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Existence and multiple solutions for a second-order difference boundary value problem via critical point theory ☆

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Abstract

In this paper, the critical point theory is employed to establish existence and multiple solutions for a second-order difference boundary value problem.

Keywords: Second-order; Difference equation; Boundary value problem; Existence and multiple solutions; Critical points

1. Introduction

Let **R** and **Z** be the sets of real numbers and integers, respectively. For any $a, b \in \mathbf{Z}$, $a < b$, denote $[a, b] = \{a, a + 1, \dots, b\}$. Assume that N is a given positive integer with $N > 2$. We consider the second-order difference boundary value problem (briefly BVP)

$$\Delta(p_{k-1}\Delta x_{k-1}) + q_k x_k + f(k, x_k) = 0, \quad k \in [1, N], \tag{1.1}$$

$$x_0 = x_N, \qquad p_0\Delta x_0 = p_N \Delta x_N. \tag{1.2}$$

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As usual, Δ denotes the forward difference operator defined by $\Delta x_k = x_{k+1} - x_k$, $\Delta^2 x_k = \Delta(\Delta x_k)$, p_k and $q_k \in \mathbf{R}$ for all $k \in [1, N]$. We will assume throughout this paper that

(A) For any $k \in [1, N]$, $f(k, \cdot) : [1, N] \times \mathbf{R} \to \mathbf{R}$ is continuous.
(B) $p_N \neq 0$.

Early in 1999, by employing a fixed point theorem in cone, the existence of positive periodic solutions for the BVP was investigated by Atici and Guseinov [1]. In 2003, by using the upper and lower solution method, Atici and Cabada [2] considered Eq. (1.1) with $p_k = 1$ subject to the boundary value condition

$$x_0 = x_N, \qquad \Delta x_0 = \Delta x_N$$

and obtained the existence and uniqueness results. See [1,2] for more details.

There are many other literature dealing with the similar second-order difference equation subject to various boundary value conditions. We refer to [1–8] and references therein. However, we note that these results were usually obtained by analytic techniques and various fixed point theorems. For example, the upper and lower solution method [5–7], the conical shell fixed point theorems [1,3], the Brouwer and Schauder fixed point theorems [2,4,7], topological degree theory [8]. As we know, the critical point theory has played an important role in dealing with the existence and multiple results for differential equations, which include the boundary value problems, please see [9–11] and references given therein. However, few research has been done to use such a powerful tool to handle the difference BVP. Very recently, in [12], Agarwal, Perera and O'Regan have employed the mountain pass lemma to study the following discrete equation

$$\Delta^2 y(k-1) + f\big(k, y(k)\big) = 0, \quad k \in [1, N],$$

under Dirichlet boundary value conditions, and have obtained the existence of multiple solutions. To the best of our knowledge, this work is the one among a few which deal with difference problems via variational method.

The aim of this paper is to apply some basic theorems in critical point theory to establish existence and multiple results. For a fuller treatment on critical point theory used here we refer the reader to [13,14].

2. Auxiliary results and variational framework

Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$. A critical point of J is a point $x_0 \in E$ where $J'(x_0) = 0$ and a critical value is a number c such that $J(x_0) = c$. The following is the definition of Palais–Smale condition.

Definition 2.1. We say that J satisfies the Palais–Smale condition if every sequence $\{x_n\} \subset E$, such that $J(x_n)$ is bounded and $J'(x_n) \to 0$ as $n \to +\infty$, has a converging subsequence.

The following lemmas play an important role in proving our main results.

Lemma 2.1. [13, p. 423] *Let E be a real reflexive Banach space, and let J be weakly lower (upper) semicontinuous such that*

$$\lim_{\|x\| \to \infty} J(x) = +\infty \qquad \Big(\lim_{\|x\| \to \infty} J(x) = -\infty\Big).$$

Then, there exists $x_0 \in E$ such that

$$J(x_0) = \inf_{x \in E} J(x) \qquad \Bigl(J(x_0) = \sup_{x \in E} J(x)\Bigr).$$

Furthermore, if J has bounded linear Gâteaux derivative, then $J'(x_0) = 0$.

Denote by θ the zero of E and by S^{n-1} the $(n-1)$-dimensional unit sphere; we have

Lemma 2.2. [14, p. 53] *Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$ with J even, bounded from below and satisfying the P–S condition. Suppose $J(\theta) = 0$, and there is a set $K \subset E$ such that K is homeomorphic to S^{n-1} by an odd map, and $\sup_K J < 0$. Then J possesses at least n distinct pairs of critical points.*

We will now establish the variational functional of BVP.

Let $\mathbf{R}^N$ be the N-dimensional Hilbert space with the usual inner product and the usual norm

$$(x, y) = \sum_{i=1}^{N} x_i y_i, \quad \|x\| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbf{R}^N.$$

Define a functional J on $\mathbf{R}^N$ as

$$J(x) = \frac{1}{2} \sum_{k=1}^{N} \bigl[-p_{k-1}|\Delta x_{k-1}|^2 + q_k |x_k|^2 + 2F(k, x_k)\bigr], \tag{2.1}$$

where $F(k, u) = \int_0^u f(k, t)\, dt$ and $x_0 = x_N$. Obviously, $J(\theta) = 0$. Let

$$E = \bigl\{\chi \mid \chi = (x_0, x_1, \ldots, x_N, x_{N+1}),\ \text{where } x_0 = x_N,\ p_N \Delta x_N = p_0 \Delta x_0\bigr\}.$$

Then by the assumption (B) it is easy to see that E is isomorphic to $\mathbf{R}^N$. In the following, when we say $x \in \mathbf{R}^N$, we always imply that x can be extended to $\chi \in E$ if it is necessary. Now we claim that *if $x^T = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$ is a critical point of J, then $\chi = (x_0, x_1, \ldots, x_N, x_{N+1}) \in E$ is precisely a solution of BVP.*

Indeed, for every $u, v \in \mathbf{R}^N$ it results

$$F(k, u + sv) = F(k, u) + \int_u^{u+sv} f(k, t)\, dt = F(k, u) + svf(k, u + \theta_k sv), \quad \theta_k \in (0, 1).$$

Moreover, for any $x, y \in \mathbf{R}^N$, we have

$$\begin{aligned} J(x + sy) = J(x) &+ s \sum_{k=1}^{N} \bigl[-p_{k-1} \Delta x_{k-1} \Delta y_{k-1} + q_k x_k y_k + y_k f(k, x_k + \theta_k s y_k)\bigr] \\ &+ \frac{s^2}{2} \sum_{k=1}^{N} \bigl[-p_{k-1} |\Delta y_{k-1}|^2 + q_k |y_k|^2\bigr]. \end{aligned}$$

Hence

$$\bigl(J'(x), y\bigr) = \sum_{k=1}^{N} \bigl[-p_{k-1} \Delta x_{k-1} \Delta y_{k-1} + q_k x_k y_k + y_k f(k, x_k)\bigr]. \tag{2.2}$$

On the other hand, we have from the boundary value condition $p_N\Delta x_N = p_0\Delta x_0$, $y_N = y_0$ that

$$\begin{aligned}\sum_{k=1}^{N}\big[\Delta(p_{k-1}\Delta x_{k-1})\big]y_k &= \sum_{k=1}^{N}(p_k\Delta x_k y_k - p_{k-1}\Delta x_{k-1}y_k)\\ &= -\sum_{k=1}^{N-1} p_k\Delta x_k\Delta y_k + p_N\Delta x_N y_N - p_0\Delta x_0 y_1\\ &= -\sum_{k=1}^{N} p_{k-1}\Delta x_{k-1}\Delta y_{k-1}.\end{aligned}$$

So, if $J'(x) = 0$, then from (2.2) we have

$$\sum_{k=1}^{N}\big[\Delta(p_{k-1}\Delta x_{k-1}) + q_k x_k + f(k, x_k)\big]y_k = 0.$$

Note that $y \in \mathbf{R}^N$ is arbitrary, hence we obtain

$$\Delta(p_{k-1}\Delta x_{k-1}) + q_k x_k + f(k, x_k) = 0, \quad k \in [1, N].$$

Therefore, if x is a critical point of J, then χ is exactly the solution of BVP.

3. Main results

Now we establish the existence of at least one solution to problem BVP.

Theorem 3.1. *Assume that there exists a constant δ with $\delta > 4p - q$ such that*

$$\inf_{k\in[1,N]}\lim_{u\to\infty}\frac{f(k,u)}{u} \geqslant \delta, \tag{3.1}$$

where

$$p = \max_{k\in[1,N]}\big\{|p_k|\big\}, \qquad q = \min_{k\in[1,N]}\{q_k\}. \tag{3.2}$$

Then BVP has at least one solution.

Proof. By (3.1), there exists a constant $M > 0$ such that $|u| \geqslant M$ implying

$$\frac{f(k,u)}{u} \geqslant \delta - \varepsilon, \quad \text{for } k \in [1, N], \tag{3.3}$$

where $\varepsilon = \frac{1}{2}(\delta - 4p + q) > 0$.

We first consider the case $u \geqslant 0$. If $u > M$, we have from (3.3) that $f(k,u) \geqslant (\delta - \varepsilon)u$, and then

$$\begin{aligned}F(k,u) = \int_0^u f(k,s)\,ds &= \int_0^M f(k,s)\,ds + \int_M^u f(k,s)\,ds\\ &\geqslant \int_0^M f(k,s)\,ds + \int_M^u (\delta-\varepsilon)s\,ds\end{aligned}$$

$$= \frac{\delta-\varepsilon}{2}\left(u^2 - M^2\right) + \int_0^M f(k,s)\,ds$$
$$\geqslant \frac{\delta-\varepsilon}{2}u^2 + C_1,$$

where

$$C_1 = \inf_{k\in[1,N]}\left\{\int_0^M f(k,s)\,ds - \frac{\delta-\varepsilon}{2}M^2\right\}.$$

If $0 \leqslant u \leqslant M$, then

$$F(k,u) = \frac{\delta-\varepsilon}{2}u^2 + F(k,u) - \frac{\delta-\varepsilon}{2}u^2$$
$$\geqslant \frac{\delta-\varepsilon}{2}u^2 + \inf_{\substack{0\leqslant u\leqslant M\\ k\in[1,N]}}\left\{\int_0^u f(k,s)\,ds - \frac{\delta-\varepsilon}{2}u^2\right\},$$

so there is a constant C_2 such that

$$F(k,u) \geqslant \frac{\delta-\varepsilon}{2}u^2 + C_2, \quad u \geqslant 0.$$

If $u \leqslant 0$, by a similar argument we also obtain that there is a constant C_3 such that

$$F(k,u) \geqslant \frac{\delta-\varepsilon}{2}u^2 + C_3.$$

Thus we get $C \in \mathbf{R}$ such that

$$F(k,u) \geqslant \frac{\delta-\varepsilon}{2}u^2 + C, \quad \forall u \in \mathbf{R},\ k \in [1,N], \tag{3.4}$$

where C is a constant.

Moreover, for every $x \in \mathbf{R}^N$, we have

$$J(x) = -\frac{1}{2}\sum_{k=1}^N p_{k-1}|\Delta x_{k-1}|^2 + \frac{1}{2}\sum_{k=1}^N q_k|x_k|^2 + \sum_{k=1}^N F(k,x_k)$$
$$\geqslant -\frac{p}{2}\sum_{k=1}^N |\Delta x_{k-1}|^2 + \frac{q}{2}\sum_{k=1}^N |x_k|^2 + \sum_{k=1}^N F(k,x_k)$$
$$\geqslant -\frac{p}{2}\sum_{k=1}^N \left(|x_k| + |x_{k-1}|\right)^2 + \frac{q}{2}\sum_{k=1}^N |x_k|^2 + \frac{\delta-\varepsilon}{2}\sum_{k=1}^N |x_k|^2 + NC$$
$$\geqslant -2p\sum_{k=1}^N |x_k|^2 + \frac{q}{2}\sum_{k=1}^N |x_k|^2 + \frac{\delta-\varepsilon}{2}\sum_{k=1}^N |x_k|^2 + NC$$
$$= \left(\frac{\delta-\varepsilon}{2} - 2p + \frac{q}{2}\right)\|x\|^2 + NC$$
$$= \frac{\varepsilon}{4}\|x\|^2 + NC.$$

Note that $\varepsilon > 0$, thus $J(x) \to +\infty$ as $\|x\| \to +\infty$, i.e., $J(x)$ is a coercive map. In view of Lemma 2.1, we know that there exists at least one $\tilde{x} \in \mathbf{R}^N$ such that $J'(\tilde{x}) = 0$, hence BVP has at least one solution. We complete the proof. □

Corollary 3.1. *Suppose that*

(1) $uf(k,u) \geqslant 0$ *holds for* $|u|$ *large sufficiently*;
(2) *there exist* $\delta_1 > 0$ *and* $\alpha > 1$ *such that*

$$\inf_{k\in[1,N]} \lim_{|u|\to+\infty} \frac{|f(k,u)|}{|u|^\alpha} \geqslant \delta_1. \tag{3.5}$$

Then BVP has at least one solution.

Proof. By conditions (1) and (2), it is easy to see that

$$\inf_{k\in[1,N]} \lim_{u\to\infty} \frac{f(k,u)}{u} = +\infty,$$

so the assumption of Theorem 3.1 is satisfied, and the conclusion comes from Theorem 3.1 immediately. □

Remark. Suppose that the conditions in Theorem 3.1 are satisfied and that there exists $k \in [1, N]$ such that $f(k,0) \neq 0$, then BVP has at least one nonzero solution.

In order to establish our next theorem, we need to give some notations. We rewrite $J(x)$ as follows:

$$J(x) = -\frac{1}{2}x^T Ax + \frac{1}{2}\sum_{k=1}^{N} q_k|x_k|^2 + \sum_{k=1}^{N} F(k,x_k), \tag{3.6}$$

where

$$A = \begin{pmatrix} p_0+p_1 & -p_1 & 0 & \cdots & 0 & -p_0 \\ -p_1 & p_1+p_2 & -p_2 & \cdots & 0 & 0 \\ 0 & -p_2 & p_2+p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{N-2}+p_{N-1} & -p_{N-1} \\ -p_0 & 0 & 0 & \cdots & -p_{N-1} & p_{N-1}+p_0 \end{pmatrix}_{N\times N}.$$

and x^T denotes the transpose of x.

It is clear that 0 is an eigenvalue of A. Let $\eta = (1,1,\ldots,1)^T$ and $Y = \operatorname{span}\{\eta\}$. Evidently the eigenspace of A associated to 0 is Y. Set

$$A_{N-1} = \begin{pmatrix} p_0+p_1 & -p_1 & 0 & \cdots & 0 & 0 \\ -p_1 & p_1+p_2 & -p_2 & \cdots & 0 & 0 \\ 0 & -p_2 & p_2+p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{N-3}+p_{N-2} & -p_{N-2} \\ 0 & 0 & 0 & \cdots & -p_{N-2} & p_{N-2}+p_{N-1} \end{pmatrix}_{(N-1)\times(N-1)}.$$

Assume that $\{p_k\}_{k=0}^N$ satisfies $p_k > 0$ for $k \in [0, N-1]$, then it is easy to check that A_{N-1} is positive-definite. Hence $\operatorname{rank} A = \operatorname{rank} A_{N-1} = N-1$, which implies that A are positive semi-definite and all eigenvalues of A is positive except for 0. Denote these eigenvalues of A by $\lambda_1, \lambda_2, \ldots, \lambda_N$, where $\lambda_i > 0$, $i = 1, 2, \ldots, N-1$; $\lambda_N = 0$.

Corresponding to each eigenvalue λ_i, $i = 1, 2, \ldots, N$, there exist eigenvectors η_i ($i = 1, 2, \ldots, N$) such that $A\eta_j = \lambda_j \eta_j$ and

$$(\eta_i, \eta_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad i, j = 1, 2, \ldots, N.$$

Let $X = \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_{N-1}\}$, then $\mathbf{R}^N = X \oplus Y$. Thus for any $x \in \mathbf{R}^N$, there exists unique $\{b_j\}_{j=1}^N$, such that $x = \sum_{j=1}^N b_j \eta_j$. Moreover, it is clear that $\|x\|^2 = \sum_{j=1}^N b_j^2$. Let $\lambda = \min_{i \in [1, N-1]}\{\lambda_i\}$, $\bar{\lambda} = \max_{i \in [1, N]}\{\lambda_i\}$, then $\bar{\lambda} \geqslant \lambda > 0$, and

$$x^T A x = \sum_{j=1}^N \lambda_j b_j^2 \leqslant \bar{\lambda}\|x\|^2. \tag{3.7}$$

Lemma 3.1. *Assume that $p_k > 0$ for $k \in [0, N-1]$, and*

(i) *there exist constants $\beta_1, \beta_2, \ldots, \beta_N$ such that* $\lim_{u \to \infty} \frac{f(k,u)}{u} \geqslant \beta_k$, $k \in [1, N]$;
(ii) $r := \min_{k \in [1, N]}\{q_k + \beta_k\} > \bar{\lambda}$.

Then J satisfies the P–S condition.

Proof. Suppose that $\{x^{(n)}\}_{n=1}^\infty \subset \mathbf{R}^N$ with $\{J(x^{(n)})\}$ is bounded and $J'(x^{(n)}) \to 0$ as $n \to +\infty$. By (2.2) we find

$$\begin{aligned}\bigl(J'(x), x\bigr) &= -\sum_{k=1}^N p_{k-1}|\Delta x_{k-1}|^2 + \sum_{k=1}^N \bigl[q_k|x_k|^2 + x_k f(k, x_k)\bigr] \\ &= -x^T A x + \sum_{k=1}^N \bigl[q_k|x_k|^2 + x_k f(k, x_k)\bigr],\end{aligned}$$

hence

$$\begin{aligned}\sum_{k=1}^N \bigl[q_k\bigl|x_k^{(n)}\bigr|^2 + x_k^{(n)} f\bigl(k, x_k^{(n)}\bigr)\bigr] &= \bigl(x^{(n)}\bigr)^T A x^{(n)} + \bigl(J'\bigl(x^{(n)}\bigr), x^{(n)}\bigr) \\ &\leqslant \bar{\lambda}\bigl\|x^{(n)}\bigr\|^2 + \bigl(J'\bigl(x^{(n)}\bigr), x^{(n)}\bigr).\end{aligned} \tag{3.8}$$

We have from (i) that there exists a constant $C > 0$ such that

$$f(k, u)u \geqslant (\beta_k - \varepsilon_0)u^2 - C, \quad \forall u \in \mathbf{R},$$

where $\varepsilon_0 = \frac{r - \bar{\lambda}}{2}$, thus

$$\begin{aligned}\sum_{k=1}^{N}\bigl[q_k\bigl|x_k^{(n)}\bigr|^2+x_k^{(n)}f\bigl(k,x_k^{(n)}\bigr)\bigr] &\geqslant \sum_{k=1}^{N}(q_k+\beta_k-\varepsilon_0)\bigl|x_k^{(n)}\bigr|^2-NC\\ &\geqslant (r-\varepsilon_0)\sum_{k=1}^{N}\bigl|x_k^{(n)}\bigr|^2-NC\\ &=\frac{r+\bar\lambda}{2}\bigl\|x^{(n)}\bigr\|^2-NC.\end{aligned}$$

In view of (3.8) we obtain

$$\frac{r+\bar\lambda}{2}\bigl\|x^{(n)}\bigr\|^2-NC\leqslant\bar\lambda\bigl\|x^{(n)}\bigr\|^2+\bigl(J'\bigl(x^{(n)}\bigr),x^{(n)}\bigr)$$

or

$$\frac{r-\bar\lambda}{2}\bigl\|x^{(n)}\bigr\|^2\leqslant\bigl(J'\bigl(x^{(n)}\bigr),x^{(n)}\bigr)+NC\leqslant\bigl\|J'\bigl(x^{(n)}\bigr)\bigr\|\bigl\|x^{(n)}\bigr\|+NC.$$

Note that $J'(x^{(n)})\to 0$ as $n\to\infty$ and $\frac{r-\bar\lambda}{2}>0$, so $\{x^{(n)}\}$ is bounded. Since $\mathbf{R}^N$ is N-dimensional Hilbert space, the above argument implies that there exists a converging subsequence of $\{x^{(n)}\}$, thus the proof of Lemma 3.1 is completed. □

Theorem 3.2. *Assume that $f(k,u)$ is odd on its second variable u, and the conditions of Lemma* 3.1 *hold. In addition, suppose that*

(iii) *there exist constants $\mu_1,\mu_2,\dots,\mu_N$ such that* $\lim_{u\to 0}\frac{f(k,u)}{u}\leqslant\mu_k$, $k\in[1,N]$;
(iv) $\bar r:=\max_{k\in[1,N]}\{q_k+\mu_k\}<\lambda$.

Then BVP has at least $2(N-1)$ *distinct solutions.*

Proof. Since $f(k,u)$ is odd in u, then J is an even functional. Moreover, according to Lemma 3.1, J satisfies the P–S condition.

Let $\varepsilon_1=\frac{\lambda-\bar r}{2}$, it follows from condition (iii) that there exists $\rho>0$ such that

$$\frac{f(k,u)}{u}\leqslant\mu_k+\varepsilon_1,\quad \text{for } |u|\leqslant\rho,$$

and therefore

$$F(k,u)\leqslant\frac{\mu_k+\varepsilon_1}{2}u^2,\quad |u|\leqslant\rho. \tag{3.9}$$

For any $x=(x_1,x_2,\dots,x_N)\in X$, we have $x=\sum_{i=1}^{N-1}b_i\eta_i$ and $\|x\|=(\sum_{i=1}^{N-1}b_i^2)^{\frac{1}{2}}$, so

$$x^TAx=\sum_{i=1}^{N-1}\lambda_ib_i^2\geqslant\lambda\sum_{i=1}^{N-1}b_i^2=\lambda\|x\|^2. \tag{3.10}$$

Consequently, if $\|x\|\leqslant\rho$, then

$$\begin{aligned} J(x) &= -\frac{1}{2}x^T Ax + \frac{1}{2}\sum_{k=1}^{N} q_k|x_k|^2 + \sum_{k=1}^{N} F(k, x_k) \\ &\leqslant -\frac{\lambda}{2}\|x\|^2 + \frac{q_k + \mu_k + \varepsilon_1}{2}\sum_{k=1}^{N}|x_k|^2 \\ &\leqslant -\frac{\lambda}{2}\|x\|^2 + \frac{\bar{r} + \varepsilon_1}{2}\|x\|^2 \\ &= -\frac{\lambda - \bar{r} - \varepsilon_1}{2}\|x\|^2 = -\frac{\varepsilon_1}{2}\|x\|^2. \end{aligned} \tag{3.11}$$

Let $\sigma = \frac{\varepsilon_1}{2}\rho^2$, then $J(x) \leqslant -\sigma < 0$ for $x \in K$, where

$$K = \{x \in X \mid \|x\| = \rho\}.$$

It is clear that K is homeomorphic to S^{N-2} by odd map.

On the other hand, by condition (i), it is easy to see that there exists a constant C such that

$$F(k, u) \geqslant \frac{\beta_k - \varepsilon_0}{2}u^2 - C, \quad \forall u \in \mathbf{R}, \tag{3.12}$$

where $\varepsilon_0 = \frac{r-\bar{\lambda}}{2}$.

For any $x \in \mathbf{R}^N$, we have from (3.12) that

$$\begin{aligned} J(x) &= -\frac{1}{2}x^T Ax + \frac{1}{2}\sum_{k=1}^{N} q_k|x_k|^2 + \sum_{k=1}^{N} F(k, x_k) \\ &= -\frac{1}{2}\sum_{k=1}^{N}\lambda_k b_k^2 + \frac{1}{2}\sum_{k=1}^{N} q_k|x_k|^2 + \sum_{k=1}^{N} F(k, x_k) \\ &\geqslant -\frac{\bar{\lambda}}{2}\sum_{k=1}^{N} b_k^2 + \frac{1}{2}\sum_{k=1}^{N}(q_k + \beta_k - \varepsilon_0)x_k^2 - NC \\ &\geqslant -\frac{\bar{\lambda}}{2}\|x\|^2 + \frac{r - \varepsilon_0}{2}\|x\|^2 - NC \\ &= \frac{\varepsilon_0}{2}\|x\|^2 - NC. \end{aligned} \tag{3.13}$$

Thus $\inf_{x \in R^N} J(x) > -\infty$. By Lemma 2.2 and the above result, J possesses at least $(N-1)$ distinct pairs of critical points, i.e., BVP has at least $2(N-1)$ distinct solutions. □

Corollary 3.2. *Assume that $f(k,-u) = -f(k,u)$ and $p_k > 0$, $k = 0, 1, \ldots, N-1$. In addition, suppose that*

(a) *conditions* (1), (2) *in Corollary* 3.1 *hold*;
(b) $\lim_{u \to 0} \frac{f(k,u)}{u} \leqslant 0$;
(c) $\max_{k \in [1,N]}\{q_k\} < \lambda$.

Then the BVP possesses at least $2(N-1)$ distinct solutions.

Proof. Let $q = \min_{k\in[1,N]}\{q_k\}$, $\beta = 2\bar{\lambda} - q$, by assumptions (1), (2) in Corollary 3.1, it is obvious that

$$\inf_{k\in[1,N]} \lim_{|u|\to+\infty} \frac{f(k,u)}{u} \geqslant \beta.$$

Let $\beta_k = \beta$ $(k = 1, 2, \ldots, N)$, then $r = \min_{k\in[1,N]}\{q_k + \beta_k\} = \beta + q = 2\bar{\lambda} > \bar{\lambda}$. So both (i) and (ii) in Lemma 3.1 are satisfied. Moreover, by conditions (b) and (c) it is easy to verify that (iii), (iv) are true. Thus the conclusion comes from Theorem 3.2 immediately. □

We can get from Corollary 3.2 directly

Corollary 3.3. *Assume that $p_k > 0$, $q_k \leqslant 0$ and $f(t,u) = \sum_{i=1}^{m} g(t)|u|^{\alpha_i} \operatorname{sgn} u$, where $g(t)$ is continuous with $g(t) > 0$ and $\alpha_i > 1$ for each $i \in [1, N]$. Then the BVP possesses at least $2(N-1)$ distinct solutions.*

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