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Concentration in a thin Euclidean shell for log-concave measures

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Abstract

A weak version of a conjecture stated by Kannan, Lovász and Simonovits claims that an isotropic log-concave probability μ on $\mathbb{R}^n$ should be concentrated in a thin Euclidean shell in the following way:

$$\forall t \in [0, n^\kappa], \quad \mu\left\{x \in \mathbb{R}^n \colon \left(1 - \frac{t}{n^\kappa}\right) \leqslant \frac{|x|}{\sqrt{n}} \leqslant \left(1 + \frac{t}{n^\kappa}\right)\right\} \geqslant 1 - Ce^{-ct} \tag{1}$$

where $\kappa = 1/2$ and c and C are positive absolute constants. For $\kappa = 1/10.02$, this inequality has been established by Klartag. By combining different approaches introduced by Klartag and by Guédon, Paouris and the author, we improve this result by showing that the inequality (1) holds with $\kappa = 1/8$.

Keywords: Log-concave measure; Concentration; Convex body

1. Introduction

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ distributed according to a log-concave density. We suppose that X is isotropic, that means its barycenter $\mathbb{E}X$ is 0 and its covariance matrix $\mathrm{cov}(X) = (\mathbb{E}X_i X_j)_{1 \leqslant i,j \leqslant n}$ is the identity matrix. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$. The isotropic assumption implies $(\mathbb{E}|X|^2)^{1/2} = \sqrt{n}$. The letters $c, C, c_1, C_1, \ldots$ stand for various positive universal constants whose value may change from one line to the next.

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Anttila, Ball and Perissinaki [1] conjectured that there exists a decreasing sequence (ε_n) converging to 0 such that, for any integer $n \geqslant 1$, all isotropic log-concave random vector X in $\mathbb{R}^n$ is concentrated within a thin Euclidean shell of width ε_n in the following sense:

$$\mathbb{P}\left\{(1-\varepsilon_n) \leqslant \frac{|X|}{\sqrt{n}} \leqslant (1+\varepsilon_n)\right\} \geqslant 1-\varepsilon_n.$$

This conjecture is mainly motivated by the central limit problem for log-concave measures. It implies indeed that most marginals of log-concave measures are approximately Gaussian [1]. A positive answer to this conjecture has been established by Klartag [8] and in a different way by Guédon, Paouris and the author [6] with ε_n decreasing logarithmically with n. To find the good dependence on the dimension remains a major question in Asymptotic Geometric Analysis.

A weak version of a conjecture stated by Kannan, Lovász and Simonovits [7] claims that $X/\sqrt{n}$ should be concentrated within a Euclidean shell of width $1/\sqrt{n}$. More precisely, the expected concentration inequality is

$$\forall t > 0, \quad \mathbb{P}\left\{\left(1-\frac{t}{\sqrt{n}}\right) \leqslant \frac{|X|}{\sqrt{n}} \leqslant \left(1+\frac{t}{\sqrt{n}}\right)\right\} \geqslant 1 - Ce^{-ct}. \tag{2}$$

As shown by Paouris [13], this inequality is known for large deviations, i.e. when $t \geqslant C'\sqrt{n}$. Note also that the inequality (2) was proved when X is uniformly distributed on a generalized Orlicz ball [5]. Up to now, the best estimate in the general case has been proved by Klartag [9] with a power-law dependence on n:

$$\forall t \in \left[0, n^{1/10.02}\right], \quad \mathbb{P}\left\{\left(1-\frac{t}{n^{1/10.02}}\right) \leqslant \frac{|X|}{\sqrt{n}} \leqslant \left(1+\frac{t}{n^{1/10.02}}\right)\right\} \geqslant 1 - Ce^{-ct^{3.33}}. \tag{3}$$

We show the following

Theorem 1. *Let X be an isotropic random vector in $\mathbb{R}^n$ distributed according to a log-concave density. For any $p \in [2, c_1 n^{1/4}]$, we have*

$$\left(\mathbb{E}|X|^2\right)^{1/2} \leqslant \left(\mathbb{E}|X|^p\right)^{1/p} \leqslant \left(1+\frac{C_1 p}{n^{1/4}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}. \tag{4}$$

In particular, X satisfies the following concentration inequality

$$\forall t \in \left[0, n^{1/8}\right], \quad \mathbb{P}\left\{\left(1-\frac{t}{n^{1/8}}\right) \leqslant \frac{|X|}{\sqrt{n}} \leqslant \left(1+\frac{t}{n^{1/8}}\right)\right\} \geqslant 1 - C_2 e^{-c_2 t}. \tag{5}$$

Moreover,

$$\forall t \in \left[0, n^{1/8}\right], \quad \mathbb{P}\left(\frac{|X|}{\sqrt{n}} \geqslant \left(1+\frac{t}{n^{1/8}}\right)\right) \leqslant C_3 e^{-c_3 t^2}. \tag{6}$$

Our proof combines the approach by the moments used by Paouris [13] and Guédon, Paouris and the author [6] with arguments developed by Klartag [9] to prove the inequality (3). We specify some notations. Let f be the log-concave density of X. For an integer $k \in [1, n]$, $G_{n,k}$ stands for the Grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$. For $F \in G_{n,k}$, we denote by P_F the orthogonal projection from $\mathbb{R}^n$ on F and by $\Pi_F f$ the marginal density of f on F, i.e. the density of $P_F X$. We denote by ν_n the unique Haar probability on the special orthogonal group $SO(n)$ invariant under both left and right translations. On $SO(n)$, d stands for the geodesic distance and $\|h\|_{\mathrm{Lip}}$ stands for the Lipschitz semi-norm of a function h on $SO(n)$ with respect to d.

To emphasize the origin of the gain of our proof, we summarize the main steps of Klartag's proof of the inequality (3).

1. Using the fact that, with large probability, the k-dimensional subspaces F satisfy, for any $x \in \mathbb{R}^n$, $\frac{1}{\sqrt{k}}|P_F x| \overset{1+\varepsilon}{\approx} \frac{1}{\sqrt{n}}|x|$, Klartag reduces the study of deviations of $\frac{1}{\sqrt{n}}|X|$ to the one of $\frac{1}{\sqrt{k}}|P_F X|$ for most $F \in G_{n,k}$.
2. A key step is to get a version of Dvoretzky's theorem [11,12] for log-concave measures: an isotropic log-concave probability on $\mathbb{R}^n$, once projected on subspaces whose dimension is a power of n, becomes approximately spherically-symmetric. To build almost radial projection measures, Klartag uses the following arguments:
   - To gain smoothness, Klartag shows that, without loss of generality, one can suppose $f = \tilde{f} * \gamma_n$ where $\tilde{f}$ is a log-concave density and γ_n is a Gaussian density. He get then a Lipschitz estimate of the function $M : u \in SO(n) \mapsto \log \Pi_{u(F_0)} f(u(x_0))$ where F_0 is a fixed k-dimensional subspace and $x_0 \in F_0$: $\|M\|_{\mathrm{Lip}} \leqslant Ck^2$.
   - Using the concentration inequality on $SO(n)$ for Lipschitz functions,
   $$\forall t > 0, \quad \nu_n\left\{u \in SO(n),\ \left|h(u) - \int h\, d\nu_n\right| \geqslant t\right\} \leqslant Ce^{-c(\frac{\sqrt{n}t}{\|h\|_{\mathrm{Lip}}})^2}, \tag{7}$$
   and the principle of Milman's proof of Dvoretzky's theorem [11,12], Klartag shows that, for $k \approx n^{1/5.01}$, with large probability, the k-dimensional projection of X are approximately spherically-symmetric in the following sense: for any $x_1 \in F$ and $x_2 \in F$ with $|x_1| = |x_2| \leqslant C'\sqrt{k}$,
   $$\frac{1}{2}\Pi_F f(x_1) \leqslant \Pi_F f(x_2) \leqslant 2\Pi_F f(x_1). \tag{8}$$
3. If Z is a random vector in $\mathbb{R}^k$ with a radial density $\rho(|.|)$ where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is log-concave, Z satisfies:
   $$\forall t > 0, \quad \mathbb{P}\left\{\left(1 - \frac{t}{\sqrt{k}}\right) \leqslant \frac{|Z|}{\sqrt{k}} \leqslant \left(1 + \frac{t}{\sqrt{k}}\right)\right\} \geqslant 1 - Ce^{-ct^2}. \tag{9}$$
   When $P_F X$ is approximately spherically-symmetric in the sense of (8), the previous inequality remains true for $P_F X$.

It gives the inequality (3) with the coefficient 2 instead of 3.33 in the right-hand side of (3). The coefficient 3.33 comes from an optimization of previous arguments.

Our proof of Theorem 1 uses also the Lipschitz estimate given in the point 2 and the concentration (7) on $SO(n)$ (through the log-Sobolev inequality on $SO(n)$). Our gain comes from the fact that we do not need to use Milman's principle to deduce the inequality (8) from the concentration inequality (7). We project the vector X on k-dimensional subspaces with $k \approx n^{1/4}$. For $k \approx n^{1/4}$, the inequality (7) with $h = M$ gives $\frac{1}{2}\Pi_{u(F)} f(u(x_0)) \leqslant \Pi_{v(F)} f(v(x_0)) \leqslant 2\Pi_{u(F)} f(u(x_0))$ for (u, v) in a subset of $SO(n)$ of probability greater than $1/2$. We use this type of estimate in average instead of the uniform estimate (8) for sub-marginals.

2. Proof

Let X be an isotropic random vector in $\mathbb{R}^n$ distributed according to a log-concave density f and let G_n be a standard Gaussian vector on $\mathbb{R}^n$ independent of X. We denote by γ_n the standard Gaussian density on $\mathbb{R}^n$. By the Prékopa–Leindler inequality [14,15,10], the density $g = f * \gamma_n$ of the random vector $Y = X + G_n$ and the marginal densities $\Pi_{F_0} g$ of $P_{F_0} Y$ for $F_0 \in G_{n,k}$, are log-concave and satisfy $\operatorname{cov}(Y) = 2I_n$ and $\operatorname{cov}(P_{F_0} Y) = 2I_k$. Remark that $(\mathbb{E}|P_{F_0} Y|^2)^{1/2} = \sqrt{2k}$. Let F be a k-dimensional random subspace in $\mathbb{R}^n$ distributed according to the unique rotationally-invariant probability on $G_{n,k}$ and let U be a random rotation distributed according to the Haar probability ν_n on $SO(n)$. Note that, if $F_0 \in G_{n,k}$ is fixed, $U(F_0)$ and F have the same law. For real numbers $\delta > \sqrt{2}$ and $p \geqslant 0$, we define the function $h_p : SO(n) \to \mathbb{R}_+$ by

$$h_p(u) = \int_0^{\delta\sqrt{k}} t^{p+k-1} \Pi_{u(F_0)} g\big(tu(\theta_0)\big)\, dt$$

where F_0 is a fixed subspace in $G_{n,k}$ and θ_0 is a fixed point in the Euclidean unit sphere of F_0.

Lemma 2. *We can choose δ equal to a universal constant in such a way that, for any $p \in [2, c\min(k, \sqrt{n})]$,*

$$\frac{(\mathbb{E}|Y|^p)^{1/p}}{(\mathbb{E}|Y|^2)^{1/2}} \leqslant \big(1 + Ce^{-c\min(k,\sqrt{n})}\big) \frac{(\mathbb{E}h_p(U))^{1/p} (\mathbb{E}h_0(U))^{1/2-1/p}}{(\mathbb{E}h_2(U))^{1/2}}$$

where c and C are absolute constants.

Recall that, as consequence of Borell's lemma [4], for every random vector Z in $\mathbb{R}^n$ distributed according to a log-concave law and for every even norm $\|.\|$ on $\mathbb{R}^n$, one has the Khintchine-type inequality

$$\forall q \geqslant r \geqslant 1, \quad \big(\mathbb{E}\|Z\|^q\big)^{1/q} \leqslant C\frac{q}{r} \big(\mathbb{E}\|Z\|^r\big)^{1/r}. \tag{10}$$

Proof. Fix an integer $k \in [1, n]$, a real $p \geqslant 2$ and a subspace $F_0 \in G_{n,k}$. There exists a real number $a_{n,k,p}$ such that, for any points $y \in \mathbb{R}^n$,

$$|y|^p = a_{n,k,p} \mathbb{E}_F |P_F y|^p.$$

Consequently, denoting by G_i a standard Gaussian vector on $\mathbb{R}^i$, we have for $q \in \{2, p\}$,

$$\frac{\mathbb{E}|Y|^q}{\mathbb{E}|G_n|^q} = \frac{\mathbb{E}_{F,Y}|P_F Y|^q}{\mathbb{E}|G_k|^q}. \tag{11}$$

Simple computations gives $\mathbb{E}|G_i|^p/(\mathbb{E}|G_i|^2)^{p/2} = \Gamma(\frac{i+p}{2})\Gamma(\frac{i}{2})^{p/2-1}/\Gamma(\frac{i+2}{2})^{p/2}$. Since the function $(\log\circ\Gamma)'$ is concave (the Euler's formula shows that $\log\circ\Gamma$ is the sum of functions which have a negative third derivative), the function $i \mapsto \mathbb{E}|G_i|^p/(\mathbb{E}|G_i|^2)^{p/2}$ is decreasing. We get thus

$$\frac{(\mathbb{E}|Y|^p)^{1/p}}{(\mathbb{E}|Y|^2)^{1/2}} = \frac{(\mathbb{E}|G_n|^p)^{1/p}}{(\mathbb{E}|G_n|^2)^{1/2}} \frac{(\mathbb{E}|G_k|^2)^{1/2}}{(\mathbb{E}|G_k|^p)^{1/p}} \frac{(\mathbb{E}_{F,Y}|P_F Y|^p)^{1/p}}{(\mathbb{E}_{F,Y}|P_F Y|^2)^{1/2}} \leqslant \frac{(\mathbb{E}_{F,Y}|P_F Y|^p)^{1/p}}{(\mathbb{E}_{F,Y}|P_F Y|^2)^{1/2}}. \tag{12}$$

It is well known that, for any fixed points $y \in \mathbb{R}^n$, $|P_F y|$ satisfies the large deviation inequality

$$\forall s \geqslant 1, \quad \mathbb{P}\left(|P_F y| \geqslant 2s\sqrt{\frac{k}{n}}|y|\right) \leqslant C_1 e^{-c_1 k s^2}.$$

On the other hand, Paouris concentration inequality [13] for the log-concave random vector Y claims that

$$\forall s \geqslant 1, \quad \mathbb{P}\big(|Y| \geqslant C_3 s\sqrt{n}\big) \leqslant C_2 e^{-c_2\sqrt{n}s}.$$

Hence, we get

$$\begin{aligned}
\mathbb{P}_{F,Y}\big(|P_F Y| \geqslant \delta\sqrt{k}\big) &= \mathbb{E}_Y\big[\mathbb{P}_F\big(|P_F Y| \geqslant \delta\sqrt{k}\big)\mathbf{1}_{|Y|\leqslant \frac{\delta}{2}\sqrt{n}}\big] + \mathbb{E}_Y\big[\mathbb{P}_F\big(|P_F Y| \geqslant \delta\sqrt{k}\big)\mathbf{1}_{|Y|\geqslant \frac{\delta}{2}\sqrt{n}}\big] \\
&\leqslant \mathbb{E}_Y\left[\mathbb{P}_F\left(|P_F Y| \geqslant 2|Y|\sqrt{\frac{k}{n}}\right)\right] + \mathbb{P}_Y\left(|Y| \geqslant \frac{\delta}{2}\sqrt{n}\right) \\
&\leqslant C_1 e^{-c_1 k} + C_2 e^{-c_2\sqrt{n}}
\end{aligned} \tag{13}$$

by taking $\delta = 2C_3$. According to the inequality (10), we have $(\mathbb{E}|Y|^{2p})^{1/2p} \leqslant C_4(\mathbb{E}|Y|^p)^{1/p}$. By the relation (11), this implies $(\mathbb{E}_{F,Y}|P_F Y|^{2p})^{1/2p} \leqslant C_5(\mathbb{E}_{F,Y}|P_F Y|^p)^{1/p}$. Consequently, by Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\mathbb{E}_{F,Y}\big[|P_F Y|^p\mathbf{1}_{|P_F Y|\geqslant\delta\sqrt{k}}\big] &\leqslant \big(\mathbb{E}_{F,Y}|P_F Y|^{2p}\big)^{1/2}\mathbb{P}_{F,Y}\big(|P_F Y| \geqslant \delta\sqrt{k}\big)^{1/2} \\
&\leqslant C_5^p e^{-c_3\min(k,\sqrt{n})}\mathbb{E}_{F,Y}|P_F Y|^p \\
&\leqslant e^{-c_4\min(k,\sqrt{n})}\mathbb{E}_{F,Y}|P_F Y|^p
\end{aligned}$$

when $p \leqslant c_5\min(k, \sqrt{n})$. Hence,

$$\mathbb{E}_{F,Y}|P_F Y|^p \leqslant \big(1 + C_6 e^{-c_4\min(k,\sqrt{n})}\big)\mathbb{E}_{F,Y}\big[|P_F Y|^p\mathbf{1}_{|P_F Y|\leqslant\delta\sqrt{k}}\big]. \tag{14}$$

To conclude, it is sufficient to remark that the inequalities (12), (13) and (14) give

$$\frac{(\mathbb{E}|Y|^p)^{1/p}}{(\mathbb{E}|Y|^2)^{1/2}} \leqslant \big(1 + C_7 e^{-c_5 \min(k,\sqrt{n})}\big) \times \frac{(\mathbb{E}_{F,Y}[|P_F Y|^p \mathbf{1}_{|P_F Y| \leqslant \delta\sqrt{k}}])^{1/p} \mathbb{P}_{F,Y}(|P_F Y| \geqslant \delta\sqrt{k})^{1/2-1/p}}{(\mathbb{E}_{F,Y}[|P_F Y|^2 \mathbf{1}_{|P_F Y| \leqslant \delta\sqrt{k}}])^{1/2}}$$

and to observe that, by polar coordinates, we have for $q \in \{0, 2, p\}$,

$$\mathbb{E}_{F,Y}\big[|P_F Y|^q \mathbf{1}_{|P_F Y| \leqslant \delta\sqrt{k}}\big] = \mathbb{E}_F\left[k v_k \int_{S_F} \sigma_F(d\theta) \int_0^{\delta\sqrt{k}} t^{q+k-1} \Pi_F g(t\theta)\, dt\right] = k v_k \mathbb{E} h_q(U)$$

where v_k is the volume of the k-dimensional Euclidean unit ball and σ_F is the uniform probability on the Euclidean unit sphere S_F of F. □

To compare h_0, h_2 and h_p, we use the following result which gives a moment bound similar to the concentration inequality (9).

Lemma 3. *(See Barlow, Marshall, Proschan [2], Borell [3].) Let ϕ be an integrable log-concave function on $[0, \infty)$. The function*

$$q \in (0, \infty) \mapsto \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} \phi(t)\, dt$$

is log-concave.

To compare the moments of $h_p(U)$, we use Herbst's argument and the log-Sobolev inequality on $SO(n)$. For any Lipschitz function $H : SO(n) \to \mathbb{R}$, we have

$$\mathrm{Ent}_U\big[H^2\big] := \mathbb{E}\big[H(U)^2 \log\big(H(U)^2\big)\big] - \mathbb{E}\big[H(U)^2\big] \log\big(\mathbb{E}\big[H(U)^2\big]\big) \leqslant \frac{C}{n} \mathbb{E}\big|\nabla H(U)\big|^2$$

where $|\nabla H(u)| = \limsup_{d(v,u)\to 0} \frac{|H(v)-H(u)|}{d(v,u)}$. Let h be a positive function on $SO(n)$. If $\log(h)$ is L-Lipschitz, applying the previous inequality with $H = h^{q/2}$, we get for any $q > 0$,

$$\frac{d}{dq}\big[\log\big(\mathbb{E} h(U)^q\big)^{1/q}\big] = \frac{1}{q^2} \frac{\mathrm{Ent}_U[h^q]}{\mathbb{E} h(U)^q} \leqslant \frac{C}{n} \frac{\mathbb{E}[h(U)^{q-2} |\nabla h(U)|^2]}{\mathbb{E}[h(U)^q]} \leqslant \frac{CL^2}{n}.$$

Consequently,

$$\forall q > r > 0, \quad \big(\mathbb{E}\big|h(U)\big|^q\big)^{1/q} \leqslant e^{\frac{CL^2}{n}(q-r)} \big(\mathbb{E}\big|h(U)\big|^r\big)^{1/r}. \tag{15}$$

According to Lemma 3.1 in [9] (with $\alpha = 0$), for a fixed subspace $F_0 \in G_{n,k}$ and a fixed point $x_0 \in F_0$ such that $|x_0| \leqslant \delta\sqrt{k}$, the function

$$M_{x_0} : u \in SO(n) \mapsto \log \Pi_{u(F_0)} g\big(u(x_0)\big)$$

is Lipschitz with respect to the geodesic distance d on $SO(n)$ and $\|M_{x_0}\|_{\mathrm{Lip}} \leqslant Ck^2$. This estimate implies

$$\forall p \geqslant 0, \quad \big\|\log(h_p)\big\|_{\mathrm{Lip}} \leqslant Ck^2 \tag{16}$$

since, for any $u \in SO(n)$,

$$\big|\nabla \log h_p(u)\big| = \frac{|\int_0^{\delta\sqrt{k}} t^{p+k-1} e^{M_{t\theta_0}(u)} \nabla M_{t\theta_0}(u)\,dt|}{\int_0^{\delta\sqrt{k}} t^{p+k-1} e^{M_{t\theta_0}(u)}\,dt} \leqslant \sup_{|x_0| \leqslant \delta\sqrt{k}} \|M_{x_0}\|_{\mathrm{Lip}}.$$

Lemma 4. *If $L := \max(\|\log(h_p)\|_{\mathrm{Lip}}, \|\log(h_0)\|_{\mathrm{Lip}}) \leqslant \sqrt{n}$ and $4 \leqslant p \leqslant k$, we have*

$$\big(\mathbb{E} h_p(U)\big)^{1/p} \big(\mathbb{E}\big[h_0(U)\big]\big)^{1/2-1/p} \leqslant \left(1 + \frac{CL^2}{np} + \frac{Cp}{k}\right) \big(\mathbb{E} h_2(U)\big)^{1/2}$$

for an absolute constant $C > 0$.

Proof. Let us fix a real $p \geqslant 4$. According to Lemma 3 with $\phi(t) = \Pi_{u(F_0)} g(tu(\theta_0)) \mathbf{1}_{[0,\delta\sqrt{k}]}(t)$, we have for any $u \in SO(n)$,

$$\frac{1}{\Gamma(k+2)} h_2(u) \geqslant \left(\frac{1}{\Gamma(k+p)} h_p(u)\right)^{2/p} \left(\frac{1}{\Gamma(k)} h_0(u)\right)^{1-2/p}.$$

By simple computations on the Γ function, we get for $p \leqslant k$,

$$h_p(u)^{2/p} h_0(u)^{1-2/p} \leqslant \left(1 + \frac{2p}{k}\right) h_2(u). \tag{17}$$

On the other hand, the inequality (15) applied respectively with $h = h_p$, $q = 1$ and $r = 2/p$, and with $h = h_0$, $q = 1$ and $r = 1 - 2/p$, gives

$$\mathbb{E}\big[h_p(U)^{2/p}\big] \geqslant e^{-\frac{C_1 L^2}{np}} \big(\mathbb{E}\big[h_p(U)\big]\big)^{2/p}, \tag{18}$$

$$\mathbb{E}\big[h_0(U)^{1-2/p}\big] \geqslant e^{-\frac{C_2 L^2}{np}} \big(\mathbb{E}\big[h_0(U)\big]\big)^{1-2/p}. \tag{19}$$

In the same way, by the inequality (15) applied with $h = h_p$, $q = 4/p$ and $r = 2/p$, and with $h = h_0$, $q = 2(1 - 2/p)$ and $r = 1 - 2/p$, we get

$$\mathbb{E}\big[h_p(U)^{4/p}\big] \leqslant e^{\frac{C_3 L^2}{np^2}} \big(\mathbb{E}\big[h_p(U)^{2/p}\big]\big)^2, \tag{20}$$

$$\mathbb{E}\big[h_0(U)^{2(1-2/p)}\big] \leqslant e^{\frac{C_4 L^2}{n}} \big(\mathbb{E}\big[h_0(U)^{1-2/p}\big]\big)^2. \tag{21}$$

Denoting by $\mathrm{cov}(Z_1, Z_2)$ the covariance between two random variables Z_1 and Z_2 and by $\mathrm{Var}(Z_1)$ the variance of Z_1, the inequalities (20) and (21) give for $L \leqslant \sqrt{n}$,

$$\begin{aligned}\left|\mathrm{cov}\left(h_p(U)^{2/p}, h_0(U)^{1-2/p}\right)\right| &\leqslant \sqrt{\mathrm{Var}\left[h_p(U)^{2/p}\right]}\sqrt{\mathrm{Var}\left[h_0(U)^{1-2/p}\right]} \\ &\leqslant \left(e^{\frac{C_3L^2}{np^2}} - 1\right)^{1/2}\left(e^{\frac{C_4L^2}{n}} - 1\right)^{1/2}\mathbb{E}\left[h_p(U)^{2/p}\right]\mathbb{E}\left[h_0(U)^{1-2/p}\right] \\ &\leqslant \frac{C_5L^2}{pn}\mathbb{E}\left[h_p(U)^{2/p}\right]\mathbb{E}\left[h_0(U)^{1-2/p}\right].\end{aligned}$$

Therefore, integrating the inequality (17) and using the inequalities (18) and (19), we get for $4 \leqslant p \leqslant k$,

$$\begin{aligned}\left(1 + \frac{2p}{k}\right)\mathbb{E}\left[h_2(U)\right] &\geqslant \mathbb{E}\left[h_p(U)^{2/p}h_0(U)^{1-2/p}\right] \\ &= \mathbb{E}\left[h_p(U)^{2/p}\right]\mathbb{E}\left[h_0(U)^{1-2/p}\right] + \mathrm{cov}\left(h_p(U)^{2/p}, h_0(U)^{1-2/p}\right) \\ &\geqslant \left(1 - \frac{C_5L^2}{np}\right)\mathbb{E}\left[h_p(U)^{2/p}\right]\mathbb{E}\left[h_0(U)^{1-2/p}\right] \\ &\geqslant \left(1 - \frac{C_6L^2}{np}\right)\left(\mathbb{E}\left[h_p(U)\right]\right)^{2/p}\left(\mathbb{E}\left[h_0(U)\right]\right)^{1-2/p}.\end{aligned}$$

Consequently, the proof is complete. □

Proof of Theorem 1. According to Lemmas 2 and 4 and the Lipschitz estimate (16), we have for $k \leqslant n^{1/4}$ and $p \in [4, k]$,

$$\left(\mathbb{E}|Y|^p\right)^{1/p} \leqslant \left(1 + \frac{C_1p}{k} + \frac{C_1k^4}{np}\right)\left(\mathbb{E}|Y|^2\right)^{1/2}.$$

Recall that $Y = X + G_n$ where G_n is a standard Gaussian vector independent of X. Expanding $|X + G_n|^{2p}$ by Newton's formula, we observe that $\mathbb{E}|Y|^{2p} \geqslant \mathbb{E}(|X|^2 + |G_n|^2)^p \geqslant 2^p\mathbb{E}|X|^p\mathbb{E}|G_n|^p \geqslant (2\sqrt{n})^p\mathbb{E}|X|^p$ for any integer $p \geqslant 2$. Consequently, for any integer $p \in [4, k/2]$,

$$\begin{aligned}\left(\mathbb{E}|X|^p\right)^{1/p} &\leqslant \frac{(\mathbb{E}|Y|^{2p})^{1/p}}{2\sqrt{n}} \\ &\leqslant \left(1 + \frac{C_2p}{k} + \frac{C_2k^4}{np}\right)\frac{\mathbb{E}|Y|^2}{2\sqrt{n}} \\ &= \left(1 + \frac{C_2p}{k} + \frac{C_2k^4}{np}\right)\left(\mathbb{E}|X|^2\right)^{1/2}.\end{aligned} \tag{22}$$

Taking $k = \lfloor n^{1/4}\rfloor$, we get the first assertion of Theorem 1 for $p \in [\sqrt{k}, k/2]$:

$$\left(\mathbb{E}|X|^p\right)^{1/p} \leqslant \left(1 + \frac{C_3p}{k}\right)\left(\mathbb{E}|X|^2\right)^{1/2}. \tag{23}$$

To show the inequality (6), we can assume that $t \in [2, \sqrt{k}\,]$. We take $p = \frac{1}{2}\sqrt{k}t$. One has $p \in [\sqrt{k}, k/2]$. We remark that, by Chebychev inequality, we have

$$\mathbb{P}\left(|X| \geqslant \left(1 + \frac{t}{\sqrt{k}}\right)\left(\mathbb{E}|X|^p\right)^{1/p}\right) \leqslant \frac{1}{(1+\frac{t}{\sqrt{k}})^p} \leqslant e^{-\frac{pt}{2\sqrt{k}}} = e^{-\frac{1}{4}t^2}.$$

By the inequality (23), we obtain

$$\mathbb{P}\left(|X| \geqslant \left(1 + C_3\frac{t}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right) \leqslant e^{-\frac{1}{4}t^2}. \tag{24}$$

To prove the inequality (5), we observe that, by the inequality (22), we can choose suitably $k \approx n^{1/4}$ and $p \approx \sqrt{k}$ in such a way that $\operatorname{Var}|X|^p \leqslant \frac{1}{16}(\mathbb{E}|X|^p)^2$. Then, Chebychev inequality gives

$$\begin{aligned}\frac{1}{4} &\geqslant \mathbb{P}\left(\big||X|^p - \mathbb{E}|X|^p\big| \geqslant \frac{1}{2}\mathbb{E}|X|^p\right)\\ &\geqslant \mathbb{P}\left(|X| \leqslant \frac{1}{2^{1/p}}\left(\mathbb{E}|X|^p\right)^{1/p}\right)\\ &\geqslant \mathbb{P}\left(|X| \leqslant \left(1 - \frac{C_4}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right).\end{aligned}$$

On the other hand, the inequality (24) gives

$$\mathbb{P}\left(|X| \leqslant \left(1 + \frac{C_5}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right) \geqslant \frac{3}{4}.$$

We assume $t \geqslant 1$, otherwise the inequality (5) is trivial. Let $C_6 := \max(C_4, C_5)$ and $\lambda \in [0, 1]$ be real numbers such that $1 - \frac{C_6}{\sqrt{k}} = \lambda(1 - \frac{C_6 t}{\sqrt{k}}) + (1-\lambda)(1 + \frac{C_6}{\sqrt{k}})$, i.e. $\lambda = 2/(1+t)$. Since the function $u \in \mathbb{R}_+ \mapsto \mathbb{P}(|X| \leqslant u)$ is log-concave, we have

$$\begin{aligned}&\mathbb{P}\left(|X| \leqslant \left(1 - \frac{C_6}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right)\\ &\quad \geqslant \mathbb{P}\left(|X| \leqslant \left(1 - \frac{C_6 t}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right)^{\lambda} \mathbb{P}\left(|X| \leqslant \left(1 + \frac{C_6}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right)^{1-\lambda}.\end{aligned}$$

Hence

$$\mathbb{P}\left(|X| \leqslant \left(1 - \frac{C_6 t}{\sqrt{k}}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right) \leqslant \left(\frac{1}{4}\right)^{1/\lambda}\left(\frac{4}{3}\right)^{1/\lambda - 1} \leqslant \left(\frac{1}{3}\right)^{1/\lambda} \leqslant e^{-ct}.$$

To conclude, it is sufficient to observe that the concentration inequality

$$\forall s > 0, \quad \mathbb{P}\left(\big||X| - \left(\mathbb{E}|X|^2\right)^{1/2}\big| \geqslant s\left(\mathbb{E}|X|^2\right)^{1/2}\right) \leqslant Ce^{-c\sqrt{k}s}\mathbf{1}_{s\leqslant 1} + Ce^{-c\sqrt{n}s}\mathbf{1}_{s\geqslant 1}$$

implies the inequality (4) for $p \in [2, \sqrt{k}\,]$ thanks to the integration by parts

$$\mathbb{E}|T|^q = q \int_0^\infty u^{q-1} \mathbb{P}\big(|T| \geqslant u\big)\, du \quad \text{with } T = \frac{|X|^2}{\mathbb{E}|X|^2} - 1$$

(see Lemma 1.4. in [5]). □

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References

[1] M. Anttila, K. Ball, I. Perissinaki, The central limit problem for convex bodies, Trans. Amer. Math. Soc. 355 (12) (2003) 4723–4735.
[2] R.E. Barlow, A.W. Marshall, F. Proschan, Properties of probability distributions with monotone hazard rate, Ann. Math. Statist. 34 (1963) 375–389.
[3] C. Borell, Complements of Lyapunov's inequality, Math. Ann. 205 (1973) 323–331.
[4] C. Borell, Convex measures on locally convex spaces, Ark. Mat. 12 (1974) 239–252.
[5] B. Fleury, Between Paouris concentration inequality and variance conjecture, Ann. Inst. H. Poincare Probab. Statist., 2010, in press.
[6] B. Fleury, O. Guédon, G. Paouris, A stability result for mean width of L_p-centroid bodies, Adv. Math. 214 (2) (2007) 865–877.
[7] R. Kannan, L. Lovász, M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, Discrete Comput. Geom. 13 (3–4) (1995) 541–559.
[8] B. Klartag, A central limit theorem for convex sets, Invent. Math. 168 (2007) 91–131.
[9] B. Klartag, Power-law estimates for the central limit theorem for convex sets, J. Funct. Anal. 245 (2007) 284–310.
[10] L. Leindler, On a certain converse of Hölder's inequality, Acta Sci. Math. 33 (1972) 217–233.
[11] V.D. Milman, A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies, Funktsional. Anal. i Prilozhen. 5 (4) (1971) 28–37 (in Russian).
[12] V.D. Milman, G. Schechtman, Asymptotic Theory of Finite-Dimensional Normed Spaces, Lecture Notes in Math., vol. 1200, Springer, Berlin, 1986.
[13] G. Paouris, Concentration of mass in convex bodies, Geom. Funct. Anal. 16 (5) (2006) 1021–1049.
[14] A. Prékopa, Logarithmic concave measure with application to stochastic programming, Acta Sci. Math. 32 (1971) 301–315.
[15] A. Prékopa, On logarithmically concave measures and functions, Acta Sci. Math. 34 (1973) 335–343.