Journal of Mathematical Analysis and Applications **233,** 484–497 (1999)
Article ID jmaa.1999.6295, available online at http://www.idealibrary.com on IDEAL®

On a New Generalization of Hardy–Hilbert's Inequality and Its Applications

Yang Bicheng

Department of Mathematics, Guangdong Education College, Guangzhou, Guangdong

and

Lokenath Debnath

Department of Mathematics, University of Central Florida, Orlando, Florida 32816

Submitted by William F. Ames

Received November 2, 1998

This paper deals with a generalization of the Hardy–Hilbert inequality with best constant factor which involves the β function. As an application, we obtain a new equivalent form of the Hardy–Hilbert inequality. © 1999 Academic Press

Key Words: Hardy–Hilbert's inequality; weight coefficient; β function.

1. INTRODUCTION

If $a_n, b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then the famous Hardy–Hilbert's inequality (Hardy *et al.* [1]) may be written in the following form:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_n b_m}{(m+n+1)} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=0}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^{\infty} b_n^q \right\}^{1/q}, \quad (1.1)$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is best possible.

In particular, when $p = q = 2$, inequality (1.1) reduces to the standard Hilbert's inequality

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_n b_m}{(m+n+1)} < \pi \left\{ \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{1/2}. \quad (1.2)$$

0022-247X/99 $30.00

Both inequalities (1.1) and (1.2) play an important role in analysis and its applications. Recently, Yang [2, 3] and Gau [4] gave some improvements of (1.1) and (1.2) by estimating the weight coefficients. Yang [5] also generalized the integral form of (1.1) by introducing some parameters. We consider the weight function

$$\omega_\lambda(x) = \int_0^\infty \frac{1}{(x+y)^\lambda}\left(\frac{x}{y}\right)^{(2-\lambda)/2} dy, \qquad x \in (0,\infty), \qquad (0 < \lambda \leq 1). \tag{1.3}$$

Recently, Yang [6] gives a generalization of Hilbert's integral inequality as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy \leq B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^\infty x^{1-\lambda} f^2(x)\,dx \int_0^\infty x^{1-\lambda} g^2(x)\,dx\right\}^{1/2}, \tag{1.4}$$

where $B(p,q)$ $(p,q > 0)$ is the β function. Following the paper by Yang [6], we introduce a parameter λ and consider the weight coefficient

$$\omega_\lambda(r,n) = \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^\lambda}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{(2-\lambda)/r}, \tag{1.5}$$

where N_0 is the set of nonnegative integers and $n \in N_0$ $(r > 1, 2-r < \lambda \leq 2)$.

The main objective of this paper is to formulate a new inequality related to the double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_n b_m}{(m+n+1)^\lambda} \tag{1.6}$$

so that inequality (1.1) can be generalized with a best constant $B((\lambda+p-2)/p,(\lambda+q-2)/q)$. As an application, we give its equivalent form and obtain some particular results.

2. SOME RESULTS AND LEMMAS

First, we need the following inequality (see Yang and Debnath [7]):

If for $r = 0,1,2,3,4$, $f^{(r)}(\infty) = 0$, $f^{(2r-1)}(x) < 0$, $f^{(2r)}(x) \geq 0$, $x \in [0,\infty)$, and $\int_0^\infty f(x)\,dx < \infty$, then

$$\sum_{m=0}^{\infty} f(m) \leq \int_0^\infty f(x)\,dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{2.1}$$

LEMMA 2.1. *If $n \in N_0$, $r > 1$, $2 - r < \lambda \leq 2$, and $\lambda > 0$, define the function $R_\lambda(r, n)$ by*

$$R_\lambda(r,n) = \left(n + \frac{1}{2}\right)^{1-\lambda-(2-\lambda)/r} \int_0^{1/(2n+1)} \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du - \frac{(3r+2-\lambda)}{6r(n+1)^\lambda} \cdot 2^{(2-\lambda)/r} - \frac{\lambda}{12(n+1)^{\lambda+1}} \cdot 2^{(2-\lambda)/r}, \quad (2.2)$$

then $R_\lambda(r, n) > 0$.

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_0^{1/(2n+1)} \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du \\
&= \frac{1}{1-(2-\lambda)/r} \int_0^{1/(2n+1)} \frac{1}{(1+u)^\lambda}\, du^{1-(2-\lambda)/r} \\
&= \frac{1}{1-(2-\lambda)/r}\left[\frac{1}{(1+u)^\lambda} u^{1-(2-\lambda)/r}\Big|_0^{1/(2n+1)} - \int_0^{1/(2n+1)} u^{1-(2-\lambda)/r}\, d\frac{1}{(1+u)^\lambda}\right] \\
&= \frac{2^{-1+(2-\lambda)/r}}{1-(2-\lambda)/r} \frac{1}{(n+1)^\lambda}\left(n+\frac{1}{2}\right)^{\lambda-1+(2-\lambda)/r} + \frac{\lambda}{1-(2-\lambda)/r}\int_0^{1/(2n+1)} \frac{1}{(1+u)^{\lambda+1}} u^{1-(2-\lambda)/r}\, du \\
&= \frac{2^{-1+(2-\lambda)/r}}{1-(2-\lambda)/r} \frac{1}{(n+1)^\lambda}\left(n+\frac{1}{2}\right)^{\lambda-1+(2-\lambda)/r} \\
&\quad + \frac{\lambda}{1-(2-\lambda)/r}\left[\frac{2^{-2+(2-\lambda)/r}}{2-(2-\lambda)r} \cdot \frac{\left(n+\frac{1}{2}\right)^{\lambda-1+(2-\lambda)/r}}{(n+1)^{\lambda+1}} + (\lambda+1)\int_0^{1/(2n+1)} \frac{1}{(1+u)^{\lambda+2}} u^{2-(2-\lambda)/r}\, du\right]
\end{aligned}$$

$$> \frac{2^{-1+(2-\lambda)/r}}{1-(2-\lambda)/r} \frac{\left(n+\frac{1}{2}\right)^{\lambda-1+(2-\lambda)/r}}{(n+1)^{\lambda}}$$

$$+ \frac{\lambda}{1-(2-\lambda)/r} \cdot \frac{\left(n+\frac{1}{2}\right)^{\lambda-1+(2-\lambda)/r}}{(n+1)^{\lambda+1}}.$$

Then, by (2.1), we find

$$R_\lambda(r,n) > \left[-\frac{3r+2-\lambda}{6r} + \frac{2^{-1}}{1-(2-\lambda)/r}\right] \frac{2^{(2-\lambda)/r}}{(n+1)^{\lambda}}$$

$$+ \left[-\frac{1}{12} + \frac{1}{1-(2-\lambda)/r} \cdot \frac{2^{-2}}{2-(2-\lambda)/r}\right] \frac{\lambda\, 2^{(2-\lambda)/r}}{(n+1)^{\lambda+1}}. \tag{2.3}$$

Since $0 < \lambda \le 2$ and $0 \le (2-\lambda)/r < 1$, inequality $2(2-\lambda) + (2-\lambda)^2/r \ge 0$ is equivalent to $-[(3r+2-\lambda)/6r] + [2^{-1}/(1-(2-\lambda)/r)] \ge 0$, and inequality $((2-\lambda)/r)^2 - 3((2-\lambda)/r) - 1 < 0$ is equivalent to $-\frac{1}{12} + [1/(1-(2-\lambda)/\tau)] \cdot 2^{-2}/(2-(2-\lambda)/r) > 0$. Hence, by (2.3), we have $R_\lambda(r,n) > 0$. The lemma is proved.

LEMMA 2.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p,q\} < \lambda \le 2$, *and* $\omega_\lambda(r,n)$ *is defined by* (1.5), *we have*

$$\omega_\lambda(r,n) < B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)\left(n+\frac{1}{2}\right)^{1-\lambda}, \tag{2.4}$$

where $n \in N_0 (r = p, q)$, $B((p-2+\lambda)/p, (q-2+\lambda)/q) = \int_0^\infty 1/(1+u)^\lambda (\frac{1}{u})^{(2-\lambda)/p}\, du$.

Proof. By (2.1), we have

$$\sum_{m=0}^{\infty} \frac{1}{(m+n+1)^{\lambda}} \left(\frac{1}{m+\frac{1}{2}}\right)^{(2-\lambda)/r}$$

$$\le \int_0^\infty \frac{1}{(t+n+1)^{\lambda}} \left(\frac{1}{t+\frac{1}{2}}\right)^{(2-\lambda)/r} dt + \frac{1}{2(n+1)^{\lambda}} \cdot 2^{(2-\lambda)/r}$$

$$+ \frac{\lambda}{12(n+1)^{\lambda+1}} \cdot 2^{(2-\lambda)/r} + \frac{2-\lambda}{12r(n+1)^{\lambda}} \cdot 2^{1+(2-\lambda)/r}$$

$$= \int_0^\infty \frac{1}{(t+n+1)^\lambda}\left(\frac{1}{t+\frac{1}{2}}\right)^{(2-\lambda)/r} dt + \frac{3r+2-\lambda}{6r(n+1)^\lambda}\cdot 2^{(2-\lambda)/r}$$
$$+ \frac{\lambda}{12(n+1)^{\lambda+1}}\cdot 2^{(2-\lambda)/r}. \tag{2.5}$$

Setting $u = (t+\frac{1}{2})/(n+\frac{1}{2})$, we find

$$\int_0^\infty \frac{1}{(t+n+1)^\lambda}\left(\frac{1}{t+\frac{1}{2}}\right)^{(2-\lambda)/r} dt$$
$$= \left(n+\frac{1}{2}\right)^{1-\lambda-(2-\lambda)/r}\int_{1/(2n+1)}^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du$$
$$= \left(n+\frac{1}{2}\right)^{1-\lambda-(2-\lambda)/r}\left[\int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du - \int_0^{1/(2n+1)} \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du\right].$$

Then, by (1.5), (2.5), and (2.2), we have

$$\omega_\lambda(r,n) \le \left(n+\frac{1}{2}\right)^{(2-\lambda)/r}\left\{\left(n+\frac{1}{2}\right)^{1-\lambda-(2-\lambda)/r} \times \int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du - R_\lambda(r,n)\right\}. \tag{2.6}$$

Since $2-\min\{p,q\} < \lambda \le 2$, for $r = p, q$, we have $2 - r < \lambda \le 2$, and $\lambda > 0$. By Lemma 1.1 and (2.6), we have

$$\omega_\lambda(r,n) < \left(n+\frac{1}{2}\right)^{1-\lambda}\int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du. \tag{2.7}$$

Since the β function has the relation (see [8, p. 117])

$$B(u,v) = \int_0^\infty \frac{t^{-1+u}}{(1+t)^{u+v}}\, dt = B(v,u) \qquad (u,v>0),$$

and $(q-2+\lambda)/q+(p-2+\lambda)/p=\lambda$, we have

$$\int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/p} du$$
$$= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+(p-2+\lambda)/p}\, du = B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)$$
$$= B\left(\frac{q-2+\lambda}{q}, \frac{p-2+\lambda}{p}\right) = \int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/q} du.$$

Hence, by (2.7), we have (2.4). This proves the lemma.

LEMMA 2.3. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, $x\geq 0$, *and* $\varepsilon>0$, *then*

$$\int_0^\infty \left(x+\frac{1}{2}\right)^{-\varepsilon-1}\int_0^{1/(2x+1)} \frac{u^{(-z+\lambda-\varepsilon)/q}}{(1+u)^\lambda}\, du\, dx = O(1) \qquad (\varepsilon\to 0^+). \tag{2.8}$$

Proof. We have

$$\int_0^{1/(2x+1)} \frac{u^{(-2+\lambda-\varepsilon)/q}}{(1+u)^\lambda}\, du$$
$$< \int_0^{1/(2x+1)} u^{(-2+\lambda-\varepsilon)/q}\, du$$
$$= \frac{1}{1+(1/q)(-2+\lambda-\varepsilon)}\left(\frac{1}{2x+1}\right)^{1+(-2+\lambda-\varepsilon)/q}.$$

Since $1+(-2+\lambda)/q>0$ and $\lambda>2-\min\{p,q\}\geq 0$, there exists $n_0\in N$, such that $\lambda/n_0<1$, for $0<\varepsilon<\lambda/n_0$, $1+(-2+\lambda-\varepsilon)/q>0$. Since for $a>1$ the function $g(y)=\frac{1}{y}(\frac{1}{a})^y$ $(y\in(0,\infty))$ is strictly decreasing, we find

$$\frac{1}{1+(-2+\lambda-\varepsilon)/q}\left(\frac{1}{2x+1}\right)^{1+(-2+\lambda-\varepsilon)/q}$$
$$\leq \frac{1}{1+(1/q)[-2+\lambda(1-1/n_0)]}\left(\frac{1}{2x+1}\right)^{1+[-2+\lambda(1-1/n_0)]/q};$$

$$0 < \int_0^\infty \left(x + \frac{1}{2}\right)^{-\varepsilon - 1} \int_0^{1/(2x+1)} \frac{u^{(-2+\lambda-\varepsilon)/q}}{(1+u)^\lambda} \, du \, dx$$

$$< \frac{2^{1+(\lambda/n_0)}}{1 + (1/q)[-2 + \lambda(1 - (1/n_0))]} \int_0^\infty \left(\frac{1}{2x+1}\right)^{2+[-2+\lambda(1-1/n_0)]/q} dx$$

$$= 2^{\lambda/n_0} \left\{ \frac{1}{1 + (1/q)[-2 + \lambda(1 - (1/n_0))]} \right\}^2 .$$

Hence relation (2.8) is valid. The lemma is proved.

3. MAIN RESULTS

THEOREM 3.1. *If* $a_n, b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, *and* $0 < \sum_{n=0}^\infty (n + \frac{1}{2})^{1-\lambda} a_n^p < \infty$, $0 < \sum_{n=0}^\infty (n + \frac{1}{2})^{1-\lambda} b_n^q < \infty$, *then*

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_n b_m}{(m+n+1)^\lambda} < B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) \times \left\{ \sum_{n=0}^\infty \left(n + \frac{1}{2}\right)^{1-\lambda} a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^\infty \left(n + \frac{1}{2}\right)^{1-\lambda} b_n^q \right\}^{1/q}, \tag{3.1}$$

where the constant factor $B((p-2+\lambda)/p, (q-2+\lambda)/q)$ *is best possible.*

In particular, when $\lambda = 2$, *we have*

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_n b_m}{(m+n+1)^2} < 2\left\{ \sum_{n=0}^\infty \frac{1}{(2n+1)} a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^\infty \frac{1}{(2n+1)} b_n^q \right\}^{1/q}, \tag{3.2}$$

where the constant 2 *is still best possible.*

Proof. By Holder's inequality, we have

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_n b_m}{(m+n+1)^\lambda} = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_n}{(m+n+1)^{\lambda/p}} \left(\frac{n + \frac{1}{2}}{m + \frac{1}{2}}\right)^{(2-\lambda)/pq}$$

$$\times \frac{b_m}{(m+n+1)^{\lambda/q}} \left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}} \right)^{(2-\lambda)/pq}$$

$$\leq \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_n^p}{(m+n+1)^{\lambda}} \left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}} \right)^{(2-\lambda)/q} \right\}^{1/p}$$

$$\times \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{b_m^q}{(m+n+1)^{\lambda}} \left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}} \right)^{(2-\lambda)/p} \right\}^{1/q}$$

$$= \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n^p}{(m+n+1)^{\lambda}} \left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}} \right)^{(2-\lambda)/q} \right\}^{1/p}$$

$$\times \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{b_m^q}{(m+n+1)^{\lambda}} \left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}} \right)^{(2-\lambda)/p} \right\}^{1/q}$$

$$= \left\{ \sum_{n=0}^{\infty} \omega_{\lambda}(q,n) a_n^p \right\}^{1/p} \left\{ \sum_{m=0}^{\infty} \omega_{\lambda}(p,m) b_m^q \right\}^{1/q}.$$

Then, by (2.4), we have (3.1).

For $0 < \varepsilon \leq \lambda/n_0 < 1$, setting $a_n = (n+\frac{1}{2})^{(\lambda-2-\varepsilon)/p}$, $b_m = (m+\frac{1}{2})^{(\lambda-2-\varepsilon)/q}$, then

$$\frac{2^{\varepsilon}}{\varepsilon} = \int_0^{\infty} \left(x+\frac{1}{2}\right)^{-1-\varepsilon} dx < \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{-1-\varepsilon}$$

$$= 2^{1+\varepsilon} + \sum_{n=1}^{\infty} \left(n+\frac{1}{2}\right)^{-1-\varepsilon} < 2^{1+\varepsilon} + \int_0^{\infty} \left(x+\frac{1}{2}\right)^{-1-\varepsilon} dx$$

$$< 4 + 2^{\varepsilon}/\varepsilon;$$

$$\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q = \frac{2^{\varepsilon}}{\varepsilon} + O(1) \qquad (\varepsilon \to 0^+). \tag{3.3}$$

Since $0 \leq 2 - \min\{p,q\} < \lambda \leq 2$, by (2.8), we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_n b_m}{(m+n+1)^{\lambda}}$$

$$> \int_0^{\infty} \left(x+\frac{1}{2}\right)^{(\lambda-2-\varepsilon)/p} \int_0^{\infty} \frac{\left(y+\frac{1}{2}\right)^{(\lambda-2-\varepsilon)/q}}{(x+y+1)^{\lambda}} \, dy \, dx$$

$$= \int_0^\infty \left(x + \frac{1}{2}\right)^{-1-\varepsilon} \left[\int_{1/(2x+1)}^\infty \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du\right] dx$$

$$= \int_0^\infty \left(x + \frac{1}{2}\right)^{-1-\varepsilon} \left[\int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du\right] dx$$

$$- \int_0^\infty \left(x + \frac{1}{2}\right)^{-1-\varepsilon} \left[\int_0^{1/(2x+1)} \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du\right] dx$$

$$= \frac{2^\varepsilon}{\varepsilon} \int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du - O(1).$$

Since we have

$$\lim_{\varepsilon \to 0+} \int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du = B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right);$$

$$\int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/q}}{(1+u)^\lambda} du = B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) + o(1) \qquad (\varepsilon \to 0^+),$$

we find

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_n b_m}{(m+n+1)^\lambda}$$
$$> \frac{2^\varepsilon}{\varepsilon}\left[B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) + o(1)\right] - O(1)$$
$$= \frac{2^\varepsilon}{\varepsilon}\left[B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) + o(1)\right], \tag{3.4}$$

where $0 < K < B((p-2+\lambda)/p, (q-2+\lambda)/q)$. Putting $\varepsilon > 0$ ($\varepsilon \leq \lambda/n_0 < 1$) small enough so that

$$\frac{2^\varepsilon}{2^\varepsilon + \varepsilon O(1)}\left[B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) + o(1)\right] > K,$$

we have, by (3.3) and (3.4),

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_n b_m}{(m+n+1)^{\lambda}}$$
$$> K\left(\frac{2^{\varepsilon}}{\varepsilon}+O(1)\right)$$
$$= K\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{1/q}.$$

It follows that the constant factor $B((p-2+\lambda)/p,(q-2+\lambda)/q)$ in (3.1) is best possible.

Since $k_p(2) = B(1,1) = 1$, by (3.1), inequality (3.2) is valid. It is obvious that the constant factor 2 in (3.2) is still best possible. The theorem is proved.

When $p = 2$, we have the following:

COROLLARY 3.1. *If* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^2 < \infty$ *and* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}b_n^2 < \infty$, *where* $0 < \lambda \le 2$, *then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_n b_m}{(m+n+1)^{\lambda}}$$
$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^2\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^2\right\}^{1/2}, \quad (3.5)$$

where the constant factor $B(\lambda/2,\lambda/2)$ *is best possible*.

In particular, for $\lambda = 2$, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_n b_m}{(m+n+1)^{2}} < 2\left\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)}a_n^2\sum_{n=0}^{\infty}\frac{1}{(2n+1)}b_n^2\right\}^{1/2}, \quad (3.6)$$

where the constant 2 is still best possible.

Remark. When $\lambda = 1$, (3.1) reduces to (1.1). Inequality (3.1) is a generalization of (1.1). It is obvious that inequality (3.5) is a generalization of (1.2).

4. SOME APPLICATIONS

THEOREM 4.1. *If* $a_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, *and* $0 < \sum_{n=0}^{\infty}(n + \frac{1}{2})^{1-\lambda} a_n^p < \infty$, *then*

$$\sum_{m=0}^{\infty}\left(m + \frac{1}{2}\right)^{(\lambda-1)(p-1)}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^{\lambda}}\right]^p < \left[B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)\right]^p \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)^{1-\lambda} a_n^p, \qquad (4.1)$$

where the constant factor $[B((p-2+\lambda)/p, (q-2+\lambda)/q)]^p$ *is best possible. Inequality* (4.1) *is equivalent to* (3.1).

In particular,

(i) *for* $\lambda = 1$, *we have*

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=0}^{\infty} a_n^p, \qquad (4.2)$$

(ii) *for* $\lambda = 2$, *we have*

$$\sum_{m=0}^{\infty}\left(m + \frac{1}{2}\right)^{p-1}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^2}\right]^p < 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)} a_n^p, \quad (4.3)$$

where the constant 2 *is still best possible.*

Proof. Since $0 < \sum_{n=0}^{\infty}(n + \frac{1}{2})^{1-\lambda} a_n^p < \infty$, there exists $k_0 \in N$ such that for any $k \geq k_0$, $0 < \sum_{n=0}^{k}(n + \frac{1}{2})^{1-\lambda} a_n^p < \infty$.

We set

$$b_m(k) = \left[\left(m + \frac{1}{2}\right)^{\lambda-1}\sum_{n=0}^{k}\frac{a_n}{(m+n+1)^{\lambda}}\right]^{p-1} > 0 \qquad (m \in N, k \geq k_0),$$

and use (3.1) to obtain

$$0 < \sum_{m=0}^{k}\left(m + \frac{1}{2}\right)^{1-\lambda} b_m^q(k)$$
$$= \sum_{m=0}^{k}\left(m + \frac{1}{2}\right)^{(\lambda-1)(p-1)}\left[\sum_{n=0}^{k}\frac{a_n}{(m+n+1)^{\lambda}}\right]^p$$

$$= \sum_{m=0}^{k} \sum_{n=0}^{k} \frac{a_n b_m(k)}{(m+n+1)^\lambda}$$

$$< B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)\left\{\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p\right\}^{1/p}$$

$$\times\left\{\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q(k)\right\}^{1/q}. \qquad (4.4)$$

Hence, we find

$$\left[\sum_{m=0}^{k}\left(m+\frac{1}{2}\right)^{1-\lambda} b_m^q(k)\right]^{1/p}$$

$$= \left\{\sum_{m=0}^{k}\left(m+\frac{1}{2}\right)^{(\lambda-1)(p-1)}\left[\sum_{n=0}^{k}\frac{a_n}{(m+n+1)^\lambda}\right]^p\right\}^{1/p}$$

$$< B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)\left\{\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p\right\}^{1/p}. \qquad (4.5)$$

It follows that $0 < \sum_{m=0}^{\infty}(m+\frac{1}{2})^{1-\lambda} b_m^q(\infty) < \infty$. Hence, (4.4) is valid as $k \to \infty$ by (3.1); so is (4.5). Thus, inequality (4.1) holds. We have proved that inequality (3.1) implies (4.1).

We next show that inequality (4.1) implies the inequality (3.1):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_n b_m}{(m+n+1)^\lambda}$$

$$= \sum_{m=0}^{\infty}\left[\frac{1}{(m+\frac{1}{2})^{(1-\lambda)/q}}\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^\lambda}\right]\left[\left(m+\frac{1}{2}\right)^{(1-\lambda)/q} b_m\right]$$

$$\le \left\{\sum_{m=0}^{\infty}\frac{1}{(m+\frac{1}{2})^{(1-\lambda)(p-1)}}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^\lambda}\right]^p\right\}^{1/p}$$

$$\times\left\{\sum_{m=0}^{\infty}\left[\left(m+\frac{1}{2}\right)^{(1-\lambda)/q} b_m\right]^q\right\}^{1/q}$$

$$= \left\{\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{(\lambda-1)(p-1)}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^\lambda}\right]^p\right\}^{1/p}$$

$$\times\left\{\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{1-\lambda}b_m^q\right\}^{1/q}. \tag{4.6}$$

Since $0 < \sum_{m=0}^{\infty}(m+\frac{1}{2})^{1-\lambda}b_m^q < \infty$, by (4.1), we obtain (3.1). It follows that (3.1) implies (4.1). Since the constant factor in (3.1) is best possible, we may show that the constant factor in (4.1) is also best possible by using (4.6). Otherwise, there exist λ $(2-\min\{p,q\} < \lambda \le 2)$ and $K > 0$ such that $K < B((p-2+\lambda)/p,(q-2+\lambda)/q)$, and

$$\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{(\lambda-1)(p-1)}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^{\lambda}}\right]^p < K^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p.$$

Then, by (4.6), we find

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_nb_m}{(m+n+1)^{\lambda}}$$
$$< K\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{1/q}.$$

This is a contradiction. Since $B((p-1)/p,(q-1)/q) = B(\frac{1}{q},\frac{1}{p}) = \pi/\sin(\pi/p)$, for $\lambda = 1$, by (4.1), we have (4.2). It is obvious that (4.3) holds for $\lambda = 2$, by (4.1). This proves the theorem.

COROLLARY 4.1. *If* $0 < \lambda \le 2$, *and* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^2 < \infty$, *then we have*

$$\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{(\lambda-1)}\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^{\lambda}}\right]^2$$
$$< \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^2\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^2, \tag{4.7}$$

where the constant factor $[B(\frac{\lambda}{2},\frac{\lambda}{2})]^2$ *is best possible. Inequality* (4.7) *is equivalent to* (3.5). *In particular,*

(i) *for* $\lambda = 1$, *we have*

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^2 < \pi^2\sum_{n=0}^{\infty}a_n^2; \tag{4.8}$$

(ii) *for* $\lambda = 2$, *we have*

$$\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)\left[\sum_{n=0}^{\infty}\frac{a_n}{(m+n+1)^2}\right]^2 < 2\sum_{n=0}^{\infty}\frac{a_n^2}{(2n+1)}, \qquad (4.9)$$

where the constant 2 *is still best possible*.

Remark. Inequality (4.2) is equivalent to (1.1), and inequality (4.1) is a generalization of (4.2), which is equivalent to (3.1). Since all the constant factors involving the β function are best possible, we obtain some new results.

REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge University Press, Cambridge, MA, 1952.
2. Yang Bicheng, A refinement of the more profound Hardy–Hilbert's inequality, *Hunan Ann. Math.* **17**, No. 2 (1997), 35–38.
3. Yang Bicheng, A refinement of Hilbert's inequality, *Huanghuai J.* **13**, No. 2 (1997), 47–51.
4. Gau Mingzhe, On Hilbert's inequality and its applications, *J. Math. Anal. Appl.* **212** (1997), 316–323.
5. Yang Bicheng, On generalizations of Hardy–Hilbert's integral inequalities, *Acta Math. Sinica* **41**, No. 4 (1998), 839–844.
6. Yang Bicheng, On Hilbert's integral inequality, *J. Math. Anal. Appl.* **220** (1998), 778–785.
7. Yang Bicheng and L. Debnath, On new strengthened Hardy-Hilbert's inequality, *Internat. J. Math. Math. Sci.* **21**, No. 2 (1998), 403–408.
8. Wang Zhuxi and Gua, Dunren, "An Introduction to Special Functions," Science Press, 1979.