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Stability of Semilinear Stochastic Evolution Equations*

AKIRA ICHIKAWA

Faculty of Engineering, Shizuoka University, Hamamatsu 432, Japan

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Stability of moments of the mild solution of a semilinear stochastic evolution equation is studied and sufficient conditions are given for the exponential stability of the pth moment in terms of Liapunov function. Sufficient conditions for sample continuity of the solution are also obtained and the exponential stability of sample paths is proved. Three examples are given to illustrate the theory.

INTRODUCTION

It is known that the semigroup theory gives a unified treatment of a wide class of parabolic, hyperbolic, and functional differential equations. So much effort has been devoted to the study of optimal control and filtering of evolution equations [2, 6]. From the sytem theory point of view, stability of stochastic evolution equations is also important. In the linear case this problem has been studied only recently [7, 11, 14, 19]. A necessary and sufficient condition for the exponential stability of the second moment is obtained in terms of a Liapunov equation in [11, 14, 19] and the asymptotic stability of sample paths is considered in [11, 7]. In this paper we shall develop our study further in two directions. First we shall consider a class of semilinear-stochastic evolution equations which is quite important in applications. Second, we shall consider not only the second moment, but more generally, higher ones.

In Section 1 we shall collect basic definitions and preliminary results on stochastic integrals and Ito's formula in Hilbert space. We take the Wiener process defined by Curtain and Falb [4, 5], but introduce a definition of stochastic integrals which is weaker than theirs. They impose uniform measurability on integrands. This is rather inconvenient when we consider the integrated version of a stochastic evolution equation such as Eq. (2.4), for we have only strong continuity for semigroups. Thus a more general definition is introduced in the case of nonrandom integrands [6]. Its extension to the case of random integrands, however, is neglected in the

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literature. So we shall introduce a definition which extends the one in [6] and which is based on the stochastic integrals with respect to real Wiener processes. The advantage of using such a definition is that we can use basic results and arguments in finite dimensions. It turns out that Ito's formula can be easily proved under a more general setting than the original one in [4]. In Section 2 we shall consider a semilinear-stochastic evolution equation whose nonlinear terms satisfy the Lipschitz conditions. We shall define two notions of a solution, strong and mild, and give the existence and uniqueness of a mild solution. We shall also prove that all moments of the solution are continuous in time if the initial condition is bounded. In Section 3 we shall take nonrandom initial conditions and consider the exponential stability of moments. We shall extend finite-dimensional results based on Liapunov functions to Hilbert spaces. We encounter a difficulty that we need strong solutions in order to use Ito's formula. We can dissolve this problem, however, by introducing approximating systems with strong solutions and using a limiting argument. The definition of a mild solution does not require its sample continuity, so in Section 4 we shall give sufficient conditions for it in terms of functions similar to Liapunov functions. Once this is guaranteed, it is meaningful to consider stability of sample paths. This is done in Section 5. We shall extend the results in [11, 7] to our model. Last, three examples are given in Section 6 to illustrate our theory.

1. Preliminaries

In this section we shall collect definitions and basic results from the probability theory in infinite dimensions. Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space. Let X and Z be Banach spaces.

DEFINITION 1.1. A map $x: \Omega \to X$ is a random variable if it is strongly measurable [12, p. 72].

Let $\mathscr{L}(X, Z)$ denote the space of bounded linear operators mapping X into Z. We write $\mathscr{L}(X)$ for $\mathscr{L}(X, X)$. Recall that there are three concepts of measurability for operator-valued functions [12]. Here we introduce two definitions.

DEFINITION 1.2. A map $R: \Omega \to \mathscr{L}(X, Z)$ is:

(a) a (strong) random variable if R is strongly measurable, i.e., Rx is a random variable for all $x \in X$ (in the sense of Definition 1.1),

(b) a uniform random variable if R is uniformly measurable, i.e., R is a random variable in $\mathscr{L}(X, Z)$ (in the sense of Definition 1.1).

Definition 1.2(b) is standard in the literature, but of course

Definition 1.2(a) is a weaker concept and in fact it turns out to be more convenient when we define stochastic integrals.

There is a notion of weak random variables based on weak measurability, but we shall not use it since it coincides with that of strong random variables in separable Banach spaces. For now we shall only consider random variables in Hilbert spaces except operator-valued ones. Let H and Y be real separable Hilbert spaces. We denote by $\langle\ ,\ \rangle$ inner products in Hilbert spaces and by $|\ |$ norms of vectors and operators. Let $y: \Omega \to Y$ be a square-integrable random variable, i.e., $y \in \mathscr{L}_2(\Omega, \mathscr{F}, \mu; Y)$. The covariance operator of y is

$$\mathrm{Cov}[y] = E[(y - Ey) \circ (y - Ey)],$$

where E denotes the expectation and $g \circ h \in \mathscr{L}(Y)$ for any $g, h \in Y$ is defined by

$$(g \circ h)\, k = g\langle h, k\rangle, \qquad k \in Y.$$

Then $\mathrm{Cov}[y]$ is a selfadjoint-nonnegative trace class (or nuclear) operator [2, 8] and $\mathrm{tr}\,\mathrm{Cov}[y] = E\,|y - Ey|^2 = E\,|y|^2 - |Ey|^2$, where tr denotes the trace. We note that $(y - Ey) \circ (y - Ey)$ is a uniform random variable. If $P \in \mathscr{L}(Y)$, then

$$\mathrm{tr}\, P\, \mathrm{Cov}[y] = \mathrm{tr}\, \mathrm{Cov}[Py, y] = E\langle P(y - Ey), y - Ey\rangle, \tag{1.1}$$

where $\mathrm{Cov}[x, y] = E[(x - Ex) \circ (y - Ey)]$ is the joint covariance of x and y. A random variable $y \in \mathscr{L}_2(\Omega, \mathscr{F}, \mu; Y)$ is Gaussian if $\langle y, e_i\rangle$ is a real Gaussian random variable for all i, where (e_i), $i = 1, 2, \ldots$, is a complete orthonormal basis for Y.

PROPOSITION 1.1. *Let y be a Gaussian random variable with $Ey = 0$ and covariance Q. Then $E\,|y|^{2n} \leqslant (2n-1)!!(\mathrm{tr}\, Q)^n$ for any integer n, where $(2n-1)!! = (2n-1)(2n-3) \cdots 5 \times 3 \times 1$ and the equality holds for $n = 1$.*

Proof. Appendix 1.

Let $\mathscr{T}$ be a subinterval of $[0, \infty)$. A stochastic process in Y is a family of random variables $y(t)$, $t \in \mathscr{T}$ in Y. A stochastic process $x(t)$, $t \in \mathscr{T}$, is a modification of $y(t)$ if for each $t \in \mathscr{T}$, $x(t) = y(t)$ with probability one. If two processes are a modification of each other, we regard them as equivalent. The process $y(t)$ is measurable if y is measurable relative to $\mathscr{B}(\mathscr{T}) \times \mathscr{F}$, where $\mathscr{B}(\mathscr{T})$ is the Borel field of subsets of $\mathscr{T}$. Let $\mathscr{F}_t$, $t \in \mathscr{T}$, be a family of increasing sub σ-fields of $\mathscr{F}$. A stochastic process $y(t)$, $t \in \mathscr{T}$, is adapted to $\mathscr{F}_t$ if $y(t)$ is $\mathscr{F}_t$ measurable for all $t \in \mathscr{T}$. It is a martingale with respect to $(\mathscr{F}_t)$ if it is adapted to $\mathscr{F}_t$ with properties:

(a) $E|y(t)| < \infty$ for all $t \in \mathcal{I}$,

(b) $E[y(t)|\mathcal{F}_s] = y(s)$ for all $s < t$, $s, t \in \mathcal{I}$, where $E[\cdot|\mathcal{F}_s]$ denotes the conditional expectation with respect to $\mathcal{F}_s$.

PROPOSITION 1.2 [18]. *If $y(t)$ is a martingale in Y relative to $(\mathcal{F}_t)$, then $|y(t)|$ is a real submartingale, i.e.,*

$$E[|y(t)|\,|\mathcal{F}(s)] \geqslant |y(s)| \qquad \text{for any } s < t, \quad s, t \in \mathcal{I}.$$

DEFINITION 1.3 [4–6]. A stochastic process $w(t)$, $t \geqslant 0$, in a real separable Hilbert space H is a Wiener process if:

(a) $w(t) \in \mathcal{L}_2(\Omega, \mathcal{F}, \mu; H)$ and $Ew(t) = 0$ for all $t \geqslant 0$,

(b) $\mathrm{Cov}[w(t) - w(s)] = (t-s)\,W$, $W \in \mathcal{L}(H)$ is a nonnegative nuclear operator,

(c) $w(t)$ has continuous sample paths,

(d) $w(t)$ has independent increments.

The operator W is the incremental covariance operator of the Wiener process $w(t)$.

Let $\sigma_t[w(\cdot)]$ be the σ-field generated by $w(s)$, $0 \leqslant s \leqslant t$, then $w(t)$ is a martingale relative to $\sigma_t[w(\cdot)]$. We have the following representation of a Wiener process [6]:

PROPOSITION 1.3. *Let $w(t)$ be a Wiener process in H with incremental covariance operator W, then*

$$w(t) = \sum_{i=1}^{\infty} \beta_i(t)\, e_i, \tag{1.2}$$

where (e_i) is an orthonormal set of eigenvectors of W, $\beta_i(t)$ are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $We_i = \lambda_i e_i$ and $\mathrm{tr}\, W = \sum_{i=1}^{\infty} \lambda_i$.

COROLLARY 1.1. *Let $w(t)$ be a Wiener process in H with incremental covariance W. Then*

$w(t)$ is Gaussian for all $t > 0$, (1.3a)

$E|w(t) - w(s)|^{2n} \leqslant (2n-1)!!(t-s)^n(\mathrm{tr}\, W)^n$, where the equality holds for $n = 1$. (1.3b)

Proof. Assertion (a) is shown in [6] and (b) follows from Proposition 1.1.

Next we introduce stochastic integrals with respect to $w(t)$. Let $\mathscr{T} = [0, T]$, $0 < T < \infty$. Let $(\mathscr{F}_t)$ be a family of increasing sub σ-fields of $\mathscr{F}$ such that

$$w(t) \text{ is measurable relative to } \mathscr{F}_t \text{ for each } t \in \mathscr{T}, \tag{1.4a}$$

$$w(t) - w(s) \text{ is independent of } \mathscr{F}_s \text{ for all } s < t,\ s, t \in \mathscr{T}. \tag{1.4b}$$

First we consider stochastic integrals with respect to $\beta_i(t)$ given in (1.2). Let $f_i(t)$ be a measurable stochastic process in Y which is adapted to $\mathscr{F}_t$ with $\int_0^T |f_i(t)|^2\,dt < \infty$ wp 1 (with probability one). Then we can define the stochastic integral $\int_0^t f_i(t)\,d\beta_i(r)$ as in the scalar case and it has a version with continuous sample paths. We can easily prove as in [9, 10]:

LEMMA 1.1. *Suppose that* $\int_0^T E\,|f_i(t)|^2\,dt < \infty$. *Then*

$$E\int_0^T f_i(t)\,d\beta_i(t) = 0, \tag{1.5a}$$

$$E\left|\int_0^T f_i(t)\,d\beta_i(t)\right|^2 = \lambda_i \int_0^T E\,|f_i(t)|^2\,dt, \tag{1.5b}$$

$$\begin{aligned} E\left[\left(\int_0^t f_i(r)\,d\beta_i(r)\right) \circ \left(\int_0^t f_j(r)\,d\beta_j(r)\right)\right] &= \lambda_i \int_0^t E[f_i(r) \circ f_i(r)]\,dr, && \text{if } i = j, \\ &= 0, && \text{if } i \neq j, \end{aligned} \tag{1.5c}$$

$$E\left[\left(\int_s^t f_i(r)\,d\beta_i(r)\right) \circ \left(\int_u^v f_j(r)\,d\beta_j(r)\right)\right] = 0 \quad \text{for all } i, j \text{ if } s \leqslant t \leqslant u \leqslant v. \tag{1.5d}$$

LEMMA 1.2. *Suppose that* $\int_0^T |f_i(t)|^2\,dt < \infty$ *wp* 1. *Then*

$$\mu\left[\sup_{0 \leqslant t \leqslant T}\left|\sum_{i \in \mathscr{I}} \int_0^t f_i(r)\,d\beta_i(r)\right| > c\right] \leqslant \mu\left[\bigcup_{i \in \mathscr{I}}\left(\int_0^T |f_i(r)|^2\,dr > N\right)\right] + \frac{N}{c^2}\sum_{i \in \mathscr{I}} \lambda_i \tag{1.6}$$

for any positive numbers c and N and for any subset $\mathscr{I}$ *of integers.*

Proof. Appendix 2.

Let $\mathcal{M}(H, Y)$ be the space of stochastic process $G(\cdot, \cdot): \mathcal{I} \times \Omega \to \mathcal{L}(H, Y)$ which are strongly measurable, i.e., $G(t, \cdot)\, h$ is a measurable stochastic process for all $h \in H$. Define also

$$\begin{aligned} \mathcal{M}_1(H, Y) &= \left\{ G \in \mathcal{M}(H, Y) \;\middle|\; E \int_0^T |G(t)|^2 \, dt < \infty \right\} \\ \mathcal{M}_2(H, Y) &= \left\{ G \in \mathcal{M}(H, Y) \;\middle|\; \int_0^T |G(t)|^2 \, dt < \infty \text{ wp } 1 \right\}. \end{aligned} \tag{1.7}$$

We now define the stochastic integral $\int_0^T G(t)\, dw(t)$ for $G \in \mathcal{M}_1(H, Y)$. Let $w_n(t) = \sum_{i=1}^n e_i \beta_i(t)$. Then in view of Lemma 1.1 the stochastic integral

$$\int_0^T G(t)\, dw_n(t) = \sum_{i=1}^n \int_0^T G(t)\, e_i \, d\beta_i(t)$$

is well defined for $G \in \mathcal{M}_1(H, Y)$. Denote by y_n the stochastic integral above. Then (y_n) is a Cauchy sequence in $\mathcal{L}_2(\Omega, \mathcal{F}, \mu; Y)$. In fact using Lema 1.1 and (1.1) we have for any integer m, n with $m < n$

$$\begin{aligned} E\, |y_m - y_n|^2 &= \sum_{i=n+1}^m \lambda_i E \int_0^T \langle G(t)\, e_i, G(t)\, e_i \rangle \, dt \\ &\leqslant \left(\sum_{i=n+1}^m \lambda_i \right) E \int_0^T |G(t)|^2 \, dt \to 0 \qquad as \quad n \to \infty. \end{aligned}$$

Hence there exists a limit and we define by it the stochastic integral $\int_0^T G(t)\, dw(t)$. If $0 \leqslant t_0 \leqslant t \leqslant T$ we define by

$$\int_{t_0}^t G(r)\, dw(r) = \int_0^T 1_{[t_0, t]}(r)\, G(r)\, dw(r),$$

where $1_{[t_0, t]}$ is the characteristic function of the set $[t_0, t]$. The derivation of the following properties is immediate from Lemma 1.1.

PROPOSITION 1.4. *Let $G, F \in \mathcal{M}_1(H, Y)$. Then*

$$E \int_0^T G(t)\, dw(t) = 0, \tag{1.8a}$$

$$E\left|\int_0^T G(t)\,dw(t)\right|^2 = \int_0^T E(\operatorname{tr} G(t)\,WG^*(t))\,dt, \qquad \leqslant \operatorname{tr} W\int_0^T E\,|G(t)|^2\,dt,$$
$$= \int_0^T E(\operatorname{tr} G^*(t)\,G(t)\,W)\,dt,$$

(1.8b)

$$E\left[\left(\int_0^t G(r)\,dw(r)\right)\circ\left(\int_0^t F(r)\,dw(r)\right)\right] = \int_0^t EG(r)\,WF^*(r)\,dr, \quad (1.8c)$$

$$E\left[\left(\int_s^t G(r)\,dw(r)\right)\circ\left(\int_u^v F(r)\,dw(r)\right)\right] = 0$$
$$\textit{for any } 0\leqslant s\leqslant t\leqslant u\leqslant v\leqslant T. \quad (1.8d)$$

PROPOSITION 1.5. *Let* $G\in\mathscr{M}_1(H, Y)$ *and let* $y(t)=\int_0^t G(r)\,dw(r)$. *Then* $(y(t),\mathscr{F}_t)$ *is a martingale and* $y(t)$ *has a modification with continuous sample paths.*

Proof. See the proof of Proposition 1.7.

PROPOSITION 1.6. *Let* $G\in\mathscr{M}_1(H, Y)$, *then*:

$$\mu\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t G(r)\,dw(r)\right| > c\right] \leqslant \frac{1}{c^2}E\left|\int_0^T G(r)\,dw(r)\right|^2$$
$$\leqslant \frac{\operatorname{tr} W}{c^2}\int_0^T E\,|G(r)|^2\,dr, \quad (1.9a)$$

$$E\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t G(r)\,dw(r)\right|^2\right] \leqslant 4E\left|\int_0^T G(r)\,dw(r)\right|^2$$
$$\leqslant 4\operatorname{tr} W\int_0^T E\,|G(r)|^2\,dr, \quad (1.9b)$$

$$E\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t G(r)\,dw(r)\right|\right] \leqslant 3E\left[\int_0^T \operatorname{tr} G(r)\,WG^*(r)\,dr\right]^{1/2}. \quad (1.9c)$$

Proof. Since $|\int_0^t G(r)\,dw(r)|^2$ is a submartingale, (a) and (b) follows from Doob's inequality. Assertion (c) is also a consequence of the general inequality for martingales [16, 17].

Finally we define a stochastic integral for $G\in\mathscr{M}_2(H, Y)$. The sequence of random variables

$$y_n = \int_0^T G(t)\,dw_n(t) = \sum_{i=1}^n \int_0^T G(t)\,e_i\,d\beta_i(t)$$

is well defined. By Lemma 1.2 we obtain for any c and N

$$\mu\left[\sum_{i=n+1}^{m}\left|\int_0^T G(t)\,e_i\,d\beta_i(t)\right|>c\right]$$
$$\leqslant\frac{N}{c^2}\sum_{i=n+1}^{m}\lambda_i+\mu\left[\bigcup_{i=n+1}^{m}\left(\int_0^T|G(r)\,e_i|^2\,dr>N\right)\right]$$
$$\leqslant\frac{N}{c^2}\sum_{i=n+1}^{m}\lambda_i+\mu\left[\int_0^T|G(t)|^2\,dt>N\right].$$

First taking N sufficiently large and then taking n large enough we can make the right-hand side arbitrarily small. Thus (y_n) is a Cauchy sequence in the sense of convergence in probability. Hence there exists a limit and we define by it the stochastic integral $\int_0^T G(t)\,dw(t)$. We can define the stochastic integral $\int_{t_0}^t G(r)\,dw(r)$ for $0\leqslant t_0\leqslant t\leqslant T$ as before. Let $C(0, T; Y)$ be the space of continuous functions in Y with sup norm.

PROPOSITION 1.7. *Let $G\in\mathscr{M}_2(H, Y)$. Then:*

(a) *Let $y_n(t)=\int_0^t G(r)\,dw_n(r)$ be a version with continuous sample paths. Then there exists a subsequence which converges in $C(0, T; Y)$ with probability one. The limit process is a modification of $y(t)=\int_0^t G(r)\,dw(r)$.*

(b) *Let G, $G_n\in\mathscr{M}_2(H, Y)$. Suppose that $G_n\to G$ strongly almost everywhere on $[0, T]\times\Omega$ and $|G_n|\leqslant k$ for some stochastic process k with $k(\cdot,\omega)\in\mathscr{L}_2(0, T)$* wp 1. *Then*

$$z_n(t)=\int_0^t G_n(r)\,dw_n(r)$$

converges to $y(t)$ in probability in $C(0, T; Y)$.

Proof. Appendix 3.

PROPOSITION 1.8. *Let $\mathscr{I}=[0, T]$ and let $G\colon\mathscr{I}\times\mathscr{I}\times\Omega\to\mathscr{L}(H, Y)$ be strongly measurable such that $G(s, t)$ is $\mathscr{F}_t$ measurable for each s and*

$$\int_0^T\int_0^T|G(t, s)|^2\,ds\,dt<\infty\qquad\text{wp}\quad 1.$$

Then

$$\int_0^T\int_0^T G(t, s)\,dw(s)\,dt=\int_0^T\int_0^T G(t, s)\,dt\,dw(s)\qquad\text{wp}\quad 1,\tag{1.10}$$

where we interpret the right-hand side as $\sum_{i=1}^{\infty}\int_0^T\int_0^T G(t, s)\,e_i\,dt\,d\beta_i(s)$.

Proof. If we replace $w(t)$ by $w_n(t) = \sum_{i=1}^n e_i \beta_i(t)$, then (1.10) follows as in [5] and the general case then follows by passing to the limit.

Remark 1.1. Curtain and Falb [4, 5] defined the stochastic integral when the integrand $G(t)$ is uniformly measurable. Their definition is a special case of ours and both integrals coincide under their assumption.

Now we introduce Ito's formula. Let $(\mathscr{F}_t)$ be an increasing family of sub σ-fields of $\mathscr{F}$ with property (1.4). A stochastic process $y(t)$ is said to have a stochastic differential on $[t_0, T]$, $t_0 \geqslant 0$ if

$$y(t) = y_0 + \int_{t_0}^{t} g(r)\, dr + \int_{t_0}^{t} G(r)\, dw(r), \tag{1.11}$$

where y_0 is $\mathscr{F}_{t_0}$ measurable, $g(t)$ is Y-valued and adapted to $\mathscr{F}_t$ with $\int_{t_0}^T |g(t)|\, dt < \infty$ wp 1 and $G \in \mathscr{M}_2(H, Y)$. Let Z be a Hilbert space and let $P(\cdot, \cdot) \in \mathscr{L}(Y \times Y, Z)$ and $G \in \mathscr{L}(H, Y)$. We define

$$\operatorname{tr} P[G; W] = \sum_{i=1}^{\infty} \lambda_i P(Ge_i, Ge_i) \in Z. \tag{1.12}$$

We have Ito's formula in Hilbert space.

THEOREM 1.1. *Let H, Y, and Z be real separable Hilbert spaces. Suppose that $v(t, y): \mathscr{I} \times Y \to Z$ is continuous with properties*:

$v(t, y)$ is differentiable in t and $v_t(t, y)$ is continuouss on $I \times Y$, (1.13a)

$v(t, y)$ is twice Fréchet differentiable in y and $v_y(t, y)\, y_1 \in Z$, $v_{yy}(t, y)(y_1, y_2) \in Z$ are continuous on $\mathscr{I} \times Y$ for all $y, y_1, y_2 \in Y$, (1.13b)

where $\mathscr{I} = [t_0, T]$. If $y(t)$ is given by (1.11), *then $z(t) = v(t, y(t))$ has the stochastic differential*

$$\begin{aligned} dz(t) = \{&v_t(t, y(t)) + v_y(t, y(t))\, g(t) \\ &+ \tfrac{1}{2} \operatorname{tr} v_{yy}(t, y(t))[G(t); W]\}\, dt \\ &+ v_y(t, y(t))\, G(t)\, dw(t). \end{aligned} \tag{1.14}$$

We first consider the case $z_n(t) = v(t, y_n(t))$, with

$$y_n(t) = y_0 + \int_{t_0}^{t} g(r)\, dr + \int_{t_0}^{t} G(r)\, dw_n(r). \tag{1.15}$$

LEMMA 1.3. *Let $v(t,y)$ satisfy* (1.13) *and let $y_n(t)$ be given by* (1.15). Then

$$dz_n(t) = \Big\{ v_t(t, y_n(t)) + v_y(t, y_n(t))\, g(t) + \tfrac{1}{2} \sum_{i=1}^{n} \lambda_i v_{yy}(t, y_n(t))(G(t)\, e_i, G(t)\, e_i) \Big\}\, dt + v_y(t, y_n(t))\, G(t)\, dw_n(t). \tag{1.16}$$

Proof. Appendix 4.

Proof of Theorem 1.1. By Proposition 1.7(a) there exists a subsequence $y_{n_k}(t)$ which converges to $y(t)$ uniformly on $\mathscr{I}$ wp 1. Since $|v_y(\cdot, y_{n_k}(\cdot))|$ is uniformly bounded on $\mathscr{I}$ wp 1 we can apply Proposition 1.7(b) to conclude that $\int_{t_0}^{t} v_y(r, y_{n_k}(r))\, G(r)\, dw_{n_k}(r)$ converges in probability to $\int_{t_0}^{t} v_y(r, y(r))\, G(r)\, dw(r)$ in $C(0, T; Y)$. Now taking a subsequence, again denoted by n_k, we can replace the convergence in probability by the convergence wp 1. Thus (1.14) follows from (1.16) by taking limits along the subsequence n_k.

COROLLARY 1.2. *If, in particular, $Z = R^1$, then*

$$dz(t) = [v_t\, y(t)) + \langle v_y(t, y(t)), g(t) \rangle + \tfrac{1}{2} \operatorname{tr} G(t)\, WG^*(t)\, v_{yy}(t, y(t))]\, dt + \langle v_y(t, y(t)), G(t)\, dw(t) \rangle. \tag{1.17}$$

In applications to stochastic evolution equations we need the following:

COROLLARY 1.3. *Let A be a closed linear operator with dense domain $\mathscr{D}(A)$ in Y. Let $v(t,y)$ satisfy the assumptions in Theorem* 1.1 *except* (a) *which is replaced by*

(a′) *$v(t,y)$ is differentiable in t for each $y \in \mathscr{D}(A)$ and $v_t(t,y)$ is continuous on $\mathscr{I} \times \mathscr{D}(A)$, where $\mathscr{D}(A)$ is equipped with the graph norm of A, i.e., $|y|^2_{\mathscr{D}(A)} = |y|^2 + |Ay|^2$.*

Let $y(t)$ be given by (1.11) *with $y_0 \in \mathscr{D}(A)$, $\int_{t_0}^{T} |Ag(t)|\, dt < \infty$* wp 1 *and $AG \in \mathscr{M}_2(H, Y)$. Then the conclusion of Theorem* 1.1 *holds.*

We can prove this using Lemma 1.3. But in this case we assume that g, $f_i \in \mathscr{D}(A)$ in Appendix 4. Then we can repeat the proof. We use the new assumption (a′) when we consider terms associated with $v_t\ (t, y_n(t))$.

We can use Ito's formula to estimate moments of a stochastic integral.

Proposition 1.9. *Let* $G \in \mathscr{M}(H, Y)$ *with* $\int_0^T E|G(t)|^p\,dt < \infty$ *some integer* $p \geq 2$, *and let* $y(t) = \int_0^t G(r)\,dw(r)$. Then

$$E|y(t)|^p \leqslant [\tfrac{1}{2}p(p-1)]^{p/2}\left[\int_0^t [E(\operatorname{tr} G(r)WG^*(r))^{p/2}]^{2/p}\,dr\right]^{p/2}$$

$$\leqslant [\tfrac{1}{2}p(p-1)]^{p/2}(\operatorname{tr} W)^{p/2}\,t^{p/2-1}\int_0^t E|G(r)|^p\,dr. \tag{1.18}$$

Proof. Note that derivatives of $v(y) = |y|^p$ are given by $v_y(y) = p|y|^{p-2}y$, $v_{yy}(y) = p|y|^{p-2}I + p(p-2)|y|^{p-4}y \circ y$, I the identity. Then by Theorem 1.1 we obtain

$$|y(t)|^p = p\int_0^t |y(r)|^{p-2}\langle y(r), G(r)\,dw(r)\rangle$$

$$+\frac{p}{2}\int_0^t [|y(r)|^{p-2}\operatorname{tr} G(r)WG^*(r)$$

$$+(p-2)|y(r)|^{p-4}\operatorname{tr} G(r)WG^*(r)y(r)\circ y(r)]\,dr.$$

Now we assume that $G(t)e_i$ are step functions with $|G(t)e_i| \leqslant c < \infty$. Then by Eq. (1.3b) we see that $E|y(t)|^n < \infty$ for all $t \in \mathscr{T}$ and all integer n. Thus we can take expectations in the above equation to obtain

$$E|y(t)|^p = \frac{p}{2}\int_0^t E[|y(r)|^{p-2}\operatorname{tr} G(r)WG^*(r)$$

$$+(p-2)|y(r)|^{p-4}\operatorname{tr} G(r)WG^*(r)y(r)\circ y(r)]\,dr$$

$$\leqslant \frac{p(p-1)}{2}\int_0^t E[|y(r)|^{p-2}\operatorname{tr} G(r)WG^*(r)]\,dr$$

$$\leqslant \frac{p(p-1)}{2}\int_0^t (E|y(r)|^p)^{1-2/p}[E(\operatorname{tr} G(r)WG^*(r))^{p/2}]^{2/p}\,dr$$

$$\leqslant \frac{p(p-1)}{2}(E|y(t)|^p]^{1-2/p}\int_0^2 [E(\operatorname{tr} G(r)WG^*(r))^{p/2}]^{2/p}\,dr$$

since $E|y(t)|^p$ is monotone increasing in t. Hence

$$E|y(t)|^p \leqslant \left[\frac{p(p-1)}{2}\right]^{p/2}\left[\int_0^t [E(\operatorname{tr} G(r)WG^*(r))^{p/2}]^{2/p}\,dr\right]^{p/2}$$

$$\leqslant \left[\frac{p(p-1)}{2}\right]^{p/2} t^{p/2-1}\int_0^t E(\operatorname{tr} G(r)WG^*(r))^{p/2}\,dr$$

$$\leqslant \left[\frac{p(p-1)}{2}\operatorname{tr} W\right]^{p/2} t^{p/2-1}\int_0^t E|G(r)|^p\,dr.$$

The general case then follows from this by a limiting argument.

We give another application of Theorem 1.1.

PROPOSITION 1.10. *Let $S(t)$ be a strongly continuous semigroup on a real Hilbert space Y with generator A and $T(t)$ a strongly continuous group on Y with generator B. We assume that $\mathscr{D}(A) \subset \mathscr{D}(B^2)$, $S(t)\colon \mathscr{D}(A) \to \mathscr{D}(A)$ and that $S(t)$ and $T(t)$ commute. Then $y(t) = S(t)\, T(\beta(t))\, y_0$, $y_0 \in \mathscr{D}(A)$ is a solution of*

$$dy(t) = (A + \tfrac{1}{2}B^2)\, y(t)\, dt + By(t)\, d\beta(t), \qquad y(0) = y_0. \tag{1.19}$$

where $\beta(t)$ is a real standard Wiener process.

Proof. We apply Theorem 1.1 to the function $v(t, x) = S(t)\, T(x)\, y_0$ and the process $\beta(t)$. Then $v_t(t, x) = AS(t)\, T(x)\, y_0$, $v_x(t, x) = BS(t)\, T(x)\, y_0$, and $v_{xx}(t, x) = B^2 S(t)\, T(x)\, y_0$ and (1.19) follows.

Remark 1.2. The general theory of stochastic integrals based on margingales is given in [15–17]. Here we have restricted ourselves to the case of stochastic integrals with respect to Wiener processes and we have tried to extend finite-dimensional results [10] to Hilbert spaces.

2. STOCHASTIC EVOLUTION EQUATIONS

Let H and Y be real separable Hilbert spaces and let $w(t)$ be a Wiener process in H with incremental covariance W. Let $(\mathscr{F}_t)$ be an increasing family of sub σ-fields of $\mathscr{F}$ with property (1.4). Consider the stochastic evolution equation

$$dy(t) = [Ay(t) + f(y(t))] + G(y(t))\, dw(t), \qquad y(0) = y_0, \tag{2.1}$$

on $\mathscr{I} = [0, T]$, where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on Y and $f\colon Y \to Y$ and $G\colon Y \to L(H, Y)$ satisfy the Lipschitz condition

$$\begin{aligned} |f(y) - f(z)| &\leqslant c_1 |y - z|, \qquad c_1 > 0, \quad y, z \in Y, \\ |G(y) - G(z)| &\leqslant c_2 |y - z|, \qquad c_2 > 0, \quad y, z \in Y, \end{aligned} \tag{2.2}$$

and y_0 is $\mathscr{F}_0$ measurable.

We introduce two notions of a solution of (2.1) (see [6]).

DEFINITION 2.1. A stochastic process $y(t)$, $t \in \mathscr{I}$, is a strong solution of (2.1) if

$$y(t) \text{ is adapted to } \mathscr{F}_t, \tag{2.3a}$$

$$y(t) \text{ is continuous in } t \text{ wp } 1, \tag{2.3b}$$

$$y(t) \in \mathscr{D}(A) \text{ almost everywhere on } \mathscr{I} \times \Omega \text{ and } \int_0^T |Ay(t)|\, dt < \infty \text{ wp } 1, \tag{2.3c}$$

$$y(t) = y_0 + \int_0^t Ay(r)\, dr + \int_0^t f(y(r))\, dr + \int_0^t G(y(r))\, dw(r) \tag{2.3d}$$

for all $t \in \mathscr{I}$ wp 1.

In general this concept is rather strong and we need a weaker one in later applications.

DEFINITION 2.2. A stochastic process $y(t)$, $t \in \mathscr{I}$, is a mild solution of (2.1) if

$$y(t) \text{ is adapted to } \mathscr{F}_t, \tag{2.4a}$$

$$y(t) \text{ is measurable and } \int_0^T |y(t)|^2\, dt < \infty \text{ wp } 1, \tag{2.4b}$$

$$y(t) = S(t)\, y_0 + \int_0^t S(t-r) f(y(r))\, dr + \int_0^t S(t-r)\, G(y(r))\, dw(r) \tag{2.4c}$$

for all $t \in \mathscr{I}$ wp 1.

Remark 2.1. The integrand of the stochastic integral in (2.3) is uniformly measurable, but generally the one in (2.4) is only strongly measurable and the definition in Section 1 is appropriate.

PROPOSITION 2.1. *If* $y(t)$, $t \in \mathscr{I}$, *is a strong solution of* (2.1), *then it is also a mild solution.*

Proof. We apply Corollary 1.3 to the map $v(t, y) = S(s-t)\, y$ and the process $y_\lambda(t) = R(\lambda, A)\, y(t)$, where $R(\lambda, A)$ is the resolvent of A (see [12]). Since $v_t(t, y) = -S(s-t)\, Ay$ for any $y \in \mathscr{D}(A)$, $v_y(t, y) = S(s-t)$, and $v_{yy}(t, y) = 0$, we have

$$v(s, y_\lambda(s)) - v(0, y_\lambda(0)) = \int_0^s S(s-r)\, R(\lambda, A) f(y(r))\, dr + \int_0^s S(s-r)\, R(\lambda, A)\, G(y(r))\, dw(r).$$

Thus

$$R(\lambda, A)\, y(s) = R(\lambda, A)\left[S(s)\, y_0 + \int_0^s S(s-r) f(y(r))\, dr + \int_0^s S(s-r)\, G(y(r))\, dw(r)\right].$$

Hence we obtain

$$y(s) = S(s)\, y_0 + \int_0^s S(s-r) f(y(r))\, dr + \int_0^s S(s-r)\, G(y(r))\, dw(r).$$

PROPOSITION 2.2. *There exists at most one mild solution of* (2.1).

Proof. Suppose that $y_i(t)$, $i = 1, 2$, is a solution of (2.1). Set $y(t) = y_1(t) - y_2(t)$, then

$$y(t) = \int_0^t S(t-r)[f(y_1(r)) - f(y_2(r))]\, dr + \int_0^t S(t-r)[G(y_1(r)) - G(y_2(r))]\, dw(r).$$

Define

$$y_i^N(t) = y_i(t), \qquad \text{if } \int_0^t |y(r)|^2\, dr \leqslant N,$$

$$= 0, \qquad \text{if } \int_0^t |y(r)|^2\, dr > N.$$

and let $y^N(t) = y_1^N(t) - y_2^N(t)$. Then

$$|y^N(t)|^2 \leqslant 2\left|\int_0^t S(t-r)[f(y_1^N(r)) - f(y_2^N(r))]\, dr\right|^2 + 2\left|\int_0^t S(t-r)G(y_1^N(r)) - G(y_2^N(r))]\, dw(r)\right|^2$$

$$\leqslant 2Mc_1\sqrt{T}\int_0^t |y^N(r)|^2\, dr + 2\left|\int_0^t S(t-r)[G(y_1^N(r)) - G(y_2^N(r))]\, dw(r)\right|,$$

where M is a constant such that $|S(t)| \leqslant M$ on $[0, T]$ and we have used (2.2). Then

$$E\,|y^N(t)|^2 \leqslant 2Mc_1\sqrt{T}\int_0^t E\,|y^N(r)|^2\, dr + 2Mc_2^2 \operatorname{tr} W \int_0^t E\,|y^N(r)|^2\, dr.$$

Hence by Gronwall's inequality $E\,|y^N(t)|^2 = 0$ for all t. Thus $y^N(t) = 0$. Since $y_i^N(t) \to y_i(t)$ wp 1, $y^N(t) \to y(t)$ wp 1, from which follows $y(t) = 0$.

Next we give sufficient conditions for a mild solution to be also a strong solution.

PROPOSITION 2.3. *Suppose that*

(a) $y_0 \in \mathscr{D}(A)$ *wp* 1, $S(t-r)f(y) \in \mathscr{D}(A)$, $S(t-r)\,G(y)\,h \in \mathscr{D}(A)$ *for each* $y \in Y$, $h \in H$, *and* $t > r$,

(b)) $|AS(t-r)f(y)| \leqslant g_1(t-r)\,|y|$, $g_1 \in \mathscr{L}_1(0, T)$,

(c) $|AS(t-r)\,G(y)| \leqslant g_2(t-r)\,|y|$, $g_2 \in \mathscr{L}_2(0, T)$.

Then a mild solution $y(t)$ *is also a strong solution.*

Proof. By the above conditions we have

$$\int_0^T \int_0^t |AS(t-r)f(y(r))|\,dr\,dt < \infty \qquad \text{wp } 1$$

and

$$\int_0^T \int_0^t |AS(t-r)\,G(y(r))|^2\,dr\,dt < \infty.$$

Thus by Fubini's theorem we have

$$\int_0^t \int_0^s AS(s-r)f(y(r))\,dr\,ds = \int_0^t \int_r^t AS(s-r)f(y(r))\,ds\,dr$$

$$= \int_0^t S(t-r)f(y(r))\,dr - \int_0^t f(y(r))\,dr.$$

By Proposition 1.8 we also have

$$\int_0^t \int_0^s AS(s-r)\,G(y(r))\,dw(r)\,ds = \int_0^t \int_r^t AS(s-r)\,G(y(r))\,ds\,dw(r)$$

$$= \int_0^t S(t-r)\,G(y(r))\,dw(r)$$

$$- \int_0^t G(y(r))\,dw(r).$$

Hence $Ay(t)$ is integrable wp 1 and

$$\int_0^t Ay(s)\,ds = S(t)\,y_0 - y_0 + \int_0^t S(t-r)f(y(r))\,dr - \int_0^t f(y(r))\,dr$$

$$+\int_0^t S(t-r)\,G(y(r))\,dw(r)-\int_0^t G(y(r))\,dw(r)$$

$$=y(t)-y_0-\int_0^t f(y(r))\,dr-\int_0^t G(y(r))\,dw(r).$$

Thus $y(t)$ satisfies (2.3).

We now give an existence theorem.

THEOREM 2.1. *Let y_0 be $\mathscr{F}_0$ measurable with $E|y_0|^p<\infty$ for some integer $p\geqslant 2$. Then there exists a unique mild solution of* (2.1) *in* $C(0,T;\mathscr{L}_p(\Omega,\mathscr{F},\mu;Y))$.

Proof. Let $\mathscr{Y}$ be the closed subspace of $C(0,T;\mathscr{L}_p(\Omega,\mathscr{F},\mu;Y))$ whose elements are adapted to $\mathscr{F}_t$. We introduce a norm in $\mathscr{Y}$

$$\|z(\cdot)\|=\max_{0\leqslant t\leqslant T} e^{-bt}(E|z(t)|^p)^{1/p},\qquad z\in\mathscr{Y},\quad b>0$$

which is clearly equivalent to the norm of $C(0,T;\mathscr{L}_p(\Omega,\mathscr{F},\mu;Y))$. Now define a map Λ on $\mathscr{Y}$

$$(\Lambda z)(t)=S(t)\,y_0+\int_0^t S(t-r)f(z(r))\,dr+\int_0^t S(t-r)\,G(z(r))\,dw(r).$$

By Proposition 1.9 and (2.2) we can easily show that Λ maps $\mathscr{Y}$ into itself. We now show that Λ is a contraction on $\mathscr{Y}$. Let $x, z\in\mathscr{Y}$, then

$$(\Lambda x)(t)-(\Lambda z)(t)=\int_0^t S(t-r)[f(x(r))-f(z(r))]\,dr$$

$$+\int_0^t S(t-r)[G(x(r))-G(z(r))]\,dw(r).$$

Thus

$$[E|(\Lambda x)(t)-(\Lambda z)(t)|^p]^{1/p}\leqslant[E|u(t)|^p]^{1/p}+[E|v(t)|^p]^{1/p},$$

where u, v denote two terms in the last equation. Recall that $|S(t)|\leqslant M$ for some $M>0$ on $[0,T]$. Thus

$$e^{-bt}(E|u(t)|^p)^{1/p}$$

$$\leqslant e^{-bt}\left\{E\left|\int_0^t S(t-r)[f(x(r))-f(z(r))]\,dr\right|^p\right\}^{1/p}$$

$$\leqslant Mc_1e^{-bt}\left\{E\left(\int_0^t|x(r)-z(r)|\,dr\right)^p\right\}^{1/p}$$

$$\leqslant Mc_1 t^{(p-1)/p} \left(\int_0^t e^{-pbt} E\,|x(r)-z(r)|^p\, dr\right)^{1/p}$$

$$\leqslant Mc_1 T^{(p-1)/p} \left[\int_0^t e^{-pb(t-r)}\, dr\right]^{1/p} [\max_{0\leqslant r\leqslant t} e^{-pbr} E\,|x(r)-z(r)|^p]^{1/p}$$

$$\leqslant \frac{Mc_1 T^{(p-1)/p}}{(pb)^{1/p}} \max_{0\leqslant t\leqslant T} e^{-bt} (E\,|x(r)-z(r)|^p)^{1/p}.$$

Similarly, we have the estimate

$$e^{-bt}(E\,|v(t)|^p)^{1/p}$$

$$= e^{-bt} \left\{E \left|\int_0^t S(t-r)[G(x(r)) - G(z(r))]\, dw(r)\right|^p\right\}^{1/p}$$

$$\leqslant Mc_2 \left[\frac{p(p-1)}{2} \operatorname{tr} W\right]^{1/2} T^{1/2-1/p} e^{-bt} \left(\int_0^t E\,|x(r)-z(r)|^p\, dr\right)^{1/p}$$

$$\leqslant \frac{Mc_2}{(pb)^{1/p}} \left[\frac{p(p-1}{2} \operatorname{tr} W\right]^{1/2} T^{1/2-1/p} \|x-z\|.$$

Combining these two estimates we obtain

$$\|\Lambda x - \Lambda z\| = \max_{0\leqslant t\leqslant T} e^{-bt} [E\,|(\Lambda x)(t) - (\Lambda z)(t)|^p]^{1/p}$$

$$\leqslant \frac{Mc_1 T^{(p-1)/p} + Mc_2[(p(p-1)/2) \operatorname{tr} W]^{1/2}\, T^{1/2-1/p}}{(pb)^{1/p}} \|x-z\|.$$

Hence for $b > 0$ sufficiently large, Λ is a contraction and thus has a unique fixed point in $\mathscr{Y}$.

Corollary 2.1. *If y_0 is nonrandom, then there exists a unique mild solution in $C(0, T; \mathscr{L}_p(\Omega, \mathscr{F}, \mu; Y))$ for all $p \geqslant 2$.*

Remark 2.2. We have assumed that f and G are defined for all $y \in Y$, but there are some cases where they are defined only on a subspace of Y(See [13, 16] and Eq. (1.19)).

Remark 2.3. Stochastic evolution equations in terms of martingales are studied in [3, 15] using semigroups and in [16] using monotone operators.

3. Stability of Moments

Consider the stochastic evolution equation

$$dy(t) = [Ay(t) + f(y(t))]\, dt + G(y(t))\, dw(t), \qquad y(0) = y_0, \tag{3.1}$$

where we take y_0 nonrandom for simplicity and assume the Lipschitz conditions in (2.2). Then there exists by Corollary 2.1 a mild solution such that pth moment $E\,|y(t)|^p$ is continuous for all $p \geqslant 2$. We now consider the stability of $E\,|y(t)|^p$.

THEOREM 3.1. *Let $v(y)\colon Y \to R$ satisfy*

$v(y)$ is twice Fréchet differentiable and $v(y)$, $v_y(y)$, and $v_{yy}(y)$ are continuous in R^1, Y and $\mathscr{L}(Y)$, respectively, (3.2a)

$$|v(y)| + |y|\,|v_y(y)| + |y|^2\,|v_{yy}(y)| \leqslant c\,|y|^p \quad \text{for some} \quad p \geqslant 2 \quad \text{and} \quad c > 0, \tag{3.2b}$$

$$\mathscr{L}v(y) + av(y) \leqslant 0 \qquad \text{for all} \quad y \in \mathscr{D}(A), \tag{3.2c}$$

where a is a real number and

$$\mathscr{L}v(y) = \langle v_y(y), Ay + f(y)\rangle + \tfrac{1}{2}\,\mathrm{tr}\, G(y)\, WG^*(y)\, v_{yy}(y).$$

Then the mild solution $y(t)$ of (2.1) *satisfies the inequality*

$$Ev(y(t)) \leqslant e^{-at} v(y_0).$$

To prove this theorem we introduce approximating systems

$$dy(t) = [Ay(t) + R(\lambda) f(y(t))]\, dt + R(\lambda)\, G(y(t))\, dw(t),$$
$$y(0) = R(\lambda)\, y_0, \tag{3.3}$$

where $\lambda \varepsilon \rho(A)$, the resolvent set of A and $R(\lambda) = \lambda R(\lambda, A)$.

LEMMA 3.1. *The stochastic differential equation* (3.3) *has a unique strong solution $y(t, \lambda)$ which lies in $C(0, T; \mathscr{L}_p(\Omega, \mathscr{F}, \mu; Y))$ for all T and $p \geqslant 2$. Moreover, $y(t, \lambda)$ converges to the mild solution of* (3.1) *in $C(0, T; \mathscr{L}_p(\Omega, \mathscr{F}, \mu; Y))$ as $\lambda \to \infty$ for all T and $p \geqslant 2$.*

Proof. The first part is an immediate consequence of Theorem 2.1 and Proposition 2.3. To prove the second part we consider

$$\begin{aligned} y(t) - y(t,\lambda) = {} & S(t)[y_0 - R(\lambda) y_0] \\ & + \int_0^t S(t-r)][f(y(r)) - R(\lambda) f(y(r,\lambda))]\, dr \\ & + \int_0^t S(t-r)[G(y(r)) - R(\lambda)\, G(y(r,\lambda))]\, dw(r) \\ = {} & \int_0^t S(t-r)\, R(\lambda)[f(y(r)) - f(y(r,\lambda))]\, dr \\ & + \int_0^t S(t-r)\, R(\lambda)[G(y(r)) - G(y(r,\lambda))]\, dw(r) \\ & + \Big\{ S(t)[y_0 - R(\lambda) y_0] \\ & + \int_0^t S(t-r)[I - R(\lambda)] f(y(r))\, dr \\ & + \int_0^t S(t-r)[I - R(\lambda)]\, G(y(r))\, dw(r) \Big\}. \end{aligned}$$

Since $|a+b+c|^p \leqslant 3^p(|a|^p + |b|^p + |c|^p)$ for any real numbers a, b, c $|S(t)| \leqslant M$, for all $t \in [0, T]$ and $|R(\lambda)| \leqslant 2$ for large λ, we have

$$E\,|y(t) - y(t,\lambda)|^p \leqslant 3^p[I_1 + I_2 + I_3],$$

where

$$\begin{aligned} I_1 &= E \left| \int_0^t S(t-r)\, R(\lambda)[f((y(r)) - f(y(r,\lambda))]\, dr \right|^p \\ &\leqslant (2Mc_1)^p\, t^{p-1} \int_0^t E\,|y(r) - y(r,\lambda)|^p\, dr \\ I_2 &= E \left| \int_0^t S(t-r)\, R(\lambda)[G(y(r)) - G(y(r,\lambda))]\, dw(r) \right|^p \\ &\leqslant (2Mc_2)^p \left[\frac{p(p-1)}{2} \operatorname{tr} W \right]^{p/2} t^{p/2-1} \int_0^t E\,|y(r) - y(r,\lambda)|^p\, dr \end{aligned}$$

and

$$\begin{aligned} I_3 = E \Big| & S(t)[y_0 - R(\lambda) y_0] + \int_0^t S(t-r)[I - R(\lambda)] f(y(r))\, dr \\ & + \int_0^t S(t-r)[I - R(\lambda)]\, G(y(r))\, dw(r) \Big|^p. \end{aligned}$$

We now estimate each term in I_3:

$$|S(t)[y_0 - R(\lambda)y_0]| \leqslant M|y_0 - R(\lambda)y_0| \to 0 \qquad \text{as} \quad \lambda \to \infty$$

and

$$E\left|\int_0^t S(t-r)[I-R(\lambda)]f(y(r))\,dr\right|^p$$

$$\leqslant M^p T^{p-1}\int_0^T E|[I-R(\lambda)]f(y(r))|^p\,dr \to 0 \qquad \text{as} \quad \lambda \to \infty,$$

$$E\left|\int_0^t S(t-r)[I-R(\lambda)]\,G(y(r))\,dw(r)\right|^p$$

$$\leqslant M^p \int_0^T E[\operatorname{tr}(I-R(\lambda))\,G(y(r))\,W[(I-R(\lambda))\,G(y(r))]^*]^{p/2}\,dr \to 0 \qquad \text{as} \quad \lambda \to \infty$$

by the dominated convergence theorem. Thus we can write

$$E|y(t) - y(t,\lambda)|^p \leqslant c\int_0^t E|y(r) - y(r,\lambda)|^p\,dr + \varepsilon(\lambda),$$

where $c = (6Mc_1)^p T^{p-1} + (6Mc_2)^p[(p(p-1)/2)\operatorname{tr} W]^{p/2}T^{p/2-1}$ and $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$. By Gronwall's inequality we have

$$E|y(t) - y(t,\lambda)|^p \leqslant \varepsilon(\lambda)\,ce^{cT} \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

Proof of Theorem 3.1. We apply Ito's formula (Corollary 1.2) to the function $v(t,y) = e^{\alpha t}v(y)$ and the process $y(t,\lambda)$:

$$e^{\alpha t}v(y(t,\lambda)) - v(y(0,\lambda))$$

$$= \int_0^t e^{\alpha r}[\alpha v(y(r,\lambda))$$

$$+ \langle v_y(y(r,\lambda)), Ay(r,\lambda) + R(\lambda)f(y(r,\lambda))\rangle$$

$$+ \tfrac{1}{2}\operatorname{tr}[R(\lambda)\,G(y(r,\lambda))\,W[R(\lambda)\,G(y(r,\lambda))]^*\,v_{yy}(y(r,\lambda))]\,dr$$

$$+ \int_0^t e^{\alpha r}\langle v_y(y(r,\lambda)), R(\lambda)\,G(y(r,\lambda))\,dw(r)\rangle$$

$$\leqslant \int_0^t e^{\alpha r}\{\langle v_y(y(r,\lambda)), (R(\lambda)-I)f(y(r,\lambda))\rangle$$

$$+ \tfrac{1}{2}\operatorname{tr}[R(\lambda)\,G(y(r,\lambda))\,W[R(\lambda)\,G(y(r,\lambda))]^*\,v_{yy}(y(r,\lambda))$$

$$- G(y(r,\lambda))\,WG^*(y(r,\lambda))\,v_{yy}(y(r,\lambda))]\}\,dr$$

$$+ \int_0^t e^{\alpha r}\langle v_y(y(r,\lambda)), R(\lambda)\,G(y(r,\lambda))\,dw(r)\rangle.$$

Taking expectations we have

$$e^{\alpha t}Ev(y(t,\lambda)) \leqslant v(R(\lambda)y_0) + \int_0^t e^{\alpha r}E\{\langle v_y(y(r,\lambda)), (R(\lambda)-I)f(y(r,\lambda))\rangle + \tfrac{1}{2}\operatorname{tr}[R(\lambda)G(y(r,\lambda))W[R(\lambda)G(y(r,\lambda))]^* \times v_{yy}(y(r,\lambda)) - G(y(r,\lambda))WG^*(y(r,\lambda))v_{yy}(y(r,\lambda))]\}\,dr.$$

By Lemma 3.1, Eqs. (3.2), and (2.2) we conclude via the dominated convergence theorem that $e^{\alpha t}Ev(y(t)) \leqslant v(y_0)$.

COROLLARY 3.1. *If* $av(y) \geqslant |y|^p$ *for some* $a > 0$ *and* $\alpha > 0$, *then* $E|y(t)|^p \leqslant ae^{-\alpha t}v(y_0)$ *and the pth moment is exponentially stable.*

Note that if, in particular, $v(y) = |y|^p$, then

$$\mathscr{L}v(y) = p|y|^{p-2}\langle y, Ay + f(y)\rangle + \tfrac{1}{2}\operatorname{tr} G(y)WG^*(y) \times [p|y|^{p-2}I + p(p-2)|y|^{p-4}y \circ y].$$

COROLLARY 3.2. *Suppose that* $\langle y, Ay + f(y)\rangle \leqslant -\beta|y|^2$ *for some* $\beta > 0$, $G(0) = 0$ *and* $\tfrac{1}{2}c_2^2(p-1)\operatorname{tr} W < \beta$.
Then the pth moment is exponentially stable.

Proof. We have the estimate

$$\mathscr{L}v(y) \leqslant -p\beta|y|^p + \tfrac{1}{2}c_2^2\,p(p-1)\operatorname{tr} W|y|^p = -p[\beta - \tfrac{1}{2}c_2^2(p-1)\operatorname{tr} W]\,|y|^p.$$

4. SAMPLE CONTINUITY

In this section we give sufficient conditions for the mild solution of (2.1) to have continuous sample paths. In view of (2.4) the solution $y(t)$ has a modification with continuous sample paths if the term $\int_0^t S(t-r)G(y(r))\,dw(r)$ has the same property. So it is sufficient to consider the sample continuity of the process

$$y(t) = \int_0^t S(t-r)G(r)\,dw(r), \tag{4.1}$$

where $G \in \mathscr{M}(H, Y)$ with $E\int_0^T |G(t)|^p\,dt < \infty$, for some integer $p \geqslant 2$.

LEMMA 4.1. *Suppose that there exists a real continuous function $v(y)$ on Y such that*

$$a_1 v(y) \geqslant |y|^p, \qquad a_1 > 0, \tag{4.2a}$$

$v(y)$ is twice Fréchet differentiable and derivatives $v_y(y)$ and $v_{yy}(y)$ are continuous in Y and $\mathscr{L}(Y)$, respectively, (4.2b)

$$v(y) + |y|\,|v_y(y)| + |y|^2\,|v_{yy}(y)| \leqslant a_2 |y|^p, \qquad a_2 > 0. \tag{4.2c}$$

$$\langle v_y(y), Ay\rangle \leqslant b_1 v(y) \qquad \text{for all } y \in \mathscr{D}(A), \quad b_1 \geqslant 0. \tag{4.2d}$$

Then

$$E \sup_{0 \leqslant t \leqslant T} v(y(t, \lambda)) \leqslant 2b_1 \int_0^T Ev(y(t, \lambda))$$

$$+ (1 + a_1)\, a_2 \operatorname{tr} W \sup_{0 \leqslant t \leqslant T} [E\,|y(t, \lambda)|^p]^{(p-2)/p} \int_0^T [E\,|G_\lambda(t)|^p]^{2/p}\, dt,$$

where $y(t, \lambda) = \int_0^t S(t-r)\, G_\lambda(r)\, dw(r)$ and $G_\lambda(t) = \lambda R(\lambda, A)\, G(t)$.

Proof. We note that $y(t, \lambda)$ is the strong solution of

$$dy(t) = Ay(t)\, dt + G_\lambda(t)\, dw(t), \qquad y(0) = 0. \tag{4.3}$$

Thus we can apply Ito's formula to obtain

$$v(y(t)) \leqslant b_1 \int_0^t v(y(r))\, dr + \tfrac{1}{2} \int_0^t \operatorname{tr} G(r)\, WG^*(r)\, v_{yy}(y(r))\, dr$$

$$+ \int_0^t \langle v_y(y(r)), G(r)\, dw(r)\rangle$$

$$\leqslant b_1 \int_0^T v(y(r))\, dr + \frac{\operatorname{tr} W}{2} \int_0^T |G(r)|^2\, |v_{yy}(y(r))|\, dr$$

$$+ \sup_{0 \leqslant t \leqslant T} \left| \int_0^t \langle v_y(y(r)), G(r)\, dw(r)\rangle \right|, \tag{4.4}$$

where we have suppressed λ. Note first that

$$E \int_0^T |G(r)|^2\, |v_{yy}(y(r))|\, dr$$

$$\leqslant \int_0^T a_2 E\,|y(r)|^{p-2}\, |G(r)|^2\, dr$$

$$\leqslant \int_0^T a_2 [E\,|\,y(r)|^p]^{(p-2)/p} [E\,|\,G(r)|^p]^{2/p}\,dr$$

$$\leqslant a_2 \sup_{0\leqslant t\leqslant T} [E\,|\,y(t)|^p]^{(p-2)/p} \int_0^T [E\,|\,G(r)|^p]^{2/p}\,dr$$

$$< \infty.$$

Now for a moment we assume that $E\int_0^T |G(t)|^{2p}\,dt < \infty$. Then $y \in C(0, T; \mathscr{L}_{2p}(\Omega, \mathscr{F}, \mu; Y))$ and by Proposition 1.6 we easily see that

$$E \sup_{0\leqslant t\leqslant T} \left| \int_0^t \langle v_y(y(r)), G(r)\,dw(r)\rangle \right| < \infty.$$

Thus in view of (4.4) we conclude that $E\sup_{0\leqslant t\leqslant T} v(y(t)) < \infty$. Furthermore, we have by Proposition 1.6(c)

$$E \sup_{0\leqslant t\leqslant T} \left| \int_0^t \langle v_y(y(r)), G(r)\,dw(r)\rangle \right|$$

$$\leqslant E\left[\operatorname{tr} W \int_0^T |v_y(y(t))|^2\,|G(t)|^2\,dt\right]^{1/2}$$

$$\leqslant E\left[a_2 \operatorname{tr} W \int_0^T |\,y(t)|^{2(p-1)}\,|G(t)|^2\,dt\right]^{1/2}$$

$$\leqslant E\left[a_1 a_2 \operatorname{tr} W \int_0^T v(y(t))\,|\,y(t)|^{p-2}\,|G(t)|^2\,dt\right]^{1/2}$$

$$\leqslant E\left[a_1 a_2 \operatorname{tr} W \sup_{0\leqslant t\leqslant T} v(y(t)) \int_0^T |\,y(t)|^{p-2}\,|G(t)|^2\,dt\right]^{1/2}$$

$$\leqslant \tfrac{1}{2} E \sup_{0\leqslant t\leqslant T} v(y(t)) + \tfrac{1}{2} E\left[a_1 a_2 \operatorname{tr} W \int_0^T |\,y(t)|^{p-2}\,|G(t)|^2\,dt\right]$$

$$\leqslant \tfrac{1}{2} E \sup_{0\leqslant t\leqslant T} v(y(t)) + \frac{a_1 a_2}{2} \operatorname{tr} W \times \int_0^T [E\,|\,y(t)|^p]^{(p-2)/p} [E\,|\,G(t)|^p]^{2/p}\,dt$$

$$\leqslant \tfrac{1}{2} E \sup_{0\leqslant t\leqslant T} v(y(t)) + \frac{a_1 a_2}{2} \operatorname{tr} W \times \sup_{0\leqslant t\leqslant T} [E\,|\,y(t)|^p]^{(p-2)/p} \int_0^T [E\,|\,G(t)|^p]^{2/p}\,dt.$$

Combining this together with (4.4) we obtain

$$E \sup_{0 \leqslant t \leqslant T} v(y(t)) \leqslant 2b_1 \int_0^T Ev(y(t))\,dt$$
$$+ (1+a_1)\,a_2 \operatorname{tr} W \sup_{0 \leqslant t \leqslant T} [E\,|\,y(t)|^p]^{(p-2)/p} \int_0^T [E\,|\,G(t)|^p]^{2/p}\,dt.$$

Since the right-hand side involves only pth moments we can easily show by a limiting argument that this inequality is valid for any G with

$$\int_0^T E\,|\,G(t)|^p\,dt < \infty.$$

THEOREM 4.1. *Suppose that there is a continuous real function $v(y)$ on Y with properties in Lemma* 4.1. *Then the stochastic process $y(t)$ given by* (4.1) *has a modification with continuous sample paths.*

Proof. Let $y_n(t) = \int_0^t S(t-r)\,G_n(r)\,dw(r)$, where $G_n(t) = nR(n, A)\,G(t)$ for sufficiently large n. Then by Lemma 4.1 we have

$$E \sup_{0 \leqslant t \leqslant T} v(y_m(t) - y_n(t)) \leqslant 2b_1 \int_0^T Ev(y_m(t) - y_n(t))\,dt$$
$$+ (1+a_1)\,a_2 \operatorname{tr} W \sup_{0 \leqslant t \leqslant T} [E\,|\,y_m(t) - y_n(t)|^p]^{(p-2)/p}$$
$$\times \int_0^T [E\,|\,G_m(t) - G_n(t)|^p]^{2/p}\,dt \to 0 \quad \text{as } n, m \to \infty.$$

Thus by (4.2a) we conclude that $E[\sup_{0 \leqslant t \leqslant T} |\,y_m(t) - y_n(t)|\,]^p \to 0$ as $n, m \to \infty$. Hence there exists a subsequence which converges to some $\tilde{y}(t)$ in $C(0, T; Y)$ wp 1. But $\tilde{y}(t)$ is obviously a modification of $y(t)$ given by (4.1).

COROLLARY 4.1. *Let $y(t)$ be the mild solution of* (2.1) *given in Theorem* 2.1. *If there exists a real continuous function $v(y)$ on Y with properties in* (4.2), *then $y(t)$ has a modification with continuous sample paths.*

5. SAMPLE PATH STABILITY

If the mild solution of (3.1) has continuous sample paths, it is reasonable to consider asymptotic stability of its sample paths. In this section we shall extend the result in [7, 11]. In fact, under the assumptions of Corollary 3.1, we can show that sample paths of the mild solution of (3.1) are exponentially stable wp 1. We remark that under the conditions in Corollary 3.1 the mild solution of (3.1) has continuous sample paths since we can apply Corollary 4.1.

LEMMA 5.1. *Suppose that there exists a function $v(y)$ with properties in Corollary* 3.1. *Let $y(t)$ be the mild solution of* (3.1):

$$dy(t) = [Ay(t) + f(y(t))]\,dt + G(y(t))\,dw(t), \qquad y(0) = y_0,$$

where f and G satisfy (2.2) *and $y_0 \in Y$ is nonrandom. Then*

(a) $v(y(t)) \leqslant v(y_0) + \int_0^t \langle v_y(y(r)), G(y(r))\,dw(r)\rangle$,

(b) $E \sup_{0 \leqslant t \leqslant T} v(y(t)) \leqslant bv(y_0)$, *where b is independent of T.*

Proof. We note that $\mathscr{L}v(y) \leqslant -av(y) \leqslant 0$ since $a > 0$. Now we apply Ito's formula to the function $v(y)$ and the process $y(t, \lambda)$ given by (3.3). Then

$$\begin{aligned} v(y(t,\lambda)) &\leqslant v(R(\lambda)\,y_0) \\ &\quad + \int_0^t \langle v_y(y(r,\lambda)), R(\lambda) f(y(r,\lambda)) - f(y(r,\lambda))\rangle\,dr \\ &\quad + \tfrac{1}{2}\int_0^t [\operatorname{tr} R(\lambda)\,G(r)\,WG^*(r)\,R^*(\lambda)\,v_{yy}(y(r,\lambda)) \\ &\quad - \operatorname{tr} G(r)\,WG^*(r)\,v_{yy}(y(r,\lambda))]\,dr \\ &\quad + \int_0^t \langle v_y(y(r,\lambda)), R(\lambda)\,G(r)\,dw(r)\rangle. \end{aligned}$$

As in Theorem 4.1 we can show that there exists a subsequence $y(\cdot, \lambda) \to y(\cdot)$ in $C(0, T; Y)$ wp 1. Thus in view of (3.2b) we can pass to the $\lim_{\lambda \to \infty}$ in the inequality above to obtain (a). To prove (b) consider

$$\begin{aligned} &E\left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^T \langle v_y(y(t)), G(y(t))\,dw(t)\rangle\right|\right] \\ &\quad \leqslant E\left[\operatorname{tr} W \int_0^T |v_y(y(t))|^2\,|G(y(t))|^2\,dt\right]^{1/2} \\ &\quad \leqslant \tfrac{1}{2} E \sup_{0 \leqslant t \leqslant T} v(y(t)) + c\int_0^T Ev(y(t))\,dt \qquad \text{for some } c > 0. \end{aligned}$$

Thus from (a) and Theorem 3.1 it follows that

$$E \sup_{0\leqslant t\leqslant T} v(y(t)) \leqslant v(y_0) + 2c\int_0^T ae^{-\alpha t} v(y_0)$$
$$\leqslant bv(y_0) \qquad \text{for some } b>0.$$

THEOREM 5.1. *Under the conditions in Lemma* 5.1, *there exist a random variable* $0<T(\omega)<\infty$ *and a constant* $c>0$ *such that for all* $t>T(\omega)$

$$v(y(t)) \leqslant cv(y_0)\, e^{-\alpha t/4} \qquad \text{wp 1.}$$

Proof. We can use arguments in [7, 11]. By a modification of Lemma 5.1(a), we have for $t\geqslant n$,

$$v(y(t)) \leqslant v(y(n)) + \int_n^t \langle v_y(y(r)), G(y(r))\, dw(r)\rangle.$$

Hence

$$\mu\,[\sup_{n\leqslant t\leqslant n+1} v(y(t)) \geqslant \varepsilon_n] \leqslant \mu\left[v(y(n)) \geqslant \frac{\varepsilon_n}{2}\right]$$
$$+\mu\left[\sup_{n\leqslant t\leqslant n+1}\left|\int_n^t \langle v_y(y(r)), G(y(r))\, dw(r)\rangle\right| \geqslant \frac{\varepsilon_n}{2}\right].$$

By Proposition 1.6(c) we have

$$\mu\left[\sup_{n\leqslant t\leqslant n+1}\left|\int_n^t \langle v_y(y(r)), G(y(r))\, dw(r)\rangle\right| \geqslant \frac{\varepsilon_n}{2}\right]$$
$$\leqslant \frac{2}{\varepsilon_n} E\left[\sup_{n\leqslant t\leqslant n+1}\left|\int_n^t \langle v_y(y(r)), G(y(r))\, dw(r)\rangle\right|\right]$$
$$\leqslant \frac{2}{\varepsilon_n} kE\left\{[\sup_{n\leqslant t\leqslant n+1} v(y(t))]^{1/2}\left[\int_n^t v(y(r))\, dr\right]^{1/2}\right\}$$
$$\text{for some } k>0 \quad \text{by} \quad (3.2b)$$
$$\leqslant \frac{2}{\varepsilon_n} k[E\sup_{n\leqslant t\leqslant n+1} v(y(t))]^{1/2}\left[\int_n^{n+1} Ev(y(r))\, dr\right]^{1/2}$$
$$\leqslant (2/\varepsilon_n)\, k\sqrt{b}\,[v(y_0)]^{1/2}\sqrt{a/\alpha}\, e^{-\alpha n/2}[v(y_0)]^{1/2}$$
$$\leqslant (k_0/\varepsilon_n)\, e^{-\alpha n/2} v(y_0) \qquad \text{for some } k_0>0.$$

Furthermore, we have

$$\mu[v(y(n)) \geqslant \varepsilon_n/2] \leqslant (2/\varepsilon_n)\, Ev(y(n)) \leqslant (2/\varepsilon_n)\, ae^{-\alpha n} v(y_0).$$

Hence

$$\mu[\sup_{n \leqslant t \leqslant n+1} v(y(t)) \geqslant \varepsilon_n] \leqslant (\tilde{c}/\varepsilon_n)\, e^{-\alpha n/2} v(y_0) \qquad \text{for some } \tilde{c} > 0.$$

If we set $\varepsilon_n = \tilde{c}e^{-\alpha n/4} v(y_0)$, then

$$\mu[\sup_{n \leqslant t \leqslant n+1} v(y(t)) \geqslant \tilde{c}e^{-\alpha n/4} v(y_0)] \leqslant e^{-\alpha n/4}.$$

Thus by the Borel–Cantelli lemma we conclude that there exists a random variable $T(\omega) < \infty$ such that

$$v(y(t)) \leqslant \tilde{c}v(y_0)\, e^{-\alpha(t-1)/4} \leqslant cv(y_0)\, e^{-\alpha t/4} \qquad \text{for} \quad t > T(\omega) \quad \text{wp } 1, \quad c > 0.$$

Remark 5.1. For further developments we can consider stability problems in Sections 3 and 5 when f and G are more general as in Remark 2.2. This extension will be reported elsewhere.

6. Examples

Example 1. Consider the stochastic heat equation

$$dy(x, t) = \frac{\partial^2}{\partial x^2} y(x, t)\, dt + \sigma y(x, t)\, d\beta(t),$$

$$y(0, t) = y(1, t) = 0, \qquad y(x, 0) = y_0(x),$$

where σ is a real number and $\beta(t)$ is a real standard Wiener process. We take $Y = \mathscr{L}_2(0, 1)$, $H = R^1$, $G(y) = \sigma y$ and $A = d^2/dx^2$ with $\mathscr{D}(A) = \{ y \in Y \mid y, y'$ are absolutely continuous with $y', y'' \in Y$, $y(0) = y(1) = 0\}$. Then $\langle Ay, y \rangle \leqslant -\pi^2 |y|^2$ and $\mathscr{L} |y|^p \leqslant -p[\pi^2 - \frac{1}{2}\sigma^2(p-1)] |y|^p$. Thus pth moment of $y(t)$ is exponentially stable for all nonrandom $y_0 \in Y$ if $p < 1 + 2\pi^2/\sigma^2$. If, e.g., $\sigma = 1$, all moments up to $p = 19$ are exponentially stable. In fact, by Proposition 1.10 we have the explicit solution

$$y(t) = e^{-\sigma^2 t/2 + \sigma\beta(t)} S(t)\, y_0,$$

where

$$S(t)\, y_0 = \sum_{i=1}^{\infty} 2e^{-n^2\pi^2 t} \sin n\pi x \int_0^1 y_0(r) \sin n\pi r\, dr.$$

EXAMPLE 2. Consider the semilinear-stochastic heat equation

$$dy(x,t) = \left[\frac{\partial^2}{\partial x^2} y(x,t) - \frac{y(x,t)}{1+|y(x,t)|}\right] dt + \frac{\sigma y(x,t)}{1+|y(x,t)|}\, d\beta(t),$$

$$y_x(0,t) = y_x(1,t) = 0, \qquad y(x,0) = y_0(x),$$

where we take Y and $\beta(t)$ as in Example 1, $H = R^1$.

$$f(y) = -\frac{G(y)}{\sigma} = -y/1+|y|$$

and $A = d^2/dx^2$ with $\mathscr{D}(A) = \{y \in Y \mid y, y'$ absolutely continuous, $y', y'' \in Y$, $y'(0) = y'(1) = 0\}$. Then $\langle y, Ay + f(y)\rangle \leqslant -|y|^2$, $y \in \mathscr{D}(A)$, and

$$\mathscr{L}\,|y|^p \leqslant -p[1 - \tfrac{1}{2}\sigma^2(p-1)]\,|y|^p.$$

Thus if $p < 1 + 2/\sigma^2$, then the pth moment is exponentially stable. In this case we also have exponential stability of sample paths with probability one.

EXAMPLE 3. We give an example of Proposition 1.10:

$$dy(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} y(x,t)\, dt + \frac{\partial}{\partial x} y(x,t)\, d\beta(t),$$

$$y(0,t) = y(1,t), \qquad y_x(0,t) = y_x(1,t), \qquad y(x,0) = y_0(x).$$

If we take $Y = \mathscr{L}_2(0,1)$, $H = R^1$, $A = 0$, and $By = d/dx$ with $\mathscr{D}(B) = \{y \in Y \mid y$ absolutely continuous, $y' \in Y$, $y(0) = y(1)\}$, then $y(t) = T(\beta(t))\, y_0$, $y_0 \in \mathbb{D}(B^2)$, is a solution, where $T(t)\, y_0 = \tilde{y}_0(x+t)$ and $\tilde{y}_0(x)$ is the periodic extension of $y_0(x)$ to the interval $[0, \infty)$.

APPENDIX 1: PROOF OF PROPOSITION 1.1

Let (e_i) be a complete orthonormal basis for Y, then $y = \sum_{i=1}^{\infty} y_i e_i$, where y_i are real Gaussian random variables. Note first that $E|y|^2 = \sum_{i=1}^{\infty} Ey_i^2 = \operatorname{tr} Q$. Next we use the following results:

(a) $E\prod_{i=1}^{m} x_i \leqslant \prod_{i=1}^{m} (Ex_i^n)^{1/n}$ for positive random variables x_i and all $1 \leqslant m \leqslant n$,

(b) If x is a real Gaussian random variable with zero mean and variance σ^2, then (see [1]),

$$Ex^{2n} \leqslant (2n-1)!!\,\sigma^{2n}.$$

Now

$$E\,|y|^{2n} = E\left(\sum_{i=1}^{\infty} y_i^2\right)^n = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} E \prod_{j=1}^{n} y_{i_j}^2$$

$$\leqslant \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \prod_{j=1}^{n} (Ey_{i_j}^{2n})^{1/n}$$

$$\leqslant \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \prod_{j=1}^{n} [(2n-1)!!]^{1/n} Ey_{i_j}^2$$

$$= (2n-1)!! \left(\sum_{i=1}^{\infty} Ey_i^2\right)^n = (2n-1)!!\,(\operatorname{tr} Q)^n.$$

APPENDIX 2: PROOF OF LEMMA 1.2

Define

$$f_i^N(t) = f_i(t), \qquad \text{if } \int_0^t |f_i(r)|^2\,dr \leqslant N$$

$$= 0, \qquad \text{if } \int_0^t |f_i(r)|^2\,dr > N.$$

Then

$$\mu\left[\sup_{0\leqslant t\leqslant T}\left|\sum_{i\in J}\int_0^t f_i(r)\,d\beta_i(r)\right| > c\right]$$

$$\leqslant \mu\left[\sup_{0\leqslant t\leqslant T}\left|\sum_{i\in J}\int_0^t f_i^N(r)\,d\beta_i(r)\right| > c\right]$$

$$+\mu\left[\sup_{0\leqslant t\leqslant T}\left|\sum_{i\in J}\int_0^t [f_i(r)-f_i^N(r)]\,d\beta_i(r)\right| > 0\right]$$

$$\leqslant \frac{1}{c^2}\sum_{i\in J}\lambda_i E\int_0^T |f_i^N(r)|^2\,dr + \mu\left[\bigcup_{i\in J}\left(\int_0^T |f_i(r)|^2\,dr > N\right)\right]$$

$$\leqslant \frac{N}{c^2}\sum_{i\in J}\lambda_i + \mu\left[\bigcup_{i\in J}\left(\int_0^T |f_i(r)|^2\,dr > N\right)\right].$$

APPENDIX 3: PROOF OF PROPOSITION 1.6

(a) By Lemma 1.2 $\sup_{0\leqslant t\leqslant T}|\sum_{i=n+1}^{m}\int_0^t G(r)\,e_i\,d\beta_i(r)| \to 0$ in probability as $n \to \infty$. Thus $y_n(t)$ is a Cauchy sequence in $C(0, T; Y)$ in the sense of convergence in probability and therefore there exists a subsequence $y_{n_k}(t)$ which converges in $C(0, T; Y)$ with probability one. Let $y(t)$ be the limit, then clearly, it is a modification of the stochastic integral $\int_0^t G(r)\,dw(r)$.

(b) Let m be a positive integer and let $n > m$. Then

$$\int_0^t G_n(r)\,dw_n(r) - \int_0^t G(r)\,dw(r) = \int_0^t G_n(r)\,dw_m(r) - \int_0^t G(r)\,dw_m(r)$$
$$+ \int_0^t G(r)\,dw_m(r) - \int_0^t G(r)\,dw(r)$$
$$+ \sum_{i=m+1}^{n} \int_0^t G_n(r)\,e_i\,d\beta_i(r).$$

By definition we know that $\int_0^t G(r)\,dw_m(r) - \int_0^t G(r)\,dw(r)$ converges to 0 in probability as $m \to \infty$. By Lemma 1.2 we have

$$\mu\left[\sup_{0\leqslant t\leqslant T}\left|\sum_{i=m+1}^{n}\int_0^t G_n(r)\,e_i\,d\beta_i(r)\right| > \delta\right]$$
$$\leqslant \left[\bigcup_{i=m+1}^{n}\left(\int_0^T |G_n(r)\,e_i|^2\,dr > N\right)\right] + \frac{N}{\delta^2}\sum_{i=m+1}^{n}\lambda_i$$
$$\leqslant \mu\left[\int_0^T |G_n(r)|^2\,dr > N\right] + \frac{N}{\delta^2}\sum_{i=m+1}^{\infty}\lambda_i$$
$$\leqslant \mu\left[\int_0^T k^2(r)\,dr > N\right] + \frac{N}{\delta^2}\sum_{i=m+1}^{\infty}\lambda_i.$$

The last expression can be made arbitrarily small by taking N and then m large enough. Hence $\sum_{i=m+1}^{n}\int_0^t G_n(r)\,e_i\,d\beta_i(r)$ converges to 0 in probability as $m \to \infty$. Finally, note that

$$\mu\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t G_n(r)\,dw_m(r) - \int_0^t G(r)\,dw_m(r)\right| > \delta\right]$$
$$\leqslant \mu\left[\bigcup_{i=1}^{m}\left(\int_0^T |G_n(r)\,e_i - G(r)\,e_i|^2\,dr > \varepsilon\right)\right] + \frac{\varepsilon}{\delta^2}\sum_{i=1}^{m}\lambda_i$$

which can be made arbitrarily small for any $\delta > 0$ and m by taking ε sufficiently small and then choosing n large enough. Thus the assertion follows from these results.

Appendix 4: Proof of Lemma 1.3

Without loss of generality, we may assume in (1.15)

$$g(t) = g, \qquad G(t)\, e_i = f_i,$$

where g, f_i are $\mathscr{F}_t$ measurable random variables, since the general case follows by a straightforward limiting argument. We suppress n in y_n and z_n in this proof. Let $t_0 < t_1 < t_2 < \cdots < t_m = t$ and set

$$\Delta t_k = t_{k+1} - t_k, \qquad y^k = y(t_k), \qquad v_k = v(t_k, y^k),$$
$$\Delta y^k = y^{k+1} - y^k, \qquad \text{and} \qquad \Delta v_k = v_{k+1} - v_k$$

for $k = 0, 1, \ldots, m-1$. Then

$$z(t) - z(t_0) = v(t, y(t)) - v(t_0, y_0) = \sum_{k=0}^{m-1} \Delta v_k.$$

Now we have by Taylor's formula

$$\begin{aligned} v_k &= v(t_{k+1}, y^{k+1}) - (v(t_k, y^{k+1}) + v(t_k, y^{k+1}) - v(t_k, y^k)) \\ &= v_t(t_k, y^{k+1})\, \Delta t_k + v_y(t_k, y^k)\, \Delta y^k + \tfrac{1}{2} v_{yy}(t_k, y^k)(\Delta y^k, \Delta y^k) + \gamma_k + \delta_k, \end{aligned}$$

where

$$\begin{aligned} \gamma_k &= [v_t(t_k + \lambda_k \Delta t_k, y^{k+1}) - v_t(t_k, y^{k+1})]\, \Delta t_k, \\ \delta_k &= \tfrac{1}{2}[v_{yy}(t_k, y^k + \mu_k \Delta y^k)(\Delta y^k, \Delta y^k) - v_{yy}(t_k, y^k)(\Delta y^k, \Delta y^k)] \end{aligned}$$

and $0 < \lambda_k, \mu_k < 1$. Thus

$$\begin{aligned} z(t) - z(t_0) = \sum_{k=0}^{m-1} \{ & v_t(t_k, y^{k+1})\, \Delta t_k + v_y(t_k, y^k) \\ & + \tfrac{1}{2} v_{yy}(t_k, y^k)(\Delta y^k, \Delta y^k)\} + \theta, \end{aligned}$$

where $\theta = \sum_{k=0}^{m-1} (\gamma_k + \delta_k)$. As in [4, 9] we can show that $|\theta| \to 0$ wp 1 as $\max_k \Delta t_k \to 0$. Now substituting $\Delta y^k = g\, \Delta t_k + \sum_{i=1}^{n} f_i\, \Delta\beta_i(t_k)$ we obtain

$$z(t) - z(t_0) = \sum_{j=1}^{5} S_j + \theta,$$

where

$$S_1 = \sum_{k=0}^{m-1} [v_t(t_k, y^{k+1}) + v_y(t_k, y^k)\, g]\, \Delta t_k$$

$$S_2 = \sum_{k=0}^{m-1} \sum_{i=1}^{n} v_y(t_k, y^k) f_i \Delta\beta_i(t_k)$$

$$S_3 = \sum_{k=0}^{m-1} \tfrac{1}{2} v_{yy}(t_k, y^k) \left(\sum_{i=1}^{n} f_i \Delta\beta_i(t_k), \sum_{i=1}^{n} f_i \Delta\beta_i(t_k) \right)$$

$$S_4 = \sum_{k=0}^{m-1} \tfrac{1}{2} v_{yy}(t_k, y^k)(g, g)(\Delta t_k)^2$$

$$S_5 = \sum_{k=0}^{m-1} v_{yy}(t_k, y^k) \left(g, \sum_{i=1}^{n} f_i \Delta\beta_i(t_k) \right) \Delta t_k.$$

By continuity assumptions in (1.13) and the boundedness of $y(t)$ wp 1 on $\mathcal{T}$ we obtain

$$S_1 \to \int_{t_0}^{t} [v_t(r, y(r)) + v_y(r, y(r))\, g]\, dr, \qquad S_4 \to 0, \qquad \text{and} \qquad S_5 \to 0$$

wp 1 as $\max_k \Delta t_k \to 0$. By definition we also have

$$S_2 \to \sum_{i=1}^{n} \int_{t_0}^{t} v_y(r, y(r)) f_i \, d\beta_i(r)$$

in probability as $\max_k \Delta t_k \to 0$. We can write S_3 as

$$S_3 = \sum_{k=0}^{m-1} \sum_{i=1}^{n} \tfrac{1}{2} v_{yy}(t_k, y^k)(f_i, f_i)[\Delta\beta_i(t_k)]^2$$
$$+ \sum_{k=0}^{m-1} \sum_{i \neq j = 1}^{n} \tfrac{1}{2} v_{yy}(t_k, y^k)(f_i, f_j)\, \Delta\beta_i(t_k)\, \Delta\beta_j(t_k)$$

and as in [9] the two terms converge to $\sum_{i=1}^{n} \lambda_i \int_{t_0}^{t} \tfrac{1}{2} v_{yy}(r, y(r))(f_i, f_i)\, dr$, and to 0 in probability as $\max_k \Delta t_k \to 0$.

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