\lambda t} \cdot t^\kappa N \phi(t) dt + \int_1^\infty t^{n-1+\kappa \delta_0} e^{-\lambda t} \cdot t^\kappa N \phi(t) dt \right) \| F \| w_{\delta_0}.
\]

The right-hand side of the last inequality is a finite constant independent of \( x \in \Omega \) by condition (C-i) and (C-ii). Thus,

\[
\text{ess.sup}_{x \in \Omega} \int_\Omega w_{\delta_0}(x, y) |G_n(\lambda; x, y)| dy < \infty.
\]

A similar argument interchanging \( x \) and \( y \) gives \( \| G_n(\lambda; \cdot, \cdot) \|_{w_{\delta_0}} < \infty \).

(b) To prove that \( G_n(\lambda; \cdot, \cdot) \) belongs to \( A_2 \), it is sufficient to estimate the integral \( \int_\Omega |G_n(\lambda; x, y)|^2 dy \). We can carry this out as follows:

\[
\left( (n - 1)! \right)^2 \int_\Omega |G_n(\lambda; x, y)|^2 dy
\]

\[
\leq \int_{\mathbb{R}^N} \left( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) F(t^{-\kappa} x, t^{-\kappa} y) dt \right)^2 dy
\]

\[
\leq \int_{\mathbb{R}^N} \left( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) dt \right) \left( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) F(t^{-\kappa} x, t^{-\kappa} y)^2 dt \right) dy
\]

\[
= \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) dt \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) \left( \int_{\mathbb{R}^N} F(t^{-\kappa} x, t^{-\kappa} y)^2 dy \right) dt
\]

\[
= \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) dt \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) \left( \int_{\mathbb{R}^N} F(t^{-\kappa} x, t^{-\kappa} y)^2 dy \right) dt
\]

\[
\leq \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) dt \int_0^\infty t^{n-1} e^{-\lambda t} \cdot t^\kappa N \phi(t) dt \times \| F \|_2^2.
\]

Thus, by conditions (C-i) and (C-ii), \( \text{ess.sup}_{x \in \Omega} \int_\Omega |G_n(\lambda; x, y)|^2 dy < \infty \). By a similar argument interchanging \( x \) and \( y \), we obtain \( \| G_n(\lambda; \cdot, \cdot) \|_2 < \infty \).

(c) We obtain that for a.e. \( x \in \Omega \) and all \( m \in \mathbb{N} \),

\[
(n - 1)! \int_{|x-y|>m} w_{\delta_0}(x, y) |G_n(\lambda; x, y)| dy
\]

\[
\leq \int_{|x-y|>m} w_{\delta_0}(x, y) \left( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) F(t^{-\kappa} x, t^{-\kappa} y) dt \right) dy
\]
\[= \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) \left( \int_{|x-y|>m} w_0(x, y) F\left(t^{-\delta} x, t^{-\delta} y\right) dy \right) dt\]

\[= \int_0^1 t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{|t^{-\delta} x-y|>t^{-\delta} m} (1 + t^{\kappa} |t^{-\delta} x-y|)^\delta F\left(t^{-\delta} x, t^{-\delta} y\right) dy \right) dt\]

\[\leq \int_0^1 t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{|t^{-\delta} x-y|>t^{-\delta} m} w_0(t^{-\delta} x, t^{-\delta} y) F(t^{-\delta} x, t^{-\delta} y) dy \right) dt\]

\[+ \int_1^\infty t^{n-1+\kappa \delta_0} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \left( \int_{|t^{-\delta} x-y|>t^{-\delta} m} w_0(t^{-\delta} x, t^{-\delta} y) F(t^{-\delta} x, t^{-\delta} y) dy \right) dt\]

\[\leq \int_0^1 t^{n-1} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) dt \times \| \chi(\Gamma[m]^{\delta}) F \|_{w_{\delta_0}} \]

\[+ \int_1^\infty t^{n-1+\kappa \delta_0} e^{-\lambda t} \cdot t^{\kappa N} \phi(t) \| \chi(\Gamma[t^{-\delta} m]^{\delta}) F \|_{w_{\delta_0}} dt.\]

(The function \(t \mapsto \| \chi(\Gamma[t^{-\delta} m]^{\delta}) F \|_{w_{\delta_0}}\) is measurable on \((0, \infty)\) since both of the functions

\[t \mapsto \text{ess. sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \chi(\Gamma[t^{-\delta} m]^{\delta})(x, y) w_0(x, y) F(x, y) dy\]

and

\[t \mapsto \text{ess. sup}_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} \chi(\Gamma[t^{-\delta} m]^{\delta})(x, y) w_0(x, y) F(x, y) dx\]

are monotone increasing, hence measurable.) The first term of this right-hand side converges to 0 as \(m \to \infty\) by \(F \in A^0_{w_{\delta_0}}\) and condition (C-ii). It is verified that the second term of this right-hand side converges to 0 as \(m \to \infty\) since we can apply Lebesgue’s convergence theorem by \(F \in A^0_{w_{\delta_0}}\) and condition (C-ii). Thus, we have

\[\lim_{m \to \infty} \text{ess. sup}_{x \in \Omega} \int_{|x-y|>m} w_0(x, y) |G_n(\lambda; x, y)| dy = 0.\]

By a similar argument interchanging \(x\) and \(y\), we conclude

\[\lim_{m \to \infty} \| \chi(\Gamma[m]^{\delta}) G_n(\lambda; \cdot, \cdot) \|_{w_{\delta}} = 0.\]

(d) Note that for a.e. \((x, y) \in \Omega \times \Omega\),
\[
(n - 1)! \left| G_n(\lambda; x, y) \right|^2 \leq \left( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) F(t^{-\kappa} x, t^{-\kappa} y) \, dt \right)^2
\]

by Schwarz’s inequality and that \( \int_0^\infty t^{n-1} e^{-\lambda t} \phi(t) \, dt < \infty \) (cf. condition (C-ii)). In a way similar to that of the proof of (c), it would be proved that \( G_n(\lambda; \cdot, \cdot) \in A^0_n \).

(ii) For each \( \lambda > 0, n \in \mathbb{N} \) with \( n > \kappa N \) and \( p \in [1, \infty) \), the equality \((n - 1)! (\lambda - A_p)^{-n} = \int_0^\infty t^{n-1} e^{-\lambda t} \exp(tA_p) \, dt \) holds. Hence \((\lambda - A_p)^{-n}\) is an \( L^p(\Omega)\)-bounded extension of \((\lambda - A)^{-n}|_{L^2(\Omega) \cap L^p(\Omega)} \). On the other hand, \( R_{n,p}(\lambda) \) is nothing but this extension by definition. Hence \( R_{n,p}(\lambda) = (\lambda - A_p)^{-n} \) for each \( \lambda > 0, n \in \mathbb{N} \) with \( n > \kappa N \) and \( p \in [1, \infty) \). \( \square \)

The next theorem is the main abstract result of this paper. The author would like to emphasize that it is a generalization of [10, Theorem 2.11]. Indeed, Assumption 3.1 is weaker than the assumption of [10, Theorem 2.11] that estimate (3.3) holds, as we saw in Example 3.3.

**Theorem 3.9.** Suppose that a \( C_0 \)-semigroup \( T = (T(t))_{t \geq 0} \) on \( L^2(\Omega) \) with generator \( A \) satisfies Assumption 3.1, and let \( A_p \) be the generator of \( T_p \) in Proposition 3.4. Then the following assertions hold.

(i) If \( T(t) \) is normal for each \( t \geq 0 \), then \( \sigma(A_p) \) is independent of \( p \in [1, \infty) \).

(ii) In general, \( \sigma(A_p) \cup \sigma(A_{p'}) \) is independent of \( p \in (1, \infty) \), where \( p' \) is the conjugate exponent of \( p \).

**Proof.** (i) Let an arbitrary \( n \in \mathbb{N} \) with \( n > \kappa N \) be fixed, where \( \kappa \) is as in Assumption 3.1. Proposition 3.8 implies that \((\lambda - A_p)^p\) has an integral kernel \( G_n(\lambda; \cdot, \cdot) \) for each \( \lambda > 0 \) and \( p \in [1, \infty) \), which is independent of \( p \in [1, \infty) \) and belongs to \( A^{0,0}_w \) for a \( \delta \in (0, 1] \). Further, it follows from the assumption that \( (\lambda - A)^n \) is normal for each \( \lambda > 0 \) and so is \( G_n(\lambda; \cdot, \cdot) \). Hence, we can apply Barnes’ theorem, and obtain the conclusion of assertion (i) by Lemma 3.7(i).

(ii) is proved by using Lemma 3.7(ii) instead of Lemma 3.7(i) in the proof above. \( \square \)

**Remark 3.10.** (i) The assumption of (i) in Theorem 3.9 that \( T(t) \) is normal for each \( t \geq 0 \) is equivalent to the following assumption: There exists a \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \) such that \((\lambda - A)^{-1}\) is normal (cf. the proof of [10, Theorem 2.11]).

(ii) Although we used Karrmann’s result in the proof above, we can give another proof of Theorem 3.9. For the details, see Section 5.

### 4. An application

In this section, we apply Theorem 3.9 to obtain the \( L^p \)-spectral independence of \((-\Delta)^{\alpha} + V\). Our result encompasses that of [13]. To state our result, we first fix a notation.

Hereafter, \( U_\alpha := (U_\alpha(t))_{t \geq 0} \) denotes the \( C_0 \)-semigroup on \( L^2(\mathbb{R}^N) \) generated by \((-\Delta)^{\alpha}\) for each \( \alpha \in (0, 1] \): \( U_\alpha(t) = \exp(-t(-\Delta)^{\alpha}) \). \( U_\alpha \) is a positive (positivity preserving) \( C_0 \)-semigroup of contractions on \( L^2(\mathbb{R}^N) \). When \( \alpha = 1 \), as is well known, the integral kernel of \( U_\alpha(t) \) \((t > 0)\) is
(4πt)^{-N/2} \exp(-|x - y|^2/(4t)). Hence, U_1 satisfies Assumption 3.1. When 0 < \alpha < 1, U_{\alpha} also satisfies Assumption 3.1 as stated in Example 3.3. Hence, for every \alpha \in (0, 1], by Proposition 3.4, there exists a unique C_0-semigroup U_{\alpha,p} := (U_{\alpha,p}(t))_{t \geq 0} on L^p(\mathbb{R}^N) that is consistent with U_{\alpha} for each p \in [1, \infty) (in general, we say that a C_0-semigroup T_p on L^p(\mathbb{R}^N) is consistent with a C_0-semigroup T_q on L^q(\mathbb{R}^N) if T_p(t)f = T_q(t)f for all t \geq 0 and f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)). U_{\alpha,p} is also a positive C_0-semigroup of contractions on L^p(\mathbb{R}^N), −H_{\alpha,p} denotes the generator of U_{\alpha,p}.

Next, we precisely define the operator H_{\alpha,p} + V. The definition depends on Voigt’s theory of absorption semigroups [15] and on the results of [13]. Let V : \mathbb{R}^N \to \mathbb{R} be a measurable function and let V_+ [respectively V_-] denote the positive [respectively negative] part of V: V_+ := V \lor 0 [respectively V_- := (−V) \lor 0]. The heart of Voigt’s theory is to consider the limit of the C_0-semigroup perturbed by “truncations” of V. We define for n \in \mathbb{N} the truncation V^{(n)} of V by V^{(n)} := (\text{sign} V)(|V| \wedge n). We simply write V^{(n)}_+ and V^{(n)}_- instead of (V_+)^{(n)} and (V_-)^{(n)}, respectively. First, we assume that

\[ H^\alpha(\mathbb{R}^N) \cap Q(V_+) \text{ is dense in } L^2(\mathbb{R}^N), \tag{4.1} \]

where H^\alpha(\mathbb{R}^N) is the usual Sobolev space of order \alpha in L^2 sense and Q(V_+) the form domain of V_. Under this condition, by Proposition 5.8(a) in [15] (if \alpha = 1) and Proposition 2.22 in [13] (if 0 < \alpha < 1), the strong limit

\[ U_{\alpha,p,V_+}(t) := s\text{-lim}_{n \to \infty} \exp\left(-t(H_{\alpha,p} + V^{(n)}_+\right)) \tag{4.2} \]

eexists on L^p(\mathbb{R}^N) for each t \geq 0. These propositions state also that U_{\alpha,p,V_+} := (U_{\alpha,p,V_+}(t))_{t \geq 0} is a positive C_0-semigroup on L^p(\mathbb{R}^N). The C_0-semigroup U_{\alpha,p,V_+} has the following properties: U_{\alpha,2,V_+}(t) is self-adjoint for all t \geq 0. U_{\alpha,p,V_+} is consistent with U_{\alpha,q,V_+} for all p, q \in [1, \infty). For the proofs of these properties, see [15, Proposition 3.2(b)] (if \alpha = 1) and [13, Proposition 2.9] (if 0 < \alpha < 1). In addition, the inequality 0 \leq U_{\alpha,p,V_+}(t) \leq U_{\alpha,p}(t) holds for all t \geq 0 and p \in [1, \infty), i.e., 0 \leq U_{\alpha,p,V_+}(t)f \leq U_{\alpha,p}(t)f for all positive f \in L^p(\mathbb{R}^N) (cf. [15, Remark 2.1(c)]). Hence, U_{\alpha,2,V_+} satisfies domination (3.3).

Secondly, to treat the negative part of V, assume that

\[ c'_{N,\alpha}(V_-) := \lim_{\eta \downarrow 0} \left\| V_- \int_0^\eta U_{\alpha,1}(t) \, dt \right\| < 1, \tag{4.3} \]

where V_- \int_0^\eta U_{\alpha,1}(t) \, dt denotes the composite of V_- (as a multiplication operator) and \int_0^\eta U_{\alpha,1}(t) \, dt. Then, since 0 \leq U_{\alpha,1,V_+}(t) \leq U_{\alpha,1}(t), we have

\[ \lim_{\eta \downarrow 0} \left\| V_- \int_0^\eta U_{\alpha,1,V_+}(t) \, dt \right\| < 1 \tag{4.4} \]

and accordingly the strong limit

\[ U_{\alpha,p,V}(t) := s\text{-lim}_{n \to \infty} \exp\left(-t(H_{\alpha,p,V_+} - V^{(n)}_-)\right) \tag{4.5} \]
exists for each \( p \in [1, \infty) \) and \( t \geq 0 \), where \(-H_{\alpha,p,V}\) is the generator of \( U_{\alpha,p,V}\). For the existence of \( U_{\alpha,p,V}(t) \), see e.g. [15, Remark 2.1(b)]. The family of the operators \( U_{\alpha,p,V} \) is proved to be a positive \( C_0 \)-semigroup on \( L^p(\mathbb{R}^N) \) (cf. [15, Remark 2.1(b)]). The operator \( U_{\alpha,p,V}(t) \) is also expressed as

\[
U_{\alpha,p,V}(t) = \lim_{n \to \infty} \exp(-t(H_{\alpha,p} + V(n)))
\]

for each \( p \in [1, \infty) \) and \( t \geq 0 \) (cf. [16, Theorem 2.6]). By Propositions 3.1(a) and 3.2(b) in [15] (if \( \alpha = 1 \)) and Proposition 2.9 in [13] (if \( 0 < \alpha < 1 \)), \( U_{\alpha,2,V} \) is self-adjoint and \( U_{\alpha,p,V} \) is consistent with \( U_{\alpha,q,V} \) for all \( p, q \in [1, \infty) \). The next proposition shows that \( U_{\alpha,1,V}(t) (t > 0) \) is an integral operator. It also gives an estimate of the corresponding integral kernel \( K_{\alpha,V}(t;x,y) \).

**Proposition 4.1.** Suppose that \( V_- \) and \( V_+ \) satisfy assumptions (4.3) and (4.1), respectively. Then, the \( C_0 \)-semigroup \( U_{\alpha,1,V}(t) \) defined by (4.5) is an integral operator for each \( t > 0 \), and for each \( v \in L^C_{N,\alpha}(V_-), 1 \) there exist constants \( C > 0 \) and \( \alpha \in \mathbb{R} \) such that the integral kernel \( K_{\alpha,V}(t;x,y) \) of \( U_{\alpha,1,V}(t) \) satisfies

\[
0 \leq K_{\alpha,V}(t;x,y) \leq C e^{\alpha t N/2\alpha} \cdot \frac{1}{(1 + t^{-1/\alpha}|x - y|^2)^{(N/2 + \alpha)(1 - \nu)}}
\]

for all \( t > 0 \) and a.e. \((x,y) \in \mathbb{R}^N \times \mathbb{R}^N\).

**Proof.** As remarked above, the \( C_0 \)-semigroup \( U_{\alpha,2,V_+} \) satisfies the domination \( 0 \leq U_{\alpha,2,V_+}(t) \leq U_{\alpha}(t) \) \((t \geq 0)\). Hence \( U_{\alpha,2,V_+} \) satisfies Assumption 3.1 and accordingly \( U_{\alpha,2,V_+}(t) (t > 0) \) has a non-negative integral kernel \( K_{\alpha,V_+}(t;x,y) \) estimated as

\[
0 \leq K_{\alpha,V_+}(t;x,y) \leq C \frac{t}{(1/\alpha + |x - y|^2)^{N/2 + \alpha}}
\]

\[
= C t^{-N/(2\alpha)} \exp\left(-\left(N/2 + \alpha \log\left(1 + t^{-1/\alpha}|x - y|^2\right)\right)\right)
\]

for a constant \( C > 0 \) independent of \( t > 0 \) and a.e. \((x,y) \in \mathbb{R}^N \times \mathbb{R}^N\) (cf. (3.4)). (This estimate is one of the Gaussian type upper bounds defined in [8, (3.5)].) This estimate and Proposition 3.4 imply that \( U_{\alpha,2,V_+} \) induces a \( C_0 \)-semigroup on \( L^1(\mathbb{R}^N) \) with the same integral kernel \( K_{\alpha,V_+}(t;x,y) \). Since \( U_{\alpha,2,V_+} \) is consistent with \( U_{\alpha,1,V_+} \), the \( C_0 \)-semigroup on \( L^1(\mathbb{R}^N) \) induced by \( U_{\alpha,2,V_+} \) coincides with \( U_{\alpha,1,V_+} \). Thus, \( U_{\alpha,1,V_+}(t) (t > 0) \) has the integral kernel \( K_{\alpha,V_+}(t;x,y) \). Since \( K_{\alpha,V_+}(t;x,y) \) satisfies estimate (4.7), it is clear that \( U_{\alpha,1,V_+} \) satisfies the following assumptions (A1)–(A3) in [8]:

(A1) \( U_{\alpha,1,V_+} \) is a positive \( C_0 \)-semigroup on \( L^1(\mathbb{R}^N) \) and there exists a constant \( M > 0 \) such that

\[
\|U_{\alpha,1,V_+}(t)\| \leq M
\]

for all \( t \geq 0 \). (In fact, we can take \( M = 1 \), cf. [10, Proposition 3.3].)

(A2) There exists a constant \( C > 0 \) such that

\[
\|U_{\alpha,1,V_+}(t)\|_{1,\infty} \leq C t^{-N/(2\alpha)}
\]
for all $t > 0$, where we define the norm $\|T\|_{p,q}$ of an operator $T$ on $L^r(\mathbb{R}^N)$ for an $r \in [1, \infty]$ by $\|T\|_{p,q} := \sup \{\|Tu\|_{L^q(\mathbb{R}^N)} : u \in L^r(\mathbb{R}^N) \cap \mathcal{L}^p(\mathbb{R}^N), \|u\|_{L^p(\mathbb{R}^N)} = 1\}$ for each $p, q \in [1, \infty]$.

(A3) $\|U_{\alpha,1,V}(t)\|_{\infty, \infty} \leq M$ for all $t \geq 0$, where $M$ is as in (A1).

In addition, since $U_{\alpha,2,V}(t)$ is positive and self-adjoint, the function $K_{\alpha,V}(t; \cdot, \cdot)$ is real-valued and $K_{\alpha,V}(t; x, y) = K_{\alpha,V}(t; y, x)$ for a.e. $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Hence, the conjugate operator $(U_{\alpha,1,V}(t))'$ coincides with $U_{1,\alpha,V}(t)$ on $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Here, we identify the dual space $(L^1(\mathbb{R}^N))^\prime$ with $L^\infty(\mathbb{R}^N)$. Accordingly $(U_{\alpha,1,V}(t))'$ satisfies also assumption (A4) in [8]:

(A4) $(U_{\alpha,1,V}(t))'$ restricted to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ has an $L^1(\mathbb{R}^N)$-bounded extension that is strongly continuous in $t \in [0, \infty)$ on $L^1(\mathbb{R}^N)$. In this case, needless to say, the $L^1(\mathbb{R}^N)$-bounded extension coincides with $U_{\alpha,1,V}(t)$.

On the other hand, by (4.4), $V_-$ satisfies

$$\left\| V_- \int_0^\eta (U_{\alpha,1,V}(t))' \, dt \right\|_{1,1} = \left\| V_- \int_0^\eta U_{\alpha,1,V}(t) \, dt \right\| < 1$$

for sufficiently small $\eta > 0$. This inequality implies that $V_-$ is a small Miyadera perturbation of both $U_{\alpha,1,V}(t)$ and $(U_{\alpha,1,V}(t))'$ (for the definition of small Miyadera perturbation, see just below [8, (1.7)]). Thus, we have seen that the assumption of Theorem 3.10 in [8] is satisfied. Therefore, Theorem 3.10 in [8] implies that $U_{\alpha,1,V}(t)$ ($t > 0$) is an integral operator and for each $\nu \in (c'_{N,\alpha}(V_-), 1)$, there exist constants $C > 0$ and $\omega \in \mathbb{R}$ such that the corresponding kernel $K_{\alpha,V}(t; x, y)$ satisfies

$$0 \leq K_{\alpha,V}(t; x, y) \leq Ce^{\omega t - N/(2\alpha)} \cdot \frac{1}{(1 + t^{-1/\alpha}|x - y|^2)^{(N/2+\alpha)(1-\nu)}}$$

for all $t > 0$ and a.e. $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. □

Now we can obtain the following $L^p$-spectral independence. Let us keep the notation explained in the 3rd and 4th paragraphs of this section.

**Theorem 4.2.** Suppose that $V_-$ satisfies $c'_{N,\alpha}(V_-) < 2\alpha/(N + 2\alpha)$ and $V_+$ satisfies assumption (4.1). Then, we obtain

$$\sigma(H_{\alpha,p,V}) = \sigma(H_{\alpha,2,V})$$

for all $p \in [1, \infty)$, where $-H_{\alpha,p,V}$ is the generator of $U_{\alpha,p,V}$.

**Proof.** As stated just above Proposition 4.1, the operator $U_{\alpha,2,V}(t)$ is self-adjoint for all $t \geq 0$ and $U_{\alpha,p,V}$ is consistent with $U_{\alpha,q,V}$ for all $p, q \in [1, \infty)$. Further, Proposition 4.1 shows that for each $\nu \in (c'_{N,\alpha}(V_-), 2\alpha/(N + 2\alpha))$ and $t > 0$, the integral kernel $K_{\alpha,V}(t; x, y)$ of $U_{\alpha,1,V}(t)$ satisfies

$$0 \leq K_{\alpha,V}(t; x, y) \leq Ce^{\omega t - N/(2\alpha)} \cdot \frac{1}{(1 + t^{-1/\alpha}|x - y|^2)^{(N/2+\alpha)(1-\nu)}}$$
for a.e. \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) (estimate (4.6)). In what follows, we may assume \(\omega = 0\) (if necessary, consider \(H_{\alpha, 1, V} + \omega\)). This estimate implies that the operator \(U_{\alpha, 2, V}(t)\) is an integral operator with the same kernel \(K_{\alpha, V}(t; x, y)\) as that of \(U_{\alpha, 1, V}(t)\) and \(U_{\alpha, 2, V}\) satisfies Assumption 3.1 with \(\phi(t) = t^{-N/(2\alpha)} F(x, y) = (1 + |x - y|^2)^{-((N/2 + \alpha)(1 - \nu) - N)}\) and \(\kappa = 1/(2\alpha)\) (note that \(F \in A_{w_{w, 2}}^{0,0}\) for each \(v \in (c'_{1, N}, (2\alpha)\) and \(\delta \in (0, (N + 2\alpha)(1 - \nu) - N)\)). Hence, it follows from Proposition 3.4 that \(U_{\alpha, 2, V}\) induces a unique \(C_0\)-semigroup on \(L^p(\mathbb{R}^N)\) for each \(p \in [1, \infty)\). Since \(U_{\alpha, 2, V}\) is consistent with \(U_{\alpha, p, V}\) for each \(p \in [1, \infty)\), the \(C_0\)-semigroup on \(L^p(\mathbb{R}^N)\) induced by \(U_{\alpha, 2, V}\) coincides with \(U_{\alpha, p, V}\) for each \(p \in [1, \infty)\). Since \(U_{\alpha, 2, V}(t)\) is self-adjoint for each \(t \geq 0\), applying Theorem 3.9(i), we conclude that the spectrum of \(H_{\alpha, p, V}\) is independent of \(p \in [1, \infty)\). \(\square\)

**Remark 4.3.** (i) In the case of \(\alpha = 1\), a more general result than this theorem is obtained in \([5]\). In fact, \([5, \text{Theorem}]\) states that if \(c'_{1, N, \alpha}(V_v) < 1\) and \(H^1(R^N) \cap Q(V_v)\) is dense in \(L^2(R^N)\), then \(\sigma(H_{\alpha, p, V})\) is independent of \(p \in [1, \infty)\). (This assumption is equivalent to that in \([5, \text{Theorem}]\), cf. \([15, \text{Proposition 4.7}]\).)

(ii) If domination (3.3) follows from the assumption of Theorem 4.2, then by using \([10, \text{Theorem} 2.11]\) that is a special case of Theorem 3.9, we obtain the same conclusion of Theorem 4.2. However, at present, the author is not able to prove or disprove domination (3.3) over \(U_{\alpha, 2, V}\). Unfortunately, estimate (4.6) does not imply the domination.

**5. Another proof of Theorem 3.9**

As stated in Remark 3.10(ii), we give another proof of Theorem 3.9. In more detail, we use Lemma 5.3 instead of Lemma 3.7, which states that the \(L^p\)-spectral independence of the generator of a \(C_0\)-semigroup is implied by the \(L^p\)-spectral independence of the \(C_0\)-semigroup generated by the fractional powers of the generator. Compared with this, Lemma 3.7 states that the \(L^p\)-spectral independence of the generator of a \(C_0\)-semigroup is implied by the \(L^p\)-spectral independence of a power of the resolvents of the generator.

To give another proof of Theorem 3.9, we recall the following function \(f_{t, \beta}\) appearing in the theory of fractional powers of closed operators \([17, \text{Chapter IX, Section 11}]\) and make several properties of \(f_{t, \beta}\) explicit, which include the asymptotic behavior of \(f_{t, \beta}\) at infinity. For each \(t > 0\) and \(\beta \in (0, 1)\), the function \(f_{t, \beta}\) is defined as follows:

\[
f_{t, \beta}(s) := \begin{cases} 
\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{zs - t z^\beta} dz & (s \geq 0, \ \sigma > 0), \\
0 & (s < 0),
\end{cases}
\]

(5.1)

where the branch of \(z^\beta\) is so taken that \(\text{Re } z^\beta > 0\) for \(\text{Re } z > 0\). According to \([17, \text{Chapter IX, Section 11}]\) and \([13, \text{Lemma 2.4}]\), \(f_{t, \beta}\) is independent of \(\sigma > 0\) and \(f_{t, \beta}\) is non-negative and infinitely differentiable.

The next lemma is needed to prove Proposition 5.4 that will be used in the proof of Theorem 3.9.

**Lemma 5.1.** Let \(t > 0\) and \(\beta \in (0, 1)\) and suppose that a function \(\phi\) and a constant \(\kappa\) satisfy condition (C-ii) and a constant \(\delta_0\) is as in condition (C-i). Then, the following assertions hold.
(i) The function $s \mapsto ft,\beta(s)\phi(s)$ is integrable on $(0, 1]$:

$$\int_0^1 ft,\beta(s)\phi(s)\,ds < \infty.$$ 

(ii) The function $s \mapsto ft,\beta(s)\phi(s)s^{\kappa(N+\delta)}$ is integrable on $[1, \infty)$ for sufficiently small $\delta \in (0, \delta_0]$:

$$\int_1^\infty ft,\beta(s)\phi(s)s^{\kappa(N+\delta)}\,ds < \infty.$$ 

Proof. (i) As is proved in [13, Lemma 2.4], $ft,\beta(s) = O(s^j)$ as $s \downarrow 0$ for each $j \in \mathbb{N} \cup \{0\}$. This fact and condition (C-ii) imply the assertion of (i).

(ii) follows from condition (C-ii) and the next lemma.

Lemma 5.2. Let $t > 0$ and $\beta \in (0, 1)$. Then there exists a constant $C > 0$ such that

$$0 \leq ft,\beta(s) \leq Cs^{-1-\beta}$$

holds for all $s \geq 1$. Hence $ft,\beta(s) = O(s^{-1-\beta})$ as $s \to \infty$.

We leave the proof of this lemma till Appendix A. Further, we can give the asymptotic expansion formula of $ft,\beta(s)$ as $s \to \infty$. Also for this formula and the proof, see Appendix A.

The next lemma is used instead of Lemma 3.7 in the proof of Theorem 3.9. The lemma depends heavily on the theory of fractional powers of a generator of a $C_0$-semigroup and the spectral mapping theorem.

Lemma 5.3. Let $T_p = (T_p(t))_{t \geq 0}$ be a bounded $C_0$-semigroup on $L^p(\Omega)$ with generator $A_p$ for each $p \in [1, \infty)$. Then the following assertions hold.

(i) Assume that there exists a $t_0 > 0$ such that for each $\beta \in (0, 1)$, the set $\sigma(\exp(-t_0(-A_p)^\beta)) \setminus \{0\}$ is independent of $p \in [1, \infty)$. Then the spectrum of $A_p$ is independent of $p \in [1, \infty)$.

(ii) Assume that there exists a $t_0 > 0$ such that for each $\beta \in (0, 1)$, the set $\{\sigma(\exp(-t_0(-A_p)^\beta)) \cup \sigma(\exp(-t_0(-A_p')^\beta))\} \setminus \{0\}$ is independent of $p \in (1, \infty)$, where $p'$ is the conjugate exponent of $p$. Then $\sigma(A_p) \cup \sigma(A_p')$ is independent of $p \in (1, \infty)$.

Proof. Although this lemma is proved in [10, Lemma 2.9], we state the proof here for the readers’ convenience.

(i) As is well known, for each $p \in [1, \infty)$ and $\beta \in (0, 1)$, the fractional power $-(-A_p)^\beta$ generates a bounded analytic semigroup $\exp(-t(-A_p)^\beta)$ with angle $\pi(1-\beta)/2$. Hence, $\sigma((-A_p)^\beta)$ is contained in the sector $\Sigma_{\pi\beta/2} := \{\mu \in \mathbb{C} \mid |\arg \mu| < \pi\beta/2\} \cup \{0\}$. Keeping this in mind, let $p, q \in [1, \infty)$ and let us choose an arbitrary $\lambda \in \sigma(A_p)$. We use a consequence of the spectral mapping theorem (Theorem 3.1 in [2] or Theorem 5.3.1 in [9]):

$$\sigma((-A_p)^\beta) = [\sigma(-A_p)]^\beta = \{(-\mu)^\beta \mid \mu \in \sigma(A_p)\},$$

(5.2)
where $\beta$ is an arbitrary number in $(0, 1)$ and $(-\mu)^{\beta}$ denotes the principal value of $\exp(\beta \log(-\mu))$ for $\mu \neq 0$ and $0$ for $\mu = 0$. This equality means that in the case of $0 \in \sigma(A_p)$, we have $0 \in \sigma((-A_p)^{\beta})$. Now, equality (5.2) implies that

$$
\exp(-t_0(-\lambda)^{\beta}) \in \exp(-t_0\sigma((-A_p)^{\beta}))
$$

for all $\beta \in (0, 1)$ (remember that we have picked $\lambda \in \sigma(A_p)$). In addition, since $\exp(-t(-A_p)^{\beta})$ is a bounded analytic semigroup as stated above, the equality by the spectral mapping theorem

$$
\exp(-t_0\sigma((-A_p)^{\beta})) = \sigma(\exp(-t_0(-A_p)^{\beta})) \setminus \{0\}
$$

(5.3)

holds for all $\beta \in (0, 1)$ (cf. Corollary 3.12 in [4]). Thus, we have

$$
\exp(-t_0(-\lambda)^{\beta}) \in \sigma(\exp(-t_0(-A_p)^{\beta})) \setminus \{0\}
$$

for all $\beta \in (0, 1)$. Since $\sigma(\exp(-t_0(-A_p)^{\beta}) \setminus \{0\}) = \sigma(\exp(-t_0(-A_q)^{\beta}) \setminus \{0\})$ by the assumption and equality (5.3) also holds if $p$ is replaced by $q$, i.e.,

$$
\exp(-t_0(-\lambda)^{\beta}) \in \exp(-t_0\sigma((-A_q)^{\beta}))
$$

for all $\beta \in (0, 1)$. Hence for all $\beta \in (0, 1)$ there exists an $n_\beta \in \mathbb{Z}$ such that

$$
(-\lambda)^{\beta} + \frac{2n_\beta \pi i}{t_0} \in \sigma((-A_q)^{\beta}).
$$

In the case of $\lambda = 0$, if $n_\beta \neq 0$, then $(-\lambda)^{\beta} + 2n_\beta \pi i/t_0 \in \mathbb{R} \setminus \{0\}$, hence $(-\lambda)^{\beta} + 2n_\beta \pi i/t_0 \notin \sigma((-A_q)^{\beta})$. Therefore $n_\beta = 0$ and hence $(-\lambda)^{\beta} \in \sigma((-A_q)^{\beta})$ holds. Since equality (5.2) also holds if $p$ is replaced by $q$, we have $\lambda \in \sigma(A_q)$ in this case. So let $\lambda \neq 0$ in what follows. We prove that there exists a $\beta \in (0, 1)$ such that $n_\beta = 0$. For this purpose, we assume that $n_\beta \neq 0$ for each $\beta \in (0, 1)$ and we show that the assumption leads us to a contradiction. Since $(-\lambda)^{\beta} + 2n_\beta \pi i/t_0 \in \sigma((-A_q)^{\beta})$ and $\sigma((-A_q)^{\beta})$ is contained in the sector $\Sigma_{\pi/2}$, we have $|\arg((-\lambda)^{\beta} + 2n_\beta \pi i/t_0)| < \pi\beta/2$ for each $\beta \in (0, 1)$. Hence the inequality

$$
\left|\text{Im}(-\lambda)^{\beta} + \frac{2n_\beta \pi i}{t_0}\right| < \text{Re}(-\lambda)^{\beta} \cdot \tan\left(\frac{\pi\beta}{2}\right)
$$

(5.4)

holds for each $\beta \in (0, 1)$. By the assumption that $n_\beta \neq 0$ for each $\beta \in (0, 1)$, it is verified that

$$
\liminf_{\beta \downarrow 0} \left|\text{Im}(-\lambda)^{\beta} + \frac{2n_\beta \pi i}{t_0}\right| \geq \frac{2\pi}{t_0} > 0.
$$

On the other hand, the right-hand side of (5.4) converges to $0$ as $\beta \downarrow 0$. This is a contradiction. Therefore $n_\beta = 0$ for a $\beta \in (0, 1)$ and hence $(-\lambda)^{\beta} \in \sigma((-A_q)^{\beta})$. Since equality (5.2) also holds if $p$ is replaced by $q$, we have $(-\lambda)^{\beta} \in \sigma((-A_q)^{\beta})$ and accordingly $\lambda \in \sigma(A_q)$ (note that $-\lambda$ belongs to the right half-plane \(\{z \in \mathbb{C} \mid \text{Re} z > 0\}\) and the function $z \mapsto z^{\beta}$ is injective there).

(ii) This assertion is proved in a way similar to that in the proof of (i). □

The next proposition corresponds to Proposition 3.8.
**Proposition 5.4.** Let $T = (T(t))_{t \geq 0}$ be a bounded $C_0$-semigroup on $L^2(\Omega)$ with generator $A$. Assume that $T$ satisfies Assumption 3.1. Then the following assertions hold.

(i) For each $t > 0$ and $\beta \in (0, 1)$, the operator $\exp(-t(-A)^\beta)$ is an integral operator and its kernel $K_{t,\beta}$ is given by

$$K_{t,\beta}(x,y) = \int_0^\infty f_{t,\beta}(s) \tilde{K}(s,x,y) \, ds \quad (a.e. \, (x,y) \in \Omega \times \Omega),$$

where $\tilde{K}$ is as in Lemma 3.5. Moreover, $K_{t,\beta} \in A^{0,0}_{w^\delta,2}$ for each $t > 0$ and $\beta \in (0, 1)$, where $\delta$ is as in Lemma 5.1(ii).

(ii) Let $T_{p,t}(t)$ be the bounded operator defined by $K_{t,\beta}$ on $L^p(\Omega)$ for each $t > 0$, $\beta \in (0, 1)$ and $p \in [1, \infty)$ (cf. Remark 2.2). Then for each $t > 0$, $\beta \in (0, 1)$ and $p \in [1, \infty)$, the operator $T_{p,t}(t)$ coincides with $\exp(-t(-A_p)^\beta)$, where $A_p$ is the generator of $T_p$ in Proposition 3.4.

**Proof.** (i) This assertion is proved by replacing $(\lambda - A)^{-n}$, $G_n(\lambda; x, y)$ and $s^{n-1}e^{-\lambda s}$ with $\exp(-t(-A)^\beta)$, $K_{t,\beta}(x,y)$ and $f_{t,\beta}(s)$, respectively in the proof of Proposition 3.8(i).

(ii) This assertion is proved in a way similar to that in the proof of Proposition 3.8(ii).

**Another proof of Theorem 3.9.** (i) Let an arbitrary $t > 0$ be fixed. Proposition 5.4 implies that $\exp(-t(-A_p)^\beta)$ has an integral kernel $K_{t,\beta}$ for each $p \in [1, \infty)$, which is independent of $p \in [1, \infty)$ and belongs to $A^{0,0}_{w^\delta,2}$ for a $\delta \in (0, 1]$. In addition, by the assumption that $T(t)$ ($t \geq 0$) is normal and formula (2) in [17, Chapter IX, Section 11], the operator $\exp(-t(-A)^\beta)$ is normal for each $\beta \in (0, 1)$ and so is $K_{t,\beta}$. Hence, we can apply Barnes’ theorem, and obtain the conclusion of assertion (i) by Lemma 5.3(i).

(ii) is proved by using Lemma 5.3(ii) instead of Lemma 5.3(i) in the proof above.

**Acknowledgment**

The author would like to thank the reviewer for his valuable advice, which helped the author to improve the manuscript.

**Appendix A. Asymptotic behavior of $f_{t,\beta}(s)$ as $s \to \infty$**

We first prove Lemma 5.2 saying that for each $t > 0$ and $\beta \in (0, 1)$, there exists a constant $C > 0$ such that $0 \leq f_{t,\beta}(s) \leq Cs^{-1-\beta}$ for all $s \geq 1$. Although this lemma immediately follows from the asymptotic expansion formula of $f_{t,\beta}(s)$ as $s \to \infty$ stated in Lemma A.1, we give a more elementary proof than that of Lemma A.1.

**Proof of Lemma 5.2.** Let an arbitrary $s \geq 1$ be fixed. First, applying integration by parts twice to the right-hand side of (5.1), we obtain

$$2\pi i f_{t,\beta}(s) = s^{1-\beta} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} \frac{d}{dz} e^{-tz^\beta} \, dz$$
\[ s^{-2} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} \frac{d^2}{dz^2} e^{-tz^\beta} \, dz = \beta (1 - \beta) t s^{-2} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{\beta-2} e^{sz-tz^\beta} \, dz + (\beta t)^2 s^{-2} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{2\beta-2} e^{sz-tz^\beta} \, dz. \]

By the change of variables \(sz = z'\) in the first term of the right-hand side of this equality, it is verified that

\[ 2\pi i f_{1,\beta}(s) = \beta (1 - \beta) t s^{-1-\beta} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{\beta-2} e^{sz-tz^\beta} \, dz + (\beta t)^2 s^{-2} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{2\beta-2} e^{sz-tz^\beta} \, dz \quad (A.1) \]

since each term of the right-hand side of this equality is independent of \(\sigma > 0\). Then, it follows that for each \(t > 0\) there exists a constant \(C > 0\) such that

\[ 0 \leq f_{1,\beta}(s) \leq Cs^{-1-\beta} \]

holds for all \(s \geq 1\). Indeed, since \(|z|^\beta / \text{Re}(z^\beta) \leq 1 / \cos(\pi \beta / 2)\) for \(\text{Re} z > 0\), the integrand of the first term of the right-hand side of (A.1) satisfies the estimate

\[ |z^{\beta-2} \exp(z-tz^\beta)z^\beta| \leq |z|^{\beta-2} \exp(\sigma - t\kappa |z|^\beta) \quad (s \geq 1), \]

where \(\kappa := \cos(\pi \beta / 2)\). Hence, the integral \(\int_{\sigma-i\infty}^{\sigma+i\infty} z^{\beta-2} \exp(z-tz^\beta) \, dz\) is bounded for \(s \geq 1\). On the other hand, since the second term of the right-hand side of (A.1) is independent of \(\sigma > 0\), we may assume \(\sigma = s^{-1}\). Then, the integrand of the second term satisfies the estimate

\[ |z^{2\beta-2} \exp(sz-tz^\beta)| \leq |z|^{2\beta-2} \exp(1-t\kappa |z|^\beta) \quad (s \geq 1). \]

Hence, the integral \(\int_{\sigma-i\infty}^{\sigma+i\infty} z^{2\beta-2} \exp(sz-tz^\beta) \, dz\) is bounded for \(s \geq 1\).

\textbf{Lemma A.1.} Let \(t > 0\) and \(\beta \in (0, 1)\). Then the asymptotic expansion formula

\[ f_{1,\beta}(s) \sim \frac{\beta}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k t^{1+k}}{k!} \sin(\pi \beta (k+1)) \Gamma(\beta (k+1)) s^{1+\beta+k} \]

holds as \(s \to \infty\).

\textbf{Proof.} Since \(f_{1,\beta}(s) = t^{-1/\beta} f_{1,\beta}(t^{-1/\beta}s)\) for all \(s > 0\), we treat \(f_{1,\beta}\) for the time being. Since \(f_{1,\beta}\) is independent of \(\sigma > 0\) in (5.1), by using Lebesgue’s convergence theorem, we have

\[ f_{1,\beta}(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{sz-z^\beta} \, dz \]
for all \( s > 0 \). Hence, it is easy to see that

\[
f_{1,\beta}(s) = \frac{1}{2\pi} \left( \int_0^\infty e^{is\xi} e^{-\xi^\beta} d\xi + \int_0^\infty e^{-is\xi} e^{-\xi^\beta} d\xi \right) = \frac{\beta}{2\pi is} \left( \xi^\beta \int_0^\infty e^{is\xi} \cdot \xi^{\beta-1} e^{-\xi^\beta} d\xi - \xi^{-\beta} \int_0^\infty e^{-is\xi} \cdot \xi^{\beta-1} e^{-\xi^\beta} d\xi \right),
\]

where \( \xi^\beta := \exp(\pi\beta i/2) \) and \( \xi^{-\beta} := \exp(-\pi\beta i/2) \). According to Theorem 2 in [12], the asymptotic expansion formulas

\[
\int_0^\infty e^{is\xi} \cdot \xi^{\beta-1} e^{-\xi^\beta} d\xi \sim \sum_{k=0}^\infty \frac{(-1)^k}{k!} \cdot \frac{e^{\pi\beta i(k+1)} \Gamma(\beta(k+1))}{s^{\beta+\beta k}},
\]

\[
\int_0^\infty e^{-is\xi} \cdot \xi^{\beta-1} e^{-\xi^\beta} d\xi \sim \sum_{k=0}^\infty \frac{(-1)^k}{k!} \cdot \frac{e^{-\pi\beta i(k+1)} \Gamma(\beta(k+1))}{s^{\beta+\beta k}}
\]

hold as \( s \to \infty \). Hence,

\[
f_{1,\beta}(s) \sim \frac{\beta}{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \cdot \frac{\sin(\pi\beta(k+1)) \Gamma(\beta(k+1))}{s^{1+\beta+\beta k}}
\]

as \( s \to \infty \). Since \( f_{t,\beta}(s) = t^{-1/\beta} f_{1,\beta}(t^{-1/\beta} s) \) for all \( s > 0 \), we obtain the asymptotic expansion formula asserted above. \( \square \)

References