Topological Stability: Some Fundamental Properties

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INTRODUCTION

The purpose of this paper is to clarify and generalize some properties and results related to topological stability and to give proofs of some results which are stated in the literature without proof.

Section 1 of this work consists of preparatory definitions and Lemmas most of which are taken from [18]. The question whether the stability property is invariant under conjugacy was partially answered in [18]. Theorem 1 in Section 2 is a general result in this direction. In [19] Walters has proved that the topological stability of diffeomorphisms on compact manifolds of dimension \( \geq 2 \) implies the pseudo orbit tracing property. Recently in [12], Morimoto generalized this result to include the case of dimension one. In Section 3, what is presented is a unified proof of the various cases by using some interesting results on suspension flows. In [10], Hurley stated as Theorem A that if \( f \) is topologically stable and \( X \) is a chain component of \( f \) and \( k \) is the least period of any periodic point in \( X \), then \( X \) consists of no more than \( k \) chain components of \( f^k \), and \( f^k \) is mixing on each. In Section 4 we show that such \( f^k \) has the specification property on each of its chain components. In Section 5 we investigate a question of Theorem 4.1 in a paper by Franke and Selgrade [9], but without limiting ourself to the case when the abstract \( \omega \)-limit sets are hyperbolic.

1. PREPARATORY DEFINITIONS AND LEMMAS

In this paper we assume that the spaces are compact metric spaces unless otherwise stated, and \((X, \phi)\) denotes a continuous real flow \( \phi \) and a compact metric space \( X \) (i.e., \( \phi: X \times \mathbb{R} \to X \) is continuous and \( \phi(x, t+s) = \))

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Let $\phi$, denote the homeomorphism of $X$ defined by $\phi(x, t)$.

Given $x, y \in X$ and $\delta, a > 0$, a $(\delta, a)$-chain from $x$ to $y$ is a collection \[ \{x_0 = x, x_1, x_2, \ldots, x_k = y; t_0, t_1, \ldots, t_{k-1}\} \] so that for $0 \leq i \leq k-1$ we have $t_i \geq a$ and $d(\phi_{t_i}x_i, x_{i+1}) < \delta$. A point $x$ is chain equivalent to $y$ (written $x \sim y$) if for every $\delta, a > 0$ there is a $(\delta, a)$-chain from $x$ to $y$ and from $y$ to $x$. The chain recurrent set of $f$ is $\text{CR}(f) = \{x \in X; x \sim x\}$ and $f$ is said to be chain recurrent if $\text{CR}(f) = M$. For more details see [5, 7, 8].

An infinite $(\delta, a)$-chain is a pair of doubly infinite sequences $(\{x_i\}_{-\infty}^\infty, \{t_i\}_{-\infty}^\infty)$ so that $t_i \geq a$ and $d(\phi_{t_i}x_i, x_{i+1}) < \delta$ for all $i$. The definition of a finite (infinite) $(\delta, a)$-pseudo orbit is the same as that of a finite (infinite) $(\delta, a)$-chain [8, 18].

Let $(\{x_i\}, \{t_i\})$ be a $(\delta, a)$-pseudo orbit. The following notation will be standard throughout this paper. $S_0 = 0$, $S_n = \sum_{i=0}^{n-1} t_i$. We always assume $\sum_{i=0}^{n-1} t_i = 0$ if $k < j$. In particular $\sum_{i=0}^{n-1} t_i = 0$.

Give $\varepsilon > 0$, the definition of $\varepsilon$-tracing is given as follows: A finite (infinite) $(\delta, a)$-pseudo orbit $(\{x_n\}, \{t_n\})$ is $\varepsilon$-traced by an orbit $(\phi, z)_{t \in R}$ if there exists an increasing homeomorphism $\phi: R \to R$ with $\phi(0) = 0$ such that

\[ d(\phi_{z(t)}z, \phi_{t-S_n}x_n) < \varepsilon \quad \text{whenever} \quad S_n \leq t \leq S_{n+1} \quad \text{for} \ n = 0, 1, 2, \ldots \]

and

\[ d(\phi_{z(t)}z, \phi_{t+S_n}x_n + n) < \varepsilon \quad \text{whenever} \quad -S_n \leq t < -S_{n+1} \quad \text{for} \ n = 1, 2, 3, \ldots \]

A reparameterization $x: R \to R$ of an orbit is an orientation preserving homeomorphism of real values fixing the origin. In the case of a finite $(\delta, a)$-pseudo orbit one can restrict to a closed interval $[x, y]$ containing the origin. When we say $(\delta, a)$-pseudo orbit we mean infinite $(\delta, a)$-pseudo orbit unless otherwise stated.

For the case of homeomorphisms on compact spaces an easier argument can be carried on for definitions of $(\delta)$-chain, chain recurrent and pseudo orbit tracing property of a homeomorphism $f$. For more details see [1, 10, 19].

**Definition 1.** A flow $(X, \phi)$ has the pseudo orbit tracing property (POTP) if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that each $(\delta, 1)$-pseudo orbit is $\varepsilon$-traced by an orbit of $\phi$.

**Definition 2** [4]. We say that a flow $(X, \phi)$ is expansive if $\forall \varepsilon > 0$, $\exists \delta > 0$ with the property that if $d(\phi_{s(t)}x, \phi_{s(t)}y) < \delta$ for all $t \in R$, for a pair of points $x, y \in X$ and a continuous map $S: R \to R$ with $S(0) = 0$, then $y = \phi_s x$, where $|t| < \varepsilon$.

These two definitions are clearly independent of the metric.
LEMMA 1 (Cf. [4, Lemma 1]). If \( \phi \) is an expansive flow, then each fixed point of \( \phi \) is an isolated point.

This lemma shows that the study of flows with expansive property can be reduced to those without fixed points.

If \( f \) and \( g \) are homeomorphisms on a compact metric space \( X \), then \( d(f, g) \) will denote \( \sup_{x \in X} d(fx, gx) \).

DEFINITION 3. A flow \((X, \phi)\) is said to be topologically stable (sometimes called lower semistable) if \( \forall \epsilon > 0, \exists \delta > 0 \) such that for every flow \( \psi \) on \( X \) with \( d(\phi_t, \psi_t) < \delta \) for all \( t \in [0, 1] \), there exists a continuous \( h : X \to X \) so that \( d(h, I) < \epsilon \) and \( h(\text{orbit of } \psi) \subseteq \text{(orbit of } \phi) \).

\( I \) stands always for identity homeomorphism of \( X \).

PROPOSITION 1 (Cf. [18, Theorem 3]). Every continuous expansive flow \((X, \phi)\) which has the POTP is topologically stable.

PROPOSITION 2 (Cf. [18, Theorem 5]). Every topologically stable \( C^1 \)-flow \( \phi \) on a compact manifold \( M \) and without fixed points has POTP.

We recall that the flows \((X, \phi)\) and \((Y, \psi)\) are conjugate if there is a homeomorphism from \( X \) onto \( Y \) mapping orbits of \( \phi \) onto orbits of \( \psi \). Throughout this paper the type of conjugacy we will consider is a conjugacy with orientation preserved on each orbit. This assumption is used elsewhere in the literature. For example this is used by Franke and Selgrade in [9] for proving Theorem 3.1.

We say that \((X, \phi)\) and \((Y, \psi)\) are isomorphic or topologically equivalent if they are conjugate and \( h_{\phi_t} x = \psi_t h x \) for every \( t \in \mathbb{R} \) and \( x \in X \), where \( h \) is the conjugate homeomorphism.

Let \( T : X \to X \) be a homeomorphism on a compact metric space \( X \) and let \( f : X \to \mathbb{R} \) be any positive real valued continuous function.

DEFINITION 4 [4]. The suspension of \( T \) under \( f \) is the flow \( \phi_f \) on the space

\[
X_f = \bigcup_{0 < t < f(x)} \{(x, t); (x, fx) \sim (Tx, 0)\}
\]

defined for small non-negative time by \( \phi_f(x, t) = (x, t + s) \), with \( 0 \leq t + s < fx \).

In order to understand more about such flows and the metric defined on \( X_f \) see [4, 16]. It is well known that the suspension flows \((X_f, \phi_f)\) and \((X_g, \phi_g)\) of \( T \) under \( f \) and \( g \), respectively, are topologically conjugate and a
homeomorphism from $X_f$ onto $X_g$ that conjugates the flows is given by $(x, t) \mapsto (x, tf(x)/g(x))$.

Remark. In the statement of Definition 3 if $X$ is a compact manifold and $\varepsilon$ is sufficiently small, then $d(h, I) < \varepsilon$ implies that $h$ maps $X$ onto $X$ [14, p. 361]. In general this need not be so. Consider the suspension $(Y, \phi)$ of a shift $\sigma: \Sigma \to \Sigma$ where $\Sigma = \prod_{n=-\infty}^{\infty} c$ and $c = \{0, 1\}$ under 1. Theorem 1 in [19] shows that $\sigma$ has the POTP. Theorem 2 in [18] and Proposition 1 imply that $(Y, \phi)$ is topologically stable. Define a map $s: \Sigma \to \Sigma$ by

$$(s(x))_n = x_n \text{ if } n < -m \text{ or } n > m,$$

$$(s(x))_n = x_{n+1} \text{ if } -m \leq n \leq m,$$

and

$$(s(x))_n = x_{-n} \text{ for } n > 193.$$ Let $(Y, \psi)$ also be the suspension flow of $(\Sigma, s)$ under 1. For $\delta > 0$, there exists $m$ large enough that $d(\phi, \psi_t) < \delta$ for all $t \in [0, 1]$. Now it is clear that any continuous map $h: Y \to Y$ with $h$ (orbit of $\phi$) $\subseteq$ (orbit of $\psi$) has its image contained in the subset of $Y$ consisting of the periodic points only (version of Walter’s example in [19]). So $h$ is not onto.

Also we can consider conditions on the perturbation $\psi$ of the flow $\phi$ to ensure that the conjugacy map $h$ is a homeomorphism. A general result in this direction is Theorem 4 in [18].

2. Topological Stability and Conjugacy

The question we want to answer here is whether the topological stability property is invariant under conjugacy (in particular velocity changes). In [18] a partial answer was given as Corollary 1 and Corollary 2 to Theorem 5.

Lemma 2. Let $\phi, \psi$ be flows on a compact metric space $X$ and let $a > 0$. Then $\forall \lambda > 0$, $\exists \delta > 0$ such that if $d(\phi, \psi_t) < \delta$ for $0 \leq t \leq a$, then $d(\phi, \psi_t) < \lambda$ for $0 \leq t \leq 1$.

Proof. In the case $a \geq 1$ there is nothing to prove. Assume $a < 1$ and let $n$ be a positive integer greater than $1/a$. Choose $0 < \delta_2 < \delta_1/2$ where $\delta_1 = \lambda$ so that for every $x, y \in X$ we have

$$d(x, y) < \delta_2 \quad \text{implies} \quad d(\phi, x, \phi, y) < \delta_1/2 \quad \text{for } 0 \leq t \leq a.$$

Also choose $0 < \delta_3 < \delta_2/2$ so that

$$d(\phi, x, \phi, y) < \delta_2/2 \quad \text{for } 0 \leq t \leq a \text{ whenever } d(x, y) < \delta_3.$$

If we carry on in the same manner we can choose $0 < \delta_n < \delta_{n-1}/2$ so that

$$d(\phi, x, \phi, y) < \frac{\delta_{n-1}}{2} \quad \text{for } 0 \leq t \leq a, \text{ whenever } d(x, y) < \delta_n.$$
Now take $\delta = \delta_n$ and assume $d(\phi_t, \psi_t) < \delta$ for $0 \leq t \leq a$. Hence $d(\phi_{t'}, \phi_t, \psi_t) < \delta_{n-1}/2$ for $0 \leq t' \leq a$ and $0 \leq t \leq a$. Therefore

$$d(\phi_{t'+t}, \psi_{t'+t}) \leq d(\phi_t, \phi_t, \phi_t, \psi_t) + d(\phi_{t'}, \psi_t, \psi_t) \leq \frac{\delta_{n-1}}{2} + \frac{\delta_{n-1}}{2} = \delta_{n-1}.$$ 

This means that $d(\phi_t, \psi_t) < \delta_{n-1}$ for $0 \leq t \leq 2a$. Repeating the process we will have $d(\phi_{t'}, \phi_t, \psi_t) < \delta_{n-2}/2$ for $0 \leq t' < a$ and $0 \leq t < 2a$. Therefore

$$d(\phi_{t'+t}, \psi_{t'+t}) \leq d(\phi_t, \phi_t, \phi_t, \psi_t) + d(\phi_{t'}, \psi_t, \psi_t) \leq \frac{\delta_{n-2}}{2} + \frac{\delta_{n-2}}{2} = \delta_{n-2}.$$ 

So $d(\phi_t, \psi_t) < \delta_{n-2}$ for $0 \leq t \leq 3a$. Carrying on in the same manner, we will have $d(\phi_t, \psi_t) < \delta_1$ for $0 \leq t \leq na$. This shows that $d(\phi_t, \psi_t) < \lambda$ for $0 \leq t \leq 1$.

**Lemma 3** (Cf. [18, Lemma 4.2]). Let $(X, \phi)$ be a flow with no fixed points. Then $\exists T_0 > 0$ such that if $0 < T < T_0$, there is a $\lambda > 0$ such that $d(\phi_T x, y) > \lambda$ whenever $d(x, y) < \lambda$.

Suppose $h: X \rightarrow Y$ is a homeomorphism mapping orbits of $\phi$ onto orbits of $\psi$ and suppose that $(X, \phi)$ has no fixed points. It is obvious that $(Y, \psi)$ has also no fixed points. Using similar arguments to Bowen's in [4, Corollary 4] one can show that for every $x \in X$, there is a unique strictly increasing homeomorphism $\sigma_x: \mathbb{R} \rightarrow \mathbb{R}$ with $\sigma_x(0) = 0$ such that $h\phi_t x = \psi_{\sigma_x(t)} h x$.

This $\sigma$ is called the cocycle of the flow $\phi$ with values in $\mathbb{R}$ [11, 16]. So $\sigma$ is a real valued function on $X \times \mathbb{R}$. Using arguments similar to Humphries [11], we choose a function $\beta: X \times \mathbb{R} \rightarrow \mathbb{R}$ given by $(x, t) \rightarrow \beta_x(t)$, where $\beta_x(t) = r$ if and only if $\sigma_x(t) = t$ and $\psi_x h x = h \phi_{\beta_x(t)} x$ for every $x \in X$. Note that $\beta_x$ is the inverse function of $\sigma_x$ for every $x \in X$. So denote $\beta = \sigma^{-1}$.

**Claim 1.** $\sigma_x(s+t) = \sigma_{\phi_t(x)}(s) + \sigma_x(t)$ for every $s, t \in \mathbb{R}$ and $x \in X$.

**Proof.** Define $\tau_x: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\tau_x(t) = \sigma_x(s+t) - \sigma_{\phi_t(x)}(s) \quad \text{for a fixed } s \in \mathbb{R}.$$ 

Hence

$$\psi_{\tau_x(t)}(h x) = \psi_{\sigma_x(s+t)} \psi_{-\sigma_{\phi_t(x)}(s)}(h x) = \psi_{-\sigma_{\phi_t(x)}(s)} h \phi_{t+s}(x)$$

$$= \psi_{-\sigma_{\phi_t(x)}(s)} h \phi_s \phi_t(x) = \psi_{-\sigma_{\phi_t(x)}(s)} \psi_{\sigma_{\phi_t(x)}(s)} h \phi_t x = h \phi_t x.$$
Because \( \sigma_x \) is the uniquely determined function satisfying the equality
\[ h_{\psi_x} x = \psi_{\sigma_x(t)} h_x, \]
therefore \( \tau_x = \sigma_x \) for every \( x \in X \). This shows that
\[ \sigma_x(s + t) = \sigma_{\varphi_x(s)}(s) + \sigma_x(t). \]

Define a function \( w: Y \times R \to R \) by \( w_y(t) = \varphi_x^{-1}(t) \), where \( h_x = y \). Hence
\[ h^{-1}_{\psi_y y} y = \varphi_{\psi_y(t)} h^{-1}_{-y} y, \]
and this means that \( w \) is the cocycle of \( \psi \) with values in \( R \). Hence
\[ w_y(s + t) = w_{\psi_y(y)}(s) + w_y(t). \]

**Claim 2.** \( \forall \delta > 0, \exists T > 0 \) such that \( |\sigma_x(t)| < \delta \) for every \( t \in [-T, T] \) and for every \( x \in X \).

**Proof.** Suppose the hypothesis is not true. (That is, \( \exists \delta > 0 \) so that for every integer \( n > 0 \), there exists \( x_n \in X \) such that \( \sigma_{x_n}(t) \geq \delta \) for some \( t \in [0, 1/n] \). We are dealing with positive \( t \) and for negative \( t \) we can use similar processes.) By continuity of \( \sigma_{x_n}: R \to R \) and \( \sigma_{x_n}(0) = 0 \) one can assume that \( \sigma_{x_n}(t) = \delta \). Using Lemma 3 and assuming \( \delta < T_0 \) (\( T_0 \) is taken from Lemma 3), there exists a \( \lambda > 0 \) such that \( d(\psi_y y, y) > \lambda \) for every \( y \in Y \). Choose \( \gamma > 0 \) so that \( d(h_x, h_{x'}) < \gamma \) whenever \( d(x, x') < \gamma \) for \( x, x' \in X \). Also without loss of generality assume \( n \) is large enough such that \( d(\psi_{x_n} y, y) < \gamma \) for every \( t \in [0, 1/n] \) and for every \( x \in X \). Hence \( d(\psi_{x_n} y, x) < \gamma \). Therefore
\[ d(\psi_{x_n}(t) h x_n, h x_n) = d(h_{\phi_x} x_n, h x_n) < \lambda. \]

This is a contradiction. 

**Claim 3.** The cocycle \( \sigma: X \times R \to R \) is continuous.

**Proof.** Take \( \delta = T_0/4 \) (\( T_0 \) is taken from Lemma 3 with respect to \( (x, \phi) \)). Using Claim 2, \( \exists T > 0 \) such that \( |\sigma_x(t) - \sigma_{x'}(t)| < T_0/2 \) for every \( x, x' \in X \) and for every \( t \in [-T, T] \). First I want to show \( \sigma: X \times [-T, T] \to R \) is continuous. For \( t \in [-T, T] \) and given \( \varepsilon > 0 \), define a function \( f: Y \times [\varepsilon, T_0/2] \to R \) by \( f(y, s) = d(\psi_s y, y) \). By continuity of \( f \) and compactness of \( Y \times [\varepsilon, T_0/2] \), there exists a \( \lambda > 0 \) such that \( d(\psi_y y, y) > \lambda \) for every \( (y, s) \in Y \times [\varepsilon, T_0/2] \). Take \( \gamma' > 0 \) satisfying the property that \( d(h x, h x') < \gamma \), whenever \( d(x, x') < \gamma ' \). Also take \( \gamma > 0 \) such that \( d(\phi_x x, \phi_{x'} x) < \gamma ' \), whenever \( d(x, x') < \gamma \). Now suppose \( d(x, x') < \gamma \) for some \( x, x' \in X \). Then
\[ d(\psi_{\sigma_x(t)} h x, \psi_{\sigma_{x'}(t)} h x) = d(\psi_{\sigma_x(t)} h x, \psi_{\sigma_{x'}(t)} h x) = d(h_{\phi_x} x, h_{\phi_{x'}} x) < \lambda. \]
Using the fact that $|\sigma_x(t) - \sigma_y(t)| < T_0/2$ we must have $|\sigma_x(t) - \sigma_y(t)| < \varepsilon$ and this shows the continuity of $\sigma$ on $X \times [-T, T]$. Since

$$\sigma_x(s + t) = \sigma_{\phi_s(x)}(s) + \sigma_x(t) \quad \text{for every } t, s \in [-T, T],$$

and since $\sigma_{\phi_s(x)}(s)$ is a composition of two continuous functions, therefore $\sigma$ is continuous on $X \times [-2T, 2T]$. Inductively $\sigma$ is continuous on $X \times \mathbb{R}$.

A similar argument can be used in order to show that the cocycle $w$ is continuous. Hence $\sigma_{\psi}^{-1}: \mathbb{R} \to \mathbb{R}$ is also continuous jointly on $X$ and $\mathbb{R}$.

This means that we have the following lemma:

**Lemma 4.** If $h: X \to Y$ is a conjugate homeomorphism (i.e., a homeomorphism which maps orbits of $\phi$ onto orbits of $\psi$) and if there are no fixed points, then there exists a unique continuous function $\sigma: X \times \mathbb{R} \to \mathbb{R}$ such that:

1. $\sigma_x(0) = 0$ and $\sigma_x: \mathbb{R} \to \mathbb{R}$ is strictly an increasing homeomorphism.
2. $h\phi_t x = \psi_{\sigma_x(t)} h x$ for every $x \in X$ and $t \in \mathbb{R}$.

**Theorem 1.** Any flow which is conjugate to a topologically stable flow without fixed points is topologically stable.

**Proof.** Assume $(X, \phi)$ is conjugate to $(Y, \psi)$ (i.e., there exists a homeomorphism $\lambda: X \to Y$ and a continuous function $\sigma: X \times \mathbb{R} \to \mathbb{R}$ such that $\sigma_x$ is strictly an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ fixing the origin and $\lambda\phi_x = \psi_{\sigma_x(t)} \lambda x$), and assume $(Y, \psi)$ is topologically stable. Given $\varepsilon > 0$, choose $0 < \varepsilon' < \varepsilon$ so that

$$d(a, b) < \varepsilon' \quad \text{implies} \quad d(\lambda^{-1}a, \lambda^{-1}b) < \varepsilon$$

for every $a, b \in Y$.

Let $\delta' > 0$ satisfy the topological stability of $\psi$ with respect to $\varepsilon'$. Choose $0 < \delta < \delta'$ such that $d(\lambda a, \lambda b) < \delta'$, whenever $d(a, b) < \delta$ for $a, b \in X$. Let $a = \inf_{x \in X} \sigma_x^{-1}(1)$ (such an $a$ exists because of compactness of $X$) and let $\phi$ be any other flow on $X$ with $d(\phi_t x, \phi'_t) < \delta$ for all $0 \leq t \leq a$. Define a flow $\psi'$ on $Y$ as $\psi'_t x = \lambda \phi_{\sigma_x^{-1}(t)} x$. Therefore

$$d(\psi'_t x, \psi'_s x) - d(\lambda \phi_{\sigma_x^{-1}(t)} x, \lambda \phi_{\sigma_x^{-1}(s)} x) < \delta' \quad \text{for } 0 \leq t \leq 1.$$ 

Using the stability property of $\psi$, there exists a continuous $h: Y \to Y$ such that $d(h, I) < \varepsilon'$ and $h(\text{orbit of } \psi') \subseteq (\text{orbit of } \psi)$. Take $\gamma = \lambda^{-1}h\lambda: X \to X$. [End of proof]
Since }d(h\lambda, \lambda) < \varepsilon',\text{ therefore } d(\lambda^{-1}h\lambda, I) < \varepsilon,\text{ which means that } d(\gamma, I) < \varepsilon.\text{ Also}

\[
\gamma(\text{orbit of } \phi') = \lambda^{-1}h\lambda(\text{orbit of } \phi') \\
= \lambda^{-1}h(\text{orbit of } \psi') \subseteq \lambda^{-1}(\text{orbit of } \psi) \\
= (\text{orbit of } \phi).
\]

Lemma 2 implies that } \phi \text{ is topologically stable.  \[\]

**Corollary.** Any flow which is obtained from a topologically stable flow by a positive continuous change of velocity is topologically stable.

**Proof.** This follows directly from the fact that a velocity change satisfies Lemma 4. For more details see [11, Lemma 1.2]. \[\]

3. **Topological Stability and Suspension Flows**

**Lemma 5** (Cf. [18, Lemma 2.4]). If } (Y, \phi) \text{ is the suspension flow of a homeomorphism } T \text{ on } X \text{ under the constant map } 1 \text{ (i.e., } X_1 = Y \text{ and } \phi_1 = \phi), \text{ and if}

\[
d((x, s), (y, t)) < \varepsilon < \frac{1}{4}
\]

for any given } \varepsilon > 0, \text{ then}

\[
|s-t| < \varepsilon \quad \text{or} \quad |1+s-t| < \varepsilon \quad \text{or} \quad |1+t-s| < \varepsilon.
\]

Moreover,

(a) \quad t = 0 \Rightarrow \text{ either } |s| < \varepsilon \text{ or } |1+s| < \varepsilon,

(b) \quad t = \frac{1}{2} \Rightarrow |s - \frac{1}{2}| < \varepsilon.

**Theorem 2.** A homeomorphism } T \text{ on a compact metric space } X \text{ is topologically stable if its suspension flow } (Y, \phi) \text{ under a positive continuous map } f: X \to R \text{ is topologically stable.}

**Proof.** Since the stability property is preserved under conjugacy, one can assume that } (Y, \phi) \text{ is the suspension flow under the constant map } 1 \text{ (i.e., } X_1 = Y). \text{ The suspension flow has no fixed points and we have Lemma 3 with } T_0 \geq 1. \text{ Choose } \gamma \text{ as in Lemma 3 with respect to } T = \frac{1}{4}. \text{ Given } \varepsilon > 0, \text{ assume without loss of generality that } \varepsilon < \min\{\gamma, \frac{1}{4}\}. \text{ Take } 0 < \varepsilon' < \varepsilon/2 \text{ in which:}
(i) \( d(a, b) < \varepsilon' \) implies \( d(\pi a, \pi b) < \varepsilon \), where \( \pi: Y \to X \) is the projection (i.e., \( \pi(x, t) = x \) for every \( x \in X \) and \( 0 \leq t \leq 1 \)).

(ii) \( d(a, b) < \varepsilon' \) implies \( d(\phi, a, \phi, b) < \varepsilon \) for \( 0 \leq t < 1 \) and \( a, b \in Y \). Now choose \( 0 < \delta' < \varepsilon' \), let \( \delta' \) satisfy the definition of stability of \( \phi \) with respect to \( \varepsilon' \) and choose \( 0 < \delta < \delta' \) so that \( d(a, b) < \delta \) implies \( d(Ta, Tb) < \delta' \) for every \( a, b \in X \). Now let \( T': X \to X \) be another homeomorphism with \( d(T, T') < \delta \). Let \( \phi' \) be the suspension flow of \( T' \) under the constant map 1. Then for \( 0 \leq t \leq 1 \) and \( 0 \leq s < 1 \) we have

\[
d(\phi(t), (x, s), (x, s + t)) = d((x, s + t), (x, s + t)) = 0 \quad \text{if } 0 \leq s + t < 1,
\]

and

\[
d(\phi(t), (x, s), (x, s)) = d((Tx, s + t - 1), (T'x, s + t - 1))
\]

\[
= (2 - s - t) d(Tx, T'x) + (s + t - 1) d(T^2x, TT'x)
\]

\[
< (2 - s - t) \delta' + (s + t - 1) \delta < \delta' \quad \text{if } 1 \leq s + t < 2.
\]

Hence \( d(\phi(t), \phi(s)) < \varepsilon' \) for all \( 0 \leq t \leq 1 \). Using the stability property there exists a continuous mapping \( h: Y \to Y \) such that \( d(h, I) < \varepsilon' \) and \( h(\text{orbit of } \phi(t)) \subseteq \text{orbit of } \phi(t) \). So for every point \( (x, s) \in Y \) there is a continuous function \( \sigma_{(x,s)}: R \to R \) such that \( \sigma_{(x,s)}(0) = 0 \) and \( h\phi'(x, s) = \phi_{\sigma_{(x,s)}(t)}(x, s) \). For a more detailed description see [4]. Hence

\[
d(\phi_\sigma(x, s), (x, s)) = d((Tx, \sigma(x, s)), (Tx, \sigma(x, s)))
\]

\[
= d(h\phi'(x, s), (x, s)) + d(\phi'(x, s), (x, s)) < \varepsilon' + \delta' < \varepsilon
\]

for \( 0 \leq t \leq 1 \).

Since \( d(h, I) < \varepsilon' \), therefore \( d(\phi, h(x, s), (x, s)) < \varepsilon \) for \( 0 \leq t \leq 1 \). Using Lemma 5 and the continuity of \( \sigma_{(x,s)} \) we have \( |\sigma_{(x,s)}(t) - t| < \frac{1}{4} \) for all \( 0 \leq t \leq 1 \) and for every point \( (x, s) \in Y \). Now define \( \lambda: X \to X \) by \( \lambda x = \pi h(x, \frac{1}{2}) \). It is obvious that \( \lambda \) is continuous. From the way we choose \( \varepsilon' \) we have

\[
d(\lambda x, x) = d(\pi h(x, \frac{1}{2}), \pi(x, \frac{1}{2})) < \varepsilon.
\]

To show \( \lambda T' = T' \lambda \), assume \( h(x, \frac{1}{2}) = (x', w) \). Lemma 5 and the fact that \( d(h, I) < \varepsilon < \frac{1}{4} \) imply that \( |w - \frac{1}{2}| < \varepsilon < \frac{1}{4} \). Therefore

\[
1 + w + \sigma_{x,1/2}(1) < 2,
\]

and because \( 0 < w < 1 \), we have

\[
\pi \phi_{\sigma_{x,1/2}(1)}(x', w) = \pi \phi_1 (x', w).
\]
Hence
\[ \lambda T^*x = \pi h(Tx, \frac{1}{2}) = \pi h\phi_1(x, \frac{1}{2}) = \pi \phi_{\sigma(x, 1/2)}(h(x, \frac{1}{2})) = \pi \phi_{\sigma(x, \lambda)}(T(x, w)) = \pi(T', w) = \pi(T'x, w) = Tx = T\pi h(x, \frac{1}{2}) = T\lambda x. \]

This completes the proof. &

Let \( \phi \) be a \( C^1 \)-flow on a compact manifold \( M \) and let \( \dot{\phi} \) denote the vector field of \( \phi \) (i.e., \( \dot{\phi} \) is a flow generated by \( \phi \)).

**Theorem 3.** Let \((M_1, \phi)\) be the suspension flow of a topologically stable diffeomorphism \( f \) on a compact manifold \( M \) under the constant map 1. Then \( \forall \epsilon > 0, \exists \delta > 0 \) such that for any other \( C^1 \)-flow \( \psi \) on \( M_1 \), if \( d(\phi, \psi) < \delta \) then there is a continuous map \( h : M_1 \to M_1 \) such that \( h(\text{orbit of } \psi) \subseteq \text{orbit of } \phi \) and \( d(h, I) < \epsilon \).

**Proof.** Given \( \epsilon > 0 \), choose \( \gamma < \epsilon/2 \) with the property that \( d(x, y) < \gamma \) implies \( d(fx, fy) < \epsilon/2 \) for all \( x, y \in M \). Take \( \delta < \gamma \) satisfying the stability property of \( f \) with respect to \( \gamma \) and choose \( \delta' < \delta \) so that \( d(\phi_t, \psi_t) < \delta \) for \( t \in [0, 2] \) if \( ||\dot{\phi} - \dot{\psi}|| < \delta' \) for every \( C^1 \)-flow \( \psi \) on \( M \). Now assume such a flow \( \psi \) exists. From the way we define the metric on \( M_1 \) [18] it follows that for every \( x \in X \) if we take a point, say, \((x', s) = \psi_w(x, 0)\), then there is a point in the segment \((\phi, x)_0 < t < 2\) which is close to \((x', s)\) within \( \delta \)-distance. So this means that
\[ d[(x', s), (x, s)] < \delta \quad \text{for all } 0 \leq s \leq 2, \]
where \((x', s) = \psi_w(x, 0)\) for \( 0 \leq w \leq \frac{5}{3} \). Now define \( g : X \to X \) by
\[ g(x) = \{ \psi_{\tau}(x, 0), 0 \leq \tau \leq \frac{1}{3} \} \cap (X \times \{0\}). \]
Then it is obvious that \( d(f, g) < \delta \). Since \( f \) is topologically stable, there is a continuous \( h : X \to X \) with \( d(h, I) < \gamma \) and \( fh = hg \). Also define \( H : M_1 \to M_1 \) by
\[ H(z', w) = (hz, w) \quad \text{for every } (z', w) \in M_1, \]
where \( \psi_{\lambda}(z, 0) = (z', w) \) for some \( z \in M \) and for some \( \lambda \in [0, 1) \). To show \( H \) is well defined assume \((y, 1) = \psi_r(z, 0)\) for some \( r \in \left[\frac{1}{2}, \frac{3}{2}\right] \). The definition of \( g \) implies \( gz = fy \). Hence
\[ H\psi_{r+\epsilon}(z, 0) = H\psi_{\epsilon}(y, 1) = H(fy, \lambda), \]
where \( \psi_{\varepsilon}(fy, 0) = (fy, \lambda) \). Therefore

\[
H \psi_{r+\varepsilon}(z, 0) = (hfy, \lambda) = (hg z, \lambda) = (h z, \lambda) = \phi_{1+\lambda}(z, 0).
\]

This means that \( H \) is well defined and

\[
H(\text{orbit of } \psi) \subseteq (\text{orbit of } \phi).
\]

Since

\[
d[\tilde{H}(z', w), (z', w)] = d[(hz, w), (x', w)],
\]

where \((z', w) = \psi_{\lambda}(z, 0)\) for some \( \lambda \in [0, 1) \), therefore

\[
d[\tilde{H}(z', w), (z', w)] \leq d[(hz, w), (x, w)] + d[(z, w), (z', w)]
\]

\[
= (1 - w) d(hz, z) + w d(fh z, f z) + d[(z, w), (z', w)]
\]

\[
< (1 - w) \varepsilon / 2 + w \varepsilon / 2 + \varepsilon / 2 = \varepsilon.
\]

So \( \rho(H, I) < \varepsilon \) and this finishes the proof of the theorem.

**Proposition 3.** Let \( \phi \) be a \( C^1 \)-flow without fixed points on a compact manifold \( M \) which satisfies the conclusion of the above theorem. Then \( \phi \) has the POTP.

**Proof.** The same as the proof of Theorem 5 in [18].

In Theorem 5 in [18] we proved that the topological stability of \( C^1 \)-flows with no fixed points implies the POTP. For the case of a diffeomorphism in [19], Walters has proved that the topological stability of diffeomorphisms on compact manifolds of dimension \( \geq 2 \) also implies the POTP. Recently in [12], Morimoto generalizes this result for the case of dimension one. In this work using the above theorem and Proposition 1, one can unify the proof for this question.

**Corollary.** If \( f \) is a topologically stable diffeomorphism on a compact manifold \( M \), then \( f \) has the POTP.

**Proof.** Let \( \phi \) on \( M_1 \) be the suspension flow of \( f \) under the constant map \( 1 \). Then \( \phi \) satisfies the hypothesis of Theorem 3. Proposition 3 implies the POTP of \( \phi \). Therefore \( f \) has the POTP following directly from Theorem 2 in [18].

4. **Specification Property of Stable Diffeomorphisms**

In this section, we will study the specification property of topologically stable diffeomorphisms on a compact manifold \( M \).
Let $\phi$ be a $C^1$-flow on a compact manifold $M$ generated by a vector field $\dot{\phi}$.

The following two propositions are direct consequences of Proposition 8 in [15].

**Proposition 4.** Let $\phi$ be a topologically stable $C^1$-flow on a compact manifold $M$. Then $\forall \varepsilon > 0, \exists \delta > 0$ such that for every $r > 1$ and for every $x \in M$ if $d(\phi^r x, x) < \delta$, then there exists a periodic point $z \in M$ and a homeomorphism $\alpha : R \to R$ so that

$$d(\phi_{\alpha(t)} z, \phi_t x) < \varepsilon$$

for $0 \leq t \leq r,$

and

$$|\alpha(t) - t| < \varepsilon t$$

for $0 \leq t \leq r,$

$z$ is of period $\alpha(r)$.

Also in [13], Morimoto has proved that the periodic points of a topologically stable $C^1$-flow on a compact $C^\infty$ Riemannian manifold are dense in $\Omega(\phi)$, where $\Omega$ is the non-wandering set.

Using the above proposition it is obvious that the periodic points of a topologically stable $C^1$-flow on a compact manifold $M$ are dense in the chain recurrent set of $\phi$. Moreover, $CR(\phi) = \Omega(\phi) = \text{closure (Per } \phi\text{)}$.

**Proposition 5.** Let $\phi$ be a $C^1$-flow on a compact manifold $M$ satisfying the conclusion of Theorem 3. Then $\phi$ will satisfy the conclusion of Proposition 4.

If $X$ is a metric space, a homeomorphism $f : X \to X$ has the closed orbit property [3] if given $\varepsilon > 0$, there exist $\delta > 0$ and a positive integer $N = N_\varepsilon$ so that for every $x \in X$, and $n > N$, $d(f^n x, x) < \delta$ implies that there exists a periodic point $z$ in $X$ (of period $n$) such that $d(f^i z, f^i x) < \varepsilon$ for $0 \leq i \leq n$.

**Proposition 6.** Every topologically stable diffeomorphism $f$ on a compact manifold $M$ has the closed orbit property. Moreover, $\forall \varepsilon > 0, \exists \delta > 0$ such that for every $x \in M$ if $d(f^n x, x) < \delta$ for some integer $n \geq 1$, then there exists a periodic point $z \in M$ (or period $n$) which satisfies the property that $d(f^i z, f^i x) < \varepsilon$ for $0 \leq i \leq n$.

For the case of dimension of $M \geq 2$ one can prove this proposition using an idea similar to the proof of Theorem 11 in [9]. In [10], Hurley mentions this as well. For the general case I would like to sketch this idea:

Let $(M', \phi)$ be the suspension flow of $(M, f)$ under the constant map 1, then $\phi$ will satisfy the conclusion of Theorem 3. So $f$ has the closed orbit property, as follows directly from Proposition 5.
In [10], Hurley stated as Theorem A that for a topologically stable diffeomorphism $f$ on a compact manifold $M$:

1. The periodic points of $f$ are dense in the chain recurrent set of $f$.
2. $f$ has only a finite number of chain components (basic sets) and each of these is the closure of a single $f$-orbit.
3. If $f$ can be $C^0$-approximated by Morse–Smale diffeomorphism, the the chain recurrent set of $f$ is finite.
4. $f$ has a finite set of periodic points the union of whose stable sets is dense in $M$.
5. If $X$ is a chain component of $f$ and $k$ is the least period of any periodic point in $X$, then $X$ consists of no more than $k$ chain components of $f^k$, and $f^k$ is mixing on each.

A homeomorphism $f: X \to X$ is said to have the specification property if for any $\varepsilon > 0$, there is a positive integer $M = M_\varepsilon$ such that for any choice of points $x_0, x_1$ in $X$ and strings $A_0 = [a_0, b_0], A_1 = [a_1, b_1]$ with $a_1 - b_0 > M$, and any integer $p > b_1 - a_0 + M$, there exists a periodic point $z$ in $X$ with period $p$ such that

\[
d(f^iz, f^ix_0) < \varepsilon \quad \text{for } i \in A_0,
\]
\[
d(f^iz, f^ix_1) < \varepsilon \quad \text{for } i \in A_1.
\]

The specification property implies the mixing property and the density of periodic points. For more details see [17].

**Theorem 4.** If $X$ is a chain component of a topologically stable diffeomorphism $f$ on a compact manifold $M$ and $k$ is the least period of any periodic point in $X$, then $X$ consists of no more than $k$ chain components of $f^k$ and $f^k$ has the specification property on each.

**Proof.** Note that $f^k$ has the POTP on each of its chain components because $f$ does. Choose $\delta$ small enough that any $\delta$-chain for $f^k$ is $\varepsilon$-traced by an orbit, where $\varepsilon$ is less than half the distance between any two chain components of $f^k$. Fix a chain component $Y$ of $f^k$ in $X$. Since $f^k$ is mixing on $Y$ [10], there exists a positive integer $N = N_\delta$ such that for every two non-empty open subsets, $U, V$ of $Y$ each of diameter $\geqslant \delta$, $f^nU \cap V \neq \emptyset$ for all $n \geqslant N$. Let $x_0, x_1$ be any two points in $Y$ and $A_0 = [a_0, b_0], A_1 = [a_1, b_1]$ be any two strings with $a_1 - b_0 > N$. Then for any integer $p > b_1 - a_0 + N$, the sets

\[A = f^{a_1 - b_0} B_\delta(f^{kb_0}x_0) \cap B_\delta(f^{ka_1}x_1),\]
and

\[ B - f^{p-b_1+a_0} B_\delta(f^{kh_1} x_1) \cap B_\delta(f^{ka_0} x_0) \]

are non-empty, where \( B_\delta(x) \) is the \( \delta \)-ball of \( x \) in \( Y \). Take two points \( z_0 \) and \( z_1 \) in \( A \) and \( B \), respectively, and consider the following \( \delta \)-pseudo orbit:

\[ \{ ... f^{k(p-b_1+a_0-1)}(z_1), f^{ka_0}(x_0), f^{k(a_0+1)}(x_0), ... , f^{k(b_0-1)}(x_0), z_0, f(z_0), ... \]

\[ ... f^{k(a_1-b_0-1)}(z_0), f^{ka_1}(x_1), f^{k(a_1+1)}(x_1), ... , f^{k(b_1-1)}(x_1), z_1, f(z_1), ... \]

\[ ... , f^{k(p-b_1+a_0-1)}(z_1), f^{ka_0}(x_0), ... \} \]

Using the POTP plus the closed orbit property, there exists a periodic point \( z \) in \( Y \) of period \( p \) whose orbit traces this \( \delta \)-pseudo orbit.

**COROLLARY.** Suppose \( f \) is a topologically stable diffeomorphism on a compact manifold \( M \). Then \( f \) has the specification property if and only if \( f \) is mixing.

Using [10] and by an argument similar to the proof of the above theorem and [3, 6] one can show:

**COROLLARY.** If \( f \) is a topologically stable diffeomorphism on a compact manifold \( M \) and \( X \) is a connected chain component of \( f \), then

(a) the restriction \( f \) to \( X \) has the specification property,

(b) the topological entropy \( h(f) \) is strictly positive unless \( X \) is a single point,

(c) \( h(f) = \lim_{n \to \infty} \frac{1}{n} \log N_n(f) \),

where \( h(f) \) is the topological entropy of \( f \) on \( X \) and \( N_n(f) \) the cardinality of the set of fixed points of \( f^n \) in \( X \).

Also using the specification property one can study not only the entropy but also the invariant measures and the equidistribution of closed orbits. For the case of Axiom A diffeomorphisms, many results in this direction can be found in [3, 6, 17].

Theorem 4 is not true for the case of a topologically stable \( C^1 \)-flow on a compact manifold \( M \) because such a flow is not necessarily mixing on each of its chain components. An example of this is the suspension flow of any hyperbolic toral automorphism under any constant map. In other words the statements analogous to (1)–(4) of Theorem A in [10] are true except statement (5).
5. Topological Stability and \( \omega \)-Limit Sets

Franke and Selgrade [9] define a flow \( \phi \) on \( \mathcal{A} \) as an abstract \( \omega \)-limit set if there is a flow \( \psi \) on \( X \), a compact metric space and an \( x \in X \) so that \( \psi/\omega(x) \) is topologically conjugate to \( \phi \), where \( \omega(x) = \{ z \in X; \psi_{t_n} \to z \text{ some } t_n \to \infty \} \). They show that if \( \phi \) is a smooth flow on \( \mathcal{M} \) and \( \mathcal{A} \) is a hyperbolic closed invariant subset with local product structure and if \( \phi/\mathcal{A} \) is an abstract \( \omega \)-set then there is an \( x \in \mathcal{A} \) such that \( \omega(x) = \omega(z) = \mathcal{A} \). In this section we investigate related questions for the case when \( \phi/\mathcal{A} \) is not necessarily hyperbolic for a topologically stable flow \( \phi \) on \( \mathcal{M} \).

**Definition 5.** A flow \( \phi \) on \( X \) is an abstract \( \omega \)-limit set if there is a flow \( \psi \) on \( Y \) a compact metric space and \( y \in Y \) so that \( \psi/\omega(y) \) is topologically equivalent to \( \phi \).

The proof of Theorem 3.1 in [9] shows that this definition is equivalent to the one given in [9].

**Lemma 6.** For a flow \( \phi \) on \( X \) which is an abstract \( \omega \)-limit set, let \( \varepsilon > 0 \). Then there is an \((\varepsilon, 1)\)-pseudo orbit \( \langle x = \{ x_i \}_{-\infty}^{\infty}, \{ t_i \}_{-\infty}^{\infty} \rangle \) with

\[
d(\phi_{t_i}x_i, x_{i+1}) \to 0 \quad \text{as } |i| \to \infty,
\]

and

\[
\omega(x) = \omega(z) = x
\]

Here

\[
\omega(x) = \{ z \in X; x_{r_n} \to z \text{ some } r_n \to \infty \},
\]

\[
\omega(z) = \{ z \in X; x_{r_n} \to z \text{ some } r_n \to -\infty \}.
\]

**Proof:** This lemma is proved in [9] and the arguments follow the versions of Bowen’s Theorem 1 [1].

Once the connection between chain recurrence and abstract \( \omega \)-limit set is known one can show easily that the suspension flow \( (X_f, \phi_f) \) of a homeomorphism \( T \) on a compact metric space \( X \) under a positive continuous map \( f: X \to \mathbb{R} \) is an abstract \( \omega \)-limit set if and only if \( T: X \to X \) is an abstract \( \omega \)-limit set.

Finally, we will finish this section by proving the following theorem:

**Theorem 5.** Let \( \phi \) be a \( C^1 \)-flow on a compact manifold \( \mathcal{M} \) and assume \( \phi \) is topologically stable and expansive and assume \( \phi/\mathcal{X} \) is an abstract \( \omega \)-limit set for some compact \( \phi \)-invariant subset \( \mathcal{X} \) of \( \mathcal{M} \). Then \( \mathcal{X} = \omega(z) = \omega(z) \) for some \( z \in \mathcal{M} \).
The arguments are similar to Franke and Selgrade's Theorem 4.1 [9] plus the following lemmas:

**Lemma 7.** Let $\phi$ on $X$ be a continuous flow. Then $\forall \eta, T > 0, \exists \lambda > 0$ such that for every $x, y \in X$ if $d(x, y) < \lambda$, then $d(\phi_t x, \phi_t y) < \eta$ for all $t \in [-T, T]$.

**Proof.** Follows directly from the compactness of $X$. \ 

**Lemma 8.** (Cf. [18, Lemma 3.11]). Let \( \{x_j\}_{j=1}^{\infty} \) be a family of real valued continuous increasing functions on \( [-w_j, w_j] \) with $x_j(0) = 0$ and assume $w_j \to \infty$ as $j \to \infty$ and that $x_j(w_j) \leq b$ for all $j$, where $b$ is a constant real number. Then $\forall \lambda, \beta > 0, \exists m$ (positive integer) and there are real numbers $t_1, t_2 \in [0, w_j]$ with $t_1 < t_2$ such that $t_2 - t_1 = \beta$ and $|x_m(t_2) - x_m(t_1)| < \lambda$.

**Lemma 9.** Let $\phi$ be an expansive flow on $X$. Then $\forall \varepsilon > 0, \exists \delta > 0$ with the property that $\forall \varepsilon_0 > 0, \exists T > 0$ such that for every $x, y \in X$ and for every continuous and increasing real valued function $s$ on closed interval $[T, T]$ with $s(0) = 0$ if $d(\phi(sx), \phi(sy)) < \delta$ for all $s \in [-T, T]$, then $d(\phi_t x, \phi_t y) < \varepsilon_0$ for some $r \in [-\varepsilon, \varepsilon]$.

**Proof.** Given $\varepsilon > 0$, choose $\delta' < \varepsilon$ satisfying expansiveness. Take $\gamma$ so that $0 < \gamma < \min \{\delta'/2, T_0\}$ ($T_0$ is taken from Lemma 3) and so that $\gamma$ also has the property:

$$y = \phi_r x \text{ with } |r| < \gamma \implies d(\phi_r x, \phi_r y) < \delta'/2.$$  

Using Lemma 3, take $\lambda < \delta'/2$ with $d(x, y) < \lambda$ and $d(\phi_x y, \phi_y x) \geq \lambda$ for all $x, y \in X$. Now choose a positive $\delta < \min \{\lambda/8, \gamma\}$ with the property that for every $x \in X$, $d(\phi_t x, x) < \lambda/8$ for all $t \in [-\delta, \delta]$. Now assume the hypothesis of the Lemma is not true with respect to this $\delta$. Hence there exists $\varepsilon_0 > 0$ such that for every positive integer $n$ there are $x_n, y_n \in X$ and increasing functions $S_n: [-n, n] \to R$ with $S_n(0) = 0$ satisfying

$$d(\phi_{S_n(t)} x_n, \phi_{S_n(t)} y_n) < \delta \quad \text{for all } t \in [-n, n].$$  

But $d(\phi_r x_n, \phi_r y_n) \geq \varepsilon_0$ for all $r \in [-\varepsilon, \varepsilon]$. So we have sequences $\{x_n\}, \{y_n\}$ of points in $X$ and sequence $\{S_n\}$ of increasing continuous functions on $[-n, n]$. Using compactness of $X$, and without loss of generality, we can assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Hence one can find an integer $N > 0$ and an increasing sequence $\{w_n\}$ of positive real numbers such that $w_n \leq n$ for all $n > 0$ and $w_n \to \infty$ as $n \to \infty$. For $n \geq N$,

(i) $d(\phi_t y_n, \phi_t y) < \delta$ for all $-w_n \leq t \leq w_n$.  

By our assumption we have for $n \geq N$

$$(ii) \ d(\phi_n(t)x_n, \phi_n(t)y) < 2\delta \text{ for all } t \in [-w_n, w_n].$$

Since $x_n \to x$ one can also find an integer $N' > 0$ and an increasing sequence 
$\{w'_n\}$ with $w'_n \to \infty$ as $n \to \infty$ and $w'_n \leq w_n$ for all $n$ and

$$(iii) \ d(\phi_n(t)x_n, \phi_n(t)y) < \delta \text{ for all } t \in [-w'_n, w'_n] \text{ and for all } n \geq N'.$$

Therefore for every $n \geq M$

$$(iv) \ d(\phi_n(t)x_n, \phi_n(t)x) < \delta \text{ for all } t \in [-s_n^{-1}(w'_n), s_n^{-1}(w'_n)], \text{ where } M = \max \{N, N'\}.$$ 

Using this together with (ii) implies

$$d(\phi_n(t)x_n, \phi_n(t)y) \leq d(\phi_n(t)y, \phi_n(t)x_n) + d(\phi_n(t)x_n, \phi_n(t)y) < 3\delta < \frac{3\gamma}{8}$$

for all $t \in [-T_n, T_n]$,

where $T_n = \min \{s_n^{-1}(w'_n), w_n\}$. Now assume there exists a real number $a > 0$ such that $s_n^{-1}(w'_n) \leq a$ for all $n$. Therefore $w'_n \leq s_n(a) \to \infty$ as $n \to \infty$. Lemma 8 implies the existence of $t_1, t_2 \in [0, a]$ with $t_2 - t_1 = \gamma$ and $|s_n(t_2) - s_n(t_1)| < \delta$. Hence $d(\phi_{t_1}y, \phi_{t_2}y) > \gamma$ and

$$d(\phi_n(t_1)x_n, \phi_{s_n(t_1)}x) < \gamma/8 \text{ for all } n.$$ 

But

$$d(\phi_{t_1}y, \phi_{t_2}y) \leq d(\phi_{t_1}y, \phi_{s_n(t_1)}x) + d(\phi_{s_n(t_1)}x, \phi_{s_n(t_2)}x) + d(\phi_{s_n(t_2)}x, \phi_{t_2}y)$$

$$< 3\gamma/8 + \gamma/8 + 3\gamma/8 = 7\gamma/8.$$ 

This is a contradiction, and hence $\{T_n\}$ can be taken to be a strictly increasing sequence with $T_n \to \infty$ as $n \to \infty$. Hence from all this and without loss of generality one can assume the existence of a positive integer $M$ with the property that each $s_m$ with $m \geq M$ has domain $[-T_m, T_m]$ with $T_m \to \infty$ as $m \to \infty$ and satisfies:

(a) $|s_{m+1}(t) - s_m(t)| < \gamma$ for all $t \in [-T_m, T_m]$.

(b) $d(\phi_{s_m(t)}x_n, \phi_t y) < 3\gamma/8$ for all $t \in [-T_m, T_m]$.

(c) The range of $s_m$ is contained in the interior of the range of $s_{m+1}$ for all $m \geq M$.

Now define a function $s: R \to R$ by $s = s_M$ on $[-T_M, T_M]$. As we know $|s_{M+1}(T_M) - s_M(T_M)| < \gamma$, so there is a continuous function $s$ on
such that $s(TM) = sM(TM)$ and $s(TM+1) = sM+1(TM+1)$ and $|s(t) - sM+L(t)| < \gamma$ for all $t \in [TM, TM+1]$. Also we have $|sM+1(-TM) - sM(-TM)| < \gamma$. There is also a continuous function (call it $s$ as well) on $[-TM+1, -TM]$ such that $s(-TM) = sM(-TM)$ and $s(-TM+1) = sM+1(-TM+1)$ and $|s(t) - sM+1(t)| < \gamma$ for all $t \in [-TM+1, -TM]$. If we carry on in the same manner, then we have such a homeomorphism $s: \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$. Now pick $t \in \mathbb{R}$, say, first $t > 0$ and $t \in [TM, TM+1]$ for $m \geq M$. Since $|s(t) - sM+1(t)| < \gamma$, therefore

$$d(\phi_{sM+1(t)}x, \phi_{s(t)}x) < \delta/2$$

and

$$d(\phi_{s(t)}x, \phi_{t+y}) \leq d(\phi_{s(t)}x, \phi_{sM+1(t)}x) + d(\phi_{sM+1(t)}x, \phi_{t+y}) < \delta'/2 + 3\delta/8 < \delta'.$$

Expansiveness implies that $y = \phi_x x$ for some $r \in [-e, e]$ and this means that $x, y$ are in the same orbit. But if $d(\phi_x x, y_i) > \varepsilon_0$ for all $i \geq 1$ and for all $r \in [-e, e]$, then $d(\phi_x x, y) > 0$ for all $r \in [-e, e]$. This is a contradiction and finishes the proof.

**Proof of the Theorem.** Using Lemma 1 and Proposition 2, $\phi$ has the POTP. Given $\varepsilon > 0$, choose $\delta > 0$ satisfying the hypothesis of Lemma 9. Let $(x = \{x_t\}_{-\infty}^{\infty}, \{t_i\}_{-\infty}^{\infty})$ be a $(\delta', 1)$-pseudo orbit for $\phi/X$ as constructed in the proof of Lemma 5 with $\delta' > 0$ so small that every $(\delta', 1)$-pseudo orbit is $\delta$-traced by some orbit in $M$. In other words there exists $z \in M$ and an increasing homeomorphism $\alpha: \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha(t)}z, \phi_{t-s_{n+1}}x_n) < \delta$$

for $s_n \leq t \leq s_{n+1}$, $n = 0, 1, 2, \ldots$

and

$$d(\phi_{\alpha(t)}z, \phi_{t+s_{n-1}}x_{n-1}) < \delta$$

for $-s_{n-1} \leq t \leq -s_{n-1}$, $n = 1, 2, 3, \ldots$

We will now show $\omega(z) = X$. Given $x \in X$ and $\lambda$ any small positive real number, choose $T > 0$ satisfying Lemma 8 with respect to $\lambda/2$. Since $d(\phi_{t_i}x_i, x_{i+1}) \to 0$ as $|i| \to \infty$, therefore there exists a positive integer $M$ such that for all $n \geq M$,

$$d(\phi_{\alpha(s_n + t)}z, \phi_{t}x_n) < \delta$$

for all $t \in [-T, T],$

and

$$d(\phi_{\alpha(t-s_n)}z, \phi_{t}x_n) < \delta$$

for $t \in [-T, T]$. Hence for all $n \geq M$ and for every $t \in [-T, T]$ we have

$$d(\phi_{\alpha(t)} - \alpha(s_n)\phi_{\alpha(s_n)}z, \phi_{t}x_n) < \delta$$
and
\[ d(\phi_{a(t-s-a)} + a(-s-a)\phi_{a(-s-a)}z, \phi_{-n}x_{-n}) < \delta. \]

Lemma 9 implies that there exists \( r \in [-\varepsilon, \varepsilon] \) such that for all \( n \geq M \),
\[ d(\phi_{a(\varepsilon_1) + r}z, x_n) < \lambda/2, \]
and
\[ d(\phi_{a(-s-a) + r}z, x_{-n}) < \lambda/2. \]

Since \( \omega(x) = X \), therefore \( d(x_m, x) < \lambda/2 \) and \( d(x_{-m}, x) < \lambda/2 \) for some \( m \geq M \). This means that there are real numbers \( t > 0 \) and \( t' < 0 \) with \( d(\phi_t z, x) < \lambda \) and \( d(\phi_{t'} z, x) < \lambda \). Hence \( X \subseteq \omega(z) \) and \( X \subseteq \alpha(z) \). Now in order to show \( \omega(z) \subseteq X \), assume \( \phi_{t_k} z \to y \) for some \( t_k \to \infty \) and let \( \lambda > 0 \), and choose \( T > 0 \) satisfying Lemma 9 with respect to \( \lambda/2 \). Since the \( (\delta', 1) \)-pseudo orbit \( (x) = (x_i)_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty} \) is \( \delta \)-traced by an orbit \( (\phi_t z)_{t \in R} \), therefore there exist a positive integer \( n_k \) and a real number \( r_k \) such that \( d(\phi_{n_k} z, \phi_{r_k} x_{n_k}) < \delta \). Using the fact that \( d(\phi_{t_k} x_i, x_{i+1}) \to 0 \) as \( |i| \to \infty \), one can find a positive integer \( M \) such that for all \( k \geq M \), \( d(\phi_{r_k} z, y) < \lambda/2 \), and
\[ d(\phi_{r_k} z, \phi_{\gamma(t)} \phi_{r_k} x_{n_k}) < \delta \quad \text{for all} \quad t \in [-T, T] \]
and for some continuous and increasing real valued function \( \gamma \) on \( [-T, T] \). Lemma 9 implies that there exists \( e_k \in [-\varepsilon, \varepsilon] \) such that \( d(\phi_{e_k} z, \phi_{e_k} x_{n_k}) < \lambda/2 \) for all \( k \geq M \). This means that \( d(y, \phi_{e_k} x_{n_k}) < \lambda \). Hence \( \omega(z) \subseteq X \). Similarly one can show that \( \alpha(z) \subseteq X \). This finishes the proof of the theorem that \( \omega(z) = \alpha(z) = X \). 

Using a proof similar to the above one can show:

**Corollary.** Let \( \phi \) be an expansive flow which has the POTP on a compact metric space \( Y \) and assume \( X \subseteq Y \) a compact \( \phi \)-invariant set which is an abstract \( \omega \)-limit set. Then \( X = \omega(z) = \alpha(z) \) for some \( z \in Y \).

Let \( \phi \) be a smooth flow on \( M \) and let \( A \) be a hyperbolic closed invariant subset. It is obvious that Theorem 4.1 in [9] follows directly from this corollary. In other words, if \( \phi \) on \( M \) satisfies Smale's Axiom A [2] and \( \phi/X \) is an abstract \( \omega \)-limit set for some compact \( \phi \)-invariant subset \( X \) of \( \Omega \) (the non-wandering set), then \( X = \omega(z) = \alpha(z) \) for some \( z \in \Omega \).
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