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# Tree unitarity and partial wave expansion in noncommutative quantum field theory

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**Abstract**

The validity of the tree-unitarity criterion for scattering amplitudes on the noncommutative space–time is considered, as a condition that can be used to shed light on the problem of unitarity violation in noncommutative quantum field theories when time is noncommutative. The unitarity constraints on the partial wave amplitudes in the noncommutative space–time are also derived.



## 1. Introduction

Recently, quantum field theories on noncommutative (NC) space–time have received a lot of attention, after it was discovered that, in some cases, they emerge naturally as low-energy limits from string theory with an antisymmetric background field [1]. On a noncommutative analog of the Minkowski space, the coordinates satisfy nontrivial commutation relations:

$$[\hat{x_\mu}, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix of dimension (length)$^2$. The inherent non-locality and the violation of Lorentz invariance in NC QFT are the main causes which lead to some peculiar features in the case of noncommutative models.

The question of unitarity of theories with time-space noncommutativity ($\theta_{0i} \neq 0$) is a topical one in NC QFT. It was first shown in [2] that such theories are not perturbatively unitary when naive Feynman rules are used, but also that they cannot be obtained as low-energy limits from the underlying string theory (see also [3] for a study of the violation of unitarity on compact space–time). The subject was approached later again in [4], in the light of the Yang–Feldman equation [5], thereby arriving at a manifestly Hermitian solution (hence unitary theory

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with $\theta_{0i} \neq 0$). The study was further pursued in [6,7], where the Wick contraction theorem was adapted to the case when time does not commute with space, hence the time-ordering procedure does not commute with the star multiplication. As a result, a noncommutative extension of the time-ordered perturbation theory (TOPT) was formulated, which gives the same results as the standard procedure (in terms of ordinary Feynman propagators, introduced in [8]) for $\theta_{0i} = 0$, but differs from it in the case when $\theta_{0i} \neq 0$. It is claimed, and checked in the few lowest orders, that this formulation leads to theories which are perturbatively unitary [6]. However, NC QED treated according to the TOPT prescription shows a "surprising result" [7] regarding the high-energy behaviour of the two-body cross-sections: it yields cross-sections, calculated in the lowest-order perturbation theory, exhibiting a growth linear in $s$.[1] It is therefore of interest to apply other criteria, such as the tree-unitarity conditions and see whether they are violated. The fact that time–space NC quantum field theories, in addition to the impossibility of their being obtained from the string theory [1,2], violate causality on both the macro- and micro-scopic levels [11–13], gives reasons to expect that this could be the case.

The scope of this Letter is two-fold: on the one hand, we would like to check if the theories with time–space noncommutativity, treated according to the TOPT prescriptions, satisfy the tree-unitarity criterion [14,15]. Such a consideration would be interesting, since in the past the requirement of mere tree-unitarity was successful in distinguishing among different models with respect to their unitarity/renormalizability [14,15]. One could hope that the same merit would hold also in the case of NC theories.

On the other hand, we would like to derive a partial wave expansion and unitarity constraints on the partial wave amplitudes in noncommutative space (actually, in any nonisotropic space–time), as tools (together with the analyticity of the scattering amplitude and the dispersion relations) for the derivation of bounds on the cross-section and the amplitudes themselves, analogous to the celebrated Froissart–Martin bound [9,10] in the usual QFT.

**Notation.** In the following we shall denote $\epsilon_i = \theta_{0i}$ and $\beta_i = (1/2)\epsilon_{ijk}\theta_{jk}$.

## 2. Tree unitarity

To begin with, we shall recall the concept of tree unitarity [14]. The unitarity of the $S$-matrix, written in the familiar way with respect to the transition amplitude [16]

$$S = 1 + iT, \tag{2.1}$$

implies the following condition on the transition amplitude:

$$T - T^\dagger = iTT^\dagger = iT^\dagger T. \tag{2.2}$$

The on-shell transition amplitude between the initial state $|i\rangle$ and the final state $|f\rangle$ is

$$\langle f|T|i\rangle = (2\pi)^4 \delta(P' - P)\langle f|A|i\rangle, \tag{2.3}$$

where $P$, $P'$ are the initial and final four-momenta. We assume that the energy-momentum dispersion relation still takes the form $E = \sqrt{\vec{k}^2 + m^2}$ in the noncommutative case. From (2.2) it follows that the $A$-matrix elements satisfy the unitarity relation:

$$-\frac{i}{2}\big(\langle f|A|i\rangle - \langle i|A|f\rangle^*\big) = \frac{1}{2}\sum_n (2\pi)^{4-3n} \int \frac{d^3k_1}{2k_1^0} \cdots \frac{d^3k_n}{2k_n^0}\, \delta\Big(\sum k_i - P\Big)\langle k_1 \cdots k_n|A|f\rangle^* \langle k_1 \cdots k_n|A|i\rangle. \tag{2.4}$$

[1] In [7] it is stated that the same phenomenon occurs when the mass of the exchanged particle is much less than the NC energy scale and the center-of-mass energy. However, a straightforward calculation (see Eq. (2.6)) shows that this is not true for the NC $\phi^3$ scalar theory, in which case the two-body cross-section tends to 0 when $E_{\mathrm{CM}} \to \infty$, although not as fast as when it is computed in the standard ("covariant") perturbation theory.

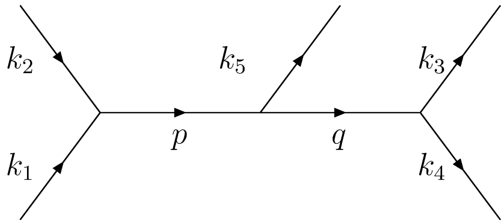


Fig. 1. Diagram corresponding to $A^s_{2\to 3}$.

Denote by $A_{n\to N-n}$ an $A$-matrix for $n$ incoming particles and $N-n$ outgoing particles. In the center-of-mass frame, one chooses *fixed* values for the incoming and outgoing momenta, so that for given values of these "fixed variables" each four-momentum $p_i$ grows as $E$, as the total center-of-mass energy ($E$) approaches infinity. A field theory will be called *tree unitary* if in the tree approximation all amplitudes $A_{n\to N-n}$ grow at most like $E^{4-N}$ as $E\to\infty$. In other words, if at high energies $A_{n\to N-n}\sim E^{\beta}$, then the requirement of tree unitarity can be expressed in the form

$$\beta \leqslant 4-N. \tag{2.5}$$

In the noncommutative quantum field theory the crossing symmetry is still holds, but it is lost when one goes to a specific reference frame and specific initial and final states (as required by the tree-unitarity criterion), so that we need to check separately if the tree unitarity is fulfilled for the $s$- and $t$-channels.

We shall begin with the $s$-channel. One typical tree-level scattering amplitude was obtained in the first paper of [6], for a two-by-two scattering $\pi(p_1)\pi(p_2)\to\chi(p_3)\chi(p_4)$ through the cubic scalar interactions defined by the Lagrangian $L_{\text{int}}=-g_\pi\pi\star\sigma\star\pi-g_\chi\chi\star\sigma\star\chi$ (the fields were taken to be nonidentical in order to reduce the number of channels to one). The expression for the $2\to 2$ scattering amplitude, in the $s$-channel and in the center-of-mass frame, can be cast into the form:

$$A^s_{2\to 2}(\vec p;\vec p\,')=\frac{2g_\pi g_\chi}{s-m_\sigma^2}\sum_{\lambda=\pm1}\biggl\{\cos\bigl[m_\sigma(\tilde p_0+\lambda\tilde p_0')\bigr]\cos\biggl[\frac{\sqrt{s}}{2}(\tilde p_0+\lambda\tilde p_0')\biggr] + \frac{\sqrt{s}}{m_\sigma}\sin\bigl[m_\sigma(\tilde p_0+\lambda\tilde p_0')\bigr]\sin\biggl[\frac{\sqrt{s}}{2}(\tilde p_0+\lambda\tilde p_0')\biggr]\biggr\}, \tag{2.6}$$

where $\tilde p_0=\theta_{0i}p^i=\vec\epsilon\cdot\vec p$ and $m_\sigma$ is the mass of the $s$-channel scalar particle.[2] The second term in the brackets, proportional to $\sqrt{s}=E$, is an element of novelty in the TOPT as compared with the usual "covariant" approach. However, when we take the limit $E\to\infty$, the $2\to 2$ amplitude ($N=4$) behaves like $E/E^2=E^{-1}$, thus fulfilling the tree-unitarity criterion, which requires it to grow not faster than $E^{(N-4)}=E^0$.

In order to be able to appreciate if the tree-unitarity criterion is satisfied in general, we shall move further to the 5-point amplitude $A^s_{2\to 3}$.

The expression of the amplitude, according to the TOPT prescription, is (see Fig. 1):

$$A^s_{2\to 3}\sim g_\pi g_\chi^2\delta(E_1+E_2-E_3-E_4-E_5)\sum_{\lambda_1,\lambda_2=\pm1}\int\frac{d^3p}{(2\pi)^3 2\omega_p}\frac{d^3q}{(2\pi)^3 2\omega_q} \times\bigl(2\pi^3\bigr)\delta(\vec k_1+\vec k_2+\vec p\,)\bigl(2\pi^3\bigr)\delta(\vec p+\vec q+\vec k_5)\bigl(2\pi^3\bigr)\delta(\vec q+\vec k_3+\vec k_4) \tag{2.7}$$

[2] To prove the affirmation of the previous footnote, one can plug the expression (2.6) into the formula of the differential cross-section calculated in CMS for external particles with equal mass, i.e., $(d\sigma/d\Omega)_{\text{CM}}=|A|^2/(64\pi^2E^2_{\text{CM}})$. It is clear that at high energies, the differential cross-section behaves at most like $1/s^2$.

$$\times \frac{[e^{-i(k_{1+},-p_{\lambda_1},k_{2+})} + (k_1 \to k_2)][e^{-i(k_{5-},p_{\lambda_1},-q_{\lambda_2})} + (q \to k_5)][e^{-i(q_{\lambda_2},k_{3-},k_{4-})} + (q \to k_4)]}{[\lambda_1(E_1+E_2) - \omega_p + i\epsilon][-\lambda_2(E_3+E_4) - \omega_q + i\epsilon]}.$$

A typical term of the amplitude is of the form:

$$\frac{1}{\omega_p\omega_q} \sum_{\lambda_1,\lambda_2=\pm 1} \frac{e^{i\lambda_1 a + \lambda_2 b + c}}{(\lambda_1 E - \omega_p)(-\lambda_2 E' - \omega_q)}, \tag{2.8}$$

where $a$, $b$ and $c$ are factors depending on the momenta of the particles involved in the interaction and on the noncommutativity parameter $\theta_{\mu\nu}$ (in such a way that $a = b = c = 0$ for $\theta = 0$), $E = E_1 + E_2$ and $E' = E_3 + E_4$. Performing the summation over $\lambda$'s, one obtains:

$$\frac{8}{(E^2-\omega_p^2)(E'^2-\omega_q^2)}\left[\left(\cos a \cos b - \frac{EE'}{\omega_p\omega_q}\sin a \sin b\right)\cos c - \left(\frac{E}{\omega_p}\sin a \cos b - \frac{E'}{\omega_q}\sin b \cos a\right)\sin c\right]. \tag{2.9}$$

In the center-of-mass frame $E^2 = (k_1+k_2)^2 = (k_3+k_4+k_5)^2$ and we shall fix the outgoing momenta (in the spirit of the tree-unitarity criterion) so that $|\vec{k}_3| = |\vec{k}_4| = |\vec{k}_5|$. Assuming for simplicity the equality of all the masses of the particles involved in the interaction, the following expression is obtained, for this *specific* phase-space configuration, in terms of the center-of-mass energy $E$:

$$\frac{24}{E^2(E^2-m^2)}\left[\left(\cos a \cos b - \frac{2E}{m}\sin a \sin b\right)\cos c - \left(\frac{E}{m}\sin a \cos b - 2\sin b \cos a\right)\sin c\right]. \tag{2.10}$$

The $E$-dependence of $a$, $b$ and $c$, which is of polynomial form, is not relevant for the high-energy behaviour, as the sine and cosine functions do not have a limit when their arguments are polynomials in $E$ for $E \to \infty$, but still they are bounded in the interval $[-1, 1]$. It becomes clear that for high energies, the typical term of the 5-point amplitude $A^s_{2\to 3}$ behaves like

$$\frac{E}{E^4} = E^{-3}.$$

According to [14], the 5-point amplitude should not grow faster then $E^{4-N} = E^{-1}$. Obviously, this requirement if fulfilled by the amplitude $A^s_{2\to 3}$.

We expect that, in the $s$-channel, the tree-amplitudes $A^s_{2\to N-2} \sim s^{\beta/2}$, with $N > 5$, will behave well at high energies, so that $\beta < (4 - N)$.

We shall now consider the tree-unitarity criterion in the $t$-channel, in which case, for a *fixed configuration*, at high energies, $t \sim s = E^2$. We have computed, according to TOPT prescriptions, the $2 \to 2$ scattering amplitude in the $t$-channel, for an interaction Lagrangian of the form $L_{\text{int}} = -g(\pi \star \sigma \star \chi + \chi \star \sigma \star \pi)$, i.e.,

$$A^t_{2\to 2} \sim g^2\left[\frac{2\cos(k_{1+}, -q_+, -k_{3+})2\cos(k_{2+}, q_+, -k_{4+})}{2\omega_q(E_1 - E_3 - \omega_q + i\epsilon)} + \frac{2\cos(k_{1+}, -q_-, -k_{3+})2\cos(k_{2+}, q_-, -k_{4+})}{2\omega_q(E_2 - E_4 - \omega_q + i\epsilon)}\right]. \tag{2.11}$$

In the center-of-mass frame, and taking for simplicity $m_\pi = m_\chi = m$, we obtained:

$$A^t_{2\to 2} \sim \frac{2g^2}{t - m_\sigma^2}\left[\cos\left(\sqrt{m_\sigma - t}\,\theta_{0i}(k_1+k_3)^i\right)\cos\left(\theta_{ij}k_1^i k_3^j\right) + \cos\left(\frac{1}{2}\sqrt{s}\,\theta_{0i}(k_1-k_3)^i\right)\right]. \tag{2.12}$$

In this case, the high-energy behaviour is governed by the first factor (as the cosines are bounded when $s \to \infty$) and is the same like in the commutative case. The amplitudes with more legs will show the same similarity with the commutative case at high energies, and we can conclude that they will satisfy the tree-unitarity criterion.

## 3. Partial wave expansion

In the commutative case, due to the rotational invariance, the $2 \to 2$ scattering amplitude depends on two variables: $s$ and $t$, i.e., the squared center-of-mass energy and the squared transferred momentum or, equivalently, $s$ and $\cos\theta$, with $\theta$ being the center-of-mass scattering angle.

The partial wave amplitudes are defined by the expansion in Legendre polynomials [16]:

$$A(s, \cos\theta) = \sum_{l=0}^{\infty} (2l+1) a_l(s) P_l(\cos\theta), \quad a_l(s) = \frac{1}{2} \int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta) A(s, \cos\theta), \tag{3.1}$$

where $A(s, \cos\theta) \equiv \mathcal{A}(s, t)$ is the scattering amplitude in terms of the Mandelstam variables $s$ and $t = -(s/2) \times (1 - \cos\theta)$ (for the equal-mass case).

In the noncommutative case the rotational invariance is lost and as a result the number of independent angular variables is increased. For the general case of space–time noncommutativity, $\theta_{\mu\nu}$ defines a plane through the vectors $\epsilon_i = \theta_{0i}$ and $\beta_i = (1/2)\epsilon_{ijk}\theta_{jk}$. The only symmetry left is then a reflection in this plane. The situation is thus close to a fully anisotropic (but translationally invariant) background, and we treat this general case in the following. The results are then generally applicable to scattering in completely anisotropic media. With respect to arbitrarily chosen axes, the directions of the three-vectors $\vec{p}_1$ and $\vec{p}_3$ are each given by two angles, $(\theta_{12}, \phi_{12})$ and $(\theta_{34}, \phi_{34})$, respectively.

However, in the case of space–space noncommutativity, when $\theta_{0i} = 0$, i.e., $\vec{\epsilon} = 0$ and in the case of lightlike noncommutativity, when $\theta^{\mu\nu}\theta_{\mu\nu} = 0$ and $\vec{\epsilon} \perp \vec{\beta}$, there are only three independent angular variables, which can be assumed to be the angles $\widehat{(\vec{\beta}, \vec{p}_1)}$, $\widehat{(\vec{\beta}, \vec{p}_3)}$ and $\widehat{(\vec{p}_1, \vec{p}_3)}$. It should be emphasized, however, that only in these latter two cases (space–space noncommutativity and lightlike noncommutativity), a NC field theory can be obtained from the string theory as the low-energy limit [1,2,17].

### 3.1. *Unitarity constraint on partial wave amplitudes*

For a 2-particles initial and final states, the on-shell amplitude is:

$$\langle p_3, p_4 | T | p_1, p_2 \rangle = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) A(\vec{p}_1, \vec{p}_2; \vec{p}_3, \vec{p}_4). \tag{3.2}$$

Next we expand $A(\vec{p}_1, \vec{p}_2; \vec{p}_3, \vec{p}_4)$ in partial waves, demanding that the amplitude is single-valued. The angular dependence will be taken into account through the spherical harmonics $Y_{lm}(\theta_{12}, \phi_{12})$ and $Y_{l'm'}(\theta_{34}, \phi_{34})$, while the dependence on $s$ will be accounted for through the partial-wave amplitudes $a_{lm,l'm'}(s)$, i.e.,

$$A(\vec{p}; \vec{p}') = 4\pi \sum_{l,l',m,m'} a_{lm,l'm'}(s) Y_{lm}(\theta_{12}, \phi_{12}) Y_{l'm'}(\theta_{34}, \phi_{34}). \tag{3.3}$$

(When there is only one preferred direction in space, e.g., $\vec{\epsilon} = 0$, $\vec{\beta} \neq 0$, invariance under rotations arround that direction implies that the scattering amplitude does not depend on, e.g., $\phi_{12}$. The expansion in that situation becomes a special case of the general formula (3.3), with only terms with $m = 0$ surviving.)

The bound can be obtained using the relation between the elastic cross-section and scattering amplitude,

$$\sigma_{\text{el}} = \frac{1}{64\pi^2 s} \int d\Omega_{34} |A|^2, \tag{3.4}$$

and the optical theorem for forward scattering (i.e., $p_1 = p_3$ and $p_2 = p_4$), written in the form

$$\operatorname{Im} A(s)_{\text{forward}} = 2\sqrt{s}\, p \sigma_{\text{tot}}, \tag{3.5}$$

when the two particles in the initial state have equal masses and $\sigma_{\text{tot}}$ is the total cross-section. Then, using the expansion (3.3), the elastic cross-section becomes:

$$\sigma_{\text{el}} = \frac{1}{4s} \sum_{l_1,l_2,l',m_1,m_2,m'} a_{l_1,m_1,l'm'}(s) a^*_{l_2,m_2,l'm'}(s) Y_{l_1m_1}(\theta_{12},\phi_{12}) Y^*_{l_2m_2}(\theta_{12},\phi_{12}) \tag{3.6}$$

and the r.h.s. of (3.5) will be:

$$\operatorname{Im} A(s)_{\text{forward}} = -2\pi i \sum_{l,l',m,m'} \bigl[a_{lm,l'm'}(s) - (-1)^{m+m'} a_{l,-m,l',-m'}(s)\bigr] Y_{lm}(\theta_{12},\phi_{12}) Y^*_{l'm'}(\theta_{12},\phi_{12}). \tag{3.7}$$

Taking into account that $\sigma_{\text{el}} \leqslant \sigma_{\text{tot}}$, it follows that

$$\begin{aligned} &(-i) \sum_{l,l',m,m'} \bigl[a_{lm,l'm'}(s) - (-1)^{m+m'} a_{l,-m,l',-m'}(s)\bigr] Y_{lm}(\theta_{12},\phi_{12}) Y_{l'm'}(\theta_{12},\phi_{12}) \\ &\quad \geqslant \frac{p}{4\pi\sqrt{s}} \sum_{l,l',l_1,m,m',m_1} (-1)^{m'} a^*_{l',-m',l_1,m_1}(s) a_{l,m,l_1,m_1}(s) Y_{lm}(\theta_{12},\phi_{12}) Y_{l'm'}(\theta_{12},\phi_{12}). \end{aligned} \tag{3.8}$$

The expression (3.8) is an exact unitarity condition on the partial-wave amplitudes. As the sign between the two sides is an inequality, one cannot use the orthonormality property of the spherical harmonics, because they do not have a definite sign on the whole domain of their arguments.

However, for energies were elastic unitarity is exact, one can obtain approximate unitarity conditions on the partial-wave amplitudes, but with an equality sign, which will make the situation easier to deal with.

With the following convention for one-particle states:

$$\langle p|p'\rangle = (2\pi)^3 2p_0 \delta(\vec{p} - \vec{p}'), \qquad 1 = \int \frac{d^3p}{2p_0(2\pi)^3} |p\rangle\langle p|, \tag{3.9}$$

we can write the *elastic* unitarity condition in terms of $A(\vec{p}_1, \vec{p}_2; \vec{p}_3, \vec{p}_4)$:

$$\begin{aligned} &A(\vec{p}_1, \vec{p}_2; \vec{p}_3, \vec{p}_4) - A^*(\vec{p}_3, \vec{p}_4; \vec{p}_1, \vec{p}_2) \\ &\quad = \frac{i}{(2\pi)^2} \int \frac{d^3k_1}{2k_1^0} \frac{d^3k_2}{2k_2^0} \delta(p_1 + p_2 - k_1 - k_2) A^*(\vec{p}_3, \vec{p}_4; \vec{k}_1, \vec{k}_2) A(\vec{p}_1, \vec{p}_2; \vec{k}_1, \vec{k}_2). \end{aligned} \tag{3.10}$$

In the center-of-mass frame, where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$, $\vec{p}_3 = -\vec{p}_4 = \vec{p}'$ and $\vec{k}_1 = -\vec{k}_2 = \vec{k}$, (3.10) becomes:

$$\begin{aligned} A(\vec{p}; \vec{p}') - A^*(\vec{p}'; \vec{p}) &= \frac{i}{(2\pi)^2} \frac{1}{8} \int d\Omega_{\vec{k}} \int_0^\infty \frac{k^2 dk}{k^2 + m^2} \delta\bigl(\sqrt{k^2 + m^2} - \sqrt{p^2 + m^2}\bigr) A^*(\vec{p}'; \vec{k}) A(\vec{p}; \vec{k}) \\ &= \frac{i}{(2\pi)^2} \frac{1}{8} \int d\Omega_{\vec{k}} \frac{p}{\sqrt{p^2 + m^2}} A^*(\vec{p}'; \vec{k}) A(\vec{p}; \vec{k}). \end{aligned} \tag{3.11}$$

Taking into account that $\sqrt{p^2 + m^2} = \sqrt{s}/2$, one obtains:

$$(-i)\bigl[A(\vec{p}; \vec{p}') - A^*(\vec{p}'; \vec{p})\bigr] = \frac{1}{16\pi^2} \frac{p}{\sqrt{s}} \int d\Omega_{\vec{k}} A^*(\vec{p}'; \vec{k}) A(\vec{p}; \vec{k}), \tag{3.12}$$

where $p$ is the magnitude of the three-momentum of the initial particles in the center-of-mass frame.

With the expansion (3.3), the integral in the r.h.s of (3.12) becomes:

$$\begin{aligned}
&\int d\Omega_{\vec{k}}\, A^*(\vec{p}\,';\vec{k})A(\vec{p};\vec{k})\\
&\quad = (4\pi)^2 \sum_{l_1,l_1',m_1,m_1'} \sum_{l_2,l_2',m_2,m_2'} a^*_{l_1,m_1,l_1',m_1'} a_{l_2,m_2,l_2',m_2'} Y^*_{l_1m_1}(\theta_{34},\phi_{34}) Y_{l_2m_2}(\theta_{12},\phi_{12})\\
&\qquad \times \int d\Omega_{\vec{k}}\, Y^*_{l_1'm_1'}(\theta_{\vec{k}},\phi_{\vec{k}}) Y_{l_2'm_2'}(\theta_{\vec{k}},\phi_{\vec{k}})\\
&\quad = (4\pi)^2 \sum_{l_1,l_2,l_1',m_1,m_2,m_1'} a^*_{l_1,m_1,l_1',m_1'} a_{l_2,m_2,l_1',m_1'} (-1)^{m_1} Y_{l_1-m_1}(\theta_{34},\phi_{34}) Y_{l_2m_2}(\theta_{12},\phi_{12}),
\end{aligned} \tag{3.13}$$

where we have used $Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$. Inserting (3.13) into (3.12), one obtains:

$$\begin{aligned}
&(-i) \sum_{l,l',m,m'} \big[a_{lm,l'm'}(s) - (-1)^{m+m'} a^*_{l',-m',l,-m}(s)\big] Y_{lm}(\theta_{12},\phi_{12}) Y_{l'm'}(\theta_{34},\phi_{34})\\
&\quad = \frac{p}{4\pi\sqrt{s}} \sum_{l,l',l_1',m,m',m_1'} a^*_{l',-m',l_1',m_1'}(s) a_{l,m,l_1',m_1'}(s) (-1)^{m'} Y_{lm}(\theta_{12},\phi_{12}) Y_{l'm'}(\theta_{34},\phi_{34}).
\end{aligned} \tag{3.14}$$

As the spherical harmonics form a complete and orthonormal set, the equality of the coefficients of the expansions follows and the elastic unitarity condition finally takes the form:

$$(-i)\big[a_{lm,l'm'}(s) - (-1)^{m+m'} a^*_{l',-m',l,-m}(s)\big] = \frac{p}{4\pi\sqrt{s}} \sum_{l_1,m_1} (-1)^{m'} a^*_{l',-m',l_1,m_1}(s) a_{l,m,l_1,m_1}(s). \tag{3.15}$$

From this expression we can get the bounds on the partial wave amplitudes. Taking, e.g., in (3.15) $m' = m = 0$ and $l = l'$, one obtains:

$$\begin{aligned}
(-i)\big[a_{l0,l0}(s) - a^*_{l0,l0}(s)\big] &= \frac{p}{4\pi\sqrt{s}} \sum_{l_1,m_1} a^*_{l0,l_1m_1}(s) a_{l0,l_1m_1}(s)\\
&= \frac{p}{4\pi\sqrt{s}} \sum_{l_1,m_1} \big|a_{l0,l_1m_1}(s)\big|^2 \geqslant \frac{p}{4\pi\sqrt{s}} \big|a_{l0,l0}(s)\big|^2.
\end{aligned} \tag{3.16}$$

Thus

$$\operatorname{Im} a_{l0,l0}(s) \geqslant \frac{p}{8\pi\sqrt{s}} \big|a_{l0,l0}(s)\big|^2, \tag{3.17}$$

which is equivalent to

$$\left| a_{l0,l0}(s) - i\frac{8\pi\sqrt{s}}{p} \right| \leqslant \frac{8\pi\sqrt{s}}{p}. \tag{3.18}$$

This is an expression of the elastic unitarity condition for the partial-wave amplitudes and it should be compared to the formula for the commutative case (the normalizations chosen in (3.1) and (3.3) correspond to each other, as $Y_{l0}(\theta,0) = \sqrt{2l+1/(4\pi)}\, P_l(\cos\theta)$)

$$\operatorname{Im} a_l(s) = \frac{p}{8\pi\sqrt{s}} \big|a_l(s)\big|^2. \tag{3.19}$$

## 4. Conclusions

We have investigated the validity of the tree-unitarity criterion [14,15] for quantum field theories with space–time noncommutativity, treated according to the noncommutative extension of the TOPT developed in [6]. We have found that the tree-unitarity condition is fullfilled by the NC $\phi^3$ scalar theory, which might have beneficial implications for its exact unitarity and renormalizability.

We have also derived the unitarity constraint on the partial wave expansion of a $2 \to 2$ scattering amplitude in the general case of noncommutative space–time with a constant noncommutativity parameter $\theta_{\mu\nu}$, which is an essential step in deriving Froissart–Martin-type of bounds on the cross-sections and scattering amplitudes on noncommutative space–time.

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