A generalization of the bounded real lemma

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Abstract

Standard bounded real problems are generalized as the quadratic comparison of two rational matrices. The new results are formulated using dissipation inequalities and linear equalities. Several new results concerning inequalities on the complex right half-plane.

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1. Introduction

In this paper we deal with special generalizations of a famous theorem of control theory which is now popularly known as the bounded real lemma. It is a significant result in its own right, but appears also as an important preparation in several results on the $H^\infty$ theory.

In recent years, $H^\infty$ control problems have become the subjects of intensive research, mainly by electrical engineers. The first version of the bounded real lemma presents simple conditions under which a transfer function is contractive on the imaginary axis. Using it, it was possible to determine the $H^\infty$ norm of a transfer function, and the lemma became a significant element of proofs of hundreds of papers (and some books). In most of these papers it is shown that the existence of feedback controllers (that results in a closed loop transfer matrix having the $H^\infty$ norm less than...
a given upper bound), is equivalent to the existence of solutions of certain algebraic Riccati equations (or inequalities). See, e.g., [2,7].

Here we present generalizations of the original bounded real lemma. We treat the question in a more general context, as a comparison of two transfer functions. The main results are formulated using the dissipation inequalities instead of Riccati equations. Difficult questions on the complex right half-plane are examined touching the topics of Douglas theorem and Nevanlinna–Pick interpolation. The paper is written as self-contained as possible and hence contains a certain amount of tutorial material.

To make the paper easier to read, the main proofs are segregated into Sections 4 and 5 and may be skipped without the loss of continuity.

2. Preliminaries and notations

2.1. On the original lemma

First, let us present the simplest form of the bounded real lemma to explain the main points of its generalization.

For this define the $H^\infty$ norm, i.e., the $H^\infty$ norm of the stable transfer matrix $G$ as

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{|G(j\omega)|\},$$

where ‘j’ denotes the imaginary unit.

Now, we have

**Lemma 1.** Let $A$ be stable, complex matrix and consider the transfer matrix

$$G(s) := C(sI - A)^{-1}B.$$  \hspace{1cm} (1)

Then

$$\|G\|_\infty < 1$$  \hspace{1cm} (2)

holds if and only if the Hamiltonian matrix

$$\begin{bmatrix} A & BB^* \\ -C^*C & -A^* \end{bmatrix}$$

has no eigenvalues in the imaginary axis.

Note that the inequality (2) is equivalent to

$$G^*(j\omega)G(j\omega) < I \quad \omega \in \mathbb{R}.$$

An idea of a generalization of the above lemma is the substitution of the identity matrix $I$ on the right-hand side for an other transfer matrix. Actually we shall compare two transfer matrices on the imaginary axis and in the complex right half-plane.
2.2. Notations

Throughout the paper, we use the Doyle convention, i.e., a transfer matrix in terms of state-space data is denoted by
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} := C(sI - A)^{-1} B + D.
\] (3)

\(X^*\) stands for the complex conjugate transpose of a matrix \(X\), \(X^\dagger\) denotes the Moore–Penrose pseudoinverse of \(X\).

0 and \(I\) means the zero and the identity matrix of fitting sizes. If the size of an identity matrix is not clear from the context then it will be indicated by index, e.g., \(I_3\) denotes the identity matrix of the size 3 \(\times\) 3.

Furthermore, introduce the notations \(C_-\), \(C_+\) and \(C_0\), for the left half-plane, the right half-plane and the imaginary axis, respectively, more exactly:
\[
\begin{align*}
C_- & := \{ s \in \mathbb{C} | \text{Re } s < 0 \}, \\
C_+ & := \{ s \in \mathbb{C} | \text{Re } s > 0 \}, \\
C_0 & := \{ s \in \mathbb{C} | \text{Re } s = 0 \}.
\end{align*}
\]

\(\nu(A)\) := the number of eigenvalues of the matrix \(A \in C_-\),
\(\pi(A)\) := the number of eigenvalues of the matrix \(A \in C_+\),
\(\delta(A)\) := the number of eigenvalues of the matrix \(A\) on \(C_0\).

3. Main results

3.1. Inequalities

In this paper we will be concerned with transfer functions of the form
\[
\begin{align*}
T_1(s) &= C_1(sI - A_1)^{-1} B_1 + D_1, \\
T_2(s) &= C_2(sI - A_2)^{-1} B_2 + D_2.
\end{align*}
\] (4)

We shall assume throughout this paper that the above realizations are minimal and that the number of their columns coincide.

Let the matrices \(A_1, B_1, C_1, D_1\); and \(A_2, B_2, C_2, D_2\) in realizations (4) be of the sizes \(n_1 \times n_1, n_1 \times m, p_1 \times n_1, p_1 \times m\), and \(n_2 \times n_2, n_2 \times m, p_2 \times n_2, p_2 \times m\), respectively, and introduce the following block-matrices
\[
\begin{align*}
A &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \\
C &= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \\
B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
D &= \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}
\end{align*}
\] (5)

of sizes \(n \times n, p \times n, n \times m, p \times m\), where \(n = n_1 + n_2, p = p_1 + p_2, n = n_1 + n_2\).
Further, introduce the matrices
\[ J = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = C(sI - A)^{-1}B + D, \]
\[ R = D^*JD = D_2^*D_2 - D_1^*D_1, \]
and assuming \( R \) is invertible, define the Hamiltonian matrix
\[ H = \begin{bmatrix} A - BR^{-1}D^*JC & BR^{-1}B^* \\ C^*JC - C^*JDR^{-1}D^*JC & -A^* + C^*JDR^{-1}B^* \end{bmatrix}. \]

Lemma 2. Assume that \( A_1 \) and \( A_2 \) have no eigenvalues on the imaginary axis (on \( C_0 \)), and that \( R \) is positive definite. Then the following two statements are equivalent:

(i) \( T_2^*(s)T_2(s) > T_1^*(s)T_1(s) \) for all \( s = j\omega, \omega \in \mathbb{R} \).

(ii) \( H \) has no eigenvalues on the imaginary axis.

Furthermore, if we assume as well that \( A_1 \) and \( A_2 \) have no common eigenvalues then the following statement is also equivalent with (i) and (ii):

(iii) There is a hermitian solution \( P \) to the algebraic Riccati equation
\[ \begin{align*}
    P(A - BR^{-1}D^*JC) + (A - BR^{-1}D^*JC)^*P \\
    -C^*JC + C^*JDR^{-1}D^*JC + PB - C^*JD & = 0,
\end{align*} \]
such that \( A_c = A + BR^{-1}(B^*P - D^*JC) \) is stable.

The above lemma remains valid if the triples \( A_1, B_1, C_1 \) or \( A_2, B_2, C_2 \) are absent, or in other words, if \( n_1 = 0 \) or \( n_2 = 0 \). Namely, in the case of \( n_2 = 0 \) the lemma gives the necessary and sufficient conditions of \( I > T_1^*(s)T_1(s) \), which corresponds to Lemma 1.

We do not assume hereafter that \( R > 0 \).

Definition 1. Introduce for square matrices \( P \) the blockmatrix
\[ K(P) := \begin{bmatrix} -P - A^*P + C^*JC & PB - C^*JD \\ B^*P - D^*JC & D^*JD \end{bmatrix}. \]

Theorem 1. Suppose that \( A_1 \) and \( A_2 \) have no common eigenvalues and have no eigenvalues on the imaginary axis. Then the following two statements are equivalent:

(i) \( T_2^*(s)T_2(s) > T_1^*(s)T_1(s) \) for all \( s = j\omega, \omega \in \mathbb{R} \).

(ii) There exists a hermitian matrix \( P \) such that \( K(P) > 0 \).

Remark. Note that the condition \( K(P) > 0 \) of (ii) in Theorem 1 includes in a natural way that \( R > 0 \).

Now, a question: is it necessary to make the assumption that \( A_1 \) and \( A_2 \) may have no common eigenvalues? As the following example shows this assumption is of primary importance in either Theorem 1 or in Theorem 2 below.
Counterexample 1. Let $T_1(s) = T_2(s) = 1/s + 1$, i.e., $C_1 = C_2 = B_1 = B_2 = 1,$ $A_1 = A_2 = -1,$ $D_1 = D_2 = 0.$

Clearly, statement (i) is valid for all $s \in \mathbb{C}$, but it can be seen easily that it is impossible to find a hermitian solution to the dissipation inequality in (ii).

Theorem 2. Consider the matrices defined in (5) and (6), assume that $A_1$ and $A_2$ have no common eigenvalues. Then the following statements are equivalent

(i) There exists a hermitian $P \geq 0$ solution to the dissipation inequality $K(P) \geq 0.$

(ii) There exists a rational transfer matrix $Q$ such that

$T_1(s) = Q(s)T_2(s)$ $\forall s \in \mathbb{C}$, \hspace{1cm} (8)

and

$\|Q(s)\| \leq 1$ $\forall s \in \mathbb{C}.$ \hspace{1cm} (9)

Furthermore, if the above relations hold, moreover $R > 0$ and $P > 0,$ then a transfer function $Q$ corresponding to (8) and (9) can be given as

$Q(s) := C_Q(sI - A_Q)^{-1}B_Q + D_Q.$

where

\[
\begin{bmatrix}
    D_Q \\
    C_Q \\
    B_Q \\
    A_Q
\end{bmatrix} :=
\begin{bmatrix}
    X \\
    [C_1 & -XC_2].
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -C_1^*X \\
    C_2^*
\end{bmatrix} + (B - P^{-1}C^*JD)R^{-1}(D_2^* - D_1^*X),
\]

and $X$ is any contractive constant matrix satisfying $XD_2 = D_1.$

Remarks.

1. According to the Douglas theorem there is always a contractive constant matrix $X$ with $XD_2 = D_1.$ Let us note, that if $R \geq 0$ then $X$ may be equal to $D_1D_2^*$.

2. The following assertion is an easy consequence of Theorem 2. If $P$ is any analytic function with the properties

$X(s)D_2 = D_1$ and $\|X(s)\| \geq 1$ $\forall s \in \mathbb{C}.$

then matrices (10) still give an analytic function with the properties (8) and (9).

If $m \leq p_2$ and there exists a point $\hat{s} \in \mathbb{C}_+$ such that rank $T_2(\hat{s}) = m$ then, clearly, rank $T_2(s) = m$ holds in almost all $s \in \mathbb{C}_+.$ Thus the following consequence of Theorem 2 can be stated. (The formulation is analogous to that of Theorem 1.)

Corollary 1. Suppose that $A_1$ and $A_2$ have no common eigenvalues and have no eigenvalues on the imaginary axis.

Assume that $m \leq p_2$ and rank $T_2(s) = m$ holds in almost all $s \in \mathbb{C}_+.$ Then the following two statements are equivalent:
(i) \( T_2^*(s)T_2(s) \geq T_1^*(s)T_1(s) \) for all \( s \in \mathbb{C}_+ \).

(ii) There exists a hermitian matrix \( P \succeq 0 \) such that \( K(P) \succeq 0 \).

As the following example shows, in general, it is a much harder problem to find a necessary and sufficient condition of the statement (i) in Corollary 1 without making any assumptions on \( T_2 \).

We shall see that the statement \( T_2^*(s)T_2(s) \geq T_1^*(s)T_1(s) \) for all \( s \in \mathbb{C}_+ \) is in general not equivalent to the existence of a non-negative hermitian solution \( P \succeq 0 \) of \( K(P) \succeq 0 \), as one expects.

**Counterexample 2.** Let

\[
T_1 := \sqrt{\frac{2}{s + 2}} \quad \text{and} \quad T_2 := \begin{bmatrix} \frac{s + 1}{s + 1} & \frac{s - 1}{s + 1} \\ \frac{s - 1}{s + 1} & \frac{s + 1}{s + 1} \end{bmatrix}.
\]

We can find the realization

\[
A_1 = -2, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_1 = 1, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = -4\sqrt{2}, \quad C_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \sqrt{2}, \quad D_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

But the corresponding dissipation inequality has the unique solution

\[
P = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix},
\]

which is obviously indefinite. Nevertheless the inequality \( T_2^*T_2 \geq T_1^*T_1 \) holds on \( \mathbb{C}_+ \).

**Theorem 3.** Assume that there exists a hermitian solution \( P \succeq 0 \) of the inequality \( K(P) \succeq 0 \). Then for any complex vectors \( v_1, v_2, \ldots, v_N \) and for any complex numbers \( s_1, s_2, \ldots, s_N \) of \( \mathbb{C}_+ \), where \( N \) is also optional, the following inequality holds

\[
\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{1}{s_k + s_l} v_k^*(T_2^*(s_k)T_2(s_l) - T_1^*(s_k)T_1(s_l))v_l \succeq 0. \tag{11}
\]

It is well-known that among analytic matrices \( Q \) the converse of Theorem 3 is also true. This is can be proved easily, e.g., using Nevanlinna–Pick interpolation with an everywhere in \( \mathbb{C}_+ \) dense sequence of points \( s_1, s_2, s_3, \ldots \). We suspect that the rational version of the converse of Theorem 3 is also true.

**Conjecture.** Assertion (11) is equivalent with (i) and (ii) in Theorem 2.
The proof seems to be more involved and it is beyond the scope of the present paper.

3.2. Equations

**Theorem 4.** Suppose that $A_1$ and $A_2$ have no common eigenvalues.

(a) The equality $T_2^*T_2 = T_1^*T_1$ is valid on the imaginary axis if and only if there exists a hermitian solution $P$ to the equation $K(P) = 0$. In this case the equation has a unique solution.

(b) If we make the additional assumption that $D_1$ and $D_2$ are invertible then $T_2^*(s)T_2(s) = T_1^*(s)T_1(s)$ for all $s \in \mathbb{C}_+$ is equivalent with $T_1(s) = D_1$, $T_2(s) = D_2$, and $D_2^*D_2 = D_1^*D_1$.

**Remarks.**

1. Obviously there is a close connection between the first part of Theorem 4 and the spectral factorization of real rational spectral densities. But (b) shows that the condition $T_j^*(s)T_j(s)$ for all $s \in \mathbb{C}_+$ is too restrictive. Note that $T_j^*(s)T_j(s)$ is not a rational function of $s$ with the exception of the case when $T_j$ is constant.

2. Counterexample 2 gives an illustrative example also to the first part of Theorem 4.

3. The case $T_2 = I$ of Theorem 4 gives the characterization of all-pass transfer matrices.

4. Preliminaries from the theory of matrix equations and inequalities

4.1. The Riccati operator

Let $A$, $Q$, $S$ be complex $n \times n$ matrices with $Q$ and $S$ hermitian. Define the $2n \times 2n$ Hamiltonian matrix

$$H = \begin{bmatrix} A & -S \\ -Q & -A^* \end{bmatrix}$$

(12)

(where in $H$ the two “$-$” signs of $Q$ and $S$ are for technical reasons). First we shall introduce the notion of Riccati operators as in [2, 7].

Assume $H$ has no eigenvalues on the imaginary axis. Then, as it is well known, it must have $n$ in $\mathbb{C}_+$ and $n$ in $\mathbb{C}_-$. First we define the domain of the function $Ric$ and afterwards the function itself.

The domain of $Ric$, denoted $\text{dom}(Ric)$, consists of Hamiltonian matrices $H$ with two properties, namely,
the matrix $H$ has no eigenvalues on the imaginary axis, and
there exists an $n \times n$ matrix $P$ such that the (graph) subspace $\text{Im} \left[ \begin{bmatrix} I \\ P \end{bmatrix} \right]$ is identical with the invariant subspace corresponding to eigenvalues of $H$ in $\mathbb{C}^-$. Then $P$ is uniquely determined by $H$, i.e., $H \rightarrow P$ is a function, which will be denoted $\text{Ric}$; thus, $P = \text{Ric}(H)$.

**Lemma 3** (Francis, 1987 [4]). Suppose $H$ has no imaginary eigenvalues, $S$ is either positive semidefinite or negative semidefinite, and $(A, S)$ is stabilizable. Then $H \in \text{dom}(\text{Ric})$.

**Lemma 4.** Suppose $H \in \text{dom}(\text{Ric})$ and $P = \text{Ric}(H)$. Then
(i) $P = P^*$ ($P$ is hermitian)
(ii) $P$ satisfies the algebraic Riccati equation
$$A^*P + PA + Q - PSP = 0.$$ (i4)
(iii) $A - SP$ is stable.

### 4.2. Existence and comparison theorems

Let $A, B, L, Q,$ and $R$ be complex matrices of sizes $n \times n, n \times m, n \times m, n \times n,$ and $m \times m$, respectively, $R$ be positive definite, and $Q$ be hermitian. Consider the algebraic Riccati inequality
$$A^*P + PA + Q - (PB + L)R^{-1}(B^*P + L^*) \geq 0,$$ (13)
and the algebraic Riccati equation
$$A^*P + PA + Q - (PB + L)R^{-1}(B^*P + L^*) = 0.$$ (14)

Naturally, an $n \times n$ matrix $P$ is called the solution of (13) (or (14)) if the left-hand side of (13) (or (14)) is a positive semidefinite (or zero) matrix. An $n \times n$ matrix $P_{\text{max}}$ is the maximal solution of (13) (or (14)), if $P_{\text{max}}$ is a solution of it, and $P_{\text{max}} \geq P$ holds for every hermitian solution $P$ of (13) (or (14)). Similarly can be defined the minimal solution.

**Lemma 5** ([7, 8]). Assume that $(A, B)$ is stabilizable, and that there exists a hermitian solution of the inequality (13). Then there exists a hermitian solution $P_{\text{max}}$ of (14) such that $P_{\text{max}} \geq P$ for every hermitian solution $P$ of (13). Moreover, all eigenvalues of $A - BR^{-1}(L^* + B^*P_{\text{max}})$ are in the closed left half-plane ($\mathbb{C}^- = \mathbb{C}_{\text{LHP}} = \mathbb{C}_{\text{LHP}} \cup \mathbb{C}_0$).

**Remark.** In the previous lemma $P_{\text{max}}$ is the maximal hermitian solution both of (13) and (14).
Define the dissipation inequality
\[
\begin{bmatrix}
P A + A^* P + Q & P B + L \\
B^* P + L^* & R
\end{bmatrix} \succeq 0.
\] (15)

It is well known and easy to see that the inequalities (13) and (15) are equivalent for positive definite \( R \). However there is an important difference between them, namely, (15) makes sense also when \( R \) is semidefinite.

Lemma 6. Assume that \((A, B)\) is controllable, \( R \) is positive semidefinite, furthermore for the given matrices \( Q_1, L_1 \) and \( R_1 > 0 \)
\[
\begin{bmatrix}
Q_1 & L_1 \\
L_1^* & R_1
\end{bmatrix} \succeq 0.
\]
holds. For \( P \in \mathbb{C}^{n \times n} \) and \( \epsilon \in \mathbb{R} \) define
\[
K_\epsilon(P) = \begin{bmatrix}
P A + A^* P + Q + \epsilon Q_1 & P B + L + L_1 \epsilon \\
B^* P + L^* + L_1^* \epsilon & R + R_1 \epsilon
\end{bmatrix}.
\] (16)

Then the following statements are equivalent:

(i) There exists a hermitian solution \( P \) to the inequality \( K_0(P) \succeq 0 \).

(ii) For any \( \epsilon > 0 \) there exists a hermitian \( P \) such that \( K_\epsilon(P) \succeq 0 \).

(iii) For any \( \epsilon \geq 0 \) there exists a maximal \( P_{\max}(\epsilon) \) and a minimal \( P_{\min}(\epsilon) \) solution of \( K_\epsilon(P) \succeq 0 \).

If (i)–(iii) holds then for any \( \epsilon_2 > \epsilon_1 \geq 0 \)

(a) \( P_{\max}(\epsilon_2) \geq P_{\max}(\epsilon_1) \)

(b) \( P_{\min}(\epsilon_2) \leq P_{\min}(\epsilon_1) \) and

(c) \( P_{\max}(0) = \lim_{\epsilon \to +0} P_{\max}(\epsilon) \),

(d) \( P_{\min}(0) = \lim_{\epsilon \to +0} P_{\min}(\epsilon) \).

Proof. The implications (iii)⇒(ii) and (iii)⇒(i) are obvious.

Proof of (i)⇒(ii). Suppose that (i) is valid with the hermitian matrix \( P \), and consider
\[
K_\epsilon(P) = K_0(P) + \epsilon \begin{bmatrix}
Q_1 & L_1 \\
L_1^* & R_1
\end{bmatrix} \succeq 0.
\]

Thus we conclude that the same \( P \) satisfies (ii) for any \( \epsilon > 0 \).

Proof of (ii)⇒(iii) supposing \( \epsilon > 0 \). Consider the equivalence
\[
K_\epsilon(P) \succeq 0 \iff PA + A^* P + Q + \epsilon Q_1 - (PB + L + \epsilon L_1)(R + \epsilon R_1)^{-1}(B^* P + L^* + \epsilon L_1^*) \succeq 0.
\]

By Lemma 5, the latter inequality has a maximal solution, which is also the maximal solution of our dissipation inequality \( K_\epsilon(P) \succeq 0 \).

The existence of the minimal solution can be proved by replacing the matrices \( A, B, P \) by \(-A, -B, Z = -P\), and by following the above argument. Hereafter we deal only with (a) and (c), the proof of (b) and (d) is analogous.
Proof of (ii) ⇒ (a) with $\epsilon_1 > 0$.

$K_{\epsilon_2}(P_{\max}(\epsilon_1)) = K_{\epsilon_1}(P_{\max}(\epsilon_1)) + (\epsilon_2 - \epsilon_1) \begin{bmatrix} \frac{Q_1}{L_1} & L_1 \\ \frac{R_1}{L_1} & -R_1 \end{bmatrix} \succeq 0$, such that $P_{\max}(\epsilon_1)$ is a solution of $K_{\epsilon_2}(P) \succeq 0$. Clearly, the maximal solution is no less than it: $P_{\max}(\epsilon_2) \succeq P_{\max}(\epsilon_1)$.

Proof of the implication \{ (a) with $\epsilon_1 > 0$

(iii) with $\epsilon > 0$ \} ⇒ (c) with $\epsilon = 0$

For any $\epsilon \in (0, 1)$, $P_{\max}(\epsilon) \succeq P_{\min}(\epsilon) \succeq P_{\min}(1)$. Thus $P_{\max}(\epsilon)$ is bounded from below by $P_{\min}(1)$ and decreasing for decreasing $\epsilon \in (0, 1)$. So we conclude that the limit

$$P_1 := \lim_{\epsilon \to 0} P_{\max}(\epsilon)$$

exists. We will prove that $P_1$ is the maximal solution of $K_0(P) \succeq 0$, i.e. $P_1 = P_{\max}(0)$.

Clearly, by the definition of $P_1$ and $K_\epsilon(P)$, we have

$$\lim_{\epsilon \to 0} K_\epsilon(P_{\max}(\epsilon)) = K_0(P_1) \succeq 0.$$

So, the maximum property of $P_1$ remains to be shown. Let $P$ be any solution of $K_0(P) \succeq 0$.

By the proof of the implication (i) ⇒ (ii) we have $K_\epsilon(P) \succeq 0$, for $\epsilon > 0$, and therefore $P \succeq P_{\max}(\epsilon)$. Since

$$\lim_{\epsilon \to +0} (P_{\max}(\epsilon) - P) = P_1 - P \succeq 0.$$

Thus we find that $P_1 \succeq P$ holds for any solution $P$ and hence $P_1 = P_{\max}(0)$. □

Remarks.

1. The technical assumption $R_1 > 0$ in the previous Lemma may be as well omitted.
2. In most of the applications we shall put $Q_1 = 0$, $L_1 = 0$ and $R_1 = I$.

5. Proof of the main results

5.1. Proof of Lemma 2

We shall first prove the equivalence of (i) and (ii). We have

$$G(s) := \begin{bmatrix} A & C^*JC & 0 \\ C^*JC & -A^* & B \\ D^*JC & -B^* & C^*JD \end{bmatrix}$$

$$= (B^*(-sI - A^*)^{-1}C^* + D^*)J(C(sI - A)^{-1}B + D)$$

$$= T_2^*(-\bar{s})T_2(s) - T_1^*(-\bar{s})T_1(s) = T^*(-\bar{s})JT(s).$$
Thus at a point \( s = j\omega (\omega \in \mathbb{R}) \) of the imaginary axis
\[
G(j\omega) = T_2^*(j\omega)T_2(j\omega) - T_1^*(j\omega)T_1(j\omega) = T^*(j\omega)JT(j\omega).
\]
Note that \( G \) is hermitian on \( \mathbb{C}_0 \). Furthermore
\[
G(s)^{-1} = \begin{bmatrix}
A - BR^{-1}D^* JC & BR^{-1}B^* \\
C^* JC - C^* JDR^{-1} D^* JC & -A^* + C^* JDR^{-1} B^* \\
R^{-1} B^* & R^{-1} \\
\end{bmatrix},
\]
so \( H \) is the \( A \) matrix of \( G^{-1} \). It is easy to see that in the above realization of \( G^{-1} \) there is no uncontrollable or unobservable modes on \( \mathbb{C}_0 \). Thus \( G^{-1} \) has no poles on \( \mathbb{C}_0 \) if and only if \( H \) has no eigenvalues on \( \mathbb{C}_0 \).

**Proof of (i) \( \Rightarrow \) (ii).** We have
\[
G(j\omega) = T_2^*(j\omega)T_2(j\omega) - T_1^*(j\omega)T_1(j\omega) > 0 \quad \text{for all } \omega \in \mathbb{R},
\]
and hence \( G(j\omega) \) is a continuous rational function on \( \mathbb{R} \). Thus \( G^{-1} \) has no poles on the imaginary axis.

**Proof of (ii) \( \Rightarrow \) (i).** The rational matrix \( G^{-1} \) has no poles and zeros on \( \mathbb{C}_0 \), thus, all the eigenvalues of \( G(j\omega) \) are nonzero, continuous, real-valued functions in \( \omega \).

This implies that \( \pi(G(j\omega)) \) is a constant function in \( \omega \), and by \( \lim_{\omega \to 0} G(j\omega) = R > 0 \) we find that \( \pi(G(j\omega)) \equiv m \).

**Proof of (ii) \( \Rightarrow \) (iii).** The pair \( (A, B) \) is controllable by the Hautus test. Combining the results of Lemmas 3 and 4 gives (iii).

**Proof of (iii) \( \Rightarrow \) (ii).** Obvious, since the eigenvalues of \( H \) coincide with the eigenvalues of \( Ac \) and \( -A^*c \), but we know that \( \delta(Ac) = 0 \).

### 5.2. Proof of Theorem 1

Assume that (ii) holds. If \( \omega \in \mathbb{R} \) then
\[
K = \begin{bmatrix}
P(\omega I - A) + (-\omega I - A^*) P & PB \\
B^* P & 0 \\
\end{bmatrix} + \begin{bmatrix}
C^* \\
-D^* \\
\end{bmatrix} \begin{bmatrix}
J \quad 0 \\
0 \\
\end{bmatrix} \succeq 0.
\]

Define \( V := \begin{bmatrix}
(\omega I - A)^{-1} B \\
- I \\
\end{bmatrix} \). This gives us
\[
0 \leq V^* KV = T^*(\omega I)JT(\omega) \quad \text{for all } \omega \in \mathbb{R}.
\]

Now assume that (i) holds. For any \( \epsilon > 0 \) introduce the matrices
\[
\widehat{D}_2 := \begin{bmatrix} D_2 \\ \epsilon I \end{bmatrix}, \quad \hat{D} := \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} D \\ \epsilon \end{bmatrix}, \quad \hat{C} := \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad \hat{J} := \begin{bmatrix} J \\ 0 \\ I \end{bmatrix}.
\]

Introduce furthermore \( \tilde{T}(s) := \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \). The inequality \( 0 \leq T^*(\omega I)JT(\omega) \) implies \( 0 < T^*(\omega I)JT(\omega) + \epsilon^2 I = \tilde{T}^*(\omega I)\tilde{J}\tilde{T}(\omega) \).
Hence, since $\tilde{D}^* \tilde{J} \tilde{C} = D^* JC$, Lemma 2 gives that the equation
\[
0 = P(A - B(R + \epsilon^2 I)^{-1} D^* JC) + (A - B(R + \epsilon^2 I)^{-1} D^* JC)^* P
\]
\[
- C^* JC + C^* JD(R + \epsilon^2 I)^{-1} D^* JC + PB(R + \epsilon^2 I)^{-1} B^* P
\]
\[
= PA + A^* P - C^* JC + (C^* JD - PB)(R + \epsilon^2 I)^{-1}(D^* JC - B^* P)
\]
has a hermitian solution $P$. Thus we find that
\[
\begin{bmatrix}
-P \epsilon A - A^* P \epsilon + C^* JC & P \epsilon B - C^* JD
\end{bmatrix}
\]
\[
\geq 0
\]
has a hermitian solution $P$ for all $\epsilon > 0$.

Combining this with Lemma 6 we get the desired result.

5.3. Some results on rational matrices

Lemma 7. Consider the transfer functions in (4)
\[
T_1(s) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad T_2(s) = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}
\]
with the same number of columns. Assume that the realizations are minimal and the matrices $A_1$ and $A_2$ have no common eigenvalues.

Assume that there exists a rational transfer function $\text{Q}(s)$ with the identity
\[
\text{Q}(s)T_2(s) = T_1(s), \quad \text{for all } s \in \mathbb{C}. \quad (18)
\]

Then there is a minimal realization of the rational matrix
\[
\text{Q}(s) = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}, \quad (19)
\]
and a constant $n_0 \times n_2$ blockmatrix $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$, where $n_0$ is the size of $A_0$, such that the partitioning of $(A_0, B_0, C_0)$ conformal with $Y$ is of the form
\[
A_0 = \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{01} \\ B_{02} \\ B_{03} \end{bmatrix}, \quad C_0 = [C_1, C_2, C_3]. \quad (20)
\]
where $\sigma(A_{33}) \subset \sigma(A_2)$ and $\sigma(A_{22}) \cap \sigma(A_2) = \emptyset$, furthermore,
\[
A_0 Y - Y A_2 + B_0 C_2 = 0, \quad (21)
\]
\[
B_0 D_2 - \begin{bmatrix} \text{I}_{n_1} & Y_1 \\ 0 & Y_2 \\ 0 & Y_3 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = 0, \quad (22)
\]
\[
C_0 Y + D_0 C_2 = 0, \quad (23)
\]
\[
D_0 D_2 = D_1. \quad (24)
\]
The converse is also true in the following sense: if (21)–(24) are valid to the matrices in (19) and (20) then (18) holds. (Here we do not assume the minimality of realizations and they may have also common poles.)

**Remark.** We note that any of the matrices may be absent in the lemma.

**Proof.** We may assume without loss of generality that $Q$ is of the form

\[
Q(s) = \begin{bmatrix}
A_q & 0 & B_q \\
0 & A_{Q33} & B_{Q3} \\
C_q & C_{Q3} & D_Q
\end{bmatrix},
\]

where $A_q$ and $A_2$ have no common eigenvalues and $\sigma(A_{Q33}) \subset \sigma(A_2)$. Denote by $n_q$ and $n_3$ the sizes of $A_q$ and $A_{Q33}$ respectively.

Therefore the Sylvester-equation

\[
A_q Y_q - Y_q A_2 + B_q C_2 = 0
\]

has a unique solution $Y_q$.

Define the similarity transformation

\[
S := \begin{bmatrix}
I_{n_q} & 0 & Y_q \\
0 & I_{n_3} & 0 \\
0 & 0 & I_{n_2}
\end{bmatrix},
\]

and note that

\[
S^{-1} := \begin{bmatrix}
I_{n_q} & 0 & -Y_q \\
0 & I_{n_3} & 0 \\
0 & 0 & I_{n_2}
\end{bmatrix}.
\]

Then (18) can be written as

\[
T_1(s) = Q(s)T_2(s) = \begin{bmatrix}
A_q & 0 & B_q C_2 & B_q D_2 \\
0 & A_{Q33} & B_{Q3} C_2 & B_{Q3} D_2 \\
C_q & C_{Q3} & D_Q C_2 & D_Q D_2
\end{bmatrix}
\]

= (using the similarity transformation “$S$”):

\[
= \begin{bmatrix}
A_q & 0 & 0 & B_q D_2 - Y_q B_2 \\
0 & A_{Q33} & B_{Q3} C_2 & B_{Q3} D_2 \\
C_q & C_{Q3} & D_Q C_2 + C_q Y_q & D_Q D_2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_q & B_q D_2 - Y_q B_2 \\
0 & A_{Q33} & B_{Q3} C_2 & B_{Q3} D_2 \\
C_q & C_{Q3} & D_Q C_2 + C_q Y_q & 0
\end{bmatrix}.
\]

So we have isolated the terms having poles in common with $T_2$. Thus

\[
T_1 = \begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
A_q & B_q D_2 - Y_q B_2 \\
C_q & D_Q D_2
\end{bmatrix}
\]

(25)
0 = \begin{bmatrix}
A_{Q333} & B_{Q3}C_2 & B_{Q3}D_2 \\
0 & A_2 & B_2 \\
C_Q & D_QC_2 + C_qY_q & 0
\end{bmatrix}.

(26)

The constant matrices in (25) give (24). On the other hand it does not necessarily follow that the realization on the right-hand side of (25) is minimal. However, we may assume without loss of generality that it is given in a control (or Kalman) normal form, i.e.

\begin{align*}
A_q &= \begin{bmatrix} A_{Q11} & A_{Q12} \\ 0 & A_{Q22} \end{bmatrix}, \\
B_qD_2 - Y_qB_2 &= \begin{bmatrix} B_{Q1}D_2 - Y_1B_2 \\ B_{Q2}D_2 - Y_2B_2 \end{bmatrix}, \\
C_q &= [C_{Q1} C_{Q2}],
\end{align*}

where $B_{Q2}D_2 - Y_2B_2 = 0$, and the pair $(A_{Q11}, B_{Q1}D_2 - Y_1B_2)$ is controllable. (The matrix $C_Q$ is partitioned according to the normal form.)

(Otherwise there is a similarity transformation which carries the system to this form.)

Thus we have

\begin{align*}
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} &= \begin{bmatrix} A_{Q11} & B_{Q1}D_2 - Y_1B_2 \\ C_Q & D_QD_2 \end{bmatrix},
\end{align*}

Since the realization (19) is minimal, the pair $(C_Q, A_Q)$ is observable and so is $(C_{Q1}, A_{Q11})$. Hence the realization on the right side is also minimal now.

On the two sides of the equality we have two minimal realizations of the same function. We may assume that

\begin{align*}
A_{Q11} &= A_1, \\
B_{Q1}D_2 - Y_1B_2 &= B_1 \\ 
C_{Q1} &= C_1.
\end{align*}

Consider now Eq. (26). Introduce

\begin{align*}
\hat{A} := \begin{bmatrix} A_{Q333} & B_{Q3}C_2 \\ 0 & A_2 \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} B_{Q3}D_2 \\ B_2 \end{bmatrix}, \quad \hat{C} := [C_{Q3} DQC_2 + C_qY_q]
\end{align*}

and the rank matrices

\begin{align*}
\Phi := [\hat{R}_Q \quad \hat{A}_Q \hat{B}_Q \quad \hat{A}_Q^2 \hat{B}_Q \quad \ldots \quad \hat{A}_Q^{n_3+1} \hat{B}_Q], \\
\Psi := \begin{bmatrix}
\hat{C}_Q \\
\hat{C}_Q \hat{A}_Q \\
\hat{C}_Q \hat{A}_Q^2 \\
\vdots \\
\hat{C}_Q \hat{A}_Q^{n_2+n_3-1}
\end{bmatrix}
\end{align*}
We collect the lower blocks of $\Phi$ as
\[ \Phi_0 := \begin{bmatrix} B_2 & A_2 B_2 & A_2^2 B_2 & \cdots & A_2^{n_2+n_3-1} B_2 \end{bmatrix}, \]
and the right-hand blocks of $\Psi$ as
\[ \Psi_0 := \begin{bmatrix} C Q_3 & C Q_3 A Q_{33} & C Q_3 A Q_{33}^2 & \cdots & C Q_3 A Q_{33}^{n_2+n_3-1} \end{bmatrix}. \]
The above realizations of $Q$ and $T_2$ are minimal, thus
\[ \text{rank } \Phi \geq \text{rank } \Phi_0 = n_2, \]
\[ \text{rank } \Psi \geq \text{rank } \Psi_0 = n_3, \]
furthermore, Eq. (26) shows us that the controllable subspace is not observable, i.e.,
\[ \Psi \cdot \Phi = 0. \quad (27) \]
Hence we conclude that
\[ \text{rank } \Phi = n_2 \quad \text{and} \quad \text{rank } \Psi = n_3, \]
or, in other words, there exist two $n_3 \times n_2$ matrices $Y_3$ and $\hat{Y}_3$ such that
\[ \Phi = \begin{bmatrix} Y_3 \Phi_0 \\ \Phi_0 \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} \Psi_0 \\ \hat{Y}_3 \Psi_0 \end{bmatrix}. \]
and we obtain
\[ 0 = \Psi \cdot \Phi = \Psi_0 \begin{bmatrix} I \\ \hat{Y}_3 \end{bmatrix} \begin{bmatrix} Y_3 \\ I \end{bmatrix} \Phi_0 = \Psi_0 (Y_3 + \hat{Y}_3) \Phi_0. \]
This gives $Y_3 + \hat{Y}_3 = 0$, or $\hat{Y}_3 = -Y_3$. Define $\Gamma := \begin{bmatrix} I_{n_3} & Y_3 \\ 0 & I_{n_2} \end{bmatrix}$. We have
\[ \Psi \Gamma = \begin{bmatrix} \Psi_0 & 0 \end{bmatrix}, \quad \Gamma^{-1} \Phi = \begin{bmatrix} 0 \\ \Phi_0 \end{bmatrix}, \]
where the first row and column give
\[ D Q C_2 + C_q Y_q = -C Q_3 Y_3, \quad \text{and} \quad B Q_3 D_2 = Y_3 B_2, \]
respectively, i.e., (23) and the last block-row of (22).
To show that
\[ A Q_{33} Y_3 - Y_3 A_2 + B Q_3 C_2 = 0, \]
define
\[ A_{\Gamma} := \Gamma^{-1} \hat{A}_Q \Gamma = \begin{bmatrix} A Q_{33} & A Q_{33} Y_3 - Y_3 A_2 + B Q_3 C_2 \\ 0 & A_2 \end{bmatrix}. \]
The controllable subspace \( \text{Im} \Phi \) is \( \hat{A}_Q \)-invariant, hence \( \text{Im} \Gamma^{-1}\Phi \) is \( \hat{A}_T \)-invariant. Thus, the latter gives

\[
\text{Im} \begin{bmatrix} 0 \\ \Phi_0 \end{bmatrix} \supset \text{Im} A_T \begin{bmatrix} 0 \\ \Phi_0 \end{bmatrix} = \begin{pmatrix} (A_Q Y_3 - Y_3 A_2 + B_Q C_2) \Phi_0 \\ \Phi_0 \end{pmatrix},
\]

and by rank \( \Phi_0 = n_2 \) we have \( A_Q Y_3 - Y_3 A_2 + B_Q C_2 = 0 \) which is the last block-row of (21).

It remains to prove that the converse of the lemma is also true. Assume that Eqs. (19)–(24) hold and let \( s \in \mathbb{C} \), such that \( s \notin \sigma(A) \cup \sigma(A_Q) \). Now (21) is equivalent to

\[
(sI - A_Q)Y - Y(IS - A_2) - B_Q C_2 = 0.
\]

Pre-multiplying by \( C_Q(sI - A_2)^{-1} \), post-multiplying by \( (sI - A_2)^{-1}B_2 \) and using relations (19)–(24), we obtain (18). □

There is another formulation of the previous lemma as follows.

**Corollary 2.** Suppose that the assumptions made in Lemma 7 are fulfilled. Define

\[
\Sigma := \begin{bmatrix} I_{n_1} & Y_1 & 0 \\ 0 & Y_2 & 0 \\ 0 & Y_3 & 0 \\ 0 & -C_2 & D_2 \end{bmatrix}, \quad \Theta := \begin{bmatrix} I_{n_1} & Y_1 & 0 \\ 0 & Y_2 & 0 \\ 0 & Y_3 & 0 \\ 0 & 0 & I_{p_1} \end{bmatrix}
\]

\[
\Sigma_Q := \begin{bmatrix} -A_Q & B_Q \\ C_Q & -D_Q \end{bmatrix} \quad \text{and} \quad \Sigma_T := \begin{bmatrix} -A & B \\ C & -D \end{bmatrix} = \begin{bmatrix} -A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & 0 & -D_1 \\ 0 & C_2 & -D_2 \end{bmatrix}.
\]

Then

\[
\Sigma_Q \Xi = \Theta \Sigma_T. \tag{28}
\]

**Proof.** Eq. (28) can be checked by simple substitution of Eqs. (19)–(24). □

**Lemma 8.** Let \( r_k : \mathbb{C} \to \mathbb{C} \) (\( k = 1, 2, \ldots \)) be a sequence of rational functions with the following properties:

(i) each \( r_k \) is contractive on \( \mathbb{C}_+ \), i.e. \( |r_k(z)| \leq 1 \) for all \( \text{Re} \ z \geq 0 \).

(ii) the degrees of the denominators do not exceed \( N \).

Then there exists a subsequence of \( r_k \) which converges pointwise on \( \mathbb{C}_+ \) to a rational contractive function.

**Proof.** First do the possible reduction of each fraction.
According to assumptions (i) and (ii) the degrees of the numerators and denominators do not exceed \( N \). Thus we can find two integers, \( m \leq n (\leq N) \), and a subsequence \( \rho_k^{(0)} \) of \( r_k \) such that the degrees of every denominators and numerators are just \( n \) and \( m \).

Thus it can be written

\[
\rho_k^{(0)}(z) = \frac{(z - \alpha_{1,k}^{(0)})(z - \alpha_{2,k}^{(0)}) \ldots (z - \alpha_{m,k}^{(0)})}{(z - \beta_{1,k}^{(0)})(z - \beta_{2,k}^{(0)}) \ldots (z - \beta_{n,k}^{(0)})}, \quad (k = 1, 2, 3, \ldots)
\]

with the suitable chosen constants

\[
\alpha_{1,k}^{(0)}, \alpha_{2,k}^{(0)}, \ldots, \alpha_{m,k}^{(0)}, \beta_{1,k}^{(0)}, \beta_{2,k}^{(0)}, \ldots, \beta_{n,k}^{(0)}, \gamma_k^{(0)}.
\]

We shall say that a sequence of complex numbers is convergent in a more general sense, or in a shortened form \( g \)-convergent, if it is either convergent or tending to \( \infty \).

It is an easy consequence of the Bolzano–Weierstrass theorem that every sequence of complex numbers has a \( g \)-convergent subsequence.

Denote a \( g \)-convergent subsequence of \( \alpha_{1,k}^{(0)} \) by \( \alpha_{1,k}^{(1)} \).

Let

\[
\rho_k^{(1)}(z) = \frac{(z - \alpha_{1,k}^{(1)})(z - \alpha_{2,k}^{(1)}) \ldots (z - \alpha_{m,k}^{(1)})}{(z - \beta_{1,k}^{(1)})(z - \beta_{2,k}^{(1)}) \ldots (z - \beta_{n,k}^{(1)})}, \quad (k = 1, 2, 3, \ldots)
\]

be the subsequence of \( \rho_k^{(0)} \) corresponding to it.

Choose now a \( g \)-convergent subsequence of \( \alpha_{2,k}^{(1)} \) and denote it by \( \alpha_{2,k}^{(2)} \).

Let

\[
\rho_k^{(2)}(z) = \frac{(z - \alpha_{1,k}^{(2)})(z - \alpha_{2,k}^{(2)}) \ldots (z - \alpha_{m,k}^{(2)})}{(z - \beta_{1,k}^{(2)})(z - \beta_{2,k}^{(2)}) \ldots (z - \beta_{n,k}^{(2)})}, \quad (k = 1, 2, 3, \ldots)
\]

be the subsequence of \( \rho_k^{(1)} \) corresponding to the choice.

We can find in \( n + m \) similar steps the subsequence \( \rho_k^{(n+m)} \) which has already \( g \)-convergent zeros and poles.

In order to simplify the notations we shall hereafter omit the indices “\( (n + m) \)” as follows:

\[
\rho_k(z) = \frac{(z - \alpha_{1,k})(z - \alpha_{2,k}) \ldots (z - \alpha_{m,k})}{(z - \beta_{1,k})(z - \beta_{2,k}) \ldots (z - \beta_{n,k})}, \quad (k = 1, 2, 3, \ldots).
\]

We may assume without loss of generality that for some \( p, q \)

\[
\alpha_{1,k}, \alpha_{2,k}, \alpha_{3,k}, \ldots, \alpha_{p,k}, \beta_{1,k}, \beta_{2,k}, \beta_{3,k}, \ldots, \beta_{q,k}
\]

are each convergent, and

\[
\alpha_{p+1,k}; \alpha_{p+2,k}; \alpha_{p+3,k}; \ldots; \alpha_{m,k}, \beta_{q+1,k}; \beta_{q+2,k}; \beta_{q+3,k}; \ldots; \beta_{n,k}
\]
are tending to infinity, or, in other words, the function \( \rho_k \) can be written in the form
\[
\rho_k(z) = f_k(z)g_k(z),
\]
where
\[
f_k(z) = \prod_{j=1}^{p} \frac{z - \alpha_{j,k}}{z - \beta_{j,k}} \quad \text{and} \quad g_k(z) = \gamma_k \prod_{j=p+1}^{m} \frac{z - \alpha_{j,k}}{z - \beta_{j,k}}.
\]

Introduce the following notations:
\[
a_i = \lim_{k \to \infty} \alpha_{i,k} \quad (i = 1, 2, \ldots, p), \quad \text{and} \quad b_j = \lim_{k \to \infty} \beta_{j,k} \quad (j = 1, 2, \ldots, q).
\]
Let \( z \in \mathbb{C}^+ \) such that \( z \notin \{a_1, a_2, \ldots, a_p\} \).

Assumption (i) gives
\[
|g_k(z)| \leq 1 \quad \text{or, equivalently,} \quad |g_k(z)| \leq \left| \frac{1}{f_k(z)} \right|, \quad (k = 1, 2, 3, \ldots).
\]
Here the right-hand side is convergent and bounded. According to the Bolzano–Weierstrass theorem one can select a convergent subsequence of \( g_k(z) \) and suppose that \( k_i \) \((i = 1, 2, \ldots, \ldots)\) is an appropriate sequence of indices. Since
\[
g_k(z) = \gamma_k \prod_{j=p+1}^{m} \frac{z - \alpha_{j,k}}{z - \beta_{j,k}} \frac{\prod_{j=p+1}^{m} \left( \frac{z}{\alpha_{j,k}} - 1 \right)}{\prod_{j=q+1}^{n} \left( \frac{z}{\beta_{j,k}} - 1 \right)},
\]
the subsequence will be convergent for all \( z \) with the limit
\[
\lim_{i \to \infty} g_{k_i}(z) = (-1)^{n+m-p-q} \lim_{i \to \infty} \gamma_{k_i} \prod_{j=p+1}^{m} \frac{\alpha_{j,k_i}}{\beta_{j,k_i}},
\]
does not depend on \( z \). Denote it by \( c \). Thus we have found a subsequence of \( r_k \) converging pointwise to a contractive rational function of the form
\[
\frac{(z - a_1)(z - a_2) \cdots (z - a_p)}{(z - b_1)(z - b_2) \cdots (z - b_q)}, \quad 0 \leq p \leq q \leq N. \quad \square
\]

**Remark.** The matrix version of the previous lemma can be seen elementwise as an easy consequence of the original one.

5.4. Proof of Theorem 2

(ii) \( \implies \) (i):

Let
\[
Q(s) = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix}, \quad (29)
\]
be a minimal realization of \( Q \).

According to Theorem 1 with the correspondence \( T_1 \leftrightarrow Q \) and \( T_2 \leftrightarrow I \) there exists a hermitian \( Z \) such that
\[
K_Q(Z) = \begin{bmatrix} -ZA_Q - A_Q^*Z - C_Q^*C_Q & ZB_Q + C_Q^*D_Q \\ B_Q^*Z + D_Q^*C_Q & I_{p_2} - D_Q^*D_Q \end{bmatrix} \succeq 0.
\]
We show that $Z > 0$.

We know that $Q$ is contractive and the realization (29) is minimal. This implies that $A_Q$ is stable and the pair $(C_Q, A_Q)$ is observable. On the other hand the 1,1 block of $K_Q(Z)$ is semidefinite, i.e.

$K_{Q11} = -ZA_Q - A_Q^* Z - C_Q^* C_Q \geq 0$, therefore

$$Z = \int_0^\infty e^{A_Q^* t} (K_{Q11} + C_Q^* C_Q) e^{A_Q^* t} \, dt \geq \int_0^\infty e^{A_Q^* t} C_Q^* C_Q e^{A_Q^* t} \, dt > 0.$$ 

Now, by Lemma 7 and Corollary 2 the matrices in (29) can be written in the form (20) and there exists a matrix

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

which satisfies (21)–(24).

As in Corollary 2 introduce

$$\Xi := \begin{bmatrix} I_{n_1} & Y_1 & 0 \\ 0 & Y_2 & 0 \\ 0 & Y_3 & 0 \end{bmatrix}$$

and

$$M := \begin{bmatrix} I_{n_1} & Y_1 \\ 0 & Y_2 \\ 0 & -C_2 & D_2 \end{bmatrix}.$$ 

Moreover, introduce $P = M^* Z M$, and note that $P \geq 0$. We claim that

$$\Xi^* K_Q(Z) \Xi = K(P). \tag{30}$$

In fact, this implies (i), since $K(P) \geq 0$ follows immediately from $K_Q(Z) \geq 0$ and (30).

In doing this we need Eq. (28) and the relationship

$$\Xi^* \begin{bmatrix} 0 & 0 \\ 0 & I_{p_2} \end{bmatrix} \Xi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_2^* C_2 & -C_2^* D_2 \\ 0 & -D_2^* C_2 & D_2^* D_2 \end{bmatrix}.$$ 

It is easy to substitute them to the left-hand side of (30) using the following partition

$$K_Q(Z)$$

$$= \begin{bmatrix} Z \\ 0 \end{bmatrix} [-A_Q & B_Q] + \begin{bmatrix} -A_Q^* \\ B_Q^* \end{bmatrix} [Z \ 0] - \begin{bmatrix} C_Q^* \\ -D_Q \end{bmatrix} [C_Q & -D_Q] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

thus

$$\Xi^* K_Q(Z) \Xi = \Xi^* \begin{bmatrix} Z \\ 0 \end{bmatrix} M [-A & B] + \begin{bmatrix} -A^* \\ B^* \end{bmatrix} M^* [Z \ 0] \Xi$$

$$- \begin{bmatrix} C_1^* C_1 & 0 & -C_1^* D_1 \\ 0 & 0 & 0 \\ -D_1^* C_1 & 0 & D_1^* D_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_2^* C_2 & -C_2^* D_2 \\ 0 & -D_2^* C_2 & D_2^* D_2 \end{bmatrix}.$$
Combining this with the identity \( Z^* M = P \) we get the desired equation (30).

(i) \( \implies \) (ii):

First we assume that \( R > 0 \) and \( P > 0 \). We deal with the problem under this conditions.

We shall prove that (10) gives an appropriate matrix \( Q \).

Note that in this case \( Y = \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \) is a solution of (19) and (19)–(24) holds. (Here \( Y_1 = 0 \) and \( \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix} \) corresponds to \( I_{n_2} \).)

Thus, according to Lemma 7, Eq. (8) also holds, i.e.

\[ T_1(s) = Q(s) T_2(s), \quad \forall s \in \mathbb{C}. \]

It remains to prove that \( Q \) will be contractive, i.e. (9) holds.

It suffices to prove that

\[
K_Q(P) = \begin{bmatrix}
-PA_Q - A_Q^* P - C_Q^* C_Q & PB_Q + C_Q^* D_Q \\
B_Q^* P + D_Q^* C_Q & I_{p_2} - D_Q^* D_Q
\end{bmatrix} \geq 0.
\]

Indeed, for any \( s \in \mathbb{C}_+ \) and \( V(s) := \begin{bmatrix} (Is - A_Q)^{-1} B_Q \\ -I \end{bmatrix} \) we have

\[
0 \leq V^* K_Q V = I - Q^* Q (s) - 2 \Re s B_Q^* (Is - A_Q^*)^{-1} P (Is - A_Q^*)^{-1} B_Q,
\]

and \( P > 0 \) implies

\[ I - Q^* Q (s) \leq 0, \quad \forall s \in \mathbb{C}_+ \]

which is just the desired contractivity (9).

Thus, we have to show that \( K_Q(P) \geq 0 \).

Our assumption \( R > 0 \) gives rank \( D_2 = m \). Let \( A \) be some constant matrix of the size \( p_2 \times (p_2 - m) \) such that

\[ A^* D_2 = 0 \quad \text{and} \quad \text{rank}[D_2 \quad A] = m. \]

Introduce the matrix

\[
\Xi := \begin{bmatrix}
I_{n_1} & 0 & 0 & 0 \\
0 & I_{n_2} & 0 & 0 \\
0 & -C_2 & D_2 & (I_{p_2} + D_2 R^{-1} D_1^* X) A
\end{bmatrix}.
\]

Obviously, rank \( \Xi = p_2 + n \), thus we have the equivalence

\[ K_Q(P) \geq 0 \iff \Xi^* K_Q(P) \Xi \geq 0 \]

We will bring the latter in a more convenient form using the following identities

\[ B_Q D_2 = B, \]

\[
[-A_Q \quad B_Q] \Xi = \begin{bmatrix}
-A \quad B \quad P^{-1} \left( \begin{bmatrix}
-C_1^* X \\
C_2^* X
\end{bmatrix} + C^* J D R^{-1} D_1^* X \end{bmatrix} \quad A
\end{bmatrix}.
\]
\[ [C_Q - D_Q] \Xi = [C_1 0 - D_1 - (I + D_1 R^{-1} D_1^*) X \Delta], \]

\[ \Xi^* \begin{bmatrix} 0 & 0 \\ 0 & I_{p_2} \end{bmatrix} \Xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_2^* C_2 & -C_2^* D_2 \\ 0 & -D_2^* C_2 & D_2^* D_2 \\ 0 & -A^* (I + X^* D_1 R^{-1} D_1^*) C_2 & A^* X^* D_1 R^{-1} D_2^* D_2 \end{bmatrix} \]

The above equations can be checked easily by using the definition of \( A_Q, B_Q, C_Q \) and \( D_Q \) in (10).

After the following preparation

\[ \Xi^* K_{Q}(P) \Xi = \Xi^* \begin{bmatrix} P & [-A_Q^* B_Q^*] [P 0] [C_Q - D_Q] + [0 0] \end{bmatrix} \Xi \]

we have

\[ \Xi^* K_{Q}(P) \Xi = \begin{bmatrix} -A^* P - P A + C^* J C & P B - C^* J D \\ B^* P - D^* J C & D^* J D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \Delta^* (I - X^* X - X^* D_1 R^{-1} D_1^* X) A \end{bmatrix} \]

\[ = \begin{bmatrix} K(P) & 0 \\ 0 & \Delta^* (I - X^* X - X^* D_1 R^{-1} D_1^* X) A \end{bmatrix}. \]

We know that \( K(P) \geq 0 \) thus it remains to show that

\[ W_1 := \Delta^* (I - X^* X - X^* D_1 R^{-1} D_1^* X) A \geq 0. \]

Define \( S := (D_2^* D_2)^{-\frac{1}{2}} \), and with this

\[ W_1 := \Delta^* (I - X^* (I + D_1 S (I - S D_1^* D_1 S)^{-1} S D_1^*) X) A = \Delta^* (I - X^* (I - D_1 (D_2^* D_2)^{-1} D_1^*)^{-1} X) A, \]

which is the Schur-complement of the 2,2 block of

\[ W := \begin{bmatrix} \Delta^* A & \Delta^* X^* \\ X A & I - D_1 (D_2^* D_2)^{-1} D_1^* \end{bmatrix}. \]

Observe that the Schur-complement of the 1,1 block is

\[ W_2 := I - D_1 (D_2^* D_2)^{-1} D_1^* - X A (\Delta^* A)^{-1} \Delta^* X = I - X X^* \geq 0. \]
Thus

\[ W_2 \geq 0 \Rightarrow W \geq 0 \Rightarrow W_1 \geq 0. \]

which gives the proof in the case when \( P > 0, R > 0 \).

Now, let us deal now with the case when \( R \) or \( P \) are not definite but semidefinite.

Let \( A \) and \( I \) be any matrices satisfying the conditions

\[ A^* A = I_m, \quad I^* I_n \quad \text{and} \quad I^* A = 0. \]

For any positive number \( \epsilon \) we define the function

\[ \hat{T}_{2\epsilon}(s) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \epsilon \Gamma + \epsilon A, \]

If we show that there exists a contactive \( Q_{\epsilon} \) for all \( \epsilon > 0 \) such that

\[ \hat{Q}_{\epsilon}(s) \hat{T}_{2\epsilon}(s) = T_1(s), \quad s \in \mathbb{C} \quad (31) \]

holds then we know by Lemma 8 with \( \epsilon = \frac{1}{k} \) that \( \hat{Q}_{\frac{1}{k}} \) \( (k = 1, 2, 3, \ldots) \) has a point-wise convergent subsequence and the limit, denoted by \( \hat{Q} \), is a rational matrix and \( \hat{Q} \) is contractive on \( \mathbb{C}_+ \).

Thus, the corresponding subsequence of \( \hat{Q}_{\frac{1}{k}} \hat{T}_{2\frac{1}{k}} \) is tending to \( \hat{Q} \left[ \begin{array}{c} T_2 \\ 0 \end{array} \right] \), and the suitable left-hand side block of \( \hat{Q} \) gives the required \( Q \) in (8) and (9).

The proof of the existence of contactive \( Q_{\epsilon} \) for which (31) holds can be reduced to the case \( P > 0 \) and \( R > 0 \).

Thus, it suffices to prove that there is a positive hermitian solution \( P \) to the following dissipation inequality of the above \( \hat{T}_{2\epsilon} \)

\[ \hat{K}(P) = K(P) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 I_{n_2} & 0 \\ 0 & 0 & \epsilon^2 I_m \end{bmatrix} \]

\[ = \begin{bmatrix} -P_{11} A_1 - A_1^* P_{11} - C_1^* C_1 & -P_{12} A_2 - A_1^* P_{12} \\ -P_{21} A_1 - A_2^* P_{21} & -P_{22} A_2 - A_2^* P_{22} + C_2^* C_2 + \epsilon^2 I \\ B_1^* P_{11} + B_2^* P_{21} + D_1^* C_1 & B_1^* P_{12} + B_2^* P_{22} - D_2^* C_2 \\ P_{13} B_1 + P_{12} B_2 + C_1^* D_1 \\ P_{23} B_1 + P_{22} B_2 - C_2^* D_2 \\ D^* J D + \epsilon^2 I \end{bmatrix}. \]

If \( P \geq 0 \) is a solution of the original inequality (18) then obviously \( P_{11} > 0 \) because \( A_1 \) is always stable.

Hence, for any \( \delta > 0 \)

\[ P_{\text{def}} := \begin{bmatrix} P_{11} & P_{12} \\ P_{11} & P_{22} + \delta I_{n_2} \end{bmatrix} > 0 \]
holds, we have
\[
\hat{K}(P_{\text{def}}) = K(P) + \begin{bmatrix}
0 & 0 & 0 \\
0 & -\delta(A_2 + A_2^*) + \epsilon^2 I_m & \delta B_2 \\
0 & \delta B_2^* & \epsilon^2 I_m
\end{bmatrix},
\]
which will be semidefinite for sufficiently small \(\delta\).

We conclude that for the maximal hermitian solution \(P_{\text{max}}\) of \(\hat{K}(P) > 0\), the inequalities \(P_{\text{max}} \geq P_{\text{def}} > 0\) hold and \(P_{\text{max}}\) is definite.

Thus the problem boils down to the case of \(R > 0, P > 0\).

5.5. Proof of Theorem 3

Introduce the rational functions
\[
L(s) := (sI - A)^{-1}B \quad \text{and} \quad M(s) := \begin{bmatrix}
(sI - A)^{-1}B \\
-I
\end{bmatrix}.
\]

We have
\[
\begin{bmatrix}
P(s_1I - A) + (s_1I - A^*)P + C^*JC & PB - C^*JD \\
B^*P - D^*JC & D^*JD
\end{bmatrix}
\]
\[
= \begin{bmatrix}
(s_1 + s_k)P & 0 \\
0 & 0
\end{bmatrix} + K(P)
\]
and by applying \(M(s_k)^*\) to the left side and \(M(s_l)\) to the right this gives
\[
1 \frac{1}{s_k + s_l} (T_2^+(s_k)T_2(s_l) - T_1^+(s_k)T_1(s_l))
= L(s_k)^* PL(s_l) + \frac{1}{s_k + s_l} M(s_k)^* K(P) M(s_l).
\]

Consequently, since \(K(P) \geq 0\) and \(\text{Re}(s_k) > 0, k = 1, 2, \ldots, N\), we find that the block-matrix with the blocks \(1 \frac{1}{s_k + s_l} K(P)\) \(k, l = 1, 2, \ldots, N\) is semidefinite.

Thus we find that the block-matrix \(S\) consisting of the blocks
\[
S_{kl} = L(s_k)^* PL(s_l) + \frac{1}{s_k + s_l} M(s_k)^* K(P) M(s_l), \quad k, l = 1, 2, \ldots, N
\]
is also positive semidefinite.

5.6. Proof of Theorem 4

The equality \(T_2^+(s)T_2(s) = T_1^+(s)T_1(s)\) is equivalent to \(T_2^+(s)T_2(s) \geq T_1^+(s)T_1(s)\) and \(T_2^+(s)T_2(s) \leq T_1^+(s)T_1(s)\).
By Theorem 1 the former holds on the imaginary axis if and only if there exists a hermitian $P_1$ such that

$$
\begin{bmatrix}
-P_1 A - A^* P_1 + C^* J C & P_1 B - C^* J D \\
B^* P_1 - D^* J C & D + J D
\end{bmatrix} \succeq 0,
$$

while the latter is equivalent to the existence of a hermitian solution $P_2$ to the inequality

$$
\begin{bmatrix}
-P_2 A - A^* P_2 - C^* J C & P_2 B + C^* J D \\
B^* P_2 + D^* J C & -D^* J D
\end{bmatrix} \succeq 0.
$$

If we add them together then we get in the inequality

$$
\begin{bmatrix}
-(P_1 + P_2) A - A^* (P_1 + P_2) & (P_1 + P_2) B \\
B^* (P_1 + P_2) & 0
\end{bmatrix} \succeq 0,
$$

which is equivalent to $Z B = 0$ and $Z A + A^* Z \succeq 0$, where $Z = P_1 + P_2$.

Since the pair $(A, B)$ is controllable, it is easy to prove that $Z = 0$.

With the notation $P = P_1 = -P_2$ we find that the above inequalities are equivalent to $K(P) \geq 0$ and $K(P) \leq 0$, concluding the proof of (a).

For the proof of (b) we note that the matrices $P_1$ and $P_2$ are positive semidefinite which implies $P = P_1 = P_2 = 0$, and $K(0) = 0$.

6. Conclusion

We have treated standard bounded real problems as the quadratic comparison of two rational matrices.

Let us note that several questions remain unanswered in this paper.

For instance: what is the necessary and sufficient condition of

$$
T^*_2(s) T_2(s) \succeq T^*_1(s) T_1(s) s \in S,
$$

where $S \subset \mathbb{C}$ is a given “simple” subset on the complex plane. We do not know the exact answer even if $S = \mathbb{C}_+$. (We have dealt with this problem by making additional assumptions on $T$.)

After all, using linear fractional mappings, the results developed in this paper are applicable to any straight line, circle and half-plane of the complex plane.

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References


