On a Hopf Algebra in Graph Theory

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We introduce and start the study of a bialgebra of graphs, which we call the 4-bialgebra, and of the dual bialgebra of 4-invariants. The 4-bialgebra is similar to the ring of graphs introduced by W. T. Tutte in 1946, but its structure is more complicated. The roots of the definition are in low dimensional topology, namely, in the recent theory of Vassiliev knot invariants. In particular, 4-invariants of graphs determine Vassiliev invariants of knots. The relation between the two notions is discussed.

1. INTRODUCTION

In paper [19] W. T. Tutte introduced a ring of graphs. The underlying module of this ring is spanned over \( \mathbb{Z} \) by finite graphs modulo the relation

\[ \Gamma - \Gamma_e^* - \Gamma_e^* = 0, \]

which we call the Tutte relation. Here \( \Gamma \) is a graph, \( e \) is a link, i.e., an edge in \( \Gamma \), which is not a loop, the graph \( \Gamma_e^* \) is obtained from \( \Gamma \) by deletion of the edge \( e \), and the graph \( \Gamma_e^* \) is the result of contraction of \( e \). Of course, Eq. (1) is in fact a huge set of relations, one for each pair (a graph, an edge in the graph). The multiplication in the ring is induced by the disjoint union of graphs.

By the time Tutte wrote his paper, a number of interesting graph invariants were known satisfying the property

\[ f(\Gamma) = f(\Gamma_e^*) + f(\Gamma_e^*). \]

Tutte called such invariants \( W \)-functions, and they are precisely the linear functions on his module. Below, we call Tutte’s \( W \)-functions Tutte

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The graph complexity, i.e., the number of spanning trees in the graph, is an example of a Tutte invariant.

He also introduced $V$-functions, that is, $W$-functions satisfying the additional multiplicativity requirement

$$f(I_1 \cdot I_2) = f(I_1) f(I_2) \tag{3}$$

for any two graphs $I_1, I_2$. In other words, $V$-functions are ring homomorphisms. The chromatic polynomial (taken with the appropriate sign) is an example of a $V$-function.

Tutte gave a complete description of all $V$-functions, which is as follows. Equation (1) allows one to present each graph as an equivalent linear combination of graphs without links (since both graphs $I_1'$ and $I_2'$ have fewer links than the graph $I$). A graph without links is a disjoint union of graphs consisting of one vertex and a number of loops, i.e., a product of such graphs. Hence, one-vertex graphs are multiplicative generators of the ring of graphs, and any $V$-function is uniquely determined by its values on one-vertex graphs with 0, 1, 2, ... loops. The main theorem of Tutte states that a $V$-function can take arbitrary values on one-vertex graphs with $n$ loops.

In other words, any graph is equivalent to a unique linear combination of linkless graphs. One more point of view is that Tutte's ring is isomorphic to the polynomial ring in an infinite number of variables, $\mathbb{Z}[s_0, s_1, s_2, ...]$, where the variable $s_n$ corresponds to the one-vertex graph with $n$ loops. A $W$-function is uniquely determined by its values on disjoint unions of one-vertex graphs.

The main goal of the present paper is to introduce and to start the study of a different, richer, ring of graphs. For convenience, we consider not a ring, but an algebra over the field $\mathbb{C}$ of complex numbers (although some remarks will be made about the case of integer coefficients). This algebra carries an additional structure of a coalgebra, whence it is a bialgebra (or a Hopf algebra, what is almost the same). The crucial difference between our algebra and Tutte's ring is that we use a different equivalence relation, a four-term relation, instead of the original Tutte relation (1) (which contains three terms). That is why we call the resulting bialgebra the 4-bialgebra of graphs.

The roots of our definition can be found in the low-dimensional topology, namely, in Vassiliev's theory of knot invariants [20]. Vassiliev introduced a class of knot invariants, which is presumably a complete class of knot invariants, and showed how these invariants can be computed by means of so-called chord diagrams. Vassiliev's invariants are, essentially, functions on chord diagrams satisfying some four-term relations. We describe these four-term relations in Section 3 below in the form suggested by Birman and Lin [3].
S. V. Duzhin started the study of the intersection graphs of chord diagrams from the Vassiliev invariants theory point of view. He noted that the chromatic polynomial of the intersection graph satisfies the four-term relation, and that it determines, therefore, a Vassiliev invariant of knots. This remark led finally to our joint paper with S. V. Chmutov [7] (see also the discussion of intersection graphs in [2]). In this paper we introduced and gave a complete description of the bialgebra of weighted graphs, which is very close to Tutte's ring in the following sense: it has precisely one generator of each order. However, it was obvious that the intersection graph of a chord diagram carries much more information than that used in the weighted bialgebra of graphs. The present paper is intended to develop a natural answer to the question, What information about the chord diagram is contained in its intersection graph?

The paper is organized as follows. In Section 2 the 4-bialgebra of graphs is defined and its structure is described. According to the general structure theorem for bialgebras, it is a polynomial bialgebra. The only problem with its structure is the number of generators of each order. Computations elaborated by E. Soboleva [17] for the first few orders show that this algebra is much richer than Tutte's ring (and therefore, the computations themselves are much more difficult). We also present some examples of 4-invariants that are not Tutte invariants and give a description of primitive elements in the 4-bialgebra based on paper [14]. However, very few steps are made in the study of this 4-bialgebra.

Section 3 treats the connection between the 4-bialgebra of graphs and the bialgebra of chord-diagrams in knot theory. Section 4 contains a number of open problems concerning the 4-bialgebra.

2. BIALGEBRA OF GRAPHS

2.1. Four-Term Relations

Let \( \Gamma \) be a graph. We restrict our consideration with graphs without loops and multiple edges (strict graphs).

A graph invariant is simply a function on (isomorphism classes of) graphs. A graph invariant can take values in an arbitrary Abelian group or in a commutative ring, although it is usually sufficient to keep in mind either the ring \( \mathbb{Z} \) of integers, or the field \( \mathbb{C} \) of complex numbers.

Denote by \( V(\Gamma) \) the set of vertices of \( \Gamma \), and by \( E(\Gamma) \) the set of its edges. Let us associate with each pair of (distinct) vertices \( A, B \in V(\Gamma) \) of a graph \( \Gamma \) two other graphs \( \Gamma'_{AB} \) and \( \Gamma_{AB} \).

The graph \( \Gamma'_{AB} \) is obtained from \( \Gamma \) by erasing the edge \( AB \in E(\Gamma) \) in the case this edge exists, and by adding the edge otherwise. In other words, we
simply change the adjacency between the vertices $A$ and $B$ in $\Gamma$. This operation is the analogue of the edge deletion, but we prefer to formulate it in a slightly more symmetric way.

The graph $\bar{\Gamma}_{AB}$ is obtained from $\Gamma$ in the following way. For any vertex $C \in V(\Gamma) \setminus \{A, B\}$ we change its adjacency with $A$ if $C$ is joined with $B$, and we do nothing otherwise. All other edges do not change. Note that the graph $\bar{\Gamma}_{AB}$ depends not only on the pair $(A, B)$, but on the order of vertices in the pair as well.

Note also that the operations $\Gamma \mapsto \Gamma'_{AB}$ and $\Gamma \mapsto \bar{\Gamma}_{AB}$ commute.

**Definition 2.1.** A graph invariant is a \textit{4-invariant} if it satisfies the \textit{four-term relation}

$$f(\Gamma) - f(\Gamma'_{AB}) = f(\bar{\Gamma}_{AB}) - f(\bar{\Gamma}'_{AB})$$

for each graph $\Gamma$ and for any pair $A, B \in V(\Gamma)$ of its vertices.

In contrast to the Tutte relation, all four graphs entering Eq. (4) have the same number of vertices (and the sets of their vertices are in a natural one-to-one correspondence).

As an immediate consequence of the definition we obtain the following statement. Let $\bar{\chi}(\Gamma) = (-1)^{\#(V(\Gamma))} \chi(\Gamma)$ denote the \textit{modified chromatic polynomial} of a graph $\Gamma$, where $\chi(\Gamma)$ is the usual chromatic polynomial and $\#(V(\Gamma))$ is the number of vertices in $\Gamma$.

**Proposition 2.2.** The modified chromatic polynomial of a graph is a \textit{4-invariant} (with values in the ring of polynomials in one variable).

**Proof.** Suppose that $\Gamma$ contains an edge $e = AB \in E(\Gamma)$. Approve the short notation $\bar{\Gamma} = \bar{\Gamma}_{AB}$. For the modified chromatic polynomial we have

$$\bar{\chi}(\Gamma) - \bar{\chi}(\Gamma'_{e}) = \bar{\chi}(\bar{\Gamma}_{e}), \quad \bar{\chi}(\bar{\Gamma}) - \bar{\chi}(\bar{\Gamma}'_{e}) = \bar{\chi}(\bar{\Gamma}'_{e}).$$

For the chromatic polynomial, the contraction of an edge requires eliminating multiple edges in the resulting graph. After eliminating possible multiple edges we obtain $\Gamma'_{e} = \bar{\Gamma}'_{e}$, which is obvious from the definitions. The proposition is proved.

However, there are many 4-invariants that are not Tutte invariants, and not each Tutte invariant is a 4-invariant. Probably the most obvious 4-invariant that is not Tutte is the number $\#(E(G))$ of edges in $G$. The verification of Eq. (4) is easy. Indeed, the difference in both sides of Eq. (4) is either 1 or $-1$ for any graph and any pair of its vertices. Other examples of 4-invariants will be presented later in this section.
2.2. **Bialgebra Structure**

Consider the (infinite dimensional) vector space over \( \mathbb{C} \) spanned by all graphs as free generators. In his original paper [19] Tutte suggested treating the disjoint union of graphs as a multiplication of the generators. This multiplication can be extended by linearity to linear combinations of graphs making the space into a commutative algebra. We denote this algebra by \( \mathcal{G} \). The empty graph plays the role of the unit in this algebra. The number of graph vertices induces a grading in \( \mathcal{G} \),

\[
\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots,
\]

where \( \mathcal{G}_k \) is the finite dimensional linear space freely spanned by graphs with \( k \) vertices, \( k = 0, 1, \ldots \). The multiplication \( m: \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \) preserves this grading:

\[
m: \mathcal{G}_m \otimes \mathcal{G}_n \to \mathcal{G}_{m+n}.
\]

A **bialgebra** is a linear space endowed with two structures, that of an algebra and that of a coalgebra. Precise definitions can be found in [18]. Throughout the paper all bialgebras are supposed to be associative and coassociative with a unit and a counit. A bialgebra is a **Hopf algebra** if it is supplied as well with an antipode, i.e., with an automorphism also satisfying some natural requirements. Nice examples of bialgebras and Hopf algebras arising in combinatorics can be found in [10].

Let us define the second operation, the **comultiplication** \( \mu: \mathcal{G} \to \mathcal{G} \otimes \mathcal{G} \) as follows. For a set \( V_1 \subseteq V(\Gamma) \) of vertices of a graph \( \Gamma \) let us denote by \( G(V_1) \) the induced subgraph of \( \Gamma \) with the set of vertices \( V_1 \); i.e., \( V_1 \) is the set of vertices of \( G(V_1) \), and \( e \in E(\Gamma) \) is an edge in \( G(V_1) \) iff both ends of \( e \) belong to \( V_1 \). We set

\[
\mu(\Gamma) = \sum_{V_1 \subseteq V(\Gamma)} G(V_1) \otimes G(V(G) \setminus V_1),
\]

where the sum is taken over all subsets \( V_1 \) of the set of vertices \( V(\Gamma) \). There are \( 2^{V(G)} \) summands in the right hand side of the previous equation. The comultiplication is extended to linear combinations of graphs by linearity. Figure 1 demonstrates an example of a comultiplication.

\[
\mu(\begin{array}{c}
\bullet \\
\end{array}) = 1 \otimes \begin{array}{c}
\bullet \\
\end{array} + 2 \bullet \otimes \begin{array}{c}
\bullet \\
\end{array} + \begin{array}{c}
\bullet \\
\end{array} \otimes \begin{array}{c}
\bullet \\
\end{array}
\]

\[
+ \begin{array}{c}
\bullet \\
\end{array} \otimes \begin{array}{c}
\bullet \\
\end{array} + 2 \begin{array}{c}
\bullet \\
\end{array} \otimes \begin{array}{c}
\bullet \\
\end{array} + \begin{array}{c}
\bullet \\
\end{array} \otimes 1
\]

**FIG. 1.** Coproduct of a graph.
The comultiplication, as well as the multiplication, respects the grading:

\[ \mu: G_n \to G_0 \otimes G_0 \otimes G_1 \otimes G_n \otimes G_{n-1} \otimes \cdots \otimes G_n \otimes G_0. \]

**Theorem 2.3.** The multiplication and the comultiplication defined above make the algebra of graphs into a commutative cocommutative bialgebra.

We omit purely technical verification of the bialgebra axioms.

The bialgebra of graphs is too complicated: it carries the same information as graphs themselves. It makes sense, therefore, to look for some reasonable quotient bialgebra, which is easier to handle.

Similarly to Tutte relation (1), one can consider the quotient space of graphs modulo the four-term relations

\[ \Gamma - \Gamma'_{AB} - \tilde{\Gamma}_{AB} + \tilde{\Gamma}'_{AB} = 0 \]  
for all graphs \( \Gamma \) and all pairs \( A, B \) of their vertices. We denote this space by \( F \). This space inherits a bialgebra structure from the bialgebra of graphs, and we call it the 4-bialgebra of graphs. The dual space \( F^* \) also carries a bialgebra structure, that of the dual bialgebra, and it will be called the bialgebra of 4-invariants.

By a slight abuse of language, below we shall not distinguish between a graph and the equivalence class it represents in \( F \).

In contrast to the case of Tutte relations, all terms in the four-term relation have the same number of vertices. Therefore, the number of vertices induces a grading in \( F \),

\[ F = F_0 \oplus F_1 \oplus F_2 \oplus \cdots, \]
where \( F_n \subset F, n = 0, 1, 2, \ldots \) is the subspace spanned by graphs with \( n \) vertices modulo the four-term relations.

**Theorem 2.4.** The multiplication and the comultiplication defined above induce a bialgebra structure on the space \( F \) of graphs modulo four-term relations.

**Proof.** The only thing we need to verify is that the multiplication and the comultiplication both respect the four-term relation (6). For the disjoint union of graphs this statement is obvious. In order to verify it for the comultiplication, it is sufficient to consider two different cases. Namely, let \( A, B \in V(\Gamma) \) be two distinct vertices of a graph \( \Gamma \). The right hand side summands in the comultiplication formula (5) split into two groups; those where both vertices \( A \) and \( B \) belong either to the subset \( V_1 \subset V(\Gamma) \) or to its complement \( V(\Gamma) \setminus V_1 \); and those where \( A \) and \( B \) belong to distinct subsets. Natural grouping of terms of the first kind for the coproduct
\( \mu(\Gamma - \Gamma'_{AB} - \tilde{\Gamma}_{AB} + \tilde{\Gamma}'_{AB}) \) provides zero as the result, while the terms of the second kind group to zero for each of the coproducts \( \mu(\Gamma - \Gamma'_{AB}) \) and \( \mu(\tilde{\Gamma}_{AB} - \tilde{\Gamma}'_{AB}) \).

The theorem is proved.

**Remark 2.5.** Yu. Vol’vovskii pointed out to me that the same multiplication and comultiplication induce a bialgebra structure on the space of graphs modulo the Tutte relations (1). Note, however, that this bialgebra is only filtered, not graded, since the terms in the Tutte relations have different numbers of vertices.

**Remark 2.6.** For a graph \( \Gamma \) denote by \( \Gamma^c \) the complementary graph to \( \Gamma \), i.e., the graph with the same set \( V(\Gamma^c) = V(\Gamma) \) of vertices, two vertices \( A, B \) being adjacent in \( \Gamma^c \) if they are disconnected in \( \Gamma \).

E. Soboleva directed my attention to the fact that passing to the complementary graph preserves the four-term relations. This fact allows one to define the complementary 4-bialgebra of graphs, isomorphic to the 4-bialgebra. The underlying space of this bialgebra is the same space \( \mathcal{F} \), and the comultiplication coincides with that for the 4-bialgebra. The multiplication \( m^c \) is given by the joint union of graphs (i.e., each pair of vertices \( A \in V(\Gamma_1), B \in V(\Gamma_2) \) in the product \( m^c(\Gamma_1, \Gamma_2) \) is connected by an edge).

According to the structure theorem (see [15, 18]), a commutative cocommutative bialgebra with unit over a field of characteristic zero is isomorphic to a polynomial bialgebra.

**Definition 2.7.** The polynomial bialgebra is the bialgebra with the algebra structure isomorphic to that of the polynomial algebra \( \mathbb{C}[x_1, x_2, \ldots] \), and the comultiplication defined on a monomial \( y_1 \cdots y_n \) (where each of the \( y_i \) is one of the variables \( x_1, x_2, \ldots \)) as follows:

\[
\mu(y_1 \cdots y_n) = 1 \otimes y_1 \cdots y_n + y_1 \otimes y_2 \cdots y_n + y_2 \otimes y_1 y_3 \cdots y_n + \cdots + y_1 \cdots y_n \otimes 1.
\]

The sum in the right hand side is taken over all subsets of the set \( \{1, \ldots, n\} \) of indices of \( y_i \).

Polynomial bialgebras are called in [10] multivariate divided powers bialgebras. Note that a polynomial bialgebra is automatically a Hopf algebra: one can define the antipode \( s: x_i \mapsto -x_i \) on all generators and extend it to the whole bialgebra by linearity.

If the bialgebra is graded, and the dimension of each grading subspace is finite, then the isomorphic polynomial bialgebra also can be chosen graded: generators are chosen to form bases in each grading subspace.
2.3. Examples of 4-Invariants

Probably the first non-trivial example of a 4-invariant that is not a Tutte invariant was suggested by Chmutov and Varchenko, in [8] (although the notion of a 4-invariant did not exist then). Let a vertex quadrangle in a graph \( I \) be a subset \( \{A_1, A_2, A_3, A_4\} \subseteq V(I) \) of the set of vertices consisting of four vertices satisfying the following requirement: there is a cyclic order \( (i, j, k, l) \) of the set of the indices \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) such that \( E(I) \) contains four edges \( A_iA_j, A_jA_k, A_kA_l, A_lA_i \). Then the number \( Q_v(I) \) of vertex quadrangles in \( I \) is a 4-invariant.

For example, one can check this statement for the four-term relation in Fig. 2: each of the graphs in the left hand side of the equality contains one vertex quadrangle, while there are no vertex quadrangles in either graph in the right hand side.

It is easy to see that the number of vertex quadrangles is not a Tutte invariant. Indeed, for the 4-cycle it equals 1, while for the other two terms in (1) it is zero.

In order to prove that the number of vertex quadrangles is indeed a 4-invariant, one can verify this first for the graphs with four vertices, which can be done by a direct computation. Denote by \( Q_v^4 \) the 4-invariant equal to the number of vertex quadrangles on all four-vertex graphs, and equal to zero on all graphs with different numbers of vertices. Now let \( U \) denote the function on graphs, identically equal to 1. This function descends to a 4-invariant since it obviously satisfies the four-term relation. Then we have \( Q_v = Q_v^4 U \); i.e., the vertex quadrangle invariant is the product of two invariants. A direct verification of the fact that the number of vertex quadrangles satisfies the four-term relation is also simple.

An edge quadrangle is a subset \( \{e_1, e_2, e_3, e_4\} \subseteq E(I) \) of the set of edges of \( I \) forming a quadrangle after an appropriate ordering. The number \( Q_e(I) \) of edge quadrangles is not a 4-invariant. However, it becomes a 4-invariant if considered modulo 2. For example, the first graph in the left hand side of Fig. 2 contains three edge quadrangles, the number of edge quadrangles in the second graph equals 1, while there are no edge quadrangles in either graph in the right hand side. Either of the proofs described above for the number of vertex quadrangles works in this case as well.

A similar statement is valid for the number \( P_e(I) \) of edge pentagons in a graph \( I \) considered mod 2. An edge pentagon is a cyclic chain of five

\[
\text{FIG. 2. Four-term relation for graphs with four vertices.}
\]
edges in $\Gamma$ with five distinct vertices. For example, the number of edge pentagons in the clique $K_5$ on five vertices is $P^*(K_5) = 12 \equiv 0 \mod 2$. Indeed, the number of different cyclic orderings on the set of five vertices is 24, and two different cyclic orderings result in the same chain of edges if they differ by reversing of the ordering.

Once again, this statement can be proved by direct computation on the set of 5-vertex graphs [17] and then the multiplication argument can be applied.

A slightly more sophisticated example of a 4-invariant, due to Yu. Vol’vovskii, is given by the following proposition.

Let us number the vertices of a graph $\Gamma$ and let $A(\Gamma)$ denote the adjacency matrix of $\Gamma$. This means that the entry $a_{ij}$ of $A(\Gamma)$ is 1 if the $i$th and the $j$th vertices are connected by an edge, and $a_{ij} = 0$ otherwise. Note that the adjacency matrix is symmetric.

**Proposition 2.8.** The rank of the adjacency matrix considered mod 2 is a 4-invariant.

**Proof.** We are going to prove a stronger statement, namely that the rank of the adjacency matrix considered modulo 2 is invariant under the mapping $\Gamma \mapsto \Gamma_{AB}$ for any graph $\Gamma$ and any pair of vertices $A, B \in V(\Gamma)$. In other words, the rank satisfies the two-term relation.

Indeed, let the vertex $A$ have the index 1 and the vertex $B$ have the index 2. Then the mapping $\Gamma \mapsto \Gamma_{AB}$ results, on the level of adjacency matrices, in the conjugation $A(\Gamma_{AB}) \equiv C^TA(\Gamma)C \mod 2$, where $C$ is the matrix

$$C = \begin{pmatrix}
1 & 1 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

and $C^T$ is the transpose matrix. In particular, this transformation does not change the rank of $A(\Gamma)$. Note, however, that this 4-invariant is not very strong.

2.4. Primitive Elements

**Definition 2.9.** An element $p$ of a bialgebra is called primitive if its coproduct is

$$\mu(p) = 1 \otimes p + p \otimes 1.$$

Primitive elements are, in a sense, the “simplest” elements of a bialgebra. For example, in a polynomial bialgebra $\mathbb{C}[x_1, x_2, \ldots]$ an element is
primitive if it is a linear combination of generators, $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$. This subsection is devoted to a description of primitive elements in the 4-bialgebra of graphs. Primitive elements form the linear space $\mathcal{P} \subset \mathcal{F}$, which is also graded,

$$\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \cdots.$$  

Note that the unit is not a primitive element since $\mu(1) = 1 \otimes 1 \neq 1 \otimes 1 + 1 \otimes 1$.

Let $P_k$ be the dimension of the space $\mathcal{P}_k$. After picking a basis $s_{1,k}$, $s_{2,k}$, ..., $s_{k,k}$ in each $\mathcal{P}_k$ we make the 4-bialgebra into the polynomial bialgebra in variables $s_{ij}$. Therefore, the crucial point in the description of the structure of the 4-bialgebra is the study of its primitive elements; especially, it is most important to find the sequence $P_1$, $P_2$, $P_3$, ....

Let us present some examples of primitive elements in the 4-bialgebra. The space $\mathcal{F}_1$ is spanned by the unique graph with one vertex. We denote this graph by $s_{1,1}$. This graph obviously is primitive, whence $P_1 = 1$.

The space $\mathcal{F}_2$ is spanned by the two two-vertex graphs: the segment and the graph consisting of two disjoint vertices. The four-term relations for two-vertex graphs are empty, whence dim $\mathcal{F}_2 = 2$. There is a unique (up to a multiplicative constant) primitive element in $\mathcal{F}_2$ (see Fig. 3(a)).

We denote this element by $s_{2,1}$. Its primitiveness can be verified directly from the definition. Therefore, $P_2 = 1$ as well. The space $\mathcal{F}_2$ is spanned by two elements $s_{2,1}$ and $s_{2,2}$.

In order 3 we meet the first non-trivial example of the 4-term relation. There are four distinct graphs with three vertices, and the only essential four-term relation is shown in Fig. 4.

Hence, dim $\mathcal{F}_3 = 3$. Note that, in contrast to the case of Tutte invariants, the graph $K_3$, the triangle, can be expressed as a linear combination of forests. The subspace $\mathcal{P}_3$ of primitive elements is spanned by the linear combination shown in Fig. 3(b).

We denote this element by $s_{3,1}$. Hence, $P_3 = 1$. The space $\mathcal{F}_3$ is thus spanned by the elements $s_{1,1}$, $s_{2,1}$, $s_{3,1}$.

\[\begin{align*}
\text{FIG. 3.} & \quad \text{Basic primitive elements (a) in } \mathcal{P}_2 \text{ and (b) in } \mathcal{P}_3. \\
\text{FIG. 4.} & \quad \text{Four-term relation for graphs with three vertices.}
\end{align*}\]
Similar computations in higher orders \[17\] give the following values of the dimensions: \( P_4 = 2, \ P_5 = 3, \ P_6 = 5 \). Computer computations elaborated by A. Kaishev show that \( P_7 = 7 \). Note that the computational complexity grows very quickly because both the number of graphs and the number of four-term relations are increasing. It is not easy even to identify the same four terms in relations obtained from different edges of different graphs.

Now we are going to describe a projection \( \pi_n : \mathcal{F}_n \to \mathcal{P}_n \) of the space of \( n \)-vertex graphs onto the subspace of primitive elements.

**Theorem 2.10.** For any graph \( \Gamma \in \mathcal{F}_n \) the element

\[
\pi_n(\Gamma) = \Gamma - 1! \sum_{V_1 \sqcup V_2 = V(\Gamma)} G(V_1) G(V_2) + 2! \sum_{V_3 \sqcup V_4 \sqcup V_5 = V(\Gamma)} G(V_1) G(V_2) G(V_3) - \cdots ,
\]

where the \( k \)th sum is taken over all partitions of the set \( V(\Gamma) \) of vertices of \( \Gamma \) into a disjoint union of nonempty subsets \( V_1, \ldots, V_k \), is primitive. The mapping \( \pi_n \) descends to a projection \( \pi_n : \mathcal{F}_n \to \mathcal{P}_n \); i.e., it is onto and \( \pi_n \circ \pi_n = \pi_n \).

The proof can be found in \[14\].

As an application of the previous theorem let us construct the primitive element \( \pi_n(C_4) \), where \( C_4 \) is the cycle on four vertices. The construction is shown in Fig. 5. Let us clarify how the coefficient \(-3\) at the edgeless graph in the right hand side of Fig. 5 can be obtained from Eq. (7). There is a unique way to split the vertices \( V(C_4) \) into two nonempty subsets \( V_1, V_2 \) such that both graphs \( G(V_1), G(V_2) \) are edgeless: each of the sets \( V_i \) must consist of opposite vertices of the square. There are two ways to split \( V(C_4) \) into three nonempty subsets so that all the three induced graphs are edgeless: the two vertices contained in one subset must be opposite vertices of the square. Finally, the unique way to split \( V(C_4) \) into four nonempty subsets leads to an edgeless graph. Hence, the coefficient at the edgeless graph in the projection must be

\[-1! \cdot 1 + 2! \cdot 2 - 3! \cdot 1 = -3.\]

Other coefficients are obtained in a similar way.

**Remark 2.11.** We can consider the 4-bialgebra over \( \mathbb{Z} \), not over a field of characteristic zero. In this case the structure theorem cannot be applied. However, since all the coefficients in (7) are integers, primitive elements of the 4-bialgebra generate it over \( \mathbb{Z} \) as well. What we do not know is whether there is torsion in the submodule of primitive elements. Up to now calculations show no torsion, but it can appear in higher orders.
3. BIALGEBRA OF ChORD DIAGRAMS

This section is devoted to a brief description of the bialgebra of chord diagrams and its connection with the 4-bialgebra of graphs. Details of the description can be found in [1, 3, 6, 12]. We also pose some problems related to the connection between the two bialgebras.

3.1. Four-Term Relation for Chord Diagrams

A chord diagram of order \( n \) is an oriented circle equipped with \( n \) pairs of distinct points considered up to orientation preserving diffeomorphisms of the circle. Points belonging to the same pair are usually connected by a chord in order to visualize the pairing (see Fig. 6(a)). On the pictures, the circle is always oriented counterclockwise.

V. A. Vassiliev invented chord diagrams for presenting singular knots in \( \mathbb{R}^3 \), i.e., knots admitting double points. Figure 7 demonstrates the four-term relation for chord diagrams. Here the dotted parts of the circles may contain arbitrary sets of chords that are, however, the same for all four diagrams.

Let \( D_n \) denote the space over \( \mathbb{C} \) spanned by all chord diagrams of order \( n \) modulo the four-term relations. Elements of the dual space \( D_n^* \) are usually called weight systems (or preinvariants, as in [6]) of order \( n \). Define the product of two chord diagrams of order \( n_1, n_2 \) as a chord diagram of order \( n = n_1 + n_2 \) that can be obtained by gluing together the two circles broken at arbitrary points distinct from the ends of the chords. The gluing must respect the orientation of the circles. It can be shown that, modulo the four-term relations, the resulting chord diagram does not depend on the choice of the breaking points.

FIG. 6. A chord diagram (a) and its intersection graph (b).
Denote the set of chords of a chord diagram $D$ by $V(D)$. Each subset $V_1 \subset V(D)$ of chords determines a chord diagram $D(V_1)$, which contains only chords belonging to $V_1$. The coproduct of a chord diagram $D \in \mathcal{D}_n$ looks like

$$\mu(D) = \sum_{V_1 \subset V(D)} D(V_1) \otimes D(V(D) \setminus V_1).$$

The following theorem belongs to Bar-Natan and Kontsevich [1].

**Theorem 3.1.** The multiplication and the comultiplication defined above make the space $\mathcal{D}$ into a commutative cocommutative bialgebra.

The structure theorem applied to this bialgebra states that it is also a polynomial bialgebra. Calculations started by Vassiliev and continued by Bar-Natan [1] and then by Kneissler [11] give the following values for the sequence $p_1, p_2, \ldots$ of dimensions of primitive elements subspaces.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_n$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>18</td>
<td>27</td>
<td>39</td>
<td>55</td>
</tr>
</tbody>
</table>

### 3.2. Intersection Graphs of Chord Diagrams

Let $v_1, v_2 \in V(D)$ be two distinct chords of a chord diagram $D$. We say that the chords $v_1, v_2$ intersect if their ends follow the circle in the alternating order.

**Definition 3.2.** The intersection graph $\gamma(D)$ of a chord diagram $D$ is specified by the following data:

- the vertices of $\gamma(D)$ are in one-to-one correspondence with the chords of the diagram $D$;
- two vertices $v_1, v_2$ are joined by an edge if the chords $v_1, v_2$ intersect.

The intersection graph for the chord diagram from Fig. 6(a) is shown on Fig. 6(b).

Note that not every graph can be obtained as the intersection graph of a chord diagram. The simplest counterexample is the 5-wheel, a graph with
Six vertices. Graphs that can be realized in this way are called circle, or alternance, graphs (see [4, 9], where algorithmic problems related to these graphs are studied). On the other hand, distinct chord diagrams can lead to the same intersection graph (see Fig. 8).

The four-term relation for chord diagrams makes their connection with intersection graphs even more complicated. Namely, explicit calculations show that, up to order 8, the equivalence class of a chord diagram modulo the four-term relation is totally determined by its intersection graph. This evidence led to our conjecture [6] that this is the case for diagrams of all orders. However, this conjecture proved to be false (T. Le, unpublished; see [5]).

The 4-bialgebra of graphs from Section 2 is invented to isolate properties of chord diagrams that are determined by their intersection graphs.

**Theorem 3.3.** The mapping \( \gamma : D \mapsto \gamma(D) \) associating to each chord diagram \( D \) its intersection graph \( \gamma(D) \) descends to a homomorphism \( \gamma : D \to \mathcal{F} \).

Note that the mapping \( \gamma \) preserves the grading and, therefore, it is the direct sum of linear mappings \( \gamma_n : D_n \to \mathcal{F}_n \).

As an immediate consequence of the theorem we obtain the following statement.

**Corollary 3.4.** Any 4-invariant \( f : \mathcal{F}_n \to A \) determines a weight system \( \gamma_n^* f : D_n \to A \).

Here \( A \) is an arbitrary Abelian group and \( \gamma_n^* f = f \circ \gamma_n \) as usual.

The proof of the theorem is obvious. The only thing we need to verify is that the mapping \( \gamma \) respects the four-term relation. But the rule for the four-term relation for graphs (6) is precisely what happens with the intersection of chords in all diagrams from Fig. 7.

In conclusion, let us state a conjecture about the properties of the mapping \( \gamma : D \to \mathcal{F} \). The statement of Le cited above shows that \( \gamma \) cannot be a monomorphism. We also know that not every graph can be obtained as the intersection graph of a chord diagram. Nevertheless, for all \( n \) the image of \( \gamma_n \) contains the \( n \)-clique (the graph with \( n \) vertices with each pair of vertices joined by an edge). The \( n \)-clique is, in a sense, “the most complicated”
Conjecture 3.5. The mapping $\gamma_n: \mathcal{D}_n \to \mathcal{F}_n$ is an epimorphism for all $n$.

Calculations up to order 7 [17] confirm this conjecture. Besides, one would expect the kernel of $\gamma_n$ to be “thin” in $\mathcal{D}_n$. Indeed, up to order 6 it is just empty, and in order 7 its dimension is one. However, up to now the evidence is too small to make a precise conjecture. Note that the study of this kernel is, probably, the most interesting part of the work. For example, only elements of this kernel (if any) may distinguish a chord diagram from its mirror image.

3.3. Weighted Invariants of Graphs

Here we explain the results from [7], where the first construction of Vassiliev invariants through the intersection graphs of chord diagrams was given. For proofs and details see [7]. A description of a bialgebra of graphs with both weighted vertices and weighted edges, which generalizes the construction from [7], can be found in [13].

Consider graphs (without loops and multiple edges) with weighted vertices, where the weight can take arbitrary positive integer values. The weight of a graph is the sum of the weights of all its vertices.

The weighted Tutte relation for weighted graphs coincides with the Tutte relation (1), where, however, the contraction $\Gamma \mapsto \Gamma'$ is understood differently. Namely, after contraction possible multiple edges are eliminated, and the weight of the resulting new vertex is set to be equal to the sum of the weights of the ends of the contracted edge $e$. All three terms in the weighted Tutte relation (1) have the same weight. Therefore, the linear space $\mathcal{W}$ generated by weighted graphs modulo the weighted Tutte relation is graded,

$$\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots,$$

with respect to the weight. The space $\mathcal{W}$ is endowed with a bialgebra structure in the same manner in which it was done in Section 2 for the 4-bialgebra of graphs.

Theorem 3.6 [7]. The bialgebra $\mathcal{W}$ is isomorphic to the polynomial bialgebra $\mathbb{C}[s_1, s_2, \ldots]$ through the mapping taking the one-vertex graph of weight $k$ to the variable $s_k$. The mapping $\Gamma_n: \mathcal{D}_n \to \mathcal{W}_n$ taking a chord diagram to its intersection graph with weights of all vertices set to 1 is a bialgebra epimorphism.
Thus, the bialgebra of weighted graphs is very small in comparison with both the bialgebra of chord diagrams and the 4-bialgebra of graphs: it contains precisely one primitive element of each order. It is a quotient bialgebra for both these bialgebras:

**Proposition 3.7.** Associate to a graph a weighted graph by assigning weight 1 to each of the vertices. Then this mapping descends to an epimorphism of the 4-bialgebra onto the bialgebra of weighted graphs.

The proof consists in showing that the contractions of the edge $AB$ in both sides of (6) lead to the same weighted graph (with the weight of the new vertex equal to 2).

Further developments in the study of the weighted bialgebra and a discussion of complexity problems arising there can be found in [16].

4. PROBLEMS

The number of open questions in the study of the 4-bialgebra exceeds the number of the solved ones dramatically. In the present section we formulate a few of them, including those that were stated above.

**Problem 1.** Describe the behavior of the sequence 1, 1, 1, 2, 3, 5, 7, ... of dimensions of the spaces $P_1, P_2, ...$ of primitive elements in the 4-algebra. In particular, find a reasonable algorithm generating this sequence.

**Problem 2.** Which of the known graph invariants are 4-invariants? Examples of such graph invariants would lead to new knot invariants.

**Problem 3.** Describe the kernel of the homomorphism $\gamma: D \rightarrow F$.

As we have seen above, this kernel is nontrivial already for $n = 7$.

**Problem 4.** If considered over $\mathbb{Z}$, does the 4-bialgebra have a torsion?

There is a construction of chord diagram invariants through Lie algebras due to Bar-Natan [1] and Kontsevich [12]. The vertex quadrangle 4-invariant $Q^v$ from Section 2.3 arose in the analysis of this construction for the Lie algebra $\text{sl}_2$ in [8]. E. Soboleva showed in [17] that the 4-invariant determined by the rank of the adjacency matrix (Proposition 2.8) coincides, essentially, with the one determined by the standard representation of the Lie algebra $\text{sl}_N$. It is known, however, that not all weight systems coming from Lie algebras determine 4-invariants (the fact that was used by Le in his disproof of the graph conjecture).
**Problem 5.** What Lie algebras determine 4-invariants of graphs? What are these 4-invariants?

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