Groups of linear isometries on poset structures

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Received 30 March 2005; accepted 30 July 2007
Available online 18 September 2007

Abstract

Let $V$ be an $n$-dimensional vector space over a finite field $\mathbb{F}_q$ and $P = \{1, 2, \ldots, n\}$ a poset. We consider on $V$ the poset-metric $d_P$. In this paper, we give a complete description of groups of linear isometries of the metric space $(V, d_P)$, for any poset-metric $d_P$.

Keywords: Poset codes; Poset metrics; Linear isometries

0. Introduction

Coding theory takes place in finite-dimensional linear spaces over finite fields. One of the main questions of the theory (classical problem) asks to find a $k$-dimensional subspace in $\mathbb{F}_q^n$, the space of $n$-tuples over the finite field $\mathbb{F}_q$, with the largest possible minimum distance. There are many possible metrics that can be defined in $\mathbb{F}_q^n$, the most common ones being the Hamming and Lee metrics.

In 1987 Niederreiter generalized the classical problem of coding theory (see [10]). Brualdi et al. (see [2]) also provided in 1995 a wider setting for the above problem: using partially ordered sets and defining the concept of poset-codes, they started to study codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics (see [1,2,5,4] and [9]).

We let $P$ be a partially ordered set (abbreviated as poset) of cardinality $n$ with order relation denoted by $\leq$. An ideal of $P$ is a subset $I \subseteq P$ with the property that $x \in I$ and $y \leq x$ implies that $y \in I$. Given $A \subseteq P$, we denote by $\langle A \rangle$ the smallest ideal of $P$ containing $A$. Without loss of generality, we assume that $P = \{1, 2, \ldots, n\}$ and that the coordinate positions of vectors in $\mathbb{F}_q^n$ are labelled by the elements of $P$.

Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, the support of $x$ is the set

\[ \text{supp}(x) := \{ i \in P : x_i \neq 0 \}, \]

\textsuperscript{1} This work has been partially supported by FPTI/PDTA, Brazil.
\textsuperscript{2} This work has been partially supported by FAPESP, Brazil.
\textsuperscript{3} This work was supported by the Basic Research Program of the Korea Science and Engineering Foundation (grant # R01-2001-000-11176-0).
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doi:10.1016/j.disc.2007.08.001
and we define the $P$-weight of $x$ to be the cardinality of the smallest ideal containing $\text{supp}(x)$:

$$w_P(x) = |\text{supp}(x)|.$$ 

The function

$$d_P : \mathbb{F}_q^n \times \mathbb{F}_q^n \rightarrow \mathbb{N}$$

defined by $d_P(x, y) = w_P(x - y)$ is a metric in $\mathbb{F}_q^n$ [12, Lemma 1.1], called a poset-metric, or a $P$-poset-metric, when it is important to stress the order taken in consideration. We denote such a metric space by $(\mathbb{F}_q^n, d_P)$.

An $[n, k, \delta_P]_q$ poset-code is a $k$-dimensional subspace $C \subseteq \mathbb{F}_q^n$, where $\mathbb{F}_q^n$ is endowed with a poset-metric $d_P$ and

$$\delta_P(C) = \min\{w_P(x) : \emptyset \neq x \in C\}$$

is the $P$-minimum distance of the code $C$. If $P$ is an antichain order, that is, an order with no comparable elements, then $P$-weight, $P$-poset-metric and $P$-minimum distance become the Hamming weight, Hamming metric and minimum distance of classical coding theory. The Rosenbloom–Tsfasman metric, introduced in [13], can be viewed as a $P$-poset-metric which corresponds to a poset consisting of a finite disjoint union of chains of equal lengths.

A linear isometry $T$ of the metric space $(\mathbb{F}_q^n, d_P)$ is a linear transformation $T : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ that preserves $P$-poset-metric,

$$d_P(T(x), T(y)) = d_P(x, y),$$

for every $x, y \in \mathbb{F}_q^n$. Equivalently, a linear transformation $T$ is an isometry if $w_P(T(x)) = w_P(x)$ for every $x \in \mathbb{F}_q^n$.

A linear isometry of $(\mathbb{F}_q^n, d_P)$ is said to be a $P$-isometry. Since an isometry must be injective, a linear isometry is an invertible map and it is easy to see that this inverse is also a linear isometry. It follows that the set of all linear isometries of the poset space $(\mathbb{F}_q^n, d_P)$ is a group. We denote it by $\text{GL}_P(\mathbb{F}_q)$ and call it the group of linear isometries of $(\mathbb{F}_q^n, d_P)$. In [3,6,8] the group of linear isometries were determined for the Rosenbloom–Tsfasman space, generalized Rosenbloom–Tsfasman space and crown space. In [11] we describe the symmetry group (not necessarily linear ones) of a poset space that is a product of Rosenbloom–Tsfasman spaces.

In this work, we give a complete description of those groups, for any given poset-metric $P$. The property of permuting chains of same length, showed in [8], corresponds, in the case of a general poset $P$, to Theorem 1.1 of the next section, which assures that every linear isometry $T$ induces an automorphism in $P$. The key to proving this is Lemma 1.2, which assures that $\langle \text{supp}(T(e_i)) \rangle$ is a prime ideal if $e_i$ is a canonical coordinate vector of $\mathbb{F}_q^n$. The characterization of linear isometries is given in Theorem 1.2: given $T \in \text{GL}_P(\mathbb{F}_q)$, there is a pair of ordered bases $\beta$ and $\beta_T$ of $\mathbb{F}_q^n$ relative to which the linear isometry $T$ is represented by an $n \times n$ upper triangular matrix $(a_{ij})_{1 \leq i, j \leq n}$ with $a_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

1. Linear isometries for general poset structures

We will present only the concepts of the theory of partially ordered sets that are strictly necessary for this work, referring the reader to [12] for more details.

A totally ordered set (or linearly ordered set) is a poset $P$ in which any two elements are comparable. A subset $C$ of a poset $P$ is called a chain if $C$ is a totally ordered set when regarded as a subposet of $P$. Two posets $P$ and $Q$ are isomorphic if there exists an order-preserving bijection $\phi : P \rightarrow Q$, called an isomorphism, whose inverse is order preserving; that is, $x \leq y$ in $P$ if and only if $\phi(x) \leq \phi(y)$ in $Q$. An isomorphism $\phi : P \rightarrow P$ is called an automorphism. Given $x, y \in P$, we say that $y$ covers $x$ if $x < y$ and if no element $z \in P$ satisfies $x < z < y$.

From here on, we denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$.

We start showing that a linear isometry $T \in \text{GL}_P(\mathbb{F}_q)$ induces an order automorphism $\phi_T$ (see Theorem 1.1) in the following way: given $i \in \{1, 2, \ldots, n\}$ there is a unique element $j \in \{1, 2, \ldots, n\}$ such that $\langle j \rangle = \langle \text{supp}(T(e_i)) \rangle$, and the map defined by $\phi_T(i) = j$ is an order automorphism.

**Lemma 1.1.** Let $T \in \text{GL}_P(\mathbb{F}_q)$ and $i \in \{1, 2, \ldots, n\}$. If $j \in \text{supp}(T(e_i))$, then we have $|\langle j \rangle| \leq |\langle i \rangle|$.

**Proof.** By assumption $\langle j \rangle \subseteq \langle \text{supp}(T(e_i)) \rangle$. It follows from this and $P$-weight preservation that $|\langle j \rangle| \leq |\langle \text{supp}(T(e_i)) \rangle| = |\langle \text{supp}(e_i) \rangle| = |\langle i \rangle|$. □
An ideal \( I \) of a poset \( P \) is said to be a prime ideal if it contains a unique maximal element.

**Lemma 1.2.** Let \( T \in \text{GL}_P(\mathbb{F}_q) \). Then, for every \( i \in \{1, 2, \ldots, n\} \), we have that \( \langle \text{supp}(T(e_i)) \rangle \) is a prime ideal.

**Proof.** We will first show that there is an element \( j \in \text{supp}(T(e_i)) \) satisfying \(|\langle j \rangle| = |\langle i \rangle|\). Assume the contrary, namely that \(|\langle j \rangle| < |\langle i \rangle| \), for all \( j \in \text{supp}(T(e_i)) \). Let \( \text{supp}(T(e_i)) = \{i_1, i_2, \ldots, i_s\} \). By assumption, \(|\langle i_u \rangle| < |\langle i \rangle|\) for \( u \in \{1, 2, \ldots, s\} \). It follows from the linearity of \( T^{-1} \) that we have \( i = \text{supp}(e_i) \subseteq \bigcup_{k=1}^{s} \text{supp}(T^{-1}(e_{i_k})) \), which implies \( i \in \text{supp}(T^{-1}(e_{i_k})) \) for some \( k \in \{1, 2, \ldots, s\} \). Using this and Lemma 1.1, we obtain \(|\langle i \rangle| \leq |\langle i_k \rangle| < |\langle i \rangle|\). This is a contradiction. By the \( P \)-weight preservation, such an element \( j \) is unique. Hence the result follows. \( \square \)

**Lemma 1.3.** If \( T \in \text{GL}_P(\mathbb{F}_q) \) and \( i \leq j \), then
\[
\langle \text{supp}(T(e_i)) \rangle \subseteq \langle \text{supp}(T(e_j)) \rangle.
\]

**Proof.** Lemma 1.2 states that \( \langle \text{supp}(T(e_i)) \rangle \) and \( \langle \text{supp}(T(e_j)) \rangle \) are prime ideals, so there are elements \( k \) and \( l \) such that \( \langle k \rangle = \langle \text{supp}(T(e_i)) \rangle \) and \( \langle l \rangle = \langle \text{supp}(T(e_j)) \rangle \). If \( k = l \), then we are done, so assume that \( k \neq l \). This means that either
\[
k \in \text{supp}(T(e_i) - T(e_j)) \quad \text{or} \quad l \in \text{supp}(T(e_i) - T(e_j)).
\]

We have three cases to consider.

*Case 1:* \( k \notin \text{supp}(T(e_i) - T(e_j)) \). In this case, \( k \in \text{supp}(T(e_j)) \). It follows that \( \langle \text{supp}(T(e_i)) \rangle = \langle k \rangle \subseteq \langle \text{supp}(T(e_j)) \rangle \).

*Case 2:* \( l \notin \text{supp}(T(e_i) - T(e_j)) \). In this case, \( l \in \text{supp}(T(e_i)) \), so \( l < k \). Hence, \( \langle \text{supp}(T(e_i)) \rangle = \langle l \rangle \subseteq \langle k \rangle = \langle \text{supp}(T(e_j)) \rangle \), so
\[
w_P(e_j) = w_P(T(e_j)) < w_P(T(e_i)) = w_P(e_i).
\]

However, the hypothesis \( i < j \) implies that \( w_P(e_i) < w_P(e_j) \), a contradiction.

*Case 3:* \( i, l \in \text{supp}(T(e_i) - T(e_j)) \). In this case,
\[
|\langle k, l \rangle| \leq |\langle \text{supp}(T(e_i) - T(e_j)) \rangle| = |\langle \text{supp}(T(e_i) - T(e_j)) \rangle| = |\langle i, j \rangle|.
\]

By hypothesis, \( i < j \), so \( |\langle k, l \rangle| \leq |\langle j \rangle| = |\langle \text{supp}(e_j) \rangle| = |\langle \text{supp}(T(e_j)) \rangle| = |\langle l \rangle| \). We conclude that \( k \subseteq \langle l \rangle \), that is, \( \langle \text{supp}(T(e_i)) \rangle \subseteq \langle \text{supp}(T(e_j)) \rangle \). \( \square \)

**Theorem 1.1.** Let \( P = \{1, 2, \ldots, n\} \) be a poset, \( \{e_1, e_2, \ldots, e_n\} \) be the canonical base of \( \mathbb{F}_q^n \) and \( T \in \text{GL}_P(\mathbb{F}_q) \) linear isometry. Then the map \( \phi_T : P \rightarrow P \) given by
\[
\phi_T(i) = \max(\text{supp}(T(e_i)))
\]
is an automorphism of \( P \).

**Proof.** The map \( \phi_T : P \rightarrow P \) given by \( \phi(i) = \max(\text{supp}(T(e_i))) \) is well-defined by Lemma 1.2. Lemma 1.3 assures that \( \phi_T \) is an order preserving map.

We claim that \( \phi_T \) is one-to-one. In fact, let us suppose that \( \phi_T(i) = \phi_T(j) \) with \( i \neq j \). Since \( \phi_T(i) = \max(\text{supp}(T(e_i))) \) and \( \phi_T(j) = \max(\text{supp}(T(e_j))) \) it follows that
\[
\langle \text{supp}(T(e_i)) \rangle = \langle \text{supp}(T(e_j)) \rangle.
\]

By the \( P \)-weight preservation and the linearity of \( T \),
\[
|\langle i, j \rangle| = |\langle \text{supp}(T(e_i) + T(e_j)) \rangle| = |\langle \text{supp}(T(e_i) + T(e_j)) \rangle|.
\]

(1)
But
\[\langle \text{supp}(T(e_i) + T(e_j)) \rangle \subseteq \langle \text{supp}(T(e_i)) \rangle \cup \langle \text{supp}(T(e_j)) \rangle\]
and since both ideals in the right hand are assumed to be equal and \(T\) is an isometry, it follows that
\[|(\text{supp}(T(e_i) + T(e_j)))| = |\langle \text{supp}(T(e_i)) \rangle| = |\langle i \rangle|\]
(2)
and
\[|(\text{supp}(T(e_i) + T(e_j)))| = |\langle \text{supp}(T(e_j)) \rangle| = |\langle j \rangle|\].
(3)
From Eqs. (1)–(3) follows that
\[|\langle i, j \rangle| = |\langle i \rangle| = |\langle j \rangle|\]
or equivalently, \(i < j\) and \(j < i\), a contradiction. Therefore we have \(i = j\), so that \(\phi_T\) is one-to-one. Since \(P\) is finite, it follows that \(\phi_T\) is bijection that preserves order, that is, an order automorphism. \(\Box\)

We now describe the main result of this work:

**Theorem 1.2.** Let \(P = \{1, 2, \ldots, n\}\) be a poset and \(\{e_1, e_2, \ldots, e_n\}\) be the canonical base of \(\mathbb{F}_q^n\). Then \(T \in \text{GL}_P(\mathbb{F}_q)\) if and only if
\[
T(e_j) = \sum_{i \leq j} x_{ij} e_{\phi_T(i)},
\]
(4)
where \(\phi_T : P \to P\) is the automorphism associated with \(T\) as in Theorem 1.1 and \(x_{ij}\) are constants with \(x_{jj} \neq 0\) for all \(j \in \{1, 2, \ldots, n\}\). Moreover, given \(T \in \text{GL}_P(\mathbb{F}_q)\), there is a pair of ordered bases \(\beta\) and \(\beta_T\) of \(\mathbb{F}_q^n\) relative to which the linear isometry \(T\) is represented by an \(n \times n\) upper triangular matrix \((a_{ij})_{1 \leq i, j \leq n}\) with \(a_{jj} \neq 0\) for every \(j \in \{1, 2, \ldots, n\}\).

**Proof.** Assume that \(T \in \text{GL}_P(\mathbb{F}_q)\) and let \(\phi_T : P \to P\) be the order automorphism induced by \(T\) as in Theorem 1.1. Then for all \(j \in \{1, 2, \ldots, n\}\),
\[\text{supp}(T(e_j)) \subseteq \langle \text{supp}(T(e_j)) \rangle = \langle \phi_T(j) \rangle = \{\phi_T(i) : i \leq j\}.\]

Hence, \(T(e_j)\) is of the form given in (4).

Conversely, assume that \(T\) is a map on \(P\) of the form given by (4). It is straightforward to show that \(T\) is a \(P\)-isometry. This proves the first part of the theorem.

Now let \(\beta := (e_{i_1}, e_{i_2}, \ldots, e_{i_n})\) be a total ordering of the canonical base such that \(e_{i_r}\) appears before \(e_{i_s}\) whenever \(w_P(e_{i_r}) \leq w_P(e_{i_s})\) for all \(i_r, i_s = 1, 2, \ldots, n\).

We define another ordered base \(\beta_T\) as the base induced by the automorphism \(\phi_T\),
\[
\beta_T := (e_{\phi_T(i_1)}, e_{\phi_T(i_2)}, \ldots, e_{\phi_T(i_n)}),
\]
and let \(A = (a_{kl})_{1 \leq k, l \leq n}\) be the matrix of \(T\) relative to the bases \(\beta\) and \(\beta_T\). So
\[
T(e_{i_1}) = x_{i_1 i_1} e_{\phi_T(i_1)},
T(e_{i_2}) = x_{i_1 i_2} e_{\phi_T(i_2)} + x_{i_2 i_2} e_{\phi_T(i_2)},
\vdots
T(e_{i_n}) = x_{i_1 i_n} e_{\phi_T(i_1)} + x_{i_2 i_n} e_{\phi_T(i_2)} + \cdots + x_{i_n i_n} e_{\phi_T(i_n)},
\]
where \(x_{ij} = 0\) if \(i \nleq j\) and \(x_{ij} \neq 0\) for all \(j \in \{1, 2, \ldots, n\}\). Therefore \(a_{kl} = x_{ij}i\) if \(k \leq l\) and \(a_{kl} = 0\) if \(k > l\). In other words, \(A\) is upper triangular matrix with \(a_{jj} = x_{ij}i_j \neq 0\) for all \(j \in \{1, 2, \ldots, n\}\). \(\Box\)
Given a poset \( P = \{1, 2, \ldots, n\} \), we denote by \( \text{Aut}(P) \) the group of order automorphisms of \( P \).

**Corollary 1.1.** Given \( T \in \text{GL}_P(\mathbb{F}_q) \) there is an ordering \( \beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\} \) of the canonical base such that \([T]_{\beta, \beta}\) is given by the product \( A \cdot U \) where \( A \) is an invertible upper triangular matrix and \( U \) is the permutation matrix corresponding to the automorphism induced by \( T \).

**Proof.** Given \( \sigma \in \text{Aut}(P) \) let \( T_\sigma \) be the linear isometry defined by \( T_\sigma(e_j) = e_{\sigma(j)} \), for \( j \in \{1, 2, \ldots, n\} \). Let \( \phi_T \) be the automorphism induced by \( T \). By Theorem 1.2,

\[
T(e_j) = \sum_{i \leq j} x_{ij} e_{\phi_T(i)}
\]

with \( x_{jj} \neq 0 \) for all \( j \in \{1, 2, \ldots, n\} \), so

\[
T \circ T_{\phi_T^{-1}}(e_j) = T(e_{\phi_T^{-1}(j)}) = \sum_{i \leq j} x_{i\phi_T^{-1}(j)} e_{\phi_T(i)}
\]

\[
= x_{\phi_T^{-1}(j)\phi_T^{-1}(j)} e_{\phi_T(j)} + \sum_{i < \phi_T^{-1}(j)} x_{i\phi_T^{-1}(j)} e_{\phi_T(i)}
\]

for all \( j \in \{1, 2, \ldots, n\} \). Since \( x_{\phi_T^{-1}(j)\phi_T^{-1}(j)} \neq 0 \) it follows that the automorphism induced by \( T \circ T_{\phi_T^{-1}} \) is the identity, so, when considering the base \( \beta_T \) as in Theorem 1.2, we find that \( \beta_T = \beta \) and the matrix of \( T \circ T_{\phi_T^{-1}} \) relative to this base is an upper triangular matrix \( A = [T \circ T_{\phi_T^{-1}}]_{\beta, \beta} \). But \( T_{\phi_T^{-1}} \) acts on \( \mathbb{F}_q^n \) as a permutation of the vectors in \( \beta \), so that in any ordered base containing those vectors, \( U^{-1} = [T_{\phi_T^{-1}}]_{\beta, \beta} \) is obtained from the permutation matrix. We note that \( T_{\phi_T} = (T_{\phi_T^{-1}})^{-1} \) and it follows that

\[
[T]_{\beta, \beta} = [T \circ T_{\phi_T^{-1}} \circ T_{\phi_T}]_{\beta, \beta}
\]

\[
= [T \circ T_{\phi_T^{-1}}]_{\beta, \beta} [T_{\phi_T}]_{\beta, \beta}
\]

\[
= A \cdot U.
\]

The \( m \)th level \( \Gamma^{(m)}(P) \) is the set of elements of \( P \) that generate prime ideals with cardinality \( m \):

\[
\Gamma^{(m)}(P) = \{ i \in P : |\langle i \rangle| = m \} = \{ i \in P : w_P(e_i) = m \}.
\]

**Corollary 1.2.** Let \( P = \{1, \ldots, n\} \) be a poset and \( k = \max \{ m : \Gamma^{(m)}(P) \neq \emptyset \} \). Then

\[
|\text{GL}_P(\mathbb{F}_q)| = (q - 1)^n \cdot \left( \prod_{i=1}^k q^{(i-1)\Gamma^{(i)}(P)} \right) \cdot |\text{Aut}(P)|.
\]

**Proof.** From Corollary 1.1, if \( T \in \text{GL}_P(\mathbb{F}_q) \) there is a re-ordering \( \beta = (e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \) of the canonical base of \( \mathbb{F}_q^n \) such that \([T]_{\beta, \beta} = A \cdot U\), where \( A = (a_{kl})_{1 \leq k, l \leq n} \) is an upper triangular matrix with \( a_{kl} = 0 \) if \( l < k \) and \( U = [T_{\phi_T}]_{\beta, \beta} \) is the permutation matrix corresponding to the automorphism \( \phi_T \) induced by linear isometry \( T \) (see Theorem 1.2). Moreover, such base \( \beta \) depends only on \( \phi_T \) and for every \( \phi \in \text{Aut}(P) \), any matrix \( A \) as in the previous corollary defines a linear \( P \)-isometry.

So for every \( l \in \{1, 2, \ldots, n\} \), there are \((q - 1)\) possible different entries for \( a_{ll} \) (since \( a_{ll} \neq 0 \)). But \( A \) is upper triangular and, given \( 1 \leq i < j \leq n \), \( a_{ij} \neq 0 \) only if \( i \leq j \), so there are at most \(|\langle j \rangle| - 1 \) possible nonzero indices \((i, j)\)
with \(1 \leq i < j \leq n\), and for each of those there are \(q\) possible different entries. Since there are exactly \(|I(i,j)|\) such indices, we find that, up to considering the automorphism of order \(\phi_T\) induced by \(T\), there are

\[
(q - 1)^n \cdot \left( \prod_{i=1}^{k} q^{(i-1)|I(i)(P)|} \right)
\]

linear \(P\)-isometries and we conclude by counting the elements of \(\text{Aut}(P)\). \(\square\)

Let \(A = (a_{ij})\) and \(B = (b_{ij})\) be elements in \(G_P\). Since

\[
(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{i \leq k \leq j} a_{ik} b_{kj},
\]

\(AB \in G_P\). We note that every element in \(G_P\) is an upper triangular matrix with nonzero diagonal entries. Hence, such elements are invertible. Since the inverse of an element in \(G_P\) is a polynomial in that element, such an element is in \(G_P\). So, we see that \(G_P\) is a subgroup of \(\text{GL}_P(\mathbb{F}_q)\). Since we already proved that \(\text{GL}_P(\mathbb{F}_q) = G_P \cdot \text{Aut}(P)\), all that is left to show is that \(G_P\) is a normal subgroup of \(\text{GL}_P(\mathbb{F}_q)\). Given \(\phi \in \text{Aut}(P)\), it is straightforward to show that \((\phi I)^{-1} = I\phi\) [3]. It follows that

\[
(\phi I)A(\phi I)^{-1} = \phi A\phi
\]

for every \(n \times n\) matrix \(A\). If \(A = (a_{ij}) \in G_P\), for each \(i = 1, 2, \ldots, n\),

\[
(\phi I)A(\phi I)^{-1}(e_i) = \phi A\phi (e_i) = \sum_{k=1}^{n} a_{\phi(k)\phi(i)} e_k
\]

\[
= \sum_{\phi(k) \leq \phi(i)} a_{\phi(k)\phi(i)} e_k
\]

\[
= \sum_{k \leq i} a_{\phi(k)\phi(i)} e_k
\]

and \(a_{\phi(i)\phi(i)} \neq 0\) for every \(i\). Thus, we find that \(G_P\) is normal in \(\text{GL}_P(\mathbb{F}_q)\). Since \(G_P \cap H_P = \{I\}\) the corollary follows. \(\square\)

2. Examples

In this section, we illustrate the results of this paper with three examples, namely the main classes of poset-metrics: the posets that are disjoint unions of chains, weak orders and the crown orders.

Example 2.1. Let \(P = P_1 \cup P_2 \cup \cdots \cup P_s\) be a poset consisting of a disjoint union of \(s\) chains. Denote by \(\mu_i\) the cardinality of the \(i\)th chain, for \(i \in \{1, 2, \ldots, s\}\). For every \(j \in \{1, 2, \ldots, n\}\) let \(\nu_j = |\{P_i : |P_i| = j\}|.\) Given \(T \in \text{GL}_P(\mathbb{F}_q)\), from
Corollary 1.1 follows that there is an ordered base $\beta$ of $F_q^n$ relative to which the linear isometry $T$ is represented by the product $A \cdot U$ of $n \times n$ matrices, where $U$ is a monomial matrix that acts by exchanging coordinate subspaces with isomorphic supports and

$$
A = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & A_t
\end{pmatrix},
$$

where each $A_i$ is a $\mu_i \times \mu_i$ upper triangular matrix with nonzero diagonal entries.

As a particular case, if $P = \{1, 2, \ldots, n\}$ is a chain, then given a linear isometry $T$ there is an ordered base $\beta$ of $F_q^n$ relative to which the linear isometry $T$ is represented by an $n \times n$ upper triangular matrix with $x_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

If $P$ consists of a finite disjoint union of chains of equal lengths, then $w_P$ becomes the Rosenbloom–Tsfasman weight defined on the linear space $M_{n \times m}(F_q)$ of all $n \times m$ matrices over $F_q$ if $(a_{ij}) \in M_{n \times m}(F_q)$, then

$$
w_P((a_{ij})) = \sum_{j=1}^{m} |\{\text{supp}(a_{1j}, a_{2j}, \ldots, a_{nj})\}|.
$$

From Corollary 1.3 [8, Theorem 1] it follows that

$$\text{GL}_P(F_q) \cong (T_n)^m \times S_m,$$

where $(T_n)^m$ denotes the direct product of $m$ copies of the group $T_n$ of all upper triangular matrices of size $n$ over $F_q$ with nonzero diagonal entries and $S_m$ is the permutation groups on a set of $m$ elements.

**Example 2.2.** Let $n_1, \ldots, n_t$ be positive integers with $n_1 + \cdots + n_t = n$. Then $W = n_1 \mathbf{1} \oplus \cdots \oplus n_t \mathbf{1}$ will denote the weak order given by the ordinal sum of the antichains $n_i \mathbf{1}$ with $n_i$ elements (see [7]). Explicitly, $W = n_1 \mathbf{1} \oplus \cdots \oplus n_t \mathbf{1}$ is the poset whose underlying set and order relation are given by

$$
\{1, 2, \ldots, n\} = n_1 \mathbf{1} \cup n_2 \mathbf{1} \cup \cdots \cup n_t \mathbf{1},
$$

$$
n_i \mathbf{1} = \{n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_{i-1} + 2, \ldots, n_1 + \cdots + n_{i-1} + n_i\}
$$

and $x < y$ if and only if $x \in n_i \mathbf{1}, y \in n_j \mathbf{1}$ for some $i, j$ with $i < j$. Notice that if $n_1 = \cdots = n_t = 1$, then $W = 11 \oplus \cdots \oplus 11$ is totally ordered with $1 < 2 < \cdots < t$ and if $t = 1$, then $W = n \mathbf{1}$ is an antichain.

For a weak order $W = n_1 \mathbf{1} \oplus \cdots \oplus n_t \mathbf{1}$ we have that $I^{(m)}(W) = n_1 \mathbf{1}$ if $m = n_1 + n_2 + \cdots + n_{i-1} + 1$, for any $s \in \{1, 2, \ldots, t\}$ and $I^{(m)}(W) = \emptyset$ otherwise. The group of the automorphisms $\text{Aut}(W)$ is isomorphic to the cartesian product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_t}$ (Aut($W$) is just the group of the maps $\phi$ that permute only the elements of each $m$th level). Corollary 1.2 assures us then that

$$|\text{GL}_W(F_q)| = (q - 1)^n \cdot \left(\prod_{i=2}^{t} q^{n_1 (n_1 + n_2 + \cdots + n_{i-1} + 1)}\right) \cdot n_1! \cdot n_2! \cdots n_t!.
$$

From Theorem 1.2 follows that there are bases $\beta$ and $\beta_T$ of $F_q^n$ such that the matrix $[T]_{\beta, \beta_T}$ has the form

$$
\begin{pmatrix}
D_{n_1 \times n_1} & \ast & \cdots & \ast \\
0 & D_{n_2 \times n_2} & \cdots & \ast \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_{n_t \times n_t}
\end{pmatrix},
$$

where $D_{n_i \times n_i}$ is a diagonal matrix for each $s = 1, 2, \ldots, t$.

**Example 2.3.** The crown is a poset with elements $C = \{1, 2, \ldots, 2n\}, n > 1,$ in which $i < n + i, i + 1 < n + i$ for each $i \in \{1, 2, \ldots, n - 1\}$, and $1 < 2n, n < 2n$ [1].
Given a crown $C = \{1, 2, \ldots, 2n\}$, $\text{Aut}(C)$ is isomorphic to the dihedral group $D_n$. We note that $\Gamma^{(1)}(C) = \{1, 2, \ldots, n\}$, $\Gamma^{(3)}(C) = \{n + 1, \ldots, 2n\}$, and $\Gamma^{(k)}(C) = \emptyset$, for $k \neq 1, 3$. From Corollary 1.2 it follows that

$$|\text{GL}_C(\mathbb{F}_q)| = (q - 1)^{2n} \cdot q^{2n} \cdot 2n.$$ 

Given $T \in \text{GL}_C(\mathbb{F}_q)$, Theorem 1.2 assures there is a pair of ordered bases $\beta$ and $\beta_T$ of $\mathbb{F}_q^{2n}$ relative to which the linear isometry $T$ is represented by the upper triangular matrix of the form

$$T = \begin{pmatrix}
a_{1,1} & 0 & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & a_{1,2n} \\
0 & a_{2,2} & 0 & \cdots & 0 & a_{2,n+1} & a_{2,n+2} & \cdots & 0 & 0 \\
0 & 0 & a_{3,3} & \cdots & 0 & 0 & a_{3,n+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n,n} & 0 & 0 & \cdots & a_{n,2n-1} & a_{n,2n} \\
0 & 0 & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & a_{n+2,n+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{2n-1,2n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n,2n}
\end{pmatrix}.$$ 

Acknowledgement

The authors thank the referee for valuable remarks which led to sensible improvements in text and proofs.

References