Rainbow paths

Domingos Dellamonica Jr. [a,*], Colton Magnant [b], Daniel M. Martin [a]

[a] *Emory University, Department of Mathematics and Computer Science, 400 Dowman Dr, Atlanta, GA 30322, USA*
[b] *Lehigh University, Department of Mathematics, 14 E. Packer Ave., Bethlehem, PA 18015, USA*

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A B S T R A C T

A k-rainbow path in a graph with colored edges is a path of length k where each edge has a different color. In this note, we settle the problem of obtaining a constructive k-coloring of the edges of K_n in which one may find, between any pair of vertices, a large number of internally disjoint k-rainbow paths. In fact, our construction obtains the largest possible number of paths. This problem was considered in a less general setting by Chartrand et al. (2007) [6].

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1. Introduction

Given an edge-colored simple graph G, a path P in G is called *rainbow* if the edges of P are assigned distinct colors. Let $l \leq k$ be integers. Suppose that the edges of G are k-colored. For $a, b \in V(G)$, denote by $p(a, b)$ the maximum number of internally disjoint rainbow paths of length l having endpoints a and b. The *rainbow (k, l)-connectivity* of G is the minimum $p(a, b)$ among all distinct $a, b \in V(G)$.

A related concept has been studied in a sequence of papers by Chartrand et al. [5,6,4] and also [3,11]. In particular, the following theorem is given.

Theorem 1 ([4]). *For any r, there exists an explicit 2-coloring of K_r in which the number of bi-chromatic paths of length 2 between any pair of vertices is at least*

$$\lfloor \sqrt{r} - 1 \rfloor.$$

Using our definitions, the theorem above is a statement about the rainbow $(2, 2)$-connectivity of a given 2-coloring of the edges of K_r. In this note, we greatly improve and generalize the above lower bound for graphs of sufficiently large order by providing a different constructive coloring. Our construction attains asymptotically the maximum rainbow connectivity possible.

Theorem 2. *For any $k \geq 2$ and $r \geq r_0 = r_0(k)$ there exists an explicit k-coloring of the edges of K_r having rainbow $(k, 2)$-connectivity*

$$\left(\frac{k-1}{k} - o(1) \right) r.$$

More generally, we shall also consider the problem of finding longer rainbow paths.

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* Corresponding author.

E-mail addresses: ddellam@mathcs.emory.edu (D. Dellamonica Jr.), com208@lehigh.edu (C. Magnant), dmarti8@mathcs.emory.edu (D.M. Martin).

Theorem 3 (*Main Result*). *For any $3 \leq l \leq k$ there exists $r_0 = r_0(k)$ such that, for every $r \geq r_0$, there is an explicit k-coloring of the edges of K_r having rainbow (k, l)-connectivity*

$$\left(1 - o(1)\right) \frac{r}{l - 1}.$$

This result is also asymptotically best possible, since any collection of internally disjoint paths of length l can contain at most $r/(l - 1)$ paths.

Our proof employs a very recent breakthrough due to Bourgain [2,12], which consists of a powerful explicit *extractor*. Roughly speaking, an (explicit) extractor is a polynomial time algorithm used to convert some special probability distributions into uniform distributions. (See [13] for a good but somewhat outdated survey on extractors.)

The application of extractors in graph constructions already appears as early as [16], where extractors are used directly to obtain good expander graphs. Similar applications followed, e.g. [7,14]. Although some constructions are obtained by simply looking at extractors from a graph perspective (extractors can be seen as graphs), the analysis of our construction is more delicate and requires additional ingredients.

It is also noteworthy that a random k-coloring of a sufficiently large complete graph has asymptotically optimal rainbow (k, l)-connectivity. Therefore, our problem is to obtain an *explicit* edge-coloring. By explicit, we mean that there is a polynomial time algorithm to compute such an edge-coloring.[1]

2. Paths of length two

A simple application of the Cauchy–Schwarz Inequality shows that one cannot hope to find a k-coloring of $E(K_r)$ in which every pair of vertices is connected by $(1 - 1/k)(r - 1)$ rainbow paths of length two.

Theorem 4. *For any k-coloring of the edges of K_r there exists a pair of vertices having at least*

$$\frac{r - 1}{k} - 1$$

monochromatic paths of length 2 between them.

Proof. Denote by $\chi : E(K_r) \to [k]$ the fixed coloring of K_r. Let us count the triples (u, v, w) of different vertices satisfying $\chi(\{u, w\}) = \chi(\{v, w\})$. For fixed $w \in V = V(K_r)$ and $l \in [k]$, denote by $N(w, l) = \left| \left\{ v \in V : \chi(\{v, w\}) = l \right\} \right|$ the number of vertices connected to w by an edge of color l. Then, the number of pairs $u \neq v \in V \setminus \{w\}$ such that uwv is a monochromatic path is given by

$$\sum_{l=1}^{k} N(w, l)\left(N(w, l) - 1\right) = \sum_{l=1}^{k} N(w, l)^2 - r + 1.$$

Applying the Cauchy–Schwarz inequality we get that the number of such pairs is at least

$$\frac{1}{k}(r - 1)^2 - (r - 1).$$

Summing over every $w \in V$ and averaging over the pairs $u \neq v$, we have $(r - 1)/k - 1$, which implies that at least one pair has at least this number of monochromatic 2-paths connecting its points. $\square$

Observe that, in view of Theorem 4, the result of Theorem 2 is asymptotically best possible. Theorem 2 follows from the proof of our main result, Theorem 3.

3. Extractors

In order to describe our construction, we need to introduce some background on the machinery of extractors, a subject intensively developed during the past two decades. In particular, we shall use Bourgain's recent breakthrough construction. We follow the presentation of [12] in the brief recollection of Bourgain's results contained in this section.

We start by defining a way to measure randomness in (discrete) probability distributions. In what follows, we shall abuse notation by using the same letter to denote a probability distribution and a random variable following that distribution.

Definition 5. A *source* is a probability distribution on binary strings of a fixed length. Let X be a source over $\{0, 1\}^n$. The *min-entropy* of X is defined as

$$H^\infty(X) = -\log\left(\max_{a \in \{0,1\}^n} \mathbf{P}[X = a]\right).$$

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[1] There are stronger notions of explicitness. For instance, one could ask for an algorithm to compute the color of an edge in polynomial time over the size of the input (the pair of vertices), which is $O(\log |V|)$.

Here, and throughout this manuscript, logarithms have base 2. We shall say that X is a *δ-source* if its *min-entropy rate* $r(X) = H^\infty(X)/n$ is at least δ.

An extractor is a function that converts some distribution into one that is close to uniform. To define precisely what it means for two distributions to be close, we state the notion of statistical difference (also known as total variation distance).

Definition 6. The *statistical difference* between two sources $X, Y \subseteq \{0, 1\}^n$, is defined as[2]

$$\frac{1}{2}\|X - Y\|_1 = \frac{1}{2}\sum_{a\in\{0,1\}^n} \left|\mathbf{P}\big[X = a\big] - \mathbf{P}\big[Y = a\big]\right|.$$

We say that X is *α-close* to Y if $\frac{1}{2}\|X - Y\|_1 \le \alpha$.

We are now ready to formally describe the extractor that will be used to define our constructive coloring. It will be convenient to denote by $\{0, 1\}^{n\times 2}$ the Cartesian product $\{0, 1\}^n \times \{0, 1\}^n$; in particular, a function with domain $\{0, 1\}^{n\times 2}$ will be thought of as a function with two parameters, each taking values in $\{0, 1\}^n$.

Definition 7. A function $E\colon \{0, 1\}^{n\times 2} \to \{0, 1\}^m$ is a *two-source-extractor* for min-entropy rate δ and error ε if, for any pair of independent sources X and Y satisfying $r(X), r(Y) \ge \delta$, the distribution of $E(X, Y)$ is ε-close to the uniform distribution over $\{0, 1\}^m$.

We say that the extractor E is *strong* if the sets

$$S = \big\{x \in \{0, 1\}^n : E(x, Y) \text{ is } \varepsilon\text{-close to uniform}\big\}$$

and

$$T = \big\{y \in \{0, 1\}^n : E(X, y) \text{ is } \varepsilon\text{-close to uniform}\big\}$$

are such that $\mathbf{P}[X \in S], \mathbf{P}[Y \in T] \ge 1 - \varepsilon$.

A classical construction of two-source-extractors is due to Vazirani, which generalized Hadamard matrices (corresponding to the case where $m = 1$ in Theorem 8).

Theorem 8 ([15,8]). *For every constant $\delta > 0$, there exists a polynomial time strong two-source-extractor* Had$\colon \{0, 1\}^{n\times 2} \to \{0, 1\}^m$, *with $m = \Omega_\delta(n)$ and error $\varepsilon = 2^{-\Omega_\delta(n)}$, that works with any independent sources X and Y having $H^\infty(X) + H^\infty(Y) \ge (1 + \delta)n$.*

Bourgain's idea was to first encode the input bits (and produce a larger, redundant string) and then apply the Hadamard extractor of Theorem 8. To describe this encoding, we shall use some basic Finite Field Theory (the reader is referred to the textbook of Dummit and Foote [9]).

Let g be a primitive element of $\mathrm{GF}(2^n)$ (*i.e.*, a generator of the multiplicative group $\mathrm{GF}(2^n)^\times$). Every element of $\mathrm{GF}(2^n)$ can be seen as an integer in $[0, 2^n - 1]$ and, clearly, also can be seen as an element of $\{0, 1\}^n$. If $x \in \mathrm{GF}(2^n)$ let $\#x$ denote the integer corresponding to x. We define $g^x = g^{\#x}$. Bourgain's extractor can be taken as

$$\mathrm{Bou}(x, y) = \mathrm{Had}\big(\bar{x}, \bar{y}\big), \tag{1}$$

where the encoding $\bar{\cdot}\colon \{0, 1\}^n \to \{0, 1\}^{2n}$ is given by

$$x \mapsto \bar{x} = (x, g^x). \tag{2}$$

The above extractor is able to inherit many useful properties of the Hadamard extractor in which it is based. In particular, Bou is strong and symmetric (that is $\mathrm{Bou}(x, y) = \mathrm{Bou}(y, x)$).

A more sophisticated analysis of the Hadamard extractor given by Bourgain showed the following. Let X and Y be independent distributions. Denote by $2X$ the distribution of a random variable that consists of summing two independent samples from X. Then the distribution $\mathrm{Had}(X, Y)$ is not very different from the distribution $\mathrm{Had}(2X, Y)$. Therefore, if X is structured[3] in such a way that $2X$ has larger min-entropy than X, then it may be possible to obtain a uniform output from Had using a source X which has a smaller min-entropy than that naturally required by Had.

We say that a source X *grows with addition* if there is $t \ge 2$ such that tX has larger min-entropy than X. The role of Bourgain's encoding (2) is to map an arbitrary distribution X into another distribution $\bar{X}$ which grows with addition (see Lemma 9). Hence, applying the Hadamard extractor over the encoded inputs is statistically equivalent to applying the same extractor over sources with higher min-entropy.

Lemma 9. *Let $3\bar{X} = \bar{X} + \bar{X} + \bar{X}$ denote the distribution induced by sampling three independent elements x_1, x_2, x_3 from X and outputting $(x_1, g^{x_1}) + (x_2, g^{x_2}) + (x_3, g^{x_3})$. Then there exists an absolute constant $\alpha > 0$ such that $H^\infty(3\bar{X}) \ge (1+\alpha)\cdot 2\cdot H^\infty(X)$. In terms of min-entropy rates, $r(3\bar{X}) \ge (1 + \alpha)\, r(X)$.*

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[2] The 1/2 factor is used to keep the statistical distance in the range $[0, 1]$.

[3] For instance, X could be a uniform distribution over a geometric progression.

The price one has to pay in order to use $3\bar{X}$ instead of $\bar{X}$ in the analysis of the extractor is an increase of the output error as described by Lemma 10.

Lemma 10. *Let X and Y be independent sources over $\{0, 1\}^n$ and suppose that $H^\infty(t_1 X) \geq k_1$, $H^\infty(t_2 Y) \geq k_2$. Let* $\text{Had}: \{0, 1\}^{n \times 2} \to \{0, 1\}^m$ *be the Hadamard extractor and*

$$\varepsilon = \exp\left\{ \frac{n - k_1 - k_2}{c} + m \right\},$$

for $c = c(t_1, t_2)$. Then $\text{Had}(X, Y)$ *is ε-close to the uniform distribution over $\{0, 1\}^m$.*

In order to have a small ε in the above lemma, one needs to have sufficient min-entropy from $t_1 X$ and $t_2 Y$ (that is, $k_1 + k_2 > n$). Moreover, the length of the output m must be sufficiently small. Given Lemma 9, after encoding (2), sources with min-entropy rate slightly smaller than 1/2 grow with addition into sources with min-entropy rate slightly greater than 1/2. Applying Lemma 10 to those sources allows us to take m linear in n, while keeping the error ε exponentially small.

Theorem 11 ([2,12]). *There exists an absolute constant $\nu > 0$ such that the (explicit) function* $\text{Bou}: \{0, 1\}^{n \times 2} \to \{0, 1\}^m$ *is a strong extractor for sources of min-entropy $(1/2 - \nu)n$ with $m = \Omega(n)$ and, furthermore, has error $\varepsilon = 2^{-\Omega(n)}$.*

Another important observation regarding the above encoding is the following.

Claim 1. *Given any X, we have $H^\infty(2\bar{X}) \geq 2H^\infty(X) - 1$.*

Proof. First, let us change variables by setting $y = g^x$ and then $(x, g^x) = (\log_g y, y)$. (We may assume that $x \neq 0$ by paying a small price in terms of min-entropy: one can alter the original distribution by shifting all the probability from 0 to all the other elements thus removing 0 from the support of the distribution without significantly decreasing the min-entropy.) Since we have $\log_g y_1 + \log_g y_2 = \log_g y_1 y_2$, an element of $2\bar{X} = \bar{X} + \bar{X}$ would be of the form $\mathbf{x} = (\log_g y_1 y_2, y_1 + y_2)$. If we have $\mathbf{x} = (\log_g y_3 y_4, y_3 + y_4)$ then the following system of equations must be satisfied

$$\begin{cases} y_1 y_2 = y_3 y_4 \\ y_1 + y_2 = y_3 + y_4. \end{cases}$$

Since this field has characteristic 2, we have $y_1 + y_3 = y_2 + y_4$ and it follows that

$$y_1^2 + y_3^2 = (y_1 + y_3)^2 = (y_1 + y_3)(y_2 + y_4) = (y_1 y_2 + y_3 y_4) + y_1 y_4 + y_2 y_3 = y_1 y_4 + y_2 y_3. \tag{3}$$

Hence, $y_1(y_1 + y_4) = y_3(y_2 + y_3)$ and we also have $y_1 + y_4 = y_2 + y_3$. Therefore either $y_1 = y_4$ (and $y_2 = y_3$) or $y_1 = y_3$ (and $y_2 = y_4$). Hence, an element of $2\bar{X}$ is the image of at most 2 pairs in $\bar{X} \times \bar{X}$. The probability of any such element is thus at most 2 times the maximum probability of a pair in $\bar{X}^2$. By the independence assumption, a pair in $\bar{X}^2$ has probability bounded by $2^{-2H^\infty(X)}$. It follows that an element in $2\bar{X}$ has probability bounded by $2 \times 2^{-2H^\infty(X)} = 2^{-2H^\infty(X)+1}$ and the claim follows. $\square$

From this we get that another interesting property of the Hadamard extractor is inherited by Bourgain's extractor.

Lemma 12. *For any positive constant γ, there is $m = \Omega(\gamma n)$ for which*

$$\text{Bou}: \{0, 1\}^{n \times 2} \to \{0, 1\}^m,$$

as defined in (1) is a strong extractor for any X and Y such that either

(A) $H^\infty(X) + H^\infty(Y) \geq (1 + \gamma)n$ or
(B) $H^\infty(X), H^\infty(Y) \geq (1/2 - \nu)n$, *where ν is the constant of Theorem 11.*

Moreover, Bou *has error $\varepsilon = 2^{-\Omega(\gamma n)}$.*

Proof. Let $m_1 = -\frac{\gamma n}{2c(2,2)}$, where $c(\cdot, \cdot)$ is the implicit function in Lemma 10. Let $m_2 = \Omega(n)$ be the output length of the Bourgain extractor of Theorem 11. Set $m = \min\{m_1, m_2\}$ and let the output length of both Had and Bou be m in what follows.
A: Given independent sources X and Y such that $H^\infty(X) + H^\infty(Y) = (1 + \gamma)n$, by Claim 1, $H^\infty(2\bar{X}) + H^\infty(2\bar{Y}) \geq (1 + \gamma)2n - 2$. By Lemma 10, $\text{Bou}(X, Y) = \text{Had}(\bar{X}, \bar{Y})$ is $2^{-\Omega(\gamma n)}$-close to the uniform distribution.
B: If X and Y satisfy $H^\infty(X), H^\infty(Y) \geq (1/2 - \nu)n$ then the conclusion is immediate from Theorem 11. $\square$

4. Coloring the edges of the complete graph

Let us define a coloring of the complete graph on 2^n vertices. Fix Bou as the extractor of Lemma 12 with $\gamma = \nu/2$ and suppose that m is the output length and ε is the error of the extractor. Let the vertex set of K_{2^n} be $V = \{0, 1\}^n$. Let $S_1, \ldots, S_k$ be a balanced partition of $\{0, 1\}^m$ (that means $\big||S_i| - |S_j|\big| \leq 1$ for every i, j). Let $f: \{0, 1\}^m \to [k]$ be defined as $f(x) = i$ iff $x \in S_i$. An edge $\{u, v\} \in \{0, 1\}^{n \times 2}$ gets color $\chi(\{u, v\}) = f\big(\text{Bou}(u, v)\big)$ (this is well defined since Bou is symmetric).

Note that we are coloring a graph on 2^n vertices. This coloring could have several local flaws (pairs of vertices which have low rainbow connectivity). We shall later take an edge-coloring induced by an appropriate subset of these vertices as the final coloring. More concretely, we have $n = O(\log r)$, where r is the order of the final graph.

Define $\Gamma(v, l) = \{w \in V : \text{Bou}(v, w) \in S_l\}$ and set $N(v, l) = |\Gamma(v, l)|$.

Lemma 13. *The above coloring satisfies the following for all but at most $k\,2^{vn/2}$ vertices v: for any color $l \in [k]$, we have $N(v, l) \geq (1 - 2k\varepsilon)2^n/k$.*

Proof. Let T_l be a set of vertices v having $N(v, l) < (1 - 2k\varepsilon)2^n/k$. Assume that $|T_l| \geq 2^{vn/2}$. Let X_l be the uniform distribution over T_l (thus having $H^\infty(X_l) \geq vn/2$). Let U_n be uniform over $V = \{0, 1\}^n$. By Lemma 12, the distribution $\text{Bou}(X_l, U_n)$ should be ε-close to uniform. But this leads to the following contradiction:

$$2^{-m}|S_l| - \varepsilon \leq \mathbf{P}\big[\text{Bou}(X_l, U_n) \in S_l\big] = \frac{1}{|T_l|} \sum_{v \in T_l} \frac{N(v, l)}{|V|} < (1 - 2k\varepsilon)\frac{1}{k}.$$

Hence $|T_l| < 2^{vn/2}$ and $\big|\bigcup_{l \in [k]} T_l\big| \leq k\,2^{vn/2}$. $\square$

From this we can also conclude the following about pairs of vertices.

Lemma 14. *For any vertex $v \in V$ such that $N(v, l) \geq (1 - 2k\varepsilon)2^n/k$ holds for all $l \in [k]$ there are at most $k^2 2^{vn/2+2k}$ vertices w such that, for some $(j, l) \in [k]^2$, we have*

$$|\Gamma(w, j) \cap \Gamma(v, l)| < (1 - 2k\varepsilon)^2 \frac{2^n}{k^2}. \tag{4}$$

Proof. The proof is similar to that of Lemma 13. For some fixed pair (j, l), let $T_{j,l}$ be the set of vertices w for which (4) holds and assume $|T_{j,l}| \geq 2^{vn/2+2k}$. Let $X_{j,l}$ be the uniform distribution over $T_{j,l}$ and Y be the uniform distribution over $\Gamma(v, l)$. Observe that $H^\infty(Y) \geq n + \log\{1/k - 2\varepsilon\} \geq n - \log 2k$ and hence, $H^\infty(X_{j,l}) + H^\infty(Y) \geq (1 + v/2)n$. It follows that $\text{Bou}(X_{j,l}, Y)$ is ε-close to uniform. On the other hand,

$$\frac{|S_j|}{2^m} - \varepsilon \leq \mathbf{P}\big[\text{Bou}(X_{j,l}, Y) \in S_j\big] = \frac{1}{|T_{j,l}|} \sum_{w \in T_{j,l}} \frac{|\Gamma(w, j) \cap \Gamma(v, l)|}{N(v, l)} < (1 - 2k\varepsilon)\frac{1}{k},$$

which is a contradiction. Therefore, $\big|\bigcup_{(j,l) \in [k]^2} T_{j,l}\big| \leq k^2\,2^{vn/2+2k}$. $\square$

Along the same lines we have the following.

Lemma 15. *Given two sets $X, Y \subseteq V$ with $|X|, |Y| \geq k\,2^{(1/2-v)n}$ all but at most $k\,2^{(1/2-v)n}$ vertices $x \in X$ have edges of all k colors going to Y.*

4.1. Randomly selecting a subgraph

Although the coloring provided by the Bourgain extractor is such that for *most* pairs (a, b), the number of k-rainbow paths between a and b is very close to the best possible, we do not have any guarantee that this holds for *all* pairs. In order to deal with this technicality, we first show that randomly selecting a small subset of vertices of K_{2^n} (much less than $\sqrt{2^n}$) the coloring induced on the edges spanned by those vertices is one satisfying the requirements. The use of the Bourgain extractor (instead of the classical Hadamard extractor) is justified if one wishes to analyse the coloring induced by such a small set of vertices. Since our final goal is a constructive coloring, we shall derandomize the vertex selection in Section 4.2.

Let $p = 2^{-(1+v)n/2}$. Assume that we pick elements from $\{0, 1\}^n$ uniformly and independently with probability p forming some set V'. Consider the coloring formed in the induced graph V'. Observe that the expected cardinality of $|V'|$ is $2^{(1-v)n/2}$. Let us say that a pair $(v, w) \in V^2$ is *bad* (with respect to V) if either v or w fails Lemma 13 for some color or if (4) holds for some $(j, l) \in [k]^2$. A pair is called *good* otherwise.

Let us estimate the expected number of bad pairs that are contained in V'. The expected number of pairs containing (at least) one vertex failing Lemma 13 is at most $p^2 \cdot k2^{vn/2} \cdot 2^n = k2^{-vn/2}$. The expected number of pairs (v, w), $v \neq w \in V$ such that neither v nor w are exceptions in Lemma 13 but (4) holds for some $(j, l) \in [k]^2$ is, by Lemma 14, $p^2 \cdot k^2 2^{vn/2+2k} \cdot 2^n = k^2 2^{-vn/2+2k}$.

We also estimate the expected number of good pairs of V that are bad with respect to V': that is, $u \neq v \in V$ is not a bad pair but, for some fixed $\gamma > 0$ and some $(j, l) \in [k]^2$,

$$\big|\Gamma(w, j) \cap \Gamma(v, l) \cap V'\big| < (1 - \gamma)(1 - 2k\varepsilon)^2 \frac{2^{(1-v)n/2}}{k^2}. \tag{5}$$

It follows by Chernoff's inequality (see [10]) and the union bound that the expected number of quadruples (w, v, j, l) satisfying (5) is at most

$$k^2 \cdot 2^{2n} \cdot \exp\left\{-cp(1 - 2k\varepsilon)^2 \frac{2^n}{k^2}\right\},$$

where $c = c(\gamma) > 0$.

The number of bad pairs with respect to V' has expectation $o(1)$. Since $|V'|$ is a binomial random variable, it is strongly concentrated around its expectation (see Chernoff's inequality [10]). By choosing an appropriate value of p, we have $|V'| \in [r, r + r^{2/3}]$ with probability $1 - o(1)$. By removing at most $r^{2/3}$ arbitrary vertices, we get the final graph with the prescribed number r of vertices. Observe that since we are removing a very small number of vertices in the end, the effect on Eq. (5) is negligible. We shall derandomize the selection of V' in order to obtain a graph in which every pair is good (with respect to V').

4.2. Derandomization

In order to obtain a constructive coloring of the edges of the complete graph, we have to derandomize the random choices made above. So far, the following procedure has been defined: some large enough n is set and a coloring of the edges K_{2^n} is given by projecting the output of Bourgain's extractor onto the set of colors. A random vertex subset of this graph is taken and, with probability $1 - o(1)$, the induced edge-coloring has the desired properties.

We would like to stress that the reason we take such a strong construction as the Bourgain extractor (instead of relying on the well-known Hadamard extractor) lies in the fact that we must select very few vertices of the initial complete graph in order to ensure that no two vertices form a bad pair. On the other hand, we must be able to say something about the distribution of the colors in subsets of the induced subgraph, and those subsets are very small relative to the original graph.

The derandomization technique that we shall use is the *Method of Conditional Expectations* [1]. The random induced subgraph is determined by picking vertices independently with probability p. Suppose that the random decisions have been made for all vertices in a subset S of the vertex set V. Namely, for each $v \in S$ a random decision has been taken and a subset $S' = S \cap V'$ has been selected. The remaining choices (for vertices in $V \setminus S$) are independent from the choices made for S and it is simple to compute expectations in the conditional space where $S' \subseteq S$ is fixed.

Let $N = N_0 + N_1 + N_2$, N_0, N_1 and N_2 be the random variables such that N_0 counts the number of bad pairs; N_1 counts the number of good pairs in V that are not good with respect to V', namely, for some $(j, l) \in [k]^2$, they satisfy (5); and $N_2 = 2 \left| |V'| - (r + r^{2/3}/2) \right| / r^{2/3}$. We showed that $\mathbf{E}[N_i] = o(1)$ for $i = 1, 2, 3$ (if we set $p = (r + r^{2/3}/2)2^{-n}$).

Note that, given $S' \subseteq S$, we can compute the conditional expectation $\mathbf{E}[N \mid V' \cap S = S']$ in polynomial time (over $2^n = \text{poly}(r)$). Indeed, computing N_0 is just a matter of enumerating all bad pairs in K_{2^n} and adding the conditional probability of each one being selected. Computing N_1 requires a somewhat similar computation: for each pair of vertices and each pair of colors, the conditional probability of satisfying (5) is readily evaluated. Clearly, N_2 is computed in constant time. The method works as follows: initially, $S = \emptyset$; given an arbitrary $v \in V \setminus S$, we decide whether vertex v will be selected by computing two conditional expectations, $E_1 = \mathbf{E}[N \mid V' \cap (S + v) = S' + v]$ and $E_2 = \mathbf{E}[N \mid V' \cap (S + v) = S']$. Note that

$$E_0 = \mathbf{E}[N \mid V' \cap S = S'] = pE_1 + (1 - p)E_2.$$

Clearly, $\min\{E_1, E_2\} \leq E_0$. Take $i \in \{1, 2\}$ such that $E_i \leq E_0$ and put v in S' if and only if $i = 1$. Update $S \leftarrow S + v$ and repeat.

Eventually the set S exhausts all elements of V. When that happens, there is a deterministically chosen set $S' = V' \subseteq V$ for which $N = 0$ (since $N = o(1)$ is an integer). It follows that $|V'| \in [r, r + r^{2/3}]$ and every pair of vertices in V' does not satisfy (5) for any $(j, l) \in [k]^2$. Remove an arbitrary set of vertices from V' so that we get $|V'| = r$ and let the induced colored graph define the coloring of K_r.

In practice, to speed up the process, once a vertex v is chosen to be part of S', every vertex which forms a bad pair with v is known not to belong to S' and we can update S to contain all these vertices while S' only gets v.

5. Internally disjoint rainbow paths

The powerful construction of Bourgain's extractor resembles a random structure in such a way that a greedy algorithm to find disjoint k-rainbow paths, which can easily be seen to work (almost surely) in a random coloring, also works with this constructive coloring.

Proof of Theorem 3. After the derandomization of Section 4.2, we obtain a coloring of K_r (with $V' = V(K_r)$) in which, for every pair $v \neq w$ and every pair of colors $(j, l) \in [k]^2$,

$$\left| \Gamma(w, j) \cap \Gamma(v, l) \cap V' \right| \geq (1 - o(1)) \frac{r}{k^2}. \tag{6}$$

Observe that this also implies an upper bound $\left| \Gamma(w, j) \cap \Gamma(v, l) \cap V' \right| \leq \left(1 + o(1)\right) r/k^2$ for all $(j, l) \in [k]^2$.

For simplicity, in this proof we shall consider the case where the length l of the rainbow paths is equal to k. The same proof yields the result for all $3 \leq l \leq k$ with obvious modifications.

Let us fix some pair $v \neq w$ and check that there are many disjoint k-rainbow paths between them. In order to find those paths we use greedy Algorithm 1. We first classify vertices according to the color of the edges connecting them to both v

and w: if the edge xv has color j and the edge xw has color l, we put x in X_{jl}. We wish to use vertices from those classes in a uniform manner, so that they always have roughly the same cardinality. This is done by choosing the $k-1$ largest classes $X_{j_1 l_1}, \ldots, X_{j_{k-1} l_{k-1}}$ and reordering them[4] so that $j_1 \neq l_{k-1}$. By Lemma 15, if all classes have at least $2k2^{(1/2-\nu)n}$ elements, we may find a sequence of vertices $v_m \in X_{j_m, l_m}$ for $m = 1, \ldots, k-1$, $v_0 = v$, $v_k = w$ such that the edges $v_i v_{i+1}$ have all distinct colors.

To simplify the notation, we denote $X \cup \{x\}$ by $X + x$ and, similarly, $X \setminus \{x\}$ is denoted $X - x$.

Algorithm 1: Finding disjoint paths

Input: vertices $v \neq w$.

1    $\mathcal{P} \leftarrow \emptyset$;

2   **foreach** $(j, l) \in [k]^2$ **do**

3      $X_{jl} \leftarrow \Gamma(v, j) \cap \Gamma(w, l) \cap V'$;

4   **while** $\min_{(j,l) \in [k]^2} |X_{jl}| \geq k2^{1+(1/2-\nu)n}$ **do**

5   let $X_{j_1 l_1}, \ldots, X_{j_{k-1} l_{k-1}}$ be some collection of $k-1$ sets satisfying $j_1 \neq l_{k-1}$ and having maximum $\sum_{m=1}^{k-1} |X_{j_m l_m}|$;

6      $Y_{k-1} \leftarrow X_{j_{k-1} l_{k-1}}$;

7   **for** $m \leftarrow k-2$ **downto** 1 **do**

8        $Y_m \leftarrow \left\{ x \in X_{j_m l_m} \ : \ \Gamma(x, j) \cap Y_{m+1} \neq \emptyset \text{ for all } j = 1, \ldots, k \right\}$;

9   **if** $|Y_m| < k2^{(1/2-\nu)n}$ **then**

10   **abort** ;

11   pick $v_1 \in Y_1$;

12      $X_{j_1 l_1} \leftarrow X_{j_1 l_1} - v_1$;

13      $K \leftarrow \{j_1, l_{k-1}\}$;

14   **for** $m = 2$ **to** $k-1$ **do**

15        $c \leftarrow \min([k] \setminus K)$;

16        $K \leftarrow K + c$;

17   pick $v_m \in \Gamma(v_{m-1}, c) \cap Y_m$;

18        $X_{j_m l_m} \leftarrow X_{l_m l_m} - v_m$;

19      $\mathcal{P} \leftarrow \mathcal{P} + vv_1 v_2 \ldots v_{k-1} w$;

It is straightforward to check that Algorithm 1 either aborts or obtains a collection of internally disjoint rainbow k-paths. Let us first prove that the algorithm does not abort (when the coloring is given by our construction).

The only way this algorithm aborts is if there are two sets, $Y = Y_{m+1}$ and $X = X_{j_m l_m}$ with $|X| > 2k2^{(1/2-\nu)n}$ and $|Y| \geq k2^{(1/2-\nu)n}$ such that less than $k2^{(1/2-\nu)n}$ vertices $x \in X$ have edges of all colors going to Y. But this is a contradiction with Lemma 15.

To prove that the path system $\mathcal{P}$ is large, we observe that by selecting the largest classes on line 5, the cardinalities of the sets X_{jl} become balanced. This is formalized, when $k \geq 4$, by Claim 2. A similar claim holds for $k = 3$ with some slight modifications on the argument.

Claim 2. *Suppose that there is a collection of k^2 positive numbers such that the maximum difference between them is at most Δ. If a procedure takes the $k-1$ largest elements and decreases them by one at each step, after at most $\Delta(k+3)$ steps, the maximum difference between any pair becomes bounded by* 1.

Proof. Let M_i be the largest element and m_i be the smallest element at the ith step. We denote by $T_{i,j}$ the jth greatest element at the ith step.

At any given step i, we have the following possibilities:

(i) $M_i \leq m_i + 1$: the next step also satisfies case (i);

(ii) $M_{i+1} = M_i \geq m_i + 2$ and $m_{i+1} = m_i$: this can only happen if $T_{i,k} = M_i$. Note that (ii) can hold for at most $k+1$ steps (since each time it happens, $k-1$ numbers equal to M_i are decreased).

(iii) $M_{i+1} = M_i - 1 \geq m_i + 1$ and $m_{i+1} = m_i - 1$: in this case we must have $T_{i,k-1} = T_{i,k} = \cdots = T_{i,k^2} = m_i$. Observe that a step in which (iii) holds cannot be followed by a step in which (ii) or (iii) holds.

(iv) $M_i \geq m_i + 2$, $T_{i,k} < M_i$ and $T_{i,k-1} > m_i$: in this case, we have $M_{i+1} = M_i - 1$ and $m_{i+1} = m_i$.

Before we reach case (i), there can be at most $k+2$ steps consecutive steps in which only either (ii) or (iii) occurs. Every time (iv) occurs, the difference between the largest and smallest element decreases by 1. Hence, in at most $\Delta(k+3)$ steps we must reach case (i). This completes the proof of Claim 2. □

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[4] The case $k = 3$ (or $l = 3$ in the general case) is slightly more complicated and we may have to replace one of the classes by a smaller one to find such ordering.

Note that, for the family of sets $\{X_{jl}\}_{(j,l)\in[k]^2}$, the initial difference is

$$\Delta = \max_{(j,l)\in[k]^2} |X_{jl}| - \min_{(j,l)\in[k]^2} |X_{jl}| = o(r/k^2).$$

Hence, after $o(r/k)$ steps, the sets are all balanced. In particular, since the procedure described by Claim 2 never increases the maximum difference, the condition of the **while** loop (line 4) remains true throughout the balancing process. This shows that, when this loop finishes, every X_{jl} has cardinality at most $k2^{1+(1/2-\nu)n} + 1$. Therefore, $o(r)$ vertices remain in $\bigcup X_{jl}$, all of the other vertices are used in (internally disjoint) k-paths of $\mathcal{P}$, thus proving the theorem. $\quad\square$

From the same constructive edge-coloring, we get Theorem 2.

Proof of Theorem 2. It suffices to observe that the construction of Theorem 3 satisfies Eq. (6) for every pair of vertices and every pair of colors. In particular, summing the left side of the inequality (6) over $j \neq l$, we are counting the number of bi-chromatic paths of length 2 between v and w. The same sum on the right side of the inequality results in $k(k-1)\left(1-o(1)\right)\frac{r}{k^2}$ and therefore we conclude the proof of the theorem. $\quad\square$

Acknowledgements

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