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Extending Irreducible Modules

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An important part of Clifford's theory does not work properly for arbitrary fields $\mathfrak{K}$. To see why, we must recall the main steps of this theory.

We are interested in extensions to $\mathfrak{K}H$-modules of a fixed H-invariant irreducible $\mathfrak{K}N$-module $\mathfrak{B}$, where N is a normal subgroup of a finite group H. We first pass to the factor algebra $\mathfrak{A} = \mathfrak{K}H/\mathfrak{J}$ modulo the ideal $\mathfrak{J} = \mathfrak{J}_1 \mathfrak{K}H = \mathfrak{K}H\mathfrak{J}_1$ generated by the annihilator $\mathfrak{J}_1$ of $\mathfrak{B}$ in $\mathfrak{K}N$. If $\mathfrak{K}H$ is given its natural grading by the factor group $G = H/N$ (see (6.2) below), then $\mathfrak{J}$ is a graded ideal, so that $\mathfrak{A}$ is a strongly G-graded ring (in the sense of [3] or Section 1 below) whose 1_G-component $\mathfrak{A}_1 \simeq \mathfrak{K}N/\mathfrak{J}_1$ is a finite-dimensional simple $\mathfrak{K}$-algebra with $\mathfrak{B}$ as its irreducible module. It follows that the extensions of $\mathfrak{B}$ to $\mathfrak{K}H$-modules are just the pull-backs of the extensions of $\mathfrak{B}$ to $\mathfrak{A}$-modules via the natural epimorphism of $\mathfrak{K}H$ onto $\mathfrak{A}$.

When $\mathfrak{K}$ is algebraically closed (or, more generally, when $\mathfrak{A}_1$ is split over $\mathfrak{K}$), then $\mathfrak{A}$ is the Kronecker product

$$\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{K}} \mathfrak{A}_1, \tag{0.1}$$

where $\mathfrak{C}$ is the centralizer $\mathrm{C}(\mathfrak{A}_1 \text{ in } \mathfrak{A})$ of $\mathfrak{A}_1$ in $\mathfrak{A}$. Here $\mathfrak{C}$ is a strongly G-graded subring of $\mathfrak{A}$ with $\mathfrak{C}_1 = \mathfrak{K}1_{\mathfrak{A}} \simeq \mathfrak{K}$. So $\mathfrak{C}$ is the twisted group algebra over $\mathfrak{K}$ of a unique *Clifford extension* $\mathrm{X}\langle\mathfrak{C}\rangle$ of the unit group $\mathrm{U}(\mathfrak{C}_1)$ of $\mathfrak{C}_1 \simeq \mathfrak{K}$ by G. Because $\mathfrak{A}_1$ splits over $\mathfrak{K}$, the decomposition (0.1) implies that tensoring with $\mathfrak{B}$ is an equivalence $\cdot \otimes_{\mathfrak{K}} \mathfrak{B}$ of the category $\mathbf{Mod}(\mathfrak{C})$ of $\mathfrak{C}$-modules with the category $\mathbf{Mod}(\mathfrak{A})$. So the extensions of $\mathfrak{B}$ to $\mathfrak{A}$-modules are isomorphic to tensor products $\mathfrak{U} \otimes_{\mathfrak{K}} \mathfrak{B}$, where $\mathfrak{U}$ is an extension of the regular $\mathfrak{C}_1$-module $\mathfrak{C}_1$ to a $\mathfrak{C}$-module. Since such $\mathfrak{U}$ exist if and only if $\mathrm{X}\langle\mathfrak{C}\rangle$ is a split group extension, we conclude that $\mathfrak{B}$ extends to a $\mathfrak{K}H$-module if and only if the Clifford extension $\mathrm{X}\langle\mathfrak{C}\rangle$ splits.

The decomposition (0.1) does not hold for arbitrary fields $\mathfrak{K}$. In general, the group G acts as $\mathfrak{K}$-automorphisms of the center $\mathfrak{Z}$ of $\mathfrak{A}_1$ via conjugation in $\mathfrak{A}$ (see (1.5) and (1.6) below), and this action may very well be non-

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trivial. In this case the product of the subalgebras $\mathfrak{C} = C(\mathfrak{A}_1 \text{ in } \mathfrak{A})$ and $\mathfrak{A}_1$ is

$$\mathfrak{C} \times_{\mathfrak{Z}} \mathfrak{A}_1 = \dot{\sum_{\sigma \in C(\mathfrak{Z} \text{ in } G)}} \mathfrak{A}_\sigma,$$

which is properly contained in $\mathfrak{A}$ if the centralizer $C(\mathfrak{Z}$ in $G)$ of $\mathfrak{Z}$ is not all of G. Thus (0.1) fails unless G centralizes $\mathfrak{Z}$. Even in that case we do not obtain an equivalence of categories when $\mathfrak{A}_1$ is not split over $\mathfrak{Z}$. Indeed, Example 7.4 below shows that $X\langle\mathfrak{C}\rangle$ can fail to split even though $\mathfrak{V}$ can be extended to a $\mathfrak{K}H$-module. This is hardly a property of a decent Clifford extension!

Of course, the above problems can be avoided by using Cline's stable Clifford theory [1], which produces a Clifford extension $X\langle\mathfrak{E}\rangle$ by giving a suitable strong G-grading to the endomorphism ring $\mathfrak{E} = \operatorname{End}_{\mathfrak{K}H}(\mathfrak{V}^H)$ of the induced $\mathfrak{K}H$-module $\mathfrak{V}^H$. In this theory there is an equivalence between $\mathbf{Mod}(\mathfrak{E})$ and $\mathbf{Mod}(\mathfrak{A})$ (see Theorem 8.2 in [3]), and $\mathfrak{V}$ does extend to an $\mathfrak{A}$-module if and only if $X\langle\mathfrak{E}\rangle$ splits (see Theorem 2.8 below). The problem with this approach is that $X\langle\mathfrak{E}\rangle$ is an extension of $U(\mathfrak{E}_1)$ by G, and $\mathfrak{E}_1 \simeq \operatorname{End}_{\mathfrak{K}N}(\mathfrak{V})$ is usually a non-commutative division ring. So none of the nice theorems about extensions of $\mathfrak{V}$, such as those of Gallagher [6], which depend upon calculations in the cohomology groups of G with coefficients in a commutative group $U(\mathfrak{C}_1)$, can be proved this way.

In view of all these problems and counterexamples it is somewhat astonishing that Gallagher's theorems do hold in general. The proof of this arose when I was refereeing the first (unpublished) version of Isaacs' paper [8]. In that version he proved that $\mathfrak{V}$ extends to a $\mathfrak{K}H$-module for arbitrary $\mathfrak{K}$ whenever N is a Hall subgroup of H, under the additional hypothesis that N is solvable when $\mathfrak{K}$ has characteristic zero. His proof in the characteristic zero case did not use Clifford's theory, but depended upon an ingenious application of techniques devised by him and Gajendragadkar [5] to compute Schur invariants. However, at one point in his prime characteristic proof he introduced certain factor sets of G in $U(\mathfrak{Z})$ *which were trivial if and only if* $\mathfrak{V}$ *could be extended to a* $\mathfrak{K}H$*-module.* Aha! Stripped of coset representatives, matrices, factor sets and other camouflage, this argument came down to the observation that, if $\mathfrak{A}_1$ splits over $\mathfrak{Z}$, then it has a decomposition

$$\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{K}} \mathfrak{D}, \tag{0.2}$$

where $\mathfrak{D}$ is a split simple $\mathfrak{K}$-algebra determined to within $U(\mathfrak{A}_1)$-conjugacy. In this case we replace (0.1) by

$$\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{K}} \mathfrak{D}, \tag{0.3}$$

where $\mathfrak{C}$ is now the centralizer, not of $\mathfrak{A}_1$, but of $\mathfrak{D}$. Here $\mathfrak{C}$ is also a strongly G-graded subring of $\mathfrak{A}$, but $\mathfrak{C}_1$ is $\mathfrak{Z}$, on which G may act non-trivially. So $\mathfrak{C}$ is the "skew twisted group algebra" or "crossed product" over $\mathfrak{Z}$ of a Clifford extension of $\mathrm{U}(\mathfrak{C}_1) \simeq \mathrm{U}(\mathfrak{Z})$ by G. As before, we have an equivalence between $\mathbf{Mod}(\mathfrak{C})$ and $\mathbf{Mod}(\mathfrak{A})$, and $\mathfrak{B}$ extends to a $\mathfrak{R}H$-module if and only if $\mathrm{X}\langle\mathfrak{C}\rangle$ splits.

With the above observation in hand, it was easy to re-prove Gallagher's results in the general case of split $\mathfrak{A}_1$, and then to deduce Isaacs' theorem in characteristic zero for arbitrary N by using tensor powers to reduce to split $\mathfrak{A}_1$. Isaacs consequently revised his paper, and the new version [8] contains the above proof with a different reduction to split $\mathfrak{A}_1$.

In the present paper we shall take a different approach, based on the key Theorem 4.4 below, which has some independent interest. Let $\mathfrak{A}_1$ be arbitrary so long as it is simple and finite-dimensional over $\mathfrak{R}$, and let $\mathfrak{F}$ be the fixed field $\mathrm{C}(G \text{ in } \mathfrak{Z})$ of G acting on the field $\mathfrak{Z}$. We assume that:

(0.4) *The order* $|G|$ *and* $\mathfrak{Z}$*-dimension* $[\mathfrak{A}_1 : \mathfrak{Z}]$ *are relatively prime.*

Then Theorem 4.4 tells us that there is a unique $\mathrm{U}(\mathfrak{A}_1)$-conjugacy class of central simple $\mathfrak{F}$-subalgebras $\mathfrak{D}$ of $\mathfrak{A}_1$ such that

$$\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{F}} \mathfrak{D}, \tag{0.5}$$

i.e., that $\mathfrak{A}_1$ is obtained from a "unique" $\mathfrak{F}$-algebra by ground field extension. As before, this gives a decomposition

$$\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}, \tag{0.6}$$

where $\mathfrak{C} = \mathrm{C}(\mathfrak{D} \text{ in } \mathfrak{A})$ is a crossed product over $\mathfrak{C}_1 = \mathfrak{Z}$ of a Clifford extension $\mathrm{X}\langle\mathfrak{C}\rangle$ of $\mathrm{U}(\mathfrak{Z})$ by G. We do not necessarily have an equivalence between $\mathbf{Mod}(\mathfrak{C})$ and $\mathbf{Mod}(\mathfrak{A})$, since $\mathfrak{D}$ need not split over $\mathfrak{F}$. Nevertheless, we can show that $\mathfrak{B}$ extends to a $\mathfrak{R}H$-module if and only if $\mathrm{X}\langle\mathfrak{C}\rangle$ splits, and we can even recapture all of Gallagher's theorems (see (6.11), Theorems 6.6 and 5.10, and Lemma 5.7 below). Of course, this gives another proof of Isaacs' theorem.

One cannot hope to extend this theory much further. An example of Janusz (see Example 7.1 below) shows that $\mathfrak{A}$ need not have a strongly G-graded subring $\mathfrak{C}$ with $\mathfrak{C}_1 = \mathfrak{Z}$ when (0.4) does not hold. So the only known approach is Cline's in the general case. This, of course, only makes more interesting the exceptions when (0.4) holds or $\mathfrak{A}_1$ is split.

1. Strongly Graded Rings

By a *ring* $\mathfrak{A}$ we always mean an associative ring with identity $1 = 1_{\mathfrak{A}}$. We denote by $\mathrm{U}(\mathfrak{A})$ the unit group of $\mathfrak{A}$, and by $\mathrm{Z}(\mathfrak{A})$ the center of $\mathfrak{A}$. Any $\mathfrak{A}$-module is understood to be right and unitary unless otherwise indicated.

We fix a multiplicative group G with identity $1 = 1_G$, and a non-zero *strongly G-graded ring* $\mathfrak{A}$. As in [3], the latter is a non-zero ring (also denoted by $\mathfrak{A}$), together with a direct sum decomposition

$$\mathfrak{A} = \sum_{\sigma \in G} \mathfrak{A}_\sigma \text{ (as additive groups)}, \tag{1.1a}$$

where the module products of the additive subgroups $\mathfrak{A}_\sigma$ satisfy

$$\mathfrak{A}_\sigma \mathfrak{A}_\tau = \mathfrak{A}_{\sigma\tau}, \quad \text{for all } \sigma, \tau \in G. \tag{1.1b}$$

The expansion (1.1a) is called the *G-grading* of $\mathfrak{A}$, while the additive subgroup $\mathfrak{A}_\sigma$ is called the *σ-component* of $\mathfrak{A}$, for any $\sigma \in G$. One proves easily (see Proposition 1.4 in [3]) that:

(1.2) *The 1_G-component $\mathfrak{A}_1$ is a subring of $\mathfrak{A}$ containing $1_{\mathfrak{A}}$.*

Since $\mathfrak{A}$ is non-zero, Proposition 5.2 of [3] tells us that:

(1.3) *The disjoint union*

$$\mathrm{GrU}(\mathfrak{A}) = \dot{\bigcup_{\sigma \in G}} (\mathfrak{A}_\sigma \cap \mathrm{U}(\mathfrak{A}))$$

is a subgroup of $\mathrm{U}(\mathfrak{A})$, *and the map* $\deg\colon \mathrm{GrU}(\mathfrak{A}) \to G$ *having the inverse images*

$$\deg^{-1}(\sigma) = \mathfrak{A}_\sigma \cap \mathrm{U}(\mathfrak{A}), \qquad \text{for all} \quad \sigma \in G,$$

is a homomorphism of groups with kernel $\mathrm{U}(\mathfrak{A}_1)$. *So the sequence of group homomorphisms*

$$\mathrm{X}\langle\mathfrak{A}\rangle\colon\ 1 \to \mathrm{U}(\mathfrak{A}_1) \xrightarrow{\subset} \mathrm{GrU}(\mathfrak{A}) \xrightarrow{\deg} G \to 1$$

is always exact except possibly at G.

The elements $u \in \mathrm{GrU}(A)$ are called the *graded units* of $\mathfrak{A}$, and $\deg(u) \in G$ is called the *degree* of any such u. Of course, deg need not be an epimorphism since $\mathfrak{A}_\sigma \cap \mathrm{U}(\mathfrak{A})$ could be empty for certain $\sigma \in G$. We know from Theorem 5.10 in [3] and the remarks following it that deg is an epimorphism, i.e., $\mathrm{X}\langle\mathfrak{A}\rangle$ is exact, if and only if $\mathfrak{A}$ is a *crossed product* of G over $\mathfrak{A}_1$ in the sense of [10].

Proposition 5.5 of [3] says that:

(1.4) *The group* $\mathrm{GrU}(\mathfrak{A})$ *acts as automorphisms of the subring* $\mathfrak{A}_1$ *by conjugation in* $\mathfrak{A}$, *with any* $u \in \mathrm{GrU}(\mathfrak{A})$ *sending any* $a_1 \in \mathfrak{A}_1$ *into*

$$a_1^u = u^{-1} a_1 u \in \mathfrak{A}_1.$$

In the language of [2] we have a "graded Clifford system $\mathfrak{A}$, $\{\mathfrak{A}_\sigma \mid \sigma \in G\}$" by (1.1) and (1.2). So [2, Sect. 2] implies that:

(1.5) *The group* G *acts as automorphisms of the subring* $\mathrm{Z}(\mathfrak{A}_1)$, *with any* $\sigma \in G$ *sending any* $z \in \mathrm{Z}(\mathfrak{A}_1)$ *into the unique element* $z^\sigma \in \mathrm{Z}(\mathfrak{A}_1)$ *satisfying*

$$z a_\sigma = a_\sigma z^\sigma, \qquad \textit{for all} \quad a_\sigma \in \mathfrak{A}_\sigma.$$

Evidently this action is related to that of (1.4) by:

(1.6) $z^u = e^{\deg(u)}$, *for all* $z \in \mathrm{Z}(\mathfrak{A}_1)$ *and* $u \in \mathrm{GrU}(\mathfrak{A})$.

2. Extension Modules

Each $\mathfrak{A}_\sigma$, $\sigma \in G$, is a two-sided $\mathfrak{A}_1$-submodule of $\mathfrak{A}$ by (1.1b). If $\mathfrak{V}$ is any non-zero $\mathfrak{A}_1$-module, then this and (1.1a) imply that the *induced* $\mathfrak{A}$-module $\mathfrak{V}^{\mathfrak{A}} = \mathfrak{V} \otimes_{\mathfrak{A}_1} \mathfrak{A}$ is the direct sum

$$\mathfrak{V}^{\mathfrak{A}} = \dot{\sum_{\sigma \in G}} (\mathfrak{V} \otimes_{\mathfrak{A}_1} \mathfrak{A}_\sigma) = \dot{\sum_{\sigma \in G}} \mathfrak{V}^\sigma \qquad (\textit{as } \mathfrak{A}_1\textit{-modules}), \tag{2.1}$$

where each *conjugate* $\mathfrak{A}_1$-module $\mathfrak{V}^\sigma = \mathfrak{V} \otimes_{\mathfrak{A}_1} \mathfrak{A}_\sigma$ has been identified naturally with its image in $\mathfrak{V}^{\mathfrak{A}}$. In view of (1.2) we may also identify $\mathfrak{V}$ with the $\mathfrak{A}_1$-submodule $\mathfrak{V}^1 = \mathfrak{V} \otimes_{\mathfrak{A}_1} \mathfrak{A}_1$ of $\mathfrak{V}^{\mathfrak{A}}$ so that:

$$v = v \otimes 1_{\mathfrak{A}} \in \mathfrak{V}^1, \qquad \textit{for all} \quad v \in \mathfrak{V}. \tag{2.2}$$

As in (4.1) of [3], we may define additive subgroups $\mathfrak{E}_\sigma$, for $\sigma \in G$, and $\mathfrak{E}$ of the ring $\mathrm{End}_{\mathfrak{A}}(\mathfrak{V}^{\mathfrak{A}})$ of all $\mathfrak{A}$-endomorphisms of $\mathfrak{V}^{\mathfrak{A}}$ by:

$$\mathfrak{E}_\sigma = \{\phi \in \mathrm{End}_{\mathfrak{A}}(\mathfrak{V}^{\mathfrak{A}}) \mid \phi(\mathfrak{V}^\tau) \subseteq \mathfrak{V}^{\sigma\tau}, \textit{for all } \tau \in G\}, \tag{2.3a}$$

$$\mathfrak{E} = \dot{\sum_{\sigma \in G}} \mathfrak{E}_\sigma. \tag{2.3b}$$

We know from (4.2) of [3] that $\mathfrak{E}$ is a subring of $\mathrm{End}_{\mathfrak{A}}(\mathfrak{V}^{\mathfrak{A}})$, that its subring $\mathfrak{E}_1$ contains the identity map $1_{\mathfrak{E}}$ of $\mathfrak{V}^{\mathfrak{A}}$ into itself, and that the grading (2.3b) makes $\mathfrak{E}$ a *G-graded ring* in the sense that

$$\mathfrak{E}_\sigma \mathfrak{E}_\tau \subseteq \mathfrak{E}_{\sigma\tau}, \qquad \textit{for all} \quad \sigma, \tau \in G.$$

Proposition 4.8 of [3] tells us that:

(2.4a) *Restriction to* $\mathfrak{B}^{\tau}$ *is an isomorphism of the additive group* $\mathfrak{E}_{\sigma}$ *onto the group* $\mathrm{Hom}_{\mathfrak{A}_1}(\mathfrak{B}^{\tau}, \mathfrak{B}^{\sigma\tau})$ *of all* $\mathfrak{A}_1$*-homomorphisms of* $\mathfrak{B}^{\tau}$ *into* $\mathfrak{B}^{\sigma\tau}$, *for any* $\sigma, \tau \in G$,

(2.4b) *Restriction to* $\mathfrak{B} = \mathfrak{B}^1$ *is an isomorphism of the subring* $\mathfrak{E}_1$ *onto* $\mathrm{End}_{\mathfrak{A}_1}(\mathfrak{B})$.

Since $\mathfrak{B}$ is non-zero, it follows from (2.4b) that $\mathfrak{E}_1$ and $\mathfrak{E}$ are non-zero. So we may define the subgroup $\mathrm{GrU}(\mathfrak{E})$ of $\mathrm{U}(\mathfrak{E})$ and the sequence of group homomorphisms

$$(2.5) \qquad \mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle = \mathrm{X}\langle \mathfrak{E} \rangle\colon\ 1 \to \mathrm{U}(\mathfrak{E}_1) \xrightarrow{\subseteq} \mathrm{GrU}(\mathfrak{E}) \xrightarrow{\deg} G \to 1,$$

by (1.3) with $\mathfrak{E}$ in place of $\mathfrak{A}$ (see Proposition 5.2 of [3]). As before, this sequence is always exact except possibly at G. Corollary 5.14 of [3] tells us that $\mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle$ is exact if and only if the $\mathfrak{A}_1$-module $\mathfrak{B}$ is *G-invariant*, i.e., is isomorphic to each of its conjugates $\mathfrak{B}^{\sigma}$, $\sigma \in G$.

A *splitting homomorphism* γ for the sequence $\mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle$ is, as usual, a homomorphism γ of the group G into $\mathrm{GrU}(\mathfrak{E})$ such that

$$(2.6) \qquad \deg(\gamma(\sigma)) = \sigma, \qquad \textit{for all} \quad \sigma \in G.$$

Evidently such γ exist if and only if the sequence $\mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle$ is both exact and split.

An *extension* $\mathfrak{B}^{\odot}$ of the $\mathfrak{A}_1$-module $\mathfrak{B}$ to an $\mathfrak{A}$-module is, of course, an $\mathfrak{A}$-module having $\mathfrak{B}$ as its restriction to an $\mathfrak{A}_1$-module. Thus the additive group of $\mathfrak{B}^{\odot}$ coincides with that of $\mathfrak{B}$, while the multiplication $\odot\colon \mathfrak{B}^{\odot} \times \mathfrak{A} \to \mathfrak{B}^{\odot}$ in $\mathfrak{B}^{\odot}$ satisfies:

$$(2.7) \qquad v \odot a_1 = v a_1 \in \mathfrak{B}, \qquad \textit{for all } v \in \mathfrak{B}^{\odot} = \mathfrak{B} \textit{ and all } a_1 \in \mathfrak{A}_1.$$

The preceding two concepts are closely related.

THEOREM 2.8. *The non-zero* $\mathfrak{A}_1$*-module* $\mathfrak{B}$ *can be extended to an* $\mathfrak{A}$*-module if and only if the sequence* $\mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle$ *is both exact and split. Indeed, there is a one-to-one correspondence between all extensions* $\mathfrak{B}^{\odot}$ *of* $\mathfrak{B}$ *to* $\mathfrak{A}$*-modules and all splitting homomorphisms* γ *for* $\mathrm{X}_{\mathfrak{A}}\langle \mathfrak{B}^{\mathfrak{A}} \rangle$, *in which* $\mathfrak{B}^{\odot}$ *corresponds to* γ *if and only if*

$$(2.9) \qquad v \odot a_{\sigma} = \gamma(\sigma^{-1})(v \otimes a_{\sigma}) \in \mathfrak{E}_{\sigma^{-1}}(\mathfrak{B}^{\sigma}) \subseteq \mathfrak{B}^1 = \mathfrak{B}, \qquad \textit{for all } v \in \mathfrak{B}, \sigma \in G, \textit{ and } a_{\sigma} \in \mathfrak{A}_{\sigma}.$$

Proof. Suppose that $\mathfrak{B}^{\odot}$ is an extension of $\mathfrak{B}$ to an $\mathfrak{A}$-module. Then (2.7) implies that there is a unique $\mathfrak{A}_1$-homomorphism of $\mathfrak{B}^{\sigma} = \mathfrak{B} \otimes_{\mathfrak{A}_1} \mathfrak{A}_{\sigma}$ into $\mathfrak{B}$

sending $v \otimes a_\sigma$ into $v \odot a_\sigma$, for any $\sigma \in G$, $v \in \mathfrak{V}$, and $a_\sigma \in \mathfrak{A}_\sigma$. In view of (2.4a), this says that there are unique elements $\gamma(\sigma^{-1}) \in \mathfrak{E}_{\sigma^{-1}}$, for $\sigma \in G$, satisfying (2.9).

By (2.2) and (2.7) the element $1_{\mathfrak{E}} \in \mathfrak{E}_1$ satisfies

$$v \odot a_1 = va_1 = v \otimes a_1 = 1_{\mathfrak{E}}(v \otimes a_1), \qquad \textit{for all } v \in \mathfrak{V} \textit{ and } a_1 \in \mathfrak{A}_1.$$

Hence the unique element $\gamma(1)$ must be $1_{\mathfrak{E}}$.

For any $\sigma, \tau \in G$, the product $\gamma(\tau^{-1})\gamma(\sigma^{-1})$ lies in $\mathfrak{E}_{\tau^{-1}}\mathfrak{E}_{\sigma^{-1}} \subseteq \mathfrak{E}_{(\sigma\tau)^{-1}}$. If $v \in \mathfrak{V}$, $a_\sigma \in \mathfrak{A}_\sigma$ and $a'_\tau \in \mathfrak{A}_\tau$, then (2.9) and the fact that $\gamma(\sigma^{-1})$ is an $\mathfrak{A}$-endomorphism of $\mathfrak{V}^{\mathfrak{A}}$ imply that

$$\begin{aligned}[\gamma(\tau^{-1})\,\gamma(\sigma^{-1})](v \otimes (a_\sigma a'_\tau)) &= \gamma(\tau^{-1})(\gamma(\sigma^{-1})([v \otimes a_\sigma]\,a'_\tau)) \\ &= \gamma(\tau^{-1})([\gamma(\sigma^{-1})(v \otimes a_\sigma)]\,a'_\tau) \\ &= \gamma(\tau^{-1})([v \odot a_\sigma]\,a'_\tau) \qquad (2.10) \\ &= \gamma(\tau^{-1})([v \odot a_\sigma] \otimes a'_\tau) \\ &= [v \odot a_\sigma] \odot a'_\tau.\end{aligned}$$

This last expression is just $v \odot (a_\sigma a'_\tau)$, since $\mathfrak{V}^{\odot}$ is an $\mathfrak{A}$-module. By (1.1b) the above products $a_\sigma a'_\tau$ generate the additive group $\mathfrak{A}_{\sigma\tau}$. So these equations force $\gamma(\tau^{-1})\,\gamma(\sigma^{-1})$ to be the unique element $\gamma((\sigma\tau)^{-1})$. This completes the proof that γ is a splitting homomorphism for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$, the only such homomorphism satisfying (2.9).

Now let γ be any splitting homomorphism for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$. For any $\sigma \in G$, $v \in \mathfrak{V}$ and $a_\sigma \in \mathfrak{A}_\sigma$ the element $\gamma(\sigma^{-1})$ of $\mathfrak{E}_{\sigma^{-1}}$ sends $v \otimes a_\sigma \in \mathfrak{V} \otimes_{\mathfrak{A}_1} \mathfrak{A}_\sigma = \mathfrak{V}^\sigma$ into an element of $\mathfrak{V}^1 = \mathfrak{V}$ by (2.3a). Thus we may use Eq. (2.9) to define a bilinear product $\odot\colon \mathfrak{V} \times \mathfrak{A}_\sigma \to \mathfrak{V}$ for any $\sigma \in G$. In view of (1.1a) there is a unique extension of these products to a bilinear product $\odot\colon \mathfrak{V} \times \mathfrak{A} \to \mathfrak{V}$. Since the homomorphism γ sends 1_G into the identity $1_{\mathfrak{E}}$ of $\mathrm{GrU}(\mathfrak{E})$, it follows from (2.2) and (2.9) that (2.7) holds. If $v \in \mathfrak{V}$, $a_\sigma \in \mathfrak{A}_\sigma$ and $a'_\tau \in \mathfrak{A}_\tau$, for any $\sigma, \tau \in G$, then (2.10) holds by (2.9) since $\gamma(\sigma^{-1})$ is an $\mathfrak{A}$-endomorphism of $\mathfrak{V}^{\mathfrak{A}}$. Because γ is a homomorphism, this implies that

$$\begin{aligned}v \odot (a_\sigma a'_\tau) &= \gamma((\sigma\tau)^{-1})(v \otimes (a_\sigma a'_\tau)) \\ &= [\gamma(\tau^{-1})\,\gamma(\sigma^{-1})](v \otimes (a_\sigma a'_\tau)) = [v \odot a_\sigma] \odot a'_\tau.\end{aligned}$$

Therefore the product $\odot$ is associative, and thus defines an extension of the $\mathfrak{A}_1$-module $\mathfrak{V}$ to an $\mathfrak{A}$-module $\mathfrak{V}^{\odot}$, the only such module satisfying (2.9). So the theorem is proved.

The group $U(\mathfrak{E}_1)$ acts naturally on the set of all splitting homomorphisms for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$, with any $u \in U(\mathfrak{E}_1)$ sending any such homomorphism γ into the *conjugate* splitting homomorphism γ^u for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$ defined by

$$\gamma^u(\sigma) = u^{-1}\gamma(\sigma)u \in \mathfrak{E}_1\mathfrak{E}_\sigma\mathfrak{E}_1 \subseteq \mathfrak{E}_\sigma, \qquad \textit{for all} \quad \sigma \in G. \tag{2.11}$$

This action is related to isomorphisms of extension modules by:

THEOREM 2.12. *Two extensions of the $\mathfrak{A}_1$-module $\mathfrak{V}$ to $\mathfrak{A}$-modules are isomorphic as $\mathfrak{A}$-modules if and only if the splitting homomorphisms for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$ corresponding to them in Theorem* 2.8 *are* $U(\mathfrak{E}_1)$*-conjugate. Thus the correspondence of Theorem* 2.8 *induces a one-to-one correspondence between all $\mathfrak{A}$-isomorphism classes of extensions of $\mathfrak{V}$ to $\mathfrak{A}$-modules and all* $U(\mathfrak{E}_1)$*-conjugacy classes of splitting homomorphisms for* $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$.

Proof. Let $\mathfrak{V}^{\odot}$ and $\mathfrak{V}^{\odot'}$ be two extensions of $\mathfrak{V}$ to $\mathfrak{A}$-modules, and γ and γ' be their respective corresponding splitting homomorphisms for $X_{\mathfrak{A}}\langle\mathfrak{V}^{\mathfrak{A}}\rangle$. Any $\mathfrak{A}$-isomorphism of $\mathfrak{V}^{\odot'}$ onto $\mathfrak{V}^{\odot}$ is also an $\mathfrak{A}_1$-automorphism of $\mathfrak{V}$, i.e., a unit of $\mathrm{End}_{\mathfrak{A}_1}(\mathfrak{V})$. By (2.4b) the $\mathfrak{A}_1$-automorphisms of $\mathfrak{V}$ are precisely the restrictions of elements $u \in U(\mathfrak{E}_1)$. In view of (1.1a) the restriction of any such u is an $\mathfrak{A}$-isomorphism of $\mathfrak{V}^{\odot'}$ onto $\mathfrak{V}^{\odot}$ if and only if

$$u(v \odot' a_\sigma) = u(v) \odot a_\sigma, \tag{2.13}$$

for any $v \in \mathfrak{V}$, $\sigma \in G$ and $a_\sigma \in \mathfrak{A}_\sigma$. From (2.9) for $\odot'$ and $\odot$, and the fact that u is an $\mathfrak{A}$-endomorphism, we obtain

$$u(v \odot' a_\sigma) = [u\gamma'(\sigma^{-1})](v \otimes a_\sigma),$$
$$u(v) \odot a_\sigma = \gamma(\sigma^{-1})(u(v) \otimes a_\sigma) = [\gamma(\sigma^{-1})u](v \otimes a_\sigma).$$

Since the products $v \otimes a_\sigma$ generate $\mathfrak{V}^{\mathfrak{A}} = \mathfrak{V} \otimes \mathfrak{A}$ by (1.1a), we conclude that (2.13) is equivalent to

$$u\gamma'(\sigma^{-1}) = \gamma(\sigma^{-1})u, \qquad \textit{for all} \quad \sigma \in G,$$

which holds if and only if $\gamma' = \gamma^u$ by (2.11). So the theorem is proved.

If we apply the above considerations to the regular $\mathfrak{A}_1$-module $\mathfrak{A}_1$, we obtain:

COROLLARY 2.14. *The regular $\mathfrak{A}_1$-module $\mathfrak{A}_1$ can be extended to an $\mathfrak{A}$-module if and only if the sequence* $X\langle\mathfrak{A}\rangle$ *is both exact and split. Indeed, there is a one-to-one correspondence between all extensions $\mathfrak{A}_1^{\odot}$ of $\mathfrak{A}_1$ to $\mathfrak{A}$-modules and all splitting homomorphisms γ for* $X\langle\mathfrak{A}\rangle$, *in which $\mathfrak{A}_1^{\odot}$ corresponds to γ if and only if*

$$a_1 \odot a_\sigma = \gamma(\sigma^{-1})a_1 a_\sigma \in \mathfrak{A}_{\sigma^{-1}}\mathfrak{A}_1\mathfrak{A}_\sigma = \mathfrak{A}_1, \qquad \textit{for all } a_1 \in \mathfrak{A}_1, \; \sigma \in G \textit{ and } a_\sigma \in \mathfrak{A}_\sigma. \tag{2.15}$$

This correspondence induces a one-to-one correspondence between all $\mathfrak{A}$-isomorphism classes of such extensions $\mathfrak{A}_1^{\bigcirc}$ and all $\mathrm{U}(\mathfrak{A}_1)$-conjugacy classes of such homomorphisms γ.

Proof. In view of (1.2) there is a natural isomorphism of the induced $\mathfrak{A}$-module $\mathfrak{A}_1^{\mathfrak{A}} = \mathfrak{A}_1 \otimes_{\mathfrak{A}_1} \mathfrak{A}$ into the regular $\mathfrak{A}$-module $\mathfrak{A}$, sending $a_1 \otimes a$ into $a_1 a$ for all $a_1 \in \mathfrak{A}_1$ and $a \in \mathfrak{A}$. Evidently this isomorphism is *grade-preserving*, i.e., sends $\mathfrak{A}_1^{\sigma} = \mathfrak{A}_1 \otimes \mathfrak{A}_\sigma$ onto $\mathfrak{A}_\sigma = \mathfrak{A}_1 \mathfrak{A}_\sigma$ for all $\sigma \in G$. Since the $\mathfrak{A}$-endomorphisms of the regular $\mathfrak{A}$-module are just left multiplications by elements of $\mathfrak{A}$, this isomorphism induces an isomorphism of the ring $\mathfrak{A}$ onto $\mathrm{End}_{\mathfrak{A}}(\mathfrak{A}_1^{\mathfrak{A}})$, sending any $a \in \mathfrak{A}$ into the endomorphism: $1_{\mathfrak{A}} \otimes a' \mapsto 1_{\mathfrak{A}} \otimes (aa')$, for all $a' \in \mathfrak{A}$. From (1.1b) and (2.3a) it is clear that this latter isomorphism maps $\mathfrak{A}_\sigma$ into $\mathfrak{E}_\sigma$ for any $\sigma \in G$. This, (1.1b) and (2.3b) imply that it is a grade-preserving isomorphism of $\mathfrak{A}$ onto $\mathfrak{E} = \mathrm{End}_{\mathfrak{A}}(\mathfrak{A}_1^{\mathfrak{A}})$. The corollary now follows directly from Theorems 2.8 and 2.12 once we translate their statements about $\mathfrak{V}^{\mathfrak{A}} = \mathfrak{A}_1^{\mathfrak{A}}$ and $\mathfrak{E}$ to statements about $\mathfrak{A}$ via the above isomorphisms.

3. Simple $\mathfrak{A}_1$

Now we assume that the subring $\mathfrak{A}_1$ of our strongly G-graded ring $\mathfrak{A}$ satisfies

(3.1a) $\mathrm{Z}(\mathfrak{A}_1)$ *is a field* $\mathfrak{Z}$,

(3.1b) $\mathfrak{A}_1$ *is simple of finite dimension* $[\mathfrak{A}_1 : \mathfrak{Z}]$ *as an algebra over* $\mathfrak{Z}$.

The action (1.5) makes $\mathfrak{Z}$ a *G-field*, i.e., a field together with an action of the group G as field automorphisms. We denote by $\mathfrak{F}$ the fixed subfield

$$\mathfrak{F} = \mathrm{C}(G \text{ in } \mathfrak{Z}) = \mathfrak{A}_1 \cap \mathrm{Z}(\mathfrak{A}) \tag{3.2}$$

under this action (here the second equality comes directly from (1.5) and (1.1a)).

We shall be interested in subrings $\mathfrak{D}$ satisfying:

(3.3a) $\mathfrak{D}$ *is an* $\mathfrak{F}$*-subalgebra of* $\mathfrak{A}_1$ *containing* $1_{\mathfrak{A}}$,

(3.3b) $\mathfrak{D}$ *is central simple of finite dimension* $[\mathfrak{D} : \mathfrak{F}]$ *as an algebra over* $\mathfrak{F}$.

For any such $\mathfrak{D}$ we have:

Lemma 3.4. *The centralizer* $\mathfrak{C} = \mathrm{C}(\mathfrak{D}$ in $\mathfrak{A})$ *of* $\mathfrak{D}$ *is a strongly G-graded subring of* $\mathfrak{A}$ *containing* $1_{\mathfrak{A}}$ *and satisfying*:

(3.5a) $\mathfrak{C}_\sigma = \mathfrak{C} \cap \mathfrak{A}_\sigma = \mathrm{C}(\mathfrak{D}$ in $\mathfrak{A}_\sigma)$, *for all* $\sigma \in G$, (3.5b) $\mathfrak{C}_1$ *is the centralizer in* $\mathfrak{A}_1$ *of the central simple* $\mathfrak{Z}$*-subalgebra* $\mathfrak{Z}\mathfrak{D}$ *of dimension* $[\mathfrak{Z}\mathfrak{D} : \mathfrak{Z}] = [\mathfrak{D} : \mathfrak{F}]$, *and hence is itself a central simple* $\mathfrak{Z}$*-algebra of finite dimension* $[\mathfrak{C}_1 : \mathfrak{Z}] = [\mathfrak{A}_1 : \mathfrak{Z}]/[\mathfrak{D} : \mathfrak{F}]$,

(3.5c) $Z(\mathfrak{C}_1) = Z(\mathfrak{A}_1) = \mathfrak{Z}$ (*as G-fields*),

(3.5d) *The* $\mathfrak{F}$*-algebra* $\mathfrak{A}$ *is the Kronecker* (*internal tensor*) *product* $\mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}$ *over* $\mathfrak{F}$ *of its subalgebras* $\mathfrak{C}$ *and* $\mathfrak{D}$, *while* $\mathfrak{A}_1$ *is* $\mathfrak{C}_1 \times_{\mathfrak{F}} \mathfrak{D}$.

Proof. It follows from (3.2) that $\mathfrak{A}$ is an algebra over its central subfield $\mathfrak{F}$. By (1.1b) each $\mathfrak{A}_\sigma$, $\sigma \in G$, is an $\mathfrak{F}$-subspace of $\mathfrak{A}$. Thus we have a "graded Clifford system $\mathfrak{A}$, $\{\mathfrak{A}_\sigma \mid \sigma \in G\}$ over $\mathfrak{F}$" in the language of [2]. So Corollary 6.5 of [2] implies that $\mathfrak{C}$ is a strongly G-graded subring of $\mathfrak{A}$ with the σ-components (3.5a).

Evidently $\mathfrak{C}_1$ equals $C(\mathfrak{Z}\mathfrak{D}$ in $\mathfrak{A}_1)$, which contains $1_{\mathfrak{A}}$. It follows from (3.3) that $\mathfrak{Z}\mathfrak{D} = \mathfrak{Z} \times_{\mathfrak{F}} \mathfrak{D}$ is a central simple $\mathfrak{Z}$-subalgebra of $\mathfrak{A}_1$ with dimension $[\mathfrak{Z}\mathfrak{D} : \mathfrak{Z}] = [\mathfrak{D} : \mathfrak{F}]$ (see Theorem V.22 [i.e., Theorem 22 of Chap. V] in [9]). Hence $\mathfrak{C}_1$ satisfies (3.5b) by Theorem V.9 of [9]).

By (3.5b) the fields $Z(\mathfrak{C}_1)$ and $Z(\mathfrak{A}_1) = \mathfrak{Z}$ coincide. If $z \in \mathfrak{Z}$ and $\sigma \in G$, then (3.5a) implies that the element $z^\sigma \in \mathfrak{Z}$ defined by the action (1.5) for $\mathfrak{A}$ satisfies

$$zc_\sigma = c_\sigma z^\sigma, \qquad \textit{for all } c_\sigma \in \mathfrak{C}_\sigma \subseteq \mathfrak{A}_\sigma.$$

Hence this z^σ coincides with the z^σ defined by the action (1.5) for $\mathfrak{C}$, and (3.5c) holds.

Finally, (3.5d) follows from Lemma 6.2 of [2]. So the present lemma is proved.

With a judicious choice of $\mathfrak{D}$ we can use the above lemma to prove that $X\langle\mathfrak{A}\rangle$ is exact.

THEOREM 3.6. *When* $\mathfrak{A}$ *satisfies* (3.1) *the sequence* $X\langle\mathfrak{A}\rangle$ *of* (1.3) *is exact.*

Proof. By Wedderburn's Theorem the simple $\mathfrak{Z}$-algebra $\mathfrak{A}_1$ is a Kronecker product $\mathfrak{C}_1 \times_{\mathfrak{Z}} \mathfrak{B}$, where $\mathfrak{C}_1$ is some division subalgebra, and the subalgebra $\mathfrak{B}$ is isomorphic to the algebra $[\mathfrak{Z}]_b$ of all $b \times b$ matrices with entries in $\mathfrak{Z}$, for some $b > 0$. Evidently the $\mathfrak{F}$-subalgebra $\mathfrak{D}$ of $\mathfrak{B}$ corresponding to the subalgebra $[\mathfrak{F}]_b$ of $[\mathfrak{Z}]_b$ satisfies (3.3). Since $[\mathfrak{Z}]_b$ is the product $\mathfrak{Z}[\mathfrak{F}]_b$, the $\mathfrak{Z}$-subalgebra $\mathfrak{Z}\mathfrak{D}$ is $\mathfrak{B}$. So $C(\mathfrak{Z}\mathfrak{D}$ in $\mathfrak{A}_1)$ is the above division algebra $\mathfrak{C}_1$ by Theorem V.9 of [9], and the sequence $X\langle\mathfrak{C}\rangle$ associated with the strongly G-graded ring $\mathfrak{C}$ of Lemma 3.4 is exact by Proposition 6.1 of [3]. Thus for any $\sigma \in G$ there is some unit $u_\sigma \in U(\mathfrak{C}) \cap \mathfrak{C}_\sigma$. From (3.5a) it is clear that u_σ also lies in $U(\mathfrak{A}) \cap \mathfrak{A}_\sigma$. Therefore $X\langle\mathfrak{A}\rangle$ is exact by (1.3), and the theorem is proved.

There are several natural homomorphisms of exact sequences involving $\mathrm{X}\langle\mathfrak{A}\rangle$. The first is:

PROPOSITION 3.7. *In the situation of Lemma* 3.4 *the sequence* $\mathrm{X}\langle\mathfrak{C}\rangle$ *is also exact, and inclusion* $\mathfrak{C} \to^{\subset} \mathfrak{A}$ *induces a homomorphism* $\mathrm{X}\langle\mathfrak{C}\rangle \to^{\subset} \mathrm{X}\langle\mathfrak{A}\rangle$ *of short exact sequences which is identity on G. That is, we have a commutative diagram*:

$$(3.8)\qquad \begin{tikzcd} \mathrm{X}\langle\mathfrak{C}\rangle\colon 1 \arrow[r] \arrow[d, "\cap"] & \mathrm{U}(\mathfrak{C}_1) \arrow[r, "\subset"] \arrow[d, "\cap"] & \mathrm{GrU}(\mathfrak{C}) \arrow[r, "\deg"] \arrow[d, "\cap"] & G \arrow[r] \arrow[d, equal] & 1 \\ \mathrm{X}\langle\mathfrak{A}\rangle\colon 1 \arrow[r] & \mathrm{U}(\mathfrak{A}_1) \arrow[r, "\subset"] & \mathrm{GrU}(\mathfrak{A}) \arrow[r, "\deg"] & G \arrow[r] & 1. \end{tikzcd}$$

Proof. The strongly G-graded ring $\mathfrak{C}$ also satisfies (3.1) by (3.5b). Hence $\mathrm{X}\langle\mathfrak{C}\rangle$ is exact by Theorem 3.6. The rest of this proposition follows directly from (3.5a) and the definition of $\mathrm{X}\langle\mathfrak{A}\rangle$ in (1.3).

The second homomorphism involving $\mathrm{X}\langle\mathfrak{A}\rangle$ comes from the action of $\mathrm{GrU}(\mathfrak{A})$ on $\mathfrak{A}_1$ in (1.4). We use exponential notation for the group $\mathrm{Aut}(\mathfrak{A}_1)$ of all automorphisms of the ring $\mathfrak{A}_1$, so that any $\alpha \in \mathrm{Aut}(\mathfrak{A}_1)$ sends any $a_1 \in \mathfrak{A}_1$ into $a_1^\alpha \in \mathfrak{A}_1$. (Note that this contrasts with our usual use of functional notation for homomorphisms.) Then we have:

(3.9) *There is a homomorphism k of the group* $\mathrm{GrU}(\mathfrak{A})$ *into* $\mathrm{Aut}(\mathfrak{A}_1)$ *sending any* $u \in \mathrm{GrU}(\mathfrak{A})$ *into* $k(u)\colon a_1 \mapsto a_1^u = u^{-1}a_1u$, *for all* $a_1 \in \mathfrak{A}_1$.

The restriction of k is the usual epimorphism of $\mathrm{U}(\mathfrak{A}_1)$ onto the subgroup $\mathrm{Inn}(\mathfrak{A}_1)$ of all inner automorphisms of $\mathfrak{A}_1$. Since $\mathrm{X}\langle\mathfrak{A}\rangle$ is exact, this implies that k induces a homomorphism, which we shall also call $k = k_G$, of G into the outer automorphism group $\mathrm{Out}(\mathfrak{A}_1) = \mathrm{Aut}(\mathfrak{A}_1)/\mathrm{Inn}(\mathfrak{A}_1)$. Thus k induces the following homomorphism of short exact sequences:

$$(3.10)\qquad \begin{tikzcd} \mathrm{X}\langle\mathfrak{A}_1\rangle\colon 1 \arrow[r] \arrow[d, "k"] & \mathrm{U}(\mathfrak{A}_1) \arrow[r, "\subset"] \arrow[d, "k"] & \mathrm{GrU}(\mathfrak{A}) \arrow[r, "\deg"] \arrow[d, "k"] & G \arrow[r] \arrow[d, "k"] & 1 \\ \mathrm{A}\langle\mathfrak{A}_1\rangle\colon 1 \arrow[r] & \mathrm{Inn}(\mathfrak{A}_1) \arrow[r, "\subset"] & \mathrm{Aut}(\mathfrak{A}_1) \arrow[r, "\mathrm{out}"] & \mathrm{Out}(\mathfrak{A}_1) \arrow[r] & 1, \end{tikzcd}$$

where "out" is the natural epimorphism of $\mathrm{Aut}(\mathfrak{A}_1)$ onto $\mathrm{Out}(\mathfrak{A}_1)$.

A final homomorphism involving $\mathrm{X}\langle\mathfrak{A}\rangle$ comes from the *principal norm* $\mathrm{Nm}\colon \mathfrak{A}_1 \to \mathfrak{Z}$ of the central simple algebra $\mathfrak{A}_1$. We recall the definition of Nm from [9, p. 113]. Choose any extension field $\mathfrak{Y}$ of $\mathfrak{Z}$ so that the $\mathfrak{Y}$-algebra $\mathfrak{Y} \otimes_{\mathfrak{Z}} \mathfrak{A}_1$ is isomorphic to the full matrix algebra $[\mathfrak{Y}]_d$, where d is the *degree* $[\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$ of $\mathfrak{A}_1$. Then $\mathrm{Nm}(a_1)$, for any $a_1 \in \mathfrak{A}_1$, is the determinant of the matrix in $[\mathfrak{Y}]_d$ corresponding to $1_{\mathfrak{Y}} \otimes a_1 \in \mathfrak{Y} \otimes_{\mathfrak{Z}} \mathfrak{A}_1$. This norm is independent of the choices of $\mathfrak{Y}$ and the isomorphism of $\mathfrak{Y} \otimes_{\mathfrak{Z}} \mathfrak{A}_1$ onto $[\mathfrak{Y}]_d$. It clearly satisfies:

(3.11a) Nm *sends the group* $\mathrm{U}(\mathfrak{A}_1)$ *homomorphically into* $\mathrm{U}(\mathfrak{Z})$,

(3..11b) $\mathrm{Nm}(z) = z^d$, *for all* $z \in \mathfrak{Z}$, *where* $d = [\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$.

The corresponding homomorphism of exact sequences is given by:

PROPOSITION 3.12. *The homomorphism* Nm: $\mathrm{U}(\mathfrak{A}_1) \to \mathrm{U}(\mathfrak{Z})$ *of groups induces a homomorphism* Nm *of the extension* $\mathrm{X}\langle\mathfrak{A}\rangle$ *of* $\mathrm{U}(\mathfrak{A}_1)$ *by G into an extension* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *of* $\mathrm{U}(\mathfrak{Z})$ *by G with the action* (1.5) *of G on* $\mathrm{U}(\mathfrak{Z})$. *That is, there exists a commutative diagram of group homomorphisms with exact rows:*

$$
\text{(3.13)}\qquad
\begin{array}{ccccccccc}
\mathrm{X}\langle\mathfrak{A}\rangle: 1 & \longrightarrow & \mathrm{U}(\mathfrak{A}_1) & \xrightarrow{\subset} & \mathrm{GrU}(\mathfrak{A}) & \xrightarrow{\deg} & G & \longrightarrow & 1 \\
\downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \| & & \\
\mathrm{NX}\langle\mathfrak{A}\rangle: 1 & \longrightarrow & \mathrm{U}(\mathfrak{Z}) & \xrightarrow{\subset} & \mathrm{NGrU}(\mathfrak{A}) & \xrightarrow{\deg} & G & \longrightarrow & 1.
\end{array}
$$

The extension $\mathrm{NX}\langle\mathfrak{A}\rangle$ *and the homomorphism* Nm: $\mathrm{X}\langle\mathfrak{A}\rangle \to \mathrm{NX}\langle\mathfrak{A}\rangle$ *are determined to within isomorphism by these conditions.*

Proof. The unicity of the principal norm implies that it is invariant under any automorphism α of the ring $\mathfrak{A}_1$:

$$
\text{(3.14)}\qquad \mathrm{Nm}(a_1^\alpha) = \mathrm{Nm}(a_1)^\alpha \in \mathfrak{Z} = \mathfrak{Z}^\alpha, \qquad \textit{for all } a_1 \in \mathfrak{A}_1.
$$

Applying this to the automorphism $k(u)$ of (3.9) and using (1.6), we obtain

$$
\mathrm{Nm}(a_1^u) = \mathrm{Nm}(a_1)^u = \mathrm{Nm}(a_1)^{\deg(u)}, \qquad \textit{for all } a_1 \in \mathfrak{A}_1 \textit{ and } u \in \mathrm{GrU}(\mathfrak{A}).
$$

The proposition follows easily from this.

4. A FIELD REDUCTION

When the ring $\mathfrak{A}_1$ satisfies (3.1), a ring automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A}_1)$ is inner if and only if it centralizes $\mathfrak{Z}$ (see Theorem V.16 in [9]). Hence:

(4.1) *The natural action of the outer automorphism group* $\mathrm{Out}(\mathfrak{A}_1)$ *as automorphisms of the field* $\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$ *(in which any* $\omega \in \mathrm{Out}(\mathfrak{A}_1)$ *sends any* $z \in \mathfrak{Z}$ *into* $z^\omega = z^\alpha \in \mathfrak{Z}$ *for any* $\alpha \in \mathrm{Aut}(\mathfrak{A}_1)$ *having* ω *as its image in* $\mathrm{Out}(\mathfrak{A}_1)$*) is faithful.*

We shall fix K and $\mathfrak{E}$ satisfying

(4.2a) *K is a subgroup of* $\mathrm{Out}(\mathfrak{A}_1)$ *having finite order* $|K|$ *relatively prime to* $[\mathfrak{A}_1 : \mathfrak{Z}]$,

(4.2b) $\mathfrak{E}$ *is the fixed field* $C(K \text{ in } \mathfrak{Z})$ *of K under the above action of* $K \leqslant \mathrm{Out}(\mathfrak{A}_1)$ *on* $\mathfrak{Z}$.

Since the finite group K acts faithfully as automorphisms of the field $\mathfrak{Z}$ by (4.1), Galois theory tells us that:

(4.3a) *The field $\mathfrak{Z}$ is a finite normal separable extension of its subfield* $\mathfrak{E}$,

(4.3b) *The action* (4.1) *induces an isomorphism of the group K onto the Galois group* $\mathrm{Gal}(\mathfrak{Z}/\mathfrak{E})$ *of $\mathfrak{Z}$ over* $\mathfrak{E}$,

(4.3c) *The $\mathfrak{E}$-dimension* $[\mathfrak{Z} : \mathfrak{E}]$ *of $\mathfrak{Z}$ equals* $|K| = |\mathrm{Gal}(\mathfrak{Z}/\mathfrak{E})|$, *and hence is relatively prime to* $[\mathfrak{A}_1 : \mathfrak{Z}]$.

Our immediate goal is:

THEOREM 4.4. *If* (3.1) *and* (4.2) *hold, then there is a unique* $U(\mathfrak{A}_1)$-*conjugacy class of central simple $\mathfrak{E}$-subalgebras $\mathfrak{D}$ of $\mathfrak{A}_1$ such that $\mathfrak{A}_1$ is the Kronecker product* $\mathfrak{Z} \times_{\mathfrak{E}} \mathfrak{D}$ *of $\mathfrak{Z}$ and $\mathfrak{D}$ over* $\mathfrak{E}$.

The proof of this theorem involves several lemmas, and will occupy the rest of this section.

Let L be the inverse image of $K \leqslant \mathrm{Out}(\mathfrak{A}_1)$ under the natural epimorphism out: $\mathrm{Aut}(\mathfrak{A}_1) \to \mathrm{Out}(\mathfrak{A}_1)$, so that we have the exact sequence

$$1 \longrightarrow \mathrm{Inn}(\mathfrak{A}_1) \xrightarrow{\subset} L \xrightarrow{\text{out}} K \longrightarrow 1.$$

For each $\sigma \in K$ we choose an element $\lambda(\sigma) \in L$ having σ as its image $\mathrm{out}(\lambda(\sigma))$ in K. Then there is a unique function i: $K \times K \to \mathrm{Inn}(\mathfrak{A}_1)$ such that

$$(4.5) \qquad \lambda(\sigma)\,\lambda(\tau) = \lambda(\sigma\tau)\, i(\sigma, \tau), \qquad \textit{for all } \sigma, \tau \in K.$$

Associativity for the product $\lambda(\rho)\,\lambda(\sigma)\,\lambda(\tau)$ in L gives the usual identity

$$(4.6) \qquad i(\rho\sigma, \tau)\, i(\rho, \sigma)^{\lambda(\tau)} = i(\rho, \sigma\tau)\, i(\sigma, \tau), \qquad \textit{for all } \rho, \sigma, \tau \in K,$$

where $i(\rho, \sigma)^{\lambda(\tau)} = \lambda(\tau)^{-1} i(\rho, \sigma)\, \lambda(\tau)$ is defined by conjugation in L.

We must lift i back along the natural epimorphism k of $U(\mathfrak{A}_1)$ onto $\mathrm{Inn}(\mathfrak{A}_1)$ defined in (3.9).

LEMMA 4.7. *There exists a function* a: $K \times K \to U(\mathfrak{A}_1)$ *satisfying*:

$$(4.8a) \qquad k(a(\sigma, \tau)) = i(\sigma, \tau), \qquad \textit{for all} \quad \sigma, \tau \in K,$$

$$(4.8b) \qquad a(\rho\sigma, \tau)\, a(\rho, \sigma)^{\lambda(\tau)} = a(\rho, \sigma\tau)\, a(\sigma, \tau), \qquad \textit{for all} \quad \rho, \sigma, \tau \in K,$$

where $a(\rho,\sigma)^{\lambda(\tau)}$ *is defined by the action of* $\lambda(\tau) \in L \leqslant \mathrm{Aut}(\mathfrak{A}_1)$ *on* $a(\rho,\sigma) \in \mathfrak{A}_1$.

Proof. We use the techniques of [4] to reduce this to a question about a 3-cocycle of K in $\mathrm{U}(\mathfrak{Z})$.

Since k: $\mathrm{U}(\mathfrak{A}_1) \to \mathrm{Inn}(\mathfrak{A}_1)$ is an epimorphism, there is certainly some function a: $K \times K \to \mathrm{U}(\mathfrak{A}_1)$ satisfying (4.8a). It follows from (3.9) that k preserves the actions of $\mathrm{Aut}(\mathfrak{A}_1)$ on $\mathrm{U}(\mathfrak{A}_1)$ and $\mathrm{Inn}(\mathfrak{A}_1)$, so that

$$k(a(\rho,\sigma)^{\lambda(\tau)}) = k(a(\rho,\sigma))^{\lambda(\tau)} = i(\rho,\sigma)^{\lambda(\tau)},$$

for any ρ, σ, $\tau \in K$. We conclude from this and (4.6) that the homomorphism k sends the element

$$z(\rho,\sigma,\tau) = a(\sigma,\tau)^{-1}\, a(\rho,\sigma\tau)^{-1}\, a(\rho\sigma,\tau)\, a(\rho,\sigma)^{\lambda(\tau)} \tag{4.9}$$

into 1. Hence $z(\rho,\sigma,\tau)$ lies in the kernel $\mathrm{U}(\mathfrak{Z})$ of k: $\mathrm{U}(\mathfrak{A}_1) \to \mathrm{Inn}(\mathfrak{A}_1)$. A straightforward computation (see Lemma 7.1 in [4]) shows that z is a 3-cocycle of K in $\mathrm{U}(\mathfrak{Z})$, where $K \leqslant \mathrm{Out}(\mathfrak{A}_1)$ acts on $\mathrm{U}(\mathfrak{Z})$ by (4.1).

We apply the principal norm Nm of $\mathfrak{A}_1$ to both sides of Eq. (4.9). From (3.14) and (4.1) we obtain

$$\mathrm{Nm}(a(\rho,\sigma)^{\lambda(\tau)}) = \mathrm{Nm}(a(\rho,\sigma))^{\lambda(\tau)} = \mathrm{Nm}(a(\rho,\sigma))^{\tau}.$$

From (3.11b) we get

$$\mathrm{Nm}(z(\rho,\sigma,\tau)) = z(\rho,\sigma,\tau)^d \in \mathrm{U}(\mathfrak{Z}),$$

where d is $[\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$. Hence (3.11a) and (4.9) give

$$z(\rho,\sigma,\tau)^d = \mathrm{Nm}(a(\sigma,\tau))^{-1}\mathrm{Nm}(a(\rho,\sigma\tau))^{-1}\mathrm{Nm}(a(\rho\sigma,\tau))\, \mathrm{Nm}(a(\rho,\sigma))^{\tau}.$$

Thus z^d is the coboundary of the 2-cochain $\mathrm{Nm}(a(\sigma,\tau))$ of K in $\mathrm{U}(\mathfrak{Z})$. Since $d = [\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$ is relatively prime to $|K|$ by (4.2a), while the exponent of the cohomology group $\mathrm{H}^3(K, \mathrm{U}(\mathfrak{Z}))$ divides $|K|$ (see Satz I.16.19 in [7]), we conclude that z is a coboundary, i.e., there is some function c: $K \times K \to \mathrm{U}(\mathfrak{Z})$ such that

$$z(\rho,\sigma,\tau) = c(\sigma,\tau)^{-1}\, c(\rho,\sigma\tau)^{-1}\, c(\rho\sigma,\tau)\, c(\rho,\sigma)^{\tau},$$

for all $\rho, \sigma, \tau \in K$. Using the centrality of $\mathrm{U}(\mathfrak{Z})$ in $\mathrm{U}(\mathfrak{A}_1)$, we deduce that a can be replaced by $c^{-1}a$ to obtain a function satisfying all the conditions of the lemma (see Theorem 8.1 of [4]).

From now on we fix a function a: $K \times K \to \mathrm{U}(\mathfrak{A}_1)$ satisfying (4.8). In view of (4.5) and (4.8a) we have

$$a_1^{\lambda(\sigma)\lambda(\tau)} = a_1^{\lambda(\sigma\tau)i(\sigma,\tau)} = a_1^{\lambda(\sigma\tau)a(\sigma,\tau)},$$

for all $a_1 \in \mathfrak{A}_1$ and $\sigma, \tau \in K$. It follows from this and (4.8b) that we may form the *crossed product* $\mathfrak{B}$ of K over $\mathfrak{A}_1$ with respect to a and λ. As in Section 1 of [10] and Section 5 of [3], this is a strongly K-graded ring $\mathfrak{B}$ generated by $\mathfrak{A}_1$ and elements $b(\sigma)$, $\sigma \in K$, satisfying:

(4.10a) $\mathfrak{A}_1$ *is the subring* $\mathfrak{B}_1$ *of* $\mathfrak{B}$,

(4.10b) $b(\sigma) \in \mathfrak{B}_\sigma \cap \mathrm{U}(\mathfrak{B})$, *for all* $\sigma \in K$,

(4.10c) $\mathfrak{B}_\sigma = b(\sigma)\mathfrak{A}_1$ *is a free rank-one* $\mathfrak{A}_1$*-module generated by* $b(\sigma)$, *for any* $\sigma \in K$,

(4.10d) $a_1 b(\sigma) = b(\sigma) a_1^{\lambda(\sigma)}$, *for any* $a_1 \in \mathfrak{A}_1$ *and* $\sigma \in K$,

(4.10e) $b(\sigma)\, b(\tau) = b(\sigma\tau)\, a(\sigma, \tau)$, *for any* $\sigma, \tau \in K$.

The field $\mathfrak{Z}$ is $\mathrm{Z}(\mathfrak{B}_1)$ by (4.10a). In view of (4.10b, d) and (1.6), the action of K on $\mathfrak{Z} = \mathrm{Z}(\mathfrak{B}_1)$ defined by (1.5) for B coincides with the action of K on $\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$ defined in (4.1):

(4.11) $z^\sigma = b(\sigma)^{-1} z b(\sigma) = z^{\lambda(\sigma)}$, *for all* $z \in \mathfrak{Z}$ *and* $\sigma \in K$.

It follows that the fixed field $\mathfrak{C} = \mathrm{C}(K \text{ in } \mathfrak{Z})$ of (4.2b) is a central subfield containing the identity $1_{\mathfrak{B}} = 1_{\mathfrak{A}}$ of $\mathfrak{B}$. Hence $\mathfrak{B}$ is an $\mathfrak{C}$-algebra.

LEMMA 4.12. *The* $\mathfrak{Z}$*-algebra* $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}$ *obtained by extending the ground field* $\mathfrak{C}$ *of* $\mathfrak{B}$ *to* $\mathfrak{Z}$ *is isomorphic to the full matrix algebra* $[\mathfrak{A}_1]_{|K|}$ *of degree* $|K|$ *over* $\mathfrak{A}_1$.

Proof. For simplicity we shall denote the $\mathfrak{Z}$-algebra $[\mathfrak{A}_1]_{|K|}$ by $\mathfrak{M}$. Instead of the integers from 1 to $|K|$ we use the elements of K itself as row and column indices for the matrices in $\mathfrak{M}$. For any $\sigma, \tau \in K$ we denote by $E_{\sigma,\tau}$ the matrix in $\mathfrak{M}$ having $1_{\mathfrak{A}}$ as its σ,τ-entry and all other entries zero. Then $\mathfrak{M}$ is the direct sum

$$\mathfrak{M} = \dot{\sum_{\sigma,\tau \in K}} \mathfrak{A}_1 E_{\sigma,\tau}$$

as an $\mathfrak{A}_1$-module, and the $E_{\sigma,\tau}$, besides commuting with the elements of $\mathfrak{A}_1$, obey the usual rules

$$\begin{aligned} E_{\pi,\rho} E_{\sigma,\tau} &= E_{\pi,\tau}, \qquad && \textit{if} \quad \rho = \sigma, \\ &= 0, && \textit{if} \quad \rho \neq \sigma, \end{aligned}$$

for all $\pi, \rho, \sigma, \tau \in K$. It follows that $\mathfrak{M}$ can be made a strongly K-graded ring by setting

$$\mathfrak{M}_\sigma = \dot{\sum_{\tau \in K}} \mathfrak{A}_1 E_{\sigma\tau,\tau}, \qquad \textit{for all} \quad \sigma \in K.$$

For any $a_1 \in \mathfrak{A}_1$ and $\sigma, \tau \in K$ we have

$$b(\sigma\tau)^{-1}(b(\sigma)a_1)b(\tau) = b(\sigma\tau)^{-1}b(\sigma)\,b(\tau)a_1^{\lambda(\tau)} = a(\sigma,\tau)a_1^{\lambda(\tau)},$$

by (4.10d) and (4.10e). Since this last expression lies in $\mathfrak{A}_1$, and $\mathfrak{B}_\sigma$ is $b(\sigma)\mathfrak{A}_1$ by (4.10c), we conclude that the map ϕ_σ defined by

$$\phi_\sigma(b_\sigma) = \sum_{\tau\in K} (b(\sigma\tau)^{-1}b_\sigma b(\tau))E_{\sigma\tau,\tau}, \qquad \textit{for all} \quad b_\sigma \in \mathfrak{B}_\sigma,$$

is a homomorphism of the additive group $\mathfrak{B}_\sigma$ into $\mathfrak{M}_\sigma$ for any $\sigma \in K$. If $b_\sigma \in \mathfrak{B}_\sigma$ and $b'_\tau \in \mathfrak{B}_\tau$, for any $\sigma, \tau \in K$, then $b_\sigma b'_\tau$ lies in $\mathfrak{B}_\sigma\mathfrak{B}_\tau = \mathfrak{B}_{\sigma\tau}$. An easy computation shows that

$$\phi_\sigma(b_\sigma)\,\phi_\tau(b'_\tau) = \phi_{\sigma\tau}(b_\sigma b'_\tau) \in \mathfrak{M}_\sigma\mathfrak{M}_\tau = \mathfrak{M}_{\sigma\tau}.$$

Hence the ϕ_σ are the restrictions to the $\mathfrak{B}_\sigma$ of a unique grade-preserving homomorphism ϕ: $\mathfrak{B} \to \mathfrak{M}$ of strongly K-graded rings.

Because $\mathfrak{Z}$ is a subring of $\mathfrak{B}_1$, Eqs. (4.11) give

$$\phi(z) = \phi_1(z) = \sum_{\tau\in K} (b(\tau)^{-1}zb(\tau))E_{\tau,\tau} = \sum_{\tau\in K} z^\tau E_{\tau,\tau},$$

for all $z \in \mathfrak{Z}$. In particular, we have

$$\phi(e) = \sum_{\tau\in K} eE_{\tau,\tau} = e1_{\mathfrak{M}},$$

for any e in the fixed subfield $\mathfrak{E}$ of K. Hence ϕ is a homomorphism of the $\mathfrak{E}$-algebra $\mathfrak{B}$ into $\mathfrak{M}$, and therefore induces a homomorphism ψ of the $\mathfrak{Z}$-algebra $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{B}$ into the $\mathfrak{Z}$-algebra $\mathfrak{M}$.

Since $\mathfrak{Z}$ and $\mathfrak{A}_1$ are $\mathfrak{E}$-subspaces of $\mathfrak{B}$, we may identify the $\mathfrak{Z}$-algebras $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{Z}$ and $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{A}_1$ naturally with their images in $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{B}$. For any z', $z \in \mathfrak{Z}$, we have

$$\psi(z' \otimes z) = z'\phi(z) = \sum_{\tau\in K} (z'z^\tau)E_{\tau,\tau} \in \dot{\sum_{\tau\in K}} \mathfrak{Z}E_{\tau,\tau}.$$

Because the subring $\dot{\sum}_{\tau\in K} \mathfrak{Z}E_{\tau,\tau}$ of $\mathfrak{M}$ is the direct sum of $|K|$ copies $\mathfrak{Z}E_{\tau,\tau}$ of $\mathfrak{Z}$, Theorem V.4 of [9] and (4.3b) above imply that ψ sends $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{Z}$ isomorphically onto this subring.

As an $\mathfrak{E}$-algebra, $\mathfrak{A}_1$ is simple of finite dimension $[\mathfrak{A}_1 : \mathfrak{E}] = [\mathfrak{A}_1 : \mathfrak{Z}][\mathfrak{Z} : \mathfrak{E}] = [\mathfrak{A}_1 : \mathfrak{Z}]\,|K|$ by (3.1) and (4.3c). Its center $\mathfrak{Z}$ is separable over $\mathfrak{E}$ by (4.3a). So Theorem V.35 of [9] tells us that the $\mathfrak{Z}$-algebra $\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{A}_1$ is semi-simple of dimension $[\mathfrak{A}_1 : \mathfrak{Z}]\,|K|$. Because ψ sends $\mathrm{Z}(\mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{A}_1) = \mathfrak{Z} \otimes_{\mathfrak{E}} \mathfrak{Z}$ monomorphically into $\mathfrak{M}$, it sends the semi-simple algebra $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{A}_1$ monomorphically into $\mathfrak{M}$. But $\psi(\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{A}_1) = \mathfrak{Z}\phi(\mathfrak{A}_1) = \mathfrak{Z}\phi_1(\mathfrak{B}_1)$ is contained in

$$\mathfrak{Z}\mathfrak{M}_1 = \mathfrak{M}_1 = \sum_{\tau \in K}^{\cdot} \mathfrak{A}_1 E_{\tau,\tau},$$

which also has $\mathfrak{Z}$-dimension $[\mathfrak{A}_1 : \mathfrak{Z}]\,|K|$. Therefore ψ sends $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{A}_1$ isomorphically onto $\mathfrak{M}_1$.

Each $\mathfrak{B}_\sigma$, $\sigma \in K$, is an $\mathfrak{C}$-subspace of $\mathfrak{B}$ by (4.10c), since $\mathfrak{C}$ is a subring of $\mathfrak{A}_1$. So $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_\sigma$ may be identified naturally with its image in $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}$. Then the $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_\sigma$ evidently form the σ-components of a K-grading making $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}$ a strongly K-graded ring. The $\mathfrak{Z}$-algebra homomorphism ψ: $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B} \to \mathfrak{M}$ is grade-preserving

$$\psi(\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_\sigma) = \mathfrak{Z}\phi(\mathfrak{B}_\sigma) \subseteq \mathfrak{Z}\mathfrak{M}_\sigma = \mathfrak{M}_\sigma, \qquad \textit{for all} \quad \sigma \in K.$$

Since $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}$ and $\mathfrak{M}$ are both strongly K-graded rings, and the restriction of ψ is an isomorphism of $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_1 = \mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{A}_1$ onto $\mathfrak{M}_1$, Lemma 13.10 of [2] tells us that ψ is an isomorphism of $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}$ onto $\mathfrak{M}$. So the present lemma is proved.

Now it is easy to complete the proof of Theorem 4.4. The $\mathfrak{Z}$-algebra $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B} \simeq [\mathfrak{A}_1]_{|K|}$ is central simple of dimension $[\mathfrak{A}_1 : \mathfrak{Z}]\,|K|^2$ by (3.1). Hence the $\mathfrak{C}$-algebra $\mathfrak{B}$ must also be central simple of dimension $[\mathfrak{A}_1 : \mathfrak{Z}]\,|K|^2$. The two numbers $[\mathfrak{A}_1 : \mathfrak{Z}]$ and $|K|^2$ are relatively prime by (4.2a). So Theorem V.31 of [9] implies that $\mathfrak{B}$ is isomorphic to the tensor product $\mathfrak{B}_1 \otimes_{\mathfrak{C}} \mathfrak{B}_2$ of two central simple $\mathfrak{C}$-algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of dimensions $[\mathfrak{A}_1 : \mathfrak{Z}]$ and $|K|^2$, respectively. This and Lemma 4.12 give us isomorphisms of $\mathfrak{Z}$-algebras

$$(\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_1) \otimes_{\mathfrak{Z}} (\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_2) \simeq \mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B} \simeq [\mathfrak{A}_1]_{|K|} \simeq \mathfrak{A}_1 \otimes_{\mathfrak{Z}} [\mathfrak{Z}]_{|K|},$$

where $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_1$ and $\mathfrak{A}_1$ are central simple $\mathfrak{Z}$-algebras of dimension $[\mathfrak{A}_1 : \mathfrak{Z}]$, while $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_2$ and $[\mathfrak{Z}]_{|K|}$ are central simple $\mathfrak{Z}$-algebras of dimension $|K|^2$. Since $[\mathfrak{A}_1 : \mathfrak{Z}]$ and $|K|^2$ are relatively prime, the unicity part of Theorem V.31 in [9] implies that $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{B}_1$ and $\mathfrak{A}_1$ are isomorphic $\mathfrak{Z}$-algebras. Therefore $\mathfrak{A}_1$ has a central simple $\mathfrak{C}$-subalgebra $\mathfrak{D}$ isomorphic to $\mathfrak{B}_1$ such that $\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{C}} \mathfrak{D}$.

Suppose that $\mathfrak{D}'$ is also a central simple $\mathfrak{C}$-subalgebra of $\mathfrak{A}_1$ such that $\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{C}} \mathfrak{D}'$. Then $\mathfrak{D}$ and $\mathfrak{D}'$ have the same dimension $[\mathfrak{A}_1 : \mathfrak{Z}]$ over $\mathfrak{C}$, as does the opposite algebra $\mathfrak{D}^{\text{opp}}$ to $\mathfrak{D}$. Let $\mathfrak{D}_0$ be a central division algebra over $\mathfrak{C}$ such that $\mathfrak{D}^{\text{opp}} \otimes_{\mathfrak{C}} \mathfrak{D}'$ is isomorphic to a full matrix algebra over $\mathfrak{D}_0$. Then $[\mathfrak{D}_0 : \mathfrak{C}]$ divides

$$[\mathfrak{D}^{\text{opp}} \otimes_{\mathfrak{C}} \mathfrak{D}' : \mathfrak{C}] = [\mathfrak{D}^{\text{opp}} : \mathfrak{C}][\mathfrak{D}' : \mathfrak{C}] = [\mathfrak{A}_1 : \mathfrak{Z}]^2,$$

and hence is relatively prime to $[\mathfrak{Z} : \mathfrak{C}] = |K|$ by (4.3c). It follows that $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D}_0$ is a division algebra over $\mathfrak{Z}$ (any left or right ideal would be simultaneously a vector space over both $\mathfrak{Z}$ and $\mathfrak{D}_0$, and thus have $\mathfrak{C}$-dimension divisible by the least common multiple $[\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D}_0 : \mathfrak{C}] = [\mathfrak{Z} : \mathfrak{C}][\mathfrak{D}_0 : \mathfrak{C}]$ of their dimensions). But the Brauer equivalence $\mathfrak{D}_0 \sim \mathfrak{D}^{\mathrm{opp}} \otimes_{\mathfrak{C}} \mathfrak{D}'$ of $\mathfrak{C}$-algebras implies the Brauer equivalences and isomorphisms of $\mathfrak{Z}$-algebras

$$\begin{aligned}\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D}_0 \sim \mathfrak{Z} \otimes_{\mathfrak{C}} (\mathfrak{D}^{\mathrm{opp}} \otimes_{\mathfrak{C}} \mathfrak{D}') &\simeq (\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D})^{\mathrm{opp}} \otimes_{\mathfrak{Z}} (\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D}') \\ &\simeq \mathfrak{A}_1^{\mathrm{opp}} \otimes_{\mathfrak{Z}} \mathfrak{A}_1 \sim \mathfrak{Z}\end{aligned}$$

(see Sect. V.13 of [9]). Thus the division $\mathfrak{Z}$-algebras $\mathfrak{Z} \otimes_{\mathfrak{C}} \mathfrak{D}_0$ and $\mathfrak{Z}$ must be isomorphic. Therefore $\mathfrak{D}_0$ is one-dimensional over $\mathfrak{C}$, and hence is isomorphic to $\mathfrak{C}$.

We now have the Brauer equivalence of $\mathfrak{C}$-algebras $\mathfrak{D}^{\mathrm{opp}} \otimes_{\mathfrak{C}} \mathfrak{D}' \sim \mathfrak{C}$, which implies that $\mathfrak{D} \sim \mathfrak{D}'$. Since $\mathfrak{D}$ and $\mathfrak{D}'$ have the same $\mathfrak{C}$-dimension $[\mathfrak{A}_1 : \mathfrak{Z}]$, this forces them to be isomorphic $\mathfrak{C}$-algebras. Any isomorphism ϕ: $\mathfrak{D} \simeq \mathfrak{D}'$ extends to an isomorphism of the $\mathfrak{Z}$-algebra $\mathfrak{Z} \times_{\mathfrak{C}} \mathfrak{D}$ onto $\mathfrak{Z} \times_{\mathfrak{C}} \mathfrak{D}'$, i.e., to an automorphism of the $\mathfrak{Z}$-Algebra $\mathfrak{A}_1$. Any such automorphism is inner by (3.1) above and Theorem V.16 of [9]. So there is some $u \in \mathrm{U}(\mathfrak{A}_1)$ such that $\mathfrak{D}' = \phi(\mathfrak{D}) = \mathfrak{D}^u$. Obviously any $\mathrm{U}(\mathfrak{A}_1)$-conjugate of $\mathfrak{D}$ also satisfies the conditions of Theorem 4.4. Therefore $\mathfrak{D}$ is unique to within $\mathrm{U}(\mathfrak{A}_1)$-conjugacy, and the theorem is proved.

5. Extending Simple Modules

We now add to (3.1) the condition that the group G have finite order relatively prime to $[\mathfrak{A}_1 : \mathfrak{Z}]$. So our entire list of hypotheses is:

(5.1a) *G is a multiplicative group of finite order $|G|$,*

(5.1b) *$\mathfrak{A}$ is a strongly G-graded ring,*

(5.1c) *$\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$ is a field,*

(5.1d) *$\mathfrak{A}_1$ is simple as a $\mathfrak{Z}$-algebra, with finite dimension $[\mathfrak{A}_1 : \mathfrak{Z}]$ relatively prime to $|G|$.*

The action (1.5) of G on the field $\mathfrak{Z}$ is equivalent to a homomorphism of the group G into the automorphism group $\mathrm{Aut}(\mathfrak{Z})$ of $\mathfrak{Z}$, with the centralizer $\mathrm{C}(\mathfrak{Z}$ in $G)$ of $\mathfrak{Z}$ in G as kernel. Since G is a finite group, Galois theory tells us that:

(5.2a) *$\mathfrak{Z}$ is a finite normal separable extension of the fixed subfield $\mathfrak{F} = \mathrm{C}(G$ in $\mathfrak{Z})$ of G,* (5.2b) *The Galois group* $\mathrm{Gal}(\mathfrak{Z}/\mathfrak{F})$ *is isomorphic to the factor group* $G/\mathrm{C}(\mathfrak{Z} \text{ in } G)$,

(5.2c) *The* $\mathfrak{F}$*-dimension* $[\mathfrak{Z} : \mathfrak{F}] = |\mathrm{Gal}(\mathfrak{Z}/\mathfrak{F})|$ *is* $|G/\mathrm{C}(\mathfrak{Z} \text{ in } G)|$, *which divides* $|G|$ *and hence is relatively prime to* $[\mathfrak{A}_1 : \mathfrak{Z}]$.

We apply Theorem 4.4 to prove:

LEMMA 5.3. *There is a unique* $\mathrm{U}(\mathfrak{A}_1)$*-conjugacy class of central simple* $\mathfrak{F}$*-subalgebras* $\mathfrak{D}$ *of* $\mathfrak{A}_1$ *such that* $\mathfrak{A}_1$ *is the Kronecker product* $\mathfrak{Z} \times_{\mathfrak{F}} \mathfrak{D}$, *and hence* $[\mathfrak{D} : \mathfrak{F}]$ *is* $[\mathfrak{A}_1 : \mathfrak{Z}]$.

Proof. Since (3.10) is a commutative diagram with exact rows, it is clear from (1.6), (3.9) and (4.1) that its homomorphism $k\colon G \to \mathrm{Out}(\mathfrak{A}_1)$ satisfies

$$z^{k(\sigma)} = z^{\sigma}, \qquad \textit{for all} \quad \sigma \in G \textit{ and } z \in \mathfrak{Z},$$

where $z^{k(\sigma)}$ is defined by (4.1) and z^{σ} by (1.5). So the image $k(G)$ of G is a finite subgroup of $\mathrm{Out}(\mathfrak{A}_1)$ with the same fixed field in $\mathfrak{Z}$ as G:

$$\mathrm{C}(k(G) \text{ in } \mathfrak{Z}) = \mathrm{C}(G \text{ in } \mathfrak{Z}) = \mathfrak{F}.$$

The order $|k(G)|$ divides $|G|$, and hence is relatively prime to $[\mathfrak{A}_1 : \mathfrak{Z}]$ by (5.1d). Therefore (4.2) holds with $K = k(G)$ and $\mathfrak{E} = \mathfrak{F}$, and Theorem 4.4 implies the present lemma.

We fix one of the subalgebras $\mathfrak{D}$ in the preceding lemma.

LEMMA 5.4. *The centralizer* $\mathfrak{C} = \mathrm{C}(\mathfrak{D} \text{ in } \mathfrak{A})$ *is a strongly* G*-graded subring of* $\mathfrak{A}$, *with the* σ*-component* $\mathfrak{C}_{\sigma} = \mathrm{C}(\mathfrak{D} \text{ in } \mathfrak{A}_{\sigma})$ *for any* $\sigma \in G$. *The* 1_G*-component* $\mathfrak{C}_1$ *is precisely* $\mathfrak{Z}$. *The sequence* $\mathrm{X}\langle\mathfrak{C}\rangle$ *is exact, and hence is a group extension*

$$\mathrm{X}\langle\mathfrak{C}\rangle\colon 1 \longrightarrow \mathrm{U}(\mathfrak{Z}) \xrightarrow{\subset} \mathrm{GrU}(\mathfrak{C}) \xrightarrow{\deg} G \longrightarrow 1$$

of $\mathrm{U}(\mathfrak{Z})$ *by* G, *where* G *acts on* $\mathrm{U}(\mathfrak{Z})$ *by* (1.5) *for* $\mathfrak{A}$. *Finally, the* $\mathfrak{F}$*-algebra* $\mathfrak{A}$ *is the Kronecker product* $\mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}$ *of its subalgebras* $\mathfrak{C}$ *and* $\mathfrak{D}$.

Proof. Conditions (3.3) are satisfied by Lemma 5.3. So Lemma 3.4 gives the first statement of the present lemma. Since $\mathfrak{Z}\mathfrak{D}$ is $\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{F}} \mathfrak{D}$, it follows from (3.5b) that

$$\mathfrak{C}_1 = \mathrm{C}(\mathfrak{Z}\mathfrak{D} \text{ in } \mathfrak{A}_1) = \mathrm{Z}(\mathfrak{A}_1) = \mathfrak{Z},$$

which is the second statement of our lemma. This and Proposition 3.7 imply that $\mathrm{X}\langle\mathfrak{C}\rangle$ is a group extension of $\mathrm{U}(\mathfrak{Z}) = \mathrm{U}(\mathfrak{C}_1)$ by G. The action of G on $\mathrm{U}(\mathfrak{Z})$ in this extension agrees with that of (1.5) for $\mathfrak{A}$ by (3.5c) and (1.6) for

$\mathfrak{C}$. Thus the third statement of the lemma holds. The rest of the lemma comes from (3.5d).

We call the group extension $X\langle\mathfrak{C}\rangle$ of $U(\mathfrak{Z})$ by G a *Clifford extension* for the strongly G-graded ring $\mathfrak{A}$. Since $\mathfrak{D}$ is determined to within $U(\mathfrak{A}_1)$-conjugacy by $\mathfrak{A}$ (see Lemma 5.3), so is $X\langle\mathfrak{C}\rangle$.

LEMMA 5.5. *Any other choice of* $\mathfrak{D}$ *has the form* $\mathfrak{D}^u$, *for some* $u \in U(\mathfrak{A}_1)$. *In this case, conjugation by u is a grade-preserving isomorphism of the strongly G-graded subring* $\mathfrak{C}$ *of* $\mathfrak{A}$ *onto* $\mathfrak{C}^u = C(\mathfrak{D}^u \text{ in } \mathfrak{A})$. *This conjugation centralizes* $\mathfrak{Z} = \mathfrak{C}_1 = \mathfrak{C}_1^u$. *Hence it sends the group* $\operatorname{GrU}(\mathfrak{C})$ *isomorphically onto* $\operatorname{GrU}(\mathfrak{C}^u)$ *so that the diagram*

$$\begin{tikzcd}
X\langle\mathfrak{C}\rangle: 1 \arrow[r] \arrow[d, "(\cdot)^u"] & U(\mathfrak{Z}) \arrow[r, "\subset"] \arrow[d, equal] & \operatorname{GrU}(\mathfrak{C}) \arrow[r, "\deg"] \arrow[d, "(\cdot)^u"] & G \arrow[r] \arrow[d, equal] & 1 \\
X\langle\mathfrak{C}^u\rangle: 1 \arrow[r] & U(\mathfrak{Z}) \arrow[r, "\subset"] & \operatorname{GrU}(\mathfrak{C}^u) \arrow[r, "\deg"] & G \arrow[r] & 1
\end{tikzcd}$$

commutes. Thus it induces an equivalence $(\cdot)^u\colon X\langle\mathfrak{C}\rangle \simeq X\langle\mathfrak{C}^u\rangle$ *of Clifford extensions of* $U(\mathfrak{Z})$ *by G.*

Proof. The first statement comes from Lemma 5.3. Evidently conjugation by $u \in U(\mathfrak{A}_1)$ is an automorphism of the ring $\mathfrak{A}$ sending the subring $\mathfrak{C} = C(\mathfrak{D} \text{ in } \mathfrak{A})$ isomorphically onto $\mathfrak{C}^u = C(\mathfrak{D}^u \text{ in } \mathfrak{A})$. Since u lies in $\mathfrak{A}_1$, it follows easily from (1.1) that conjugation by it leaves invariant each $\mathfrak{A}_\sigma$, $\sigma \in G$. This and the definition of $\mathfrak{C}_\sigma$ in Lemma 5.4 imply that

$$(\mathfrak{C}_\sigma)^u = C(\mathfrak{D} \text{ in } \mathfrak{A}_\sigma)^u = C(\mathfrak{D}^u \text{ in } \mathfrak{A}_\sigma^u) = C(\mathfrak{D}^u \text{ in } \mathfrak{A}_\sigma) = (\mathfrak{C}^u)_\sigma,$$

for any $\sigma \in G$. Thus the second statement of the lemma holds. The rest of the lemma follows immediately from this since $\mathfrak{Z} = Z(\mathfrak{A}_1)$ is centralized by $u \in \mathfrak{A}_1$.

In view of (3.11b) the homomorphisms of exact sequences $X\langle\mathfrak{C}\rangle \to^{\subset} X\langle\mathfrak{A}\rangle$ and $X\langle\mathfrak{A}\rangle \to^{\mathrm{Nm}} NX\langle\mathfrak{A}\rangle$ of (3.8) and (3.13) can now be composed to yield a homomorphism of extensions of $U(\mathfrak{Z})$ by G,

(5.6)
$$\begin{tikzcd}
X\langle\mathfrak{C}\rangle: 1 \arrow[r] \arrow[d, "\mathrm{Nm}"] & U(\mathfrak{Z}) \arrow[r, "\subset"] \arrow[d, "(\cdot)^d"] & \operatorname{GrU}(\mathfrak{C}) \arrow[r, "\deg"] \arrow[d, "\mathrm{Nm}"] & G \arrow[r] \arrow[d, equal] & 1 \\
NX\langle\mathfrak{A}\rangle: 1 \arrow[r] & U(\mathfrak{Z}) \arrow[r, "\subset"] & \operatorname{NGrU}(\mathfrak{A}) \arrow[r, "\deg"] & G \arrow[r] & 1
\end{tikzcd}$$

lying over the endomorphism $(\cdot)^d$ of $U(\mathfrak{Z})$ sending each element z into its dth power z^d, where d is the integer $[\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$.

LEMMA 5.7. *The Clifford extension* $X\langle\mathfrak{C}\rangle$ *splits if and only if the* *extension* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *splits. Indeed, there is a one-to-one correspondence between all* $\mathrm{U}(\mathfrak{Z})$*-conjugacy classes of splitting homomorphisms* γ *for* $\mathrm{X}\langle\mathfrak{C}\rangle$ *and all* $\mathrm{U}(\mathfrak{Z})$*-conjugacy classes of splitting homomorphisms* δ *for* $\mathrm{NX}\langle\mathfrak{A}\rangle$*, in which the class of* γ *corresponds to that of* δ *if and only if* δ *is* $\mathrm{U}(\mathfrak{Z})$*-conjugate to the splitting homomorphism* $\mathrm{Nm}\gamma\colon \sigma \mapsto \mathrm{Nm}(\gamma(\sigma))$ *for* $\mathrm{NX}\langle\mathfrak{A}\rangle$.

Proof. The endomorphism $(\cdot)^d$ of $\mathrm{U}(\mathfrak{Z})$ induces the similar "dth power endomorphism" of the cohomology group $\mathrm{H}^n(G, \mathrm{U}(\mathfrak{Z}))$, for any integer n. Since $d = [\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$ is relatively prime to $|G|$ by (5.1d), while the exponent of $\mathrm{H}^n(G, \mathrm{U}(\mathfrak{Z}))$ divides $|G|$ by Satz I.16.19 of [7], the latter endomorphism is an automorphism of $\mathrm{H}^n(G, \mathrm{U}(\mathfrak{Z}))$. If $\alpha \in \mathrm{H}^2(G, \mathrm{U}(\mathfrak{Z}))$ corresponds to the Clifford extension $\mathrm{X}\langle\mathfrak{C}\rangle$, then the commutivity of (5.6) implies that $\alpha^d \in \mathrm{H}^2(G, \mathrm{U}(\mathfrak{Z}))$ corresponds to the extension $\mathrm{NX}\langle\mathfrak{A}\rangle$. Because $(\cdot)^d$ is an automorphism of $\mathrm{H}^2(G, \mathrm{U}(\mathfrak{Z}))$, the element α is trivial if and only if α^d is trivial. This is equivalent to the first statement of the lemma by Satz I.17.2b of [7].

Now suppose that one of the extensions $\mathrm{X}\langle\mathfrak{C}\rangle$ and $\mathrm{NX}\langle\mathfrak{A}\rangle$ splits. Then they both split, so that we may fix a splitting homomorphism γ_0 for $\mathrm{X}\langle\mathfrak{C}\rangle$. Since (5.6) commutes, the composition $\mathrm{Nm}\gamma_0$ is a splitting homomorphism for $\mathrm{NX}\langle\mathfrak{A}\rangle$. By Satz I.17.3 of [7] there is a one-to-one correspondence between all $\mathrm{U}(\mathfrak{Z})$-conjugacy classes of splitting homomorphisms γ for $\mathrm{X}\langle\mathfrak{C}\rangle$ and the elements β of $\mathrm{H}^1(G, \mathrm{U}(\mathfrak{Z}))$, in which the class of γ corresponds to β if and only if β is the image of the unique 1-cocycle c of G in $\mathrm{U}(\mathfrak{Z})$ such that

$$\gamma(\sigma) = c(\sigma)\,\gamma_0(\sigma), \qquad \textit{for all} \quad \sigma \in G.$$

Because (5.6) commutes and c has values in $\mathrm{U}(\mathfrak{Z})$, this equation implies that

$$\mathrm{Nm}\gamma(\sigma) = c(\sigma)^d\, \mathrm{Nm}\gamma_0(\sigma), \qquad \textit{for all} \quad \sigma \in G.$$

It follows that the class of $\mathrm{Nm}\gamma$ corresponds to β^d under the similar one-to-one correspondence, between all $\mathrm{U}(\mathfrak{Z})$-conjugacy classes of splitting homomorphisms δ for $\mathrm{NX}\langle\mathfrak{A}\rangle$ and the elements of $\mathrm{H}^1(G, \mathrm{U}(\mathfrak{Z}))$, determined by the fixed splitting homomorphism $\mathrm{Nm}\gamma_0$. This and the fact that $(\cdot)^d$ is an automorphism of $\mathrm{H}^1(G, \mathrm{U}(\mathfrak{Z}))$ imply the rest of the lemma.

Corollary 2.14 for the strongly G-graded ring $\mathfrak{C}$ tells us that any splitting homomorphism γ for $\mathrm{X}\langle\mathfrak{C}\rangle$ defines an extension $\mathfrak{Z}^{\odot}$ of the regular module $\mathfrak{Z}$ over $\mathfrak{C}_1 = \mathfrak{Z}$ to a $\mathfrak{C}$-module such that

(5.8)

$$z \odot c_\sigma = \gamma(\sigma^{-1})\, z c_\sigma \in \mathfrak{C}_{\sigma^{-1}} \mathfrak{C}_1 \mathfrak{C}_\sigma = \mathfrak{C}_1, \qquad \textit{for all} \quad z \in \mathfrak{Z}, \sigma \in G \textit{ and } c_\sigma \in \mathfrak{C}_\sigma.$$

If $\mathfrak{W}$ is any $\mathfrak{D}$-module, then Lemmas 5.4 and 5.3 imply that $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{z}} \mathfrak{W}$ is naturally a module over $\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{z}} \mathfrak{D}$ having $\mathfrak{Z} \otimes_{\mathfrak{z}} \mathfrak{W}$ as its restriction to a module over $\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{z}} \mathfrak{D}$.

LEMMA 5.9. *The $\mathfrak{A}_1$-module $\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$ is simple if $\mathfrak{W}$ is an irreducible $\mathfrak{D}$-module.*

Proof. Any $\mathfrak{A}_1$-submodule $\mathfrak{U}$ of $\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$ is simultaneously a $\mathfrak{Z}$-submodule and a $\mathfrak{D}$-submodule. Since $\mathfrak{Z}$ and $\mathfrak{D}$ are both simple $\mathfrak{F}$-algebras, this forces the $\mathfrak{F}$-dimension $[\mathfrak{U} : \mathfrak{F}]$ to be divisible by both the dimension $[\mathfrak{Z} : \mathfrak{F}]$ of the irreducible module $\mathfrak{Z}$ over the field $\mathfrak{Z}$ and by the dimension $[\mathfrak{W} : \mathfrak{F}]$ of the irreducible module $\mathfrak{W}$ over $\mathfrak{D}$. But $[\mathfrak{W} : \mathfrak{F}]$ divides $[\mathfrak{D} : \mathfrak{F}]$, which equals $[\mathfrak{A}_1 : \mathfrak{Z}]$ by Lemma 5.3, and hence is relatively prime to $[\mathfrak{Z} : \mathfrak{F}]$ by (5.2c). Therefore $[\mathfrak{U} : \mathfrak{F}]$ is divisible by the least common multiple $[\mathfrak{Z} : \mathfrak{F}][\mathfrak{W} : \mathfrak{F}] = [\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W} : \mathfrak{F}]$ of $[\mathfrak{Z} : \mathfrak{F}]$ and $[\mathfrak{W} : \mathfrak{F}]$. Hence $\mathfrak{U}$ is either 0 or $\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$, and the lemma is proved.

We now have all the necessary pieces for the proof of our main result:

THEOREM 5.10. *If* (5.1) *holds, then an irreducible $\mathfrak{A}_1$-module $\mathfrak{V}$ can be extended to an $\mathfrak{A}$-module $\mathfrak{V}^{\odot}$ if and only if the exact sequence* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *of Proposition* 3.12 *splits. Indeed, if δ is any splitting homomorphism for* $\mathrm{NX}\langle\mathfrak{A}\rangle$, *and if $\mathfrak{D}$ is chosen in Lemma* 5.3 *and $\mathfrak{C}$ defined by Lemma* 5.4, *then Lemma* 5.7 *gives us a splitting homomorphism γ for the Clifford extension* $\mathrm{X}\langle\mathfrak{C}\rangle$ *such that the splitting homomorphism* $\mathrm{Nm}\gamma$ *for* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *is* $\mathrm{U}(\mathfrak{Z})$-*conjugate to δ, while Corollary* 2.14 *for the strongly G-graded ring $\mathfrak{C}$ gives us a unique extension $\mathfrak{Z}^{\odot}$ of the regular module over $\mathfrak{Z} = \mathfrak{C}_1$ to a $\mathfrak{C}$-module satisfying* (5.8), *and Lemma* 5.9 *implies that the module $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}$ over $\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}$ is an extension of the irreducible $\mathfrak{A}_1$-module $\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$ for any simple $\mathfrak{D}$-module $\mathfrak{W}$. The isomorphism class of the resulting $\mathfrak{A}$-module $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}$ depends only on the* $\mathrm{U}(\mathfrak{Z})$-*conjugacy class of δ, and not on the choices of $\mathfrak{D}$, γ and $\mathfrak{W}$ made in its definition.*

Proof. The regular module $\mathfrak{A}_1$ over the finite-dimensional simple $\mathfrak{Z}$-algebra $\mathfrak{A}_1$ is a direct sum of copies of the irreducible $\mathfrak{A}_1$-module $\mathfrak{V}$. Hence $\mathfrak{A}_1$ extends to an $\mathfrak{A}$-module if $\mathfrak{V}$ so extends. In that case the sequence $\mathrm{X}\langle\mathfrak{A}\rangle$ splits by Corollary 2.14. Because diagram (3.13) commutes, this implies that $\mathrm{NX}\langle\mathfrak{A}\rangle$ splits.

Conversely, any splitting homomorphism δ for $\mathrm{NX}\langle\mathfrak{A}\rangle$ leads via the construction in the second statement of the theorem to an extension $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}$ of an irreducible $\mathfrak{A}_1$-module $\mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$ to an $\mathfrak{A}$-module. Since the finite-dimensional simple $\mathfrak{Z}$-algebra $\mathfrak{A}_1$ has just one isomorphism class of irreducible modules, the extension $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}$ is isomorphic to an extension of $\mathfrak{V} \simeq \mathfrak{Z} \otimes_{\mathfrak{F}} \mathfrak{W}$ to an $\mathfrak{A}$-module. Thus the first two statements of the theorem hold.

Any other choice of the irreducible module $\mathfrak{W}$ over the finite-dimensional simple $\mathfrak{F}$-algebra $\mathfrak{D}$ must be an isomorphic module $\mathfrak{W}'$. Evidently the module $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}'$ over $\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}$ is then isomorphic to $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{F}} \mathfrak{W}$. So the isomorphism class of the latter module does not depend on the choice of $\mathfrak{W}$.

Any other choice of the splitting homomorphism γ must be a conjugate γ^u by some $u \in \mathrm{U}(\mathfrak{Z})$ in view of Lemma 5.7. Corollary 2.14 for $\mathfrak{C}$ tells us that the corresponding extension $\mathfrak{Z}^{\odot^u}$ of $\mathfrak{Z}$ to a $\mathfrak{C}$-module is isomorphic to $\mathfrak{Z}^{\odot}$. Hence the modules $\mathfrak{Z}^{\odot^u} \otimes_{\mathfrak{Z}} \mathfrak{W}$ and $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{Z}} \mathfrak{W}$ over $\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{Z}} \mathfrak{D}$ are isomorphic. So the isomorphism class of the latter $\mathfrak{A}$-module does not depend on the choice of γ.

Any other choice of $\mathfrak{D}$ is a conjugate $\mathfrak{D}^u$ by some $u \in \mathrm{U}(\mathfrak{A}_1)$ in view of Lemma 5.3. It is evident from Lemma 5.5 that the conjugate γ^u is a splitting homomorphism for the Clifford extension $\mathrm{X}\langle \mathfrak{C}^u \rangle$ associated with $\mathfrak{D}^u$. The homomorphism Nm: $\mathrm{GrU}(\mathfrak{A}) \to \mathrm{NGrU}(\mathfrak{A})$ of (3.13) carries γ^u into the splitting homomorphism $\mathrm{Nm}(\gamma^u)$ for $\mathrm{NX}\langle \mathfrak{A} \rangle$ given by

$$[\mathrm{Nm}(\gamma^u)](\sigma) = \mathrm{Nm}(u^{-1}\gamma(\sigma)u) = \mathrm{Nm}(u)^{-1}\mathrm{Nm}\gamma(\sigma)\,\mathrm{Nm}(u), \qquad \textit{for all } \sigma \in G.$$

So $\mathrm{Nm}(\gamma^u)$ is the conjugate of $\mathrm{Nm}\gamma$ by the element $\mathrm{Nm}(u)$ of $\mathrm{Nm}(\mathrm{U}(\mathfrak{A}_1)) \subseteq \mathrm{U}(\mathfrak{Z})$. Since $\mathrm{Nm}\gamma$ is $\mathrm{U}(\mathfrak{Z})$-conjugate to δ, so is $\mathrm{Nm}(\gamma^u)$. Thus $\mathfrak{D}^u$, $\mathfrak{C}^u$ and γ^u may be used in place of $\mathfrak{D}$, $\mathfrak{C}$ and γ in the construction of the theorem.

From the simple $\mathfrak{D}$-module $\mathfrak{W}$ we construct the *conjugate* simple $\mathfrak{D}^u$-module $\mathfrak{W}^u$ whose elements are the formal expressions w^u for $w \in \mathfrak{W}$, and whose operations are determined by:

(5.11a) *The map* $w \mapsto w^u$ *is an isomorphism of the additive group of* $\mathfrak{W}$ *onto that of* $\mathfrak{W}^u$,

(5.11b) $w^u d^u = (wd)^u$, *for all* $w \in \mathfrak{W}$ *and* $d \in \mathfrak{D}$.

Let $\mathfrak{Z}^{\odot^u}$ be the extension of $\mathfrak{Z}$ to a $\mathfrak{C}^u$-module corresponding to γ^u in Corollary 2.14 for $\mathfrak{C}^u$. Then (5.8) and its equivalent for $\odot^u$ give

$$z \odot^u c_\sigma^u = \gamma^u(\sigma^{-1}) z^u c_\sigma^u = [\gamma(\sigma^{-1}) z c_\sigma]^u = [z \odot c_\sigma]^u = z \odot c_\sigma,$$

for all $z \in \mathfrak{Z}$, $\sigma \in G$ and $c_\sigma \in \mathfrak{C}_\sigma$, since $u \in \mathrm{U}(\mathfrak{A}_1)$ centralizes both elements z and $z \odot c_\sigma$ of $\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$. It follows that $\mathfrak{Z}^{\odot^u}$ is isomorphic to the conjugate $\mathfrak{C}^u$-module $(\mathfrak{Z}^{\odot})^u$. Evidently the module $(\mathfrak{Z}^{\odot})^u \otimes_{\mathfrak{Z}} \mathfrak{W}^u$ over $\mathfrak{A} = \mathfrak{A}^u = \mathfrak{C}^u \times_{\mathfrak{Z}} \mathfrak{D}^u$ is isomorphic to the conjugate module $(\mathfrak{Z}^{\odot} \otimes_{\mathfrak{Z}} \mathfrak{W})^u$. It follows easily from (5.11) that the map $y^u \mapsto yu$, which is defined since $u \in \mathrm{U}(\mathfrak{A}_1)$ is a unit of $\mathfrak{A}$, sends the $\mathfrak{A}$-module $(\mathfrak{Z}^{\odot} \otimes_{\mathfrak{Z}} \mathfrak{W})^u$ isomorphically onto $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{Z}} \mathfrak{W}$. Therefore the $\mathfrak{A}$-modules $\mathfrak{Z}^{\odot^u} \otimes_{\mathfrak{Z}} \mathfrak{W}^u$ and $\mathfrak{Z}^{\odot} \otimes_{\mathfrak{Z}} \mathfrak{W}$ are isomorphic. So the isomorphism class of the latter $\mathfrak{A}$-module does not depend on the choice of $\mathfrak{D}$ either, and the theorem is proved.

6. Group Representations

We start with an epimorphism π of a multiplicative group H onto our earlier group G. So we have an exact sequence of group homomorphisms

$$(6.1) \qquad X\colon 1 \longrightarrow N \xrightarrow{\subset} H \xrightarrow{\pi} G \longrightarrow 1,$$

where N is the kernel of π. The group algebra $\mathfrak{R}H$ of H over any commutative ring $\mathfrak{R}$ is then a strongly G-graded ring with the σ-components

$$(6.2) \qquad (\mathfrak{R}H)_\sigma = \sum_{\rho \in H, \pi(\rho)=\sigma} \mathfrak{R}\rho, \qquad \textit{for all} \quad \sigma \in G.$$

Notice that the 1_G-component of $\mathfrak{R}H$ is just the group algebra of the normal subgroup N:

$$(6.3) \qquad (\mathfrak{R}H)_1 = \mathfrak{R}N \qquad (\textit{as } \mathfrak{R}\textit{-subalgebras of } \mathfrak{R}H).$$

Hence any $\mathfrak{R}N$-module $\mathfrak{V}$ determines the usual induced $\mathfrak{R}H$-module $\mathfrak{V}^H = \mathfrak{V}^{\mathfrak{R}H} = \mathfrak{V} \otimes_{\mathfrak{R}N} \mathfrak{R}H$ and conjugate $\mathfrak{R}N$-modules $\mathfrak{V}^\sigma = \mathfrak{V} \otimes_{\mathfrak{R}N} (\mathfrak{R}H)_\sigma$, for $\sigma \in G$, as in (2.1).

We now fix $\mathfrak{V}$ satisfying:

(6.4) *$\mathfrak{V}$ is a G-invariant simple $\mathfrak{R}N$-module which is finitely generated as an $\mathfrak{R}$-module.*

The factor ring of $\mathfrak{R}H$ modulo the annihilator $\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H)$ of $\mathfrak{V}^H$ in $\mathfrak{R}H$ will be our earlier strongly G-graded ring $\mathfrak{A}$.

LEMMA 6.5. *If* (6.4) *holds, then* $\mathfrak{A} = \mathrm{A}(\mathfrak{V}) = \mathfrak{R}H/\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H)$ *is a strongly G-graded ring such that the natural epimorphism of $\mathfrak{R}H$ onto $\mathfrak{A}$ is grade-preserving. This epimorphism induces an isomorphism of the factor ring* $\mathfrak{R}N/\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R}N)$ *onto $\mathfrak{A}_1$. Hence $\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$ is a field, $\mathfrak{A}_1$ is a simple algebra of finite dimension $[\mathfrak{A}_1 : \mathfrak{Z}]$ over $\mathfrak{Z}$, and $\mathfrak{V}$ is naturally a simple $\mathfrak{A}_1$-module.*

Proof. From (1.1b) we obtain

$$\mathfrak{V}^\sigma(\mathfrak{R}H)_\tau = \mathfrak{V}^{\sigma\tau}, \qquad \textit{for all} \quad \sigma, \tau \in G.$$

This, (2.1) and (1.1a) imply that $\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H)$ is a graded two-sided ideal of $\mathfrak{R}H$ with the G-grading

$$\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H) = \dot{\sum_{\sigma \in G}} \mathrm{ann}(\mathfrak{V}^H \text{ in } (\mathfrak{R}H)_\sigma).$$

The first statement of the lemma follows directly from this.

In view of (6.3) the natural epimorphism of $\mathfrak{R}H$ onto $\mathfrak{A}$ induces an isomorphism of $\mathfrak{R}H/\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}N)$ onto $\mathfrak{A}_1$. By (2.1) we have

$$\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}N) = \mathrm{ann}\left(\dot{\sum_{\sigma \in G}} \mathfrak{V}^\sigma \text{ in } \mathfrak{R}N\right) = \bigcap_{\sigma \in G} \mathrm{ann}(\mathfrak{V}^\sigma \text{ in } \mathfrak{R}N).$$

But $\mathrm{ann}(\mathfrak{V}^\sigma \text{ in } \mathfrak{R}N)$ is $\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R}N)$ for any $\sigma \in G$, since $\mathfrak{V} \simeq \mathfrak{V}^\sigma$ is G-invariant. Thus $\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}N)$ is $\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R}N)$, and the second statement of the lemma holds.

It follows from (6.4) that $\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R})$ is a maximal ideal of the commutative ring $\mathfrak{R}$. Hence $\mathfrak{A}_1 \simeq \mathfrak{R}N/\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R}N)$ is an algebra over the field $\mathfrak{K} = \mathfrak{R}/\mathrm{ann}(\mathfrak{V} \text{ in } \mathfrak{R})$, and $\mathfrak{V}$ is a faithful simple $\mathfrak{A}_1$-module of finite $\mathfrak{K}$-dimension. The rest of the lemma is an immediate consequence of this.

Conditions (3.1) are now satisfied by the strongly G-graded ring $\mathfrak{A}$, so that we have the exact sequence $\mathrm{NX}\langle\mathfrak{A}\rangle$ of Proposition 3.12. The degree $d = [\mathfrak{A}_1 : \mathfrak{Z}]^{1/2}$ of $\mathfrak{A}_1$ is evidently the dimension $[\mathfrak{V}' : \mathfrak{Z}']$ of any $\mathfrak{Z}' \otimes_{\mathfrak{Z}} \mathfrak{A}_1$-composition factor $\mathfrak{V}'$ of $\mathfrak{Z}' \otimes_{\mathfrak{Z}} \mathfrak{V}$, where $\mathfrak{Z}'$ is any extension of $\mathfrak{Z}$ to a splitting field for $\mathfrak{A}_1$. So we may call d the *absolute degree* of the irreducible $\mathfrak{R}N$-module $\mathfrak{V}$.

THEOREM 6.6. *If, in the above situation, G has finite order relatively prime to the absolute degree d of $\mathfrak{V}$, then $\mathfrak{V}$ can be extended to an $\mathfrak{R}H$-module if and only if the exact sequence* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *splits. In that case, any* $\mathrm{U}(\mathfrak{Z})$*-conjugacy class of splitting homomorphisms δ for* $\mathrm{NX}\langle\mathfrak{A}\rangle$ *determines a unique isomorphism class of extensions of $\mathfrak{V}$ to $\mathfrak{A}$-modules, and hence to $\mathfrak{R}H$-modules, by the construction of Theorem* 5.10.

Proof. It follows from the formula (2.9) of Theorem 2.8 that any extension of $\mathfrak{V}$ to an $\mathfrak{R}H$-module is annihilated by $\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H)$, and hence is an extension of $\mathfrak{V}$ to a module over $\mathfrak{A} = \mathfrak{R}H/\mathrm{ann}(\mathfrak{V}^H \text{ in } \mathfrak{R}H)$. Since any extension of $\mathfrak{V}$ to an $\mathfrak{A}$-module is automatically an $\mathfrak{R}H$-module, this and Theorem 5.10 imply the present theorem.

It follows from (1.3) and (6.2) that inclusion induces a homomorphism of the extension X of N by G into the extension $\mathrm{X}\langle\mathfrak{R}H\rangle$ of $\mathrm{U}(\mathfrak{R}N)$ by G. The natural grade-preserving epimorphism of $\mathfrak{R}H$ onto $\mathfrak{A}$ induces a homomorphism of the extension $\mathrm{X}\langle\mathfrak{R}H\rangle$ into the extension $\mathrm{X}\langle\mathfrak{A}\rangle$ of $\mathrm{U}(\mathfrak{A}_1)$ by G. Composing these homomorphisms with the homomorphism Nm: $\mathrm{X}\langle\mathfrak{A}\rangle \to \mathrm{NX}\langle\mathfrak{A}\rangle$ of (3.13), we obtain a homomorphism $\mathrm{Nm}'\colon X \to \mathrm{NX}\langle\mathfrak{A}\rangle$, i.e., a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccccc}
X & :1 & \longrightarrow & N & \xrightarrow{\subset} & H & \xrightarrow{\pi} & G & \longrightarrow 1\\
\downarrow{\scriptstyle \mathrm{Nm}'} & & & \downarrow{\scriptstyle \mathrm{Nm}'} & & \downarrow{\scriptstyle \mathrm{Nm}'} & & \| & \\
\mathrm{NX}\langle\mathfrak{A}\rangle & :1 & \longrightarrow & \mathrm{U}(\mathfrak{Z}) & \xrightarrow{\subset} & \mathrm{NGrU}(\mathfrak{A}) & \xrightarrow{\deg} & G & \longrightarrow 1.
\end{array}$$

The kernel L of $\mathrm{Nm}'\colon H \to \mathrm{NGrU}(\mathfrak{A})$ is a normal subgroup of H contained in N, and $\mathrm{Nm}'\colon X \to \mathrm{NX}\langle\mathfrak{A}\rangle$ induces a monomorphism

$$(6.7)\qquad \begin{array}{ccccccccc} X/L & :1 & \longrightarrow & N/L & \xrightarrow{\subset} & H/L & \xrightarrow{\pi'} & G & \longrightarrow 1 \\ \big\downarrow{\scriptstyle \mathrm{Nm}''} & & & \big\downarrow{\scriptstyle \mathrm{Nm}''} & & \big\downarrow{\scriptstyle \mathrm{Nm}''} & & \Vert & \\ \mathrm{NX}\langle\mathfrak{A}\rangle & :1 & \longrightarrow & \mathrm{U}(\mathfrak{Z}) & \xrightarrow{\subset} & \mathrm{NGrU}(\mathfrak{A}) & \xrightarrow{\deg} & G & \longrightarrow 1 \end{array}$$

of the factor sequence X/L into $\mathrm{NX}\langle\mathfrak{A}\rangle$.

COROLLARY 6.8. *If the sequence X/L splits in the situation of Theorem 6.6, then $\mathfrak{B}$ can be extended to an $\mathfrak{R}H$-module. Indeed, any N/L-conjugacy class of splitting homomorphisms γ for X/L determines a unique isomorphism class of extensions of $\mathfrak{B}$ to $\mathfrak{R}H$-modules corresponding to the splitting homomorphisms $\mathrm{Nm}''\gamma$ for $\mathrm{NX}\langle\mathfrak{A}\rangle$ in Theorem 6.6.*

Proof. This is an immediate consequence of Theorem 6.6 and the commutativity of (6.7).

A few special cases of this corollary are worth noting.

COROLLARY 6.9. *If the homomorphism $\mathrm{Nm}'\colon N \to \mathrm{U}(\mathfrak{Z})$ is trivial (e.g., if N is perfect), then N/L is 1 and $(\pi')^{-1}$ is the only splitting homomorphism for X/L. So a unique isomorphism class of extensions of $\mathfrak{B}$ to $\mathfrak{R}H$-modules is determined in this case.*

COROLLARY 6.10. *If N/L has finite order relatively prime to $|G|$ (e.g., if $|N|$ is finite and relatively prime to $|G|$), then there is a unique N/L-conjugacy class of splitting homomorphisms γ for X/L. So a unique isomorphism class of extensions of $\mathfrak{B}$ to $\mathfrak{R}H$-modules is determined in this case.*

Proofs. Corollary 6.9 is obvious, and Corollary 6.10 follows from Satz I.17.5 in [7].

Remarks. We can make the regular $\mathfrak{Z}$-module $\mathfrak{Z}$ into a module $\mathrm{Nm}(\mathfrak{B})$ over the group algebra $\mathfrak{Z}N$ by letting any $\rho \in N$ act on $\mathrm{Nm}(\mathfrak{B}) = \mathfrak{Z}$ as multiplication by $\mathrm{Nm}'(\rho) \in \mathrm{U}(\mathfrak{Z})$ in the ring $\mathfrak{Z}$. It is possible to rephrase Theorem 6.6 in terms of extensions of the $\mathfrak{Z}N$-module $\mathrm{Nm}(\mathfrak{B})$, not to $\mathfrak{Z}H$-modules, but to modules over the *skew group algebra* $\mathfrak{Z}^sH$, the $\mathfrak{Z}$-vector space having the elements of H as a basis, in which the product of elements of H agrees with their product in the group H, and we have

$$z\rho = \rho z^{\pi(\rho)}, \qquad \textit{for all} \quad z \in \mathfrak{Z} \textit{ and } \rho \in H,$$

where $\pi(\rho) \in G$ acts on $\mathfrak{Z} = \mathrm{Z}(\mathfrak{A}_1)$ by (1.5) for $\mathfrak{A}$. As in (6.2) and (6.3), the ring $\mathfrak{Z}^sH$ is strongly G-graded with 1_G-component $\mathfrak{Z}^sN = \mathfrak{Z}N$. So we may speak about extensions of the $\mathfrak{Z}N$-module $\mathrm{Nm}(\mathfrak{B})$ to $\mathfrak{Z}^sH$-modules.

If $\mathrm{Nm}(\mathfrak{B})^H$ is the induced $\mathfrak{Z}^sH$-module $\mathrm{Nm}(\mathfrak{B}) \otimes_{\mathfrak{Z}N} \mathfrak{Z}^sH$, then we may show, as in Lemma 6.5, that the factor ring $\mathfrak{A}' = \mathfrak{Z}^s H/\operatorname{ann}(\operatorname{Nm}(\mathfrak{B})^H \text{ in } \mathfrak{Z}^s H)$ is strongly G-graded, and that $\mathfrak{A}'_1$ is isomorphic to the field $\mathfrak{Z} \simeq \mathfrak{Z}N/\operatorname{ann}(\operatorname{Nm}(\mathfrak{B}) \text{ in } \mathfrak{Z}N)$. So Corollary 2.14 gives us a one-to-one correspondence between all isomorphism classes of extensions of $\operatorname{Nm}(\mathfrak{B})$ to $\mathfrak{Z}^s H$-modules and all $\mathrm{U}(\mathfrak{A}'_1)$-conjugacy classes of splitting homomorphisms for $\mathrm{X}\langle\mathfrak{A}'\rangle$. The homomorphism $\operatorname{Nm}'\colon X \to \mathrm{NX}\langle\mathfrak{A}\rangle$ induces an isomorphism of $\mathrm{X}\langle\mathfrak{A}'\rangle$ onto $\mathrm{NX}\langle\mathfrak{A}\rangle$ as extensions of $\mathrm{U}(\mathfrak{A}'_1) \simeq \mathrm{U}(\mathfrak{Z})$ by G. This and Theorem 6.6 tell us that (in the situation of that theorem):

(6.11) *$\mathfrak{B}$ can be extended to an $\mathfrak{R}H$-module if and only if* $\operatorname{Nm}(\mathfrak{B})$ *can be extended to a $\mathfrak{Z}^s H$-module. Indeed, any isomorphism class of extensions of* $\operatorname{Nm}(\mathfrak{B})$ *to $\mathfrak{Z}^s H$-modules determines a unique isomorphism class of extensions of $\mathfrak{B}$ to $\mathfrak{R}H$-modules.*

7. Counterexamples

An arbitrary strongly G-graded ring $\mathfrak{A}$ satisfying (3.1), even one coming from a group algebra of a finite group H as in Lemma 6.5, need not have any strongly G-graded subring $\mathfrak{C}$ with $\mathfrak{C}_1 = \mathfrak{Z}$. So it need not have anything resembling the Clifford extension $\mathrm{X}\langle\mathfrak{C}\rangle$ of Lemma 5.4. The following example, which was kindly provided by Janusz, illustrates this.

Example 7.1 (Janusz). Pick two primes p and q such that

$$p \equiv 1 \pmod{q^2}. \tag{7.2}$$

The residue class field $\bar{\mathfrak{Q}}_p$ of the p-adic completion $\mathfrak{Q}_p$ of the rational number field has order p. So we may choose a primitive $(p-1)$st root of unity θ in $\mathfrak{Q}_p$ whose image $\bar{\theta}$ in $\bar{\mathfrak{Q}}_p$ generates the cyclic group $\mathrm{U}(\bar{\mathfrak{Q}}_p)$ of order $p-1$.

Let $\mathfrak{K}$ be the extension field $\mathfrak{Q}_p(\omega)$ obtained by adjoining a primitive pth root of unity ω to $\mathfrak{Q}_p$. Then the Galois group $\operatorname{Gal}(\mathfrak{K}/\mathfrak{Q}_p)$ is cyclic of order $p-1$, and thus contains an element α of order q^2 by (7.2). We define $\mathfrak{A}$ to be the crossed product over $\mathfrak{K}$ of the cyclic group $\langle\alpha\rangle$ of order q^2 generated by α, determined by

(7.3a) $\mathfrak{A} = \dot{\sum}_{i=1}^{q^2} \rho^i \mathfrak{K}$ *(as $\mathfrak{K}$-modules),*

(7.3b) $k\rho = \rho k^{\alpha}$, *for all* $k \in \mathfrak{K}$,

(7.3c) ρ^{q^2} *is the element θ of the subring $\mathfrak{K}$ of $\mathfrak{A}$,*

for some element ρ. Then $\mathfrak{A}$ is a central simple algebra of degree

$$[\mathfrak{A} : \mathfrak{F}]^{1/2} = |\langle\alpha\rangle| = q^2$$

over the fixed field $\mathfrak{F} = \mathrm{C}(\langle\alpha\rangle \text{ in } \mathfrak{K})$ of $\langle\alpha\rangle$.

Because $\mathfrak{K} = \mathfrak{Q}_p(\omega)$ is a totally ramified extension of $\mathfrak{Q}_p$, the inclusions $Q_p \subseteq \mathfrak{F} \subseteq \mathfrak{K}$ induce isomorphisms between the residue class fields $\bar{\mathfrak{Q}}_p, \bar{\mathfrak{F}}$ and $\bar{\mathfrak{K}}$ of $\mathfrak{Q}_p$, $\mathfrak{F}$ and $\mathfrak{K}$, respectively. If we identify $\bar{\mathfrak{Q}}_p, \bar{\mathfrak{F}}$ and $\bar{\mathfrak{K}}$ by these isomorphisms, then the map $\bar{\mathrm{N}}_{\mathfrak{K}\to\mathfrak{F}}: \bar{\mathfrak{K}} \to \bar{\mathfrak{F}}$ induced by the field norm $\mathrm{N}_{\mathfrak{K}\to\mathfrak{F}}$: $\mathfrak{K} \to \mathfrak{F}$ just sends any $\bar{k} \in \bar{\mathfrak{K}}$ into its q^2 power. So (7.2) implies that the generator $\bar{\theta}$ of $\mathrm{U}(\bar{\mathfrak{F}})$ has multiplicative order q^2 modulo the image $\mathrm{U}(\bar{\mathfrak{F}})^{q^2} = \bar{\mathrm{N}}_{\mathfrak{K}\to\mathfrak{F}}(\mathrm{U}(\bar{\mathfrak{K}}))$ of $\mathrm{U}(\bar{\mathfrak{K}})$. It follows that θ has multiplicative order q^2 modulo the image $\mathrm{N}_{\mathfrak{K}\to\mathfrak{F}}(\mathrm{U}(\mathfrak{K}))$ of $\mathrm{U}(\mathfrak{K})$. This and (7.3c) imply that the class $\{\mathfrak{A}\}$ of $\mathfrak{A}$ has order q^2 in the Brauer group $\mathrm{Br}(\mathfrak{F})$ of the field $\mathfrak{F}$. Since q^2 is also the degree of $\mathfrak{A}$, we conclude from this and Theorem V.28 of [9] that $\mathfrak{A}$ is a division algebra.

The crossed product $\mathfrak{A}$ is naturally strongly $\langle\alpha\rangle$-graded, with (7.3a) as its grading. Hence it is also strongly graded with respect to the factor group $G = \langle\alpha\rangle/\langle\alpha^q\rangle$ of $\langle\alpha\rangle$, with the σ-components

$$\mathfrak{A}_\sigma = \sum_{\alpha^i \in \sigma} \rho^i \mathfrak{K}, \qquad \textit{for all} \quad \sigma \in G = \langle\alpha\rangle/\langle\alpha^q\rangle.$$

In this grading the subalgebra $\mathfrak{A}_1$ is the crossed product of $\langle\alpha^q\rangle$ over $\mathfrak{K}$ generated by ρ^q and hence, in view of (7.3b), is a central simple algebra over the fixed field $\mathfrak{Z} = \mathrm{C}(\langle\alpha^q\rangle \text{ in } \mathfrak{K})$.

It follows from (1.6) above and Corollary 6.2 of [3] that any strongly G-graded subring $\mathfrak{C}$ of $\mathfrak{A}$ with $\mathfrak{C}_1 = \mathfrak{Z}$ must be a crossed product over $\mathfrak{Z}$ of the Galois group $\mathrm{Gal}(\mathfrak{Z}/\mathfrak{F})$, which is isomorphic to G. Hence $\mathfrak{C}$ is a central simple $\mathfrak{F}$-subalgebra of $\mathfrak{A}$ with degree $[\mathfrak{C} : \mathfrak{F}]^{1/2} = |G| = q$. Thus $\mathfrak{A}$ is the Kronecker product $\mathfrak{C} \times_{\mathfrak{F}} \mathfrak{D}$ of $\mathfrak{C}$ with its centralizer $\mathfrak{D}$, which is another central simple $\mathfrak{F}$-subalgebra of $\mathfrak{A}$ with degree $[\mathfrak{A} : \mathfrak{F}]^{1/2}/[\mathfrak{C} : \mathfrak{F}]^{1/2} = q^2/q = q$. Since $\mathfrak{C}$ and $\mathfrak{D}$ both have degree q, their classes $\{\mathfrak{C}\}$ and $\{\mathfrak{D}\}$ in $\mathrm{Br}(\mathfrak{F})$ both have orders dividing q by Theorem V.28 of [9]. Hence $\{\mathfrak{A}\} = \{\mathfrak{C}\}\{\mathfrak{D}\}$ also has order dividing q, contradicting the fact that its order is q^2. Therefore $\mathfrak{A}$ has no strongly G-graded subring $\mathfrak{C}$ with $\mathfrak{C}_1 = \mathfrak{Z}$. In view of Lemma 3.4, this implies that $\mathfrak{A}_1$ has no central simple $\mathfrak{F}$-subalgebra $\mathfrak{D}$ such that $\mathfrak{A}_1 = \mathfrak{Z} \times_{\mathfrak{F}} \mathfrak{D}$.

It is clear from (7.3) that the algebra $\mathfrak{A}$ is generated over $\mathfrak{Q}_p$ by its finite multiplicative subgroup $H = \langle\omega, \rho\rangle$ of order $pq^2(p-1)$, while $\mathfrak{A}_1$ is generated by the normal subgroup $N = \langle\omega, \rho^q\rangle$ of index q in H. So the strongly G-graded ring $\mathfrak{A}$ is a factor ring of the strongly G-graded group algebra $\mathfrak{Q}_p H$ as in Lemma 6.5.

Our next example shows that the Clifford extensions $\mathrm{X}\langle\mathfrak{C}\rangle$ of Lemma 5.4, even when they exist, need not have the vital property of splitting if and only if the irreducible $\mathfrak{A}_1$-module extends to an $\mathfrak{A}$-module. Again our counterexample comes from a group algebra of a finite group.

EXAMPLE 7.4. Now let $\mathfrak{A}$ be the Kronecker product $\mathfrak{C} \times_{\mathfrak{Q}} \mathfrak{A}_1$ of two quaternion algebras $\mathfrak{C}$ and $\mathfrak{A}_1$ over the rational number field $\mathfrak{Q}$. If 1, i, j, k is the usual quaternionic basic for $\mathfrak{C}$, then the decomposition:

$$\mathfrak{C} = 1\mathfrak{Q} \dotplus i\mathfrak{Q} \dotplus j\mathfrak{Q} \dotplus k\mathfrak{Q}$$

makes $\mathfrak{C}$ a strongly G-graded algebra for a four-group G. Hence the decomposition

$$\mathfrak{A} = 1\mathfrak{A}_1 \dotplus i\mathfrak{A}_1 \dotplus j\mathfrak{A}_1 \dotplus k\mathfrak{A}_1$$

makes $\mathfrak{A}$ a strongly G-graded ring. The quaternion algebra $\mathfrak{A}_1$ satisfies (3.1), and G centralizes its center $\mathfrak{Z} = 1\mathfrak{Q} = \mathfrak{F}$ under the action (1.5). So Lemma 5.3 holds with $\mathfrak{D} = \mathfrak{A}_1$, and $\mathfrak{C} = C(\mathfrak{A}_1 \text{ in } \mathfrak{A})$ has the properties of Lemma 5.4. Any splitting homomorphism γ for the Clifford extension $X\langle\mathfrak{C}\rangle$ would provided an isomorphism of the group algebra $\mathfrak{Q}G$ onto $\mathfrak{C}$, an isomorphism that cannot exist since G is abelian and $\mathfrak{C}$ is non-commutative. Hence no such γ exists. But $\mathfrak{A} = \mathfrak{C} \times_{\mathfrak{Q}} \mathfrak{A}_1 \simeq [\mathfrak{Q}]_4$ has a 4-dimensional irreducible module whose restriction is the irreducible regular $\mathfrak{A}_1$-module $\mathfrak{A}_1$. Hence the latter can be extended to $\mathfrak{A}$ even though $X\langle\mathfrak{C}\rangle$ does not split.

Evidently the above $\mathfrak{A}$ is an epimorphic image of the group algebra $\mathfrak{Q}H$ of the central product H of two quaternion groups M and N in such a way that $\mathfrak{A}_1$ is the image of the group algebra $\mathfrak{Q}N$ of the normal subgroup N and $\mathfrak{C}$ is the image of $\mathfrak{Q}M$. So this $\mathfrak{A}$ also comes from a group algebra via Lemma 6.5.

Even in the "best possible" case when (5.1) holds, there can exist extensions of an irreducible $\mathfrak{A}_1$-module $\mathfrak{B}$ to $\mathfrak{A}$-modules which do not come from splitting homomorphisms for $NX\langle\mathfrak{A}\rangle$ or for a Clifford extension $X\langle\mathfrak{C}\rangle$ via the construction of Theorem 5.10. The following example illustrates this for group rings.

EXAMPLE 7.5. We take for $\mathfrak{A}$ the Kronecker product $\mathfrak{C} \times_{\mathfrak{Q}} \mathfrak{A}_1$ of a quaternion algebra $\mathfrak{A}_1$ over the rational number field $\mathfrak{Q}$ with the group algebra $\mathfrak{C} = \mathfrak{Q}\langle\sigma\rangle$ of a cyclic group $\langle\sigma\rangle$ of order 3. Evidently $\mathfrak{A}$ is naturally a strongly $\langle\sigma\rangle$-graded ring with the grading in which

$$\mathfrak{A}_{\sigma^i} = \sigma^i\mathfrak{A}_1, \qquad \textit{for} \quad i = 1, 2, 3.$$

Here all the conditions (5.1) are satisfied with $G = \langle\sigma\rangle$ of order 3, and with $[\mathfrak{A}_1 : \mathfrak{Z}] = [\mathfrak{A}_1 : \mathfrak{Q}] = 4$. Clearly $\mathfrak{F} = \mathfrak{Z} = 1\mathfrak{Q}$ in (5.2), so that $\mathfrak{D} = \mathfrak{A}_1$ in Lemma 5.3, and $\mathfrak{C}$, with the G-grading

$$\mathfrak{C} = 1\mathfrak{Q} \dotplus \sigma\mathfrak{Q} \dotplus \sigma^2\mathfrak{Q},$$

is the strongly G-graded subring of Lemma 5.4. Because 1 is the only cube root of unity in $\mathfrak{Q}$, the only splitting homomorphism γ for the Clifford extension $X\langle\mathfrak{C}\rangle$ is inclusion: $G \subseteq \mathrm{GrU}(\mathfrak{C})$. But the exact sequence $X\langle\mathfrak{A}\rangle$ also has a splitting homomorphism γ' sending σ into its product with the cube root $(-1+i+j+k)/2$ of unity in the quaternion algebra $\mathfrak{A}_1$. Evidently the two splitting homomorphisms γ and γ' for $X\langle\mathfrak{A}\rangle$ are not $U(\mathfrak{A}_1)$-conjugate. So they lead to non-isomorphic extensions of the irreducible regular $\mathfrak{A}_1$-module to $\mathfrak{A}$-modules in Corollary 2.14.

Of course, the present $\mathfrak{A}$ also comes from a group algebra $\mathfrak{Q}H$ via Lemma 6.5, where H is the direct product of the group G of order 3 with a quaternion group of order 4.

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