Bull. Sci. math.
1998, 122, p. 243-250.

INTEGRABILITY AND GEOMETRIC PREQUANTIZATION OF THE MAXWELL-BLOCH EQUATIONS

BY

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ABSTRACT. – In this paper we discuss the integrability and geometric prequantization of the 3-dimensional real valued Maxwell-Bloch equations and point out some of their properties. © Elsevier, Paris

1. Introduction

It is well known that the description of the interaction between laser light and a material sample composed of two-level atoms begins with Maxwell's equations of the electric field and Schrödinger's equations for the probability amplitudes of the atomic levels. The resulting dynamics is given by the following equations usually called Maxwell-Schrödinger equations:

$$
(1.1) \qquad \begin{aligned} \dot{E} &= ib_+b_-^* \\ \dot{b}_+ &= iEb_- \\ \dot{b}_- &= iE^*b_+ \end{aligned}
$$

(*) Manuscript presented by J.-P. Françoise, received in December 1996.

MOS Subject classification: 58F06.

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BULLETIN DES SCIENCES MATHÉMATIQUES – 0007-4497/98/03

where E denotes the self-consistent electric field and $b_{\pm}(t)$ are the probability amplitudes. They have a Hamiltonian formulation and moreover there exists a homoclinic chaos [3].

Starting with the equations (1.1), let us introduce the Stokes variables

$$\widetilde{E} = 2E; \qquad \widetilde{P} = 2ib_+b_-^*; \qquad \widetilde{D} = |b_+|^2 - |b_-|^2, \tag{1.2}$$

They inherit the chaotic dynamics of the equations(1.1) simply as a result of the above change of variables and moreover the chaotic dynamics remains near two homoclinic orbits which lie in the real subspace $E = 0$, $P = 0$. The last property suggests modelling the chaotic dynamics by taking the real parts of all quantities in (1.2). The resulting approximate dynamics is governed by the following equations, usually called 3-dimensional real valued Maxwell-Bloch equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1x_3 \\ \dot{x}_3 &= -x_1x_2 \end{aligned} \tag{1.3}$$

where

$$x_1 = \mathrm{Re}(E); \qquad x_2 = \mathrm{Re}(P); \qquad x_3 = D.$$

Their Hamilton-Poisson formulation, stability and control have been extensively studied in [2], [4], [5].

The goal of our paper is to discuss their integrability via a Weierstrass function, their numerical integration via the Lie-Trotter formula and mid-point rule, and their prequantization from the geometric prequantization point of view.

2. Integrability

To begin with, let us introduce the matrices

$$L = \begin{bmatrix} x_3 & \frac{1}{\sqrt{2}}\left(x_2 - \frac{1}{2}x_1^2\right) \\ \frac{1}{\sqrt{2}}\left(x_2 + \frac{1}{2}x_1^2\right) & \frac{1}{2}x_1^2 \end{bmatrix}$$

$$B = \begin{bmatrix} x_1 & \frac{x_1}{\sqrt{2}} \\ -\frac{x_1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Then an easy computation proves the following result:

THEOREM 2.1. – *The equations* (1.3) *have a Lax formulation, i.e., they can be put in the equivalent form:*

$$\dot{L} = [L, B].$$

As a consequence of the above theorem the following statements hold:

(i) The flow of L is isospectral, i.e., it does not depend on t.

(ii)

$$H = \text{Trace}\,(L) = x_3 + \frac{1}{2}x_1^2 \tag{2.1}$$

and

$$C = \frac{1}{2}\,\text{Trace}\,(L^2) = \frac{1}{2}\,(x_2^2 + x_3^2) \tag{2.2}$$

are constants of notion.

We can now prove:

THEOREM 2.2. – *The equations* (1.3) *can be explicitly integrated via a Weierstrass function. More exactly we have:*

$$x_1 = \pm\sqrt{\frac{5H}{3} - 2\mathcal{P}}$$

$$x_2 = \pm\sqrt{2C - \left(2\mathcal{P} + \frac{1}{3}\,H\right)^2}$$

$$x_3 = 2\mathcal{P} + \frac{H}{3}$$

where $\mathcal{P}$ *is the Weierstrass function given by*

$$(\dot{\mathcal{P}})^2 = 4\mathcal{P}^3 - \frac{4C - H}{3}\,\mathcal{P} + \frac{H^3 + 18CH}{27}.$$

Proof. – We have successively:

$$2H - 2x_3 = x_1^2,$$

$$2C - x_3^2 = x_2^2,$$

$$x_1^2 x_2^2 = 2x_3^3 - 2Hx_3^2 - 4Cx_3 + 4CH,$$

and then

$$(\dot{x}_3)^2 = 2x_3^3 - 2Hx_3^2 - 4Cx_3 + 4CH. \tag{2.3}$$

Let us make now the change of variables

$$x_3 = a\mathcal{P} + b,$$

where a, b will be determined later. Then (2.3) becomes

$$(\dot{\mathcal{P}})^2 = 2a\,\mathcal{P}^3 + 2\,(3b - H)\,\mathcal{P}^2 + \frac{6b^2 - 4bH - 4C}{a}\,\mathcal{P} + \frac{2b^3 - 2b^2H - 4bC + 4CH}{a^2}.$$

If we impose now the conditions

$$2a = 4$$

$$3b - H = 0$$

then we obtain immediately the desired result.

Q.E.D.

3. Numerical integration

In this section we shall discuss the numerical integration of the equations (1.3) via the Lie-Trotter formula and the mid-point rule. To begin with, let us remind that the equations (1.3) have a Hamilton-Poisson formulation, [2], with the phase space $\mathbf{R}^3$, the Poisson structure given by the matrix

$$\Pi_{MB} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix}$$

and the Hamiltonian H given by (2.1). Moreover, the Casimir of our configuration $(\mathbf{R}^3, \Pi_{MB})$ is given by (2.2), i.e.

$$(\nabla C)^t \cdot \Pi_{MB} = 0.$$

Now, the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_3},$$

where

$$H_1 = \frac{1}{2}x_1^2, \qquad H_3 = x_3.$$

Their flows can be explicitly computed and we obtain

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x_1(0)\,t & \sin x_1(0)\,t \\ 0 & -\sin x_1(0)\,t & \cos x_1(0)\,t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}.$$

Then the Lie-Trotter formula [6], [8] is given by:

$$\begin{aligned} x_1^{k+1} &= x_1^k + t x_2^k \\ x_2^{k+1} &= x_2^k \cos x_1(0)\, t + x_3^k \sin x_1(0)\, t \\ x_3^{k+1} &= -x_2^k \sin x_1(0)\, t + x_3^k \cos x_1(0)\, t \end{aligned} \tag{3.1}$$

THEOREM 3.1. – *The first order integrator* (3.1) *has the following properties:*

(i) *It is a Poisson integrator.*

(ii) *Its restriction to the coadjoint orbit* (b_k, ω_k), *where*

$$b_k = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_2^2 + x_3^2 = k^2\}$$

and

$$\omega_k = \frac{1}{k}\,(x_3\, dx_1 \wedge dx_2 - x_2\, dx_1 \wedge dx_3)$$

is a symplectic integrator.

(iii) *It does not preserve the Hamiltonian* (2.1).

The proof is a straightforward computation and we shall omit any other details.

The mid-point rule is an implicit integrator which in this particular case can be written in the following form:

$$\begin{aligned} X_1^{k+1} - x_1^k &= \frac{h}{2}\,(x_2^{k+1} + x_2^k) \\ x_2^{k+1} - x_2^k &= \frac{h}{4}\,(x_1^{k+1} + x_1^k)\,(x_2^{k+1} + x_2^k) \\ x_3^{k+1} - x_3^k &= -\frac{h}{4}\,(x_1^{k+1} + x_1^k)\,(x_2^{k+1} + x_2^k) \end{aligned} \tag{3.2}$$

Using the same arguments as in [1] we can prove

THEOREM 3.2. – *The first order integrator* (3.2) *has the following properties:*

(i) *It preserves the Hamiltonian* (2.1) *and the Casimir* (2.2).

(ii) *It is not a Poisson integrator.*

(iii) *Its restriction to the coadjoint orbit* (b_k, ω_k) *is not a symplectic integrator.*

4. Geometric prequantization

In this last section we shall discuss the geometric prequantization of the 3-dimensional real valued Maxwell-Bloch equations. To this aim, some auxiliar results have to be invoked.

THEOREM 4.1. – *The equations* (1.3) *have a Hamilton-Poisson formulation with the phase space* $\mathbf{R}^3$, *the Poisson bracket given by the matrix*

$$\Pi'_{MB} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x_1 \\ 0 & -x_1 & 0 \end{bmatrix} \tag{4.1}$$

and the Hamiltonian

$$H' = \frac{1}{2}(x_2^2 + x_3^2). \tag{4.2}$$

Moreover, a Casimir of our configuration $(\mathbf{R}^3, \Pi'_{MB})$ *is given by*

$$C' = \frac{1}{2}x_3^2 + x_1. \tag{4.3}$$

Proof. – One easily check that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x_1 \\ 0 & -x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$$

which proves the theorem.

Q.E.D.

THEOREM 4.2. – *The Hamilton-Poisson mechanical system* $(\mathbf{R}^3, \Pi'_{MB}, H')$ *has a full realization on the canonical symplectic manifold* $(\mathbf{R}^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$.

Proof. – Let us take in $\mathbf{R}^4$ the Hamiltonian

$$H'' = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8}q_1^4 - \frac{1}{2}q_1^2 p_2.$$

Then the corresponding Hamilton's equations are given by

$$
(4.4) \qquad \begin{aligned}
\dot q_1 &= p_1 \\
\dot q_2 &= p_2 - \frac{1}{2} q_1^2 \\
\dot p_1 &= q_1 p_2 - \frac{1}{2} q_1^3 \\
\dot p_2 &= 0
\end{aligned}
$$

If we define now:

$$
\begin{aligned}
&\Phi = \mathbf{R}^4 \to \mathbf{R}^3 \\
&\Phi(q_1, q_2, p_1, p_2) = (x_1, x_2, x_3) = \left(q_1, p_1, p_2 - \frac{1}{2} q_1^2\right),
\end{aligned}
$$

then it is easy to see that Φ is a surjective submersion, the equations (1.3) are mapped onto the equations (4.4), and the Poisson structure (4.1) is mapped onto the Poisson structure $\{\cdot, \cdot\}_\omega$ which is canonically induced by ω. It follows that $(\mathbf{R}^4, \omega, H'')$ is a full symplectic realization of the Hamilton-Poisson mechanical system $(\mathbf{R}^3, \Pi''_{MB}, H')$ as required.

Q.E.D.

THEOREM 4.3. – *The Hamiltonian mechanical system* $(\mathbf{R}^4, \omega, H'')$ *given above is completely integrable on* $\mathbf{R}^4 \backslash \{p_1 = 0,\ q_1 p_2 - \frac{1}{2} q_0^3 = 0\}$.

Proof. – Let us take now

$$
\begin{aligned}
K_1 &= p_2 \\
K_2 &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8} q_1^4 - \frac{1}{2} q_1^2 p_2.
\end{aligned}
$$

Then an easy computation shows us that K_1 and K_2 are constants of motion, they are in involution and moreover dK_1 and dK_2 are linearly independent on $\mathbf{R}^4 \backslash \{p_1 = 0,\ q_1 p_2 - \frac{1}{2} q_1^3 = 0\}$. Therefore $(\mathbf{R}^4, \omega, H'')$ is a completely integrable Hamiltonian mechanical system on $\mathbf{R}^4 \backslash \{p_1 = 0,\ q_1 p_2 - \frac{1}{2} q_1^3 = 0\}$.

Q.E.D.

It is clear that our symplectic manifold $(\mathbf{R}^4, \omega)$ is quantizable from the geometric quantization point of view [7], [9], with the Hilbert representation space $\mathcal{H}_\omega = L^2(\mathbf{R}^4, C)$ and the prequantum operator δ_ω given by

$$\delta_f^\omega = i\hbar \sum_{i=1}^{2} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - \sum_{i=1}^{2} p_i \frac{\partial f}{\partial p_i} + f,$$

for each $f \in C^\infty(\mathbf{R}^4, \mathbf{R})$ and where $\hbar$ is the Planck constant divided by 2π.

If we take now $\mathcal{H} = \mathcal{H}_\omega$ and for each $f \in C^\infty(\mathbf{R}^3, \mathbf{R})$, $\delta_f = \delta^\omega_{f \circ \Phi}$, then we have:

THEOREM 4.4. – *The pair* $(\mathcal{H}, \delta)$ *gives a prequantization of the Poisson manifold* $(\mathbf{R}^3, \Pi'_{MB})$.

Proof. – One easily check that Dirac's conditions are all satisfied as required.

Q.E.D.

Acknowledgement

This work was finalized during the author's stay at the Mathematisches Institut der Technische Universität München. I want to express here all my gratitude to Professor Dr. Klaus BUCHNER for his support and hospitality.

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